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The Martin boundary of the half disk with rotation free densities

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The purpose of this paper is to determine the Martin compactification $(\Omega^+)_P^*$ of the upper half unit disk $\Omega^+=\{|z|<1, \operatorname{Im} z>0\}$ with respect to the equation $\Delta u=Pu$ with a rotation free nonnegative locally Hölder continuous coefficient $P(z)$ on $\{0<|z|\leqq 1\}$.

Before stating our result more precisely we start by fixing terminologies. We denote by Ω the punctured unit disk $\{0<|z|<1\}$. By a density P on Ω we mean a nonnegative locally Hölder continuous function on $\bar{\Omega}-\{0\}$ $(\bar{\Omega}=\{|z|\leqq 1\})$. For a density P on Ω we consider the Martin compactification Ω_P^* ($(\Omega^+)_P^*$, resp.) of Ω (Ω^+, resp.) with respect to the equation

$$\Delta u = Pu \ (\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2) \tag{1}$$

on Ω (Ω^+, resp.). We also denote by Γ_P (Γ_P^+, resp.) the ideal boundary $\Omega_P^*-\Omega$ ($(\Omega^+)_P^*-\Omega^+$, resp.).

We are interested in the ideal boundaries $\Gamma_P(0)$ and $\Gamma_P^+(0)$ over $z=0$, i.e. the set of points ζ^* in Γ_P and Γ_P^+, respectively, satisfying that there exists a sequence $\{\zeta_n\}_1^\infty$ in Ω and Ω^+, respectively, with $\lim|\zeta_n|=0$ and $\lim\zeta_n=\zeta^*$. If $\Gamma_P(0)$ ($\Gamma_P^+(0)$, resp.) consists of a single point, we say that the *Picard principle* on Ω (Ω^+, resp.) is valied for P at $z=0$. Sufficient conditions for Picard principle on Ω at $z=0$ are given in [7], [8], [5], [9], [4], [10], [11], [14] and sufficient conditions for Picard principle on Ω^+ at $z=0$ are given in [3], [16], [1], [2], [13], [15]. We remark (cf. e.g. [1]) that the ideal boundaries $\Gamma_P^+(\xi)$ over ξ in $\partial\Omega^+=\bar{\Omega}^+-\Omega^+$ $(\bar{\Omega}^+=\{|z|\leqq 1, \operatorname{Im} z\geqq 0\})$ are pairwise disjoint and $\Gamma_P^+(\xi)$ consists of a single point for every ξ in $\partial\Omega^+-\{0\}$ since P is Hölder continuous on a neighbourhood of ξ. In the case that P is a *rotation free* density on Ω, i.e. a density P on Ω satisfying $P(z)=P(|z|)$ $(z\in\Omega)$, the ideal boundary $\Gamma_P(0)$ over $z=0$ is characterized completely by Nakai [7] in terms of the *singularity index* $\alpha(P)$ of P at $z=0$ which is a value in $[0, 1)$ depending on the singular behavior of P at $z=0$ (cf. §1):

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$$(2)\quad \begin{cases} \textit{The homeomorphism } \pi_P \textit{ from } \Omega \textit{ to } \{\alpha(P)<|z|<1\} \textit{ defined by } \pi_P(z)= \\ (\alpha(P)+(1-\alpha(P))|z|)z/|z| \ (z\in\Omega) \textit{ can be extended to a homeomorphism} \\ \textit{from } \Omega_P^* \textit{ to } \{\alpha(P)\leqq|z|\leqq 1\} \textit{ and every point in } \Gamma_P(0)=\pi_P^{-1}(\{|z|=\alpha(P)\}) \\ \textit{is minimal.} \end{cases}$$

The purpose of this paper is to show that a similar characterization is valid for Ω^+. Namely we will prove the following:

THEOREM. *Let P be a rotation free density on Ω. Then the homeomorphism π_P^+ from Ω^+ to $\{\alpha(P)<|z|<1, \operatorname{Im} z>0\}$ defined by $\pi_P^+(z)=(\alpha(P)+(1-\alpha(P))|z|)z/|z|$ can be extended to a homeomorphism from $(\Omega^+)_P^*$ to $\{\alpha(P)\leqq|z|\leqq 1, \operatorname{Im} z\geqq 0\}$ and every point in $\Gamma_P^+(0)=(\pi_P^+)^{-1}(\{|z|=\alpha(P), \operatorname{Im} z\geqq 0\})$ is minimal.*

We will recall in §1 the proof of (2) according to [7], but partly modified suitable for our purpose, which plays an essential role in the proof of the above theorem. The proof of the above theorem will be given in §2 and the characterization of the Martin compactification $(\Omega_\theta)_P^*$ of $\Omega_\theta=\{0<|z|<1, 0<\arg z<\theta\}$ $(\theta\in(0, 2\pi))$ will be given in §3 by using the above theorem.

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§1. The Martin compactification Ω_P^* of Ω

1. We give in this section an outline of the proof of (2) in [7]. Let P be a rotation free density on Ω. The unique bounded solution $e_P(z;\rho)$ $(\rho\in(0, 1])$ of (1) on $\Omega(\rho)=\{0<|z|<\rho\}$ with boundary values 1 on $\partial\Omega(\rho)=\{|z|=\rho\}$ is referred to as the *P-unit* on $\Omega(\rho)$. We consider rotation free densities P_n $(n=0, 1,\cdots)$ on Ω defined by $P_n(z)=P(z)+n^2|z|^{-2}$ $(z\in\bar{\Omega}-\{0\})$ and denote by $e_n(z;\rho)$ the P_n-unit on $\Omega(\rho)$, where we follow the convention $P_0=P$ and $e_0(z;\rho)=e_P(z;\rho)$. We simply denote by $e_n(z)$ the P_n-unit $e_n(z; 1)$ on $\Omega(1)=\Omega$. Since P_n is rotation free, $e_n(z;\rho)$ is also rotation free and the function $e_n(r;\rho)$ of r in $(0,\rho)$ is the unique bounded solution of

$$\frac{d^2}{dr^2}\phi(r)+\frac{1}{r}\frac{d}{dr}\phi(r)-P_n(r)\phi(r)=0(n=0, 1,\cdots)$$

on $(0,\rho)$ with boundary value 1 at $r=\rho$. Then P_n-units have the following fundamental properties ([7]):

$$0<e_n(r;\rho)<1 \quad (0<r<\rho\leqq 1;\ n=0, 1,\cdots),$$

$$(3)\qquad e_n(r;\rho)=e_n(r)/e_n(\rho) \quad (0<r\leqq\rho\leqq 1;\ n=0, 1,\cdots),$$

$$(4)\qquad \{e_{n+1}(r;\rho)/e_n(r;\rho)\}^3\leqq e_{n+2}(r;\rho)/e_{n+1}(r;\rho)\leqq e_{n+1}(r;\rho)/e_n(r;\rho)$$

$$(0<r\leqq\rho\leqq 1;\ n=0,1,\cdots),$$

(5) $$e_n(r;\rho)/e_0(r;\rho) < e_n(s;\rho)/e_0(s;\rho)<1 \quad (0<r<s<\rho\leqq 1;\ n=1,2,\cdots).$$

In view of (5) the limit

$$\alpha_n(P) = \lim_{r\to 0} e_n(r)/e_0(r)\ (n=1,2,\cdots)$$

exists and is referred to as the n^{th} *singularity index* of P at $z=0$. In particular we denote by $\alpha(P)$ the 1^{st} singularity index $\alpha_1(P)$ and call it simply the *singularity index* of P at $z=0$. By (4) singularity indices satisfy

(6) $$0\leqq \alpha(P)^{(3^n-1)/2}\leqq \alpha_n(P)\leqq \alpha(P)^n<1 \quad (n=1,2,\cdots).$$

2. Let $G(z,\zeta)=G_P^\Omega(z,\zeta)$ be the P-Green's function on Ω, i.e. the Green's function on Ω with respect to (1). The function $G(z, re^{i\theta})$ $(z\in\Omega,\ r\in(0,|z|))$ of θ in $[0,2\pi)$ is expanded into its Fourier series

$$G(z,re^{i\theta}) = c_0(z;r)/2+\sum_{n=1}^\infty\{c_n(z;r)\cos n\theta+s_n(z;r)\sin n\theta\}$$

by using its Fourier coefficients

$$\begin{cases} c_n(z;r)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} G(z,re^{i\theta})\cos n\theta d\theta & (n=0,1,\cdots),\\[2ex] s_n(z;r)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} G(z,re^{\theta i})\sin n\theta d\theta & (n=1,2,\cdots).\end{cases}$$

For $0<r\leqq\rho<|z|<1$, the expansion of $G(z,re^{i\theta})$ is rewritten as

(7) $$G(z,re^{i\theta})=\frac{1}{2}\frac{c_0(z;\rho)}{e_0(\rho)}e_0(r) + \sum_{n=1}^\infty\left\{\frac{c_n(z;\rho)}{e_n(\rho)}\cos n\theta+\frac{s_n(z;\rho)}{e_n(\rho)}\sin n\theta\right\}e_n(r).$$

The class $\{c_n(z;r), s_n(z;r)\}$ of functions of z in $\{r<|z|<1\}$ is linearly independent in the following sense:

LEMMA 1 ([7]). *If r is an arbitrarily fixed number in* $(0,1)$ *and* $\sum_0^\infty a_n$ *and* $\sum_1^\infty b_n$ *are absolutely convergent series with*

$$\sum_{n=0}^\infty a_nc_n(z;r)+\sum_{n=1}^\infty b_ns_n(z;r)=0$$

on a nonempty open subset of $\{r<|z|<1\}$, *then* $a_0=a_n=b_n=0$ $(n=1,2,\cdots)$.

3. Consider the function

$$L(z,\zeta)=G(z,\zeta)/e_0(\zeta)\quad (z,\zeta\in\Omega).$$

By (7), in view of (3), (4), (5) and (6), $L(z, \zeta)$ converges to the positive solution

$$L(z) = c_0(z;\rho)/2e_0(\rho) \quad (z \in \Omega,\ \rho \in (0, |z|))$$

of (1) on Ω as $\zeta \to 0$ uniformly on every compact subset of Ω if $\alpha(P)=0$ and $L(z, re^{i\theta})$ converges to the positive solution

$$(8)\quad L(z;\sigma) = \frac{1}{2}\frac{c_0(z;\rho)}{e_0(\rho)} + \sum_{n=1}^{\infty}\left\{\frac{c_n(z;\rho)}{e_n(\rho)}\cos n\sigma + \frac{s_n(z;\rho)}{e_n(\rho)}\sin n\sigma\right\}\alpha_n(P)$$

$$(z \in \Omega,\ \rho \in (0, |z|),\ \sigma \in [0, 2\pi))$$

of (1) on Ω as $r \to 0$ and $\theta \to \sigma$ uniformly on every compact subset of Ω if $\alpha(P)>0$. Now the P-Martin kernel

$$K(z, \zeta) = K_P^{\Omega}(z, \zeta) = G(z, \zeta)/G(i/2, \zeta) \quad (z, \zeta \in \Omega)$$

on Ω converges to the positive solution

$$k(z) = L(z)/L(i/2) \quad (z \in \Omega)$$

of (1) on Ω as $\zeta \to 0$ uniformly on every compact subset of Ω if $\alpha(P)=0$ and converges to the positive solution

$$k(z;\sigma) = L(z;\sigma)/L(i/2;\sigma) \quad (z \in \Omega,\ \sigma \in [0, 2\pi))$$

of (1) on Ω as $|\zeta| \to 0$ and $\arg\zeta \to \sigma$ uniformly on every compact subset of Ω if $\alpha(P)>0$. By Lemma 1 $k(z;\sigma) \not\equiv k(z;\tau)$ $(\sigma \neq \tau)$.

4. The homeomorphism

$$\pi_P(z) = (\alpha(P) + (1-\alpha(P))|z|)z/|z| \quad (z \in \Omega)$$

from Ω to $\{\alpha(P)<|z|<1\}$ can be extended to a homeomorphism from Ω_P^* to $\{\alpha(P) \leqq |z| \leqq 1\}$ and the functions $k(z)$ and $k(z;\sigma)$ $(\sigma \in [0, 2\pi))$ are the P-Martin kernels with pole at ideal boundary points $\pi_P^{-1}(0)$ and $\pi_P^{-1}(\alpha(P)e^{i\sigma})$, respectively. The supports of representing measures (cf. e.g. [6], [12], [3]) of $k(z)$ and $k(z;\sigma)$ are contained in $\Gamma_P(0)$. If $\alpha(P)=0$, then $\Gamma_P(0)$ consists of a single point so that $k(z)$ is minimal. If $\alpha(P)>0$, then $k(z;\sigma)$ $(\sigma \in [0, 2\pi))$ are all simultaneously minimal or nonminimal by the equality

$$k(z;\sigma) = k(ze^{-i\sigma};0)/k(ie^{-i\sigma}/2;0)$$

so that $k(z;\sigma)$ are minimal.

§ 2. The Martin compactification $(\Omega^+)_P^*$ of Ω^+

5. We give in this section the proof of Theorem. Since the P-Green's function $G^+(z, \zeta)=G_P^{\Omega^+}(z, \zeta)$ on Ω^+, i.e. the Green's function on Ω^+ with respect to (1), is given by

$$G^+(z, \zeta) = G(z, \zeta) - G(z, \bar{\zeta}) \quad (z, \zeta \in \Omega^+),$$

the function $G^+(z, re^{i\theta})$ of θ in $[0, \pi]$ has by (7) the following Fourier series expansion:

$$G^+(z, re^{i\theta}) = 2 \sum_{n=1}^{\infty} \frac{s_n(z; \rho)}{e_n(\rho)} e_n(r) \sin n\theta \quad (z \in \Omega^+, 0<r\leqq\rho<|z|).$$

6. We consider the function

$$L^+(z, \zeta) = \frac{G^+(z, \zeta)}{e_1(\zeta) \sin \arg \zeta} \, (z, \zeta \in \Omega^+)$$

which is expanded into the series

$$L^+(z, re^{i\theta}) = 2 \frac{s_1(z; \rho)}{e_1(\rho)} + 2 \sum_{n=2}^{\infty} \frac{s_n(z; \rho)}{e_n(\rho)} \frac{\sin n\theta}{\sin \theta} \frac{e_n(r)}{e_1(r)}$$

$$(z \in \Omega^+, \rho \in (0, |z|), r \in (0, \rho], \theta \in (0, \pi)).$$

Each term of the above series satisfies by (3) and (4) that

$$\left| \frac{s_n(z; \rho)}{e_n(\rho)} \frac{\sin n\theta}{\sin \theta} \frac{e_n(r)}{e_1(r)} \right| \leqq n \frac{c_0(z; \rho)}{e_1(\rho)} \left\{ \frac{e_1(r; \rho)}{e_0(r; \rho)} \right\}^{n-1} \tag{9}$$

Therefore in view of (5) $L^+(z, \zeta)$ converges to the nonnegative solution

$$L^+(z) = 2s_1(z; \rho)/e_1(\rho) \quad (z \in \Omega^+, \rho \in (0, |z|))$$

of (1) on Ω^+ as $\zeta \to 0$ uniformly on every compact subset of Ω^+ if $\alpha(P)=0$. The function $L^+(z)$ is positive on Ω^+ by Lemma 1. Then the P-Martin kernel

$$K^+(z, \zeta) = K_P^{\Omega^+}(z, \zeta) = G^+(z, \zeta)/G^+(i/2, \zeta) \quad (z, \zeta \in \Omega^+)$$

on Ω^+ converges to the positive solution

$$k^+(z) = L^+(z)/L^+(i/2) \quad (z \in \Omega^+)$$

of (1) on Ω^+ as $\zeta \to 0$ uniformly on every compact subset of Ω^+ if $\alpha(P)=0$. In the case that $\alpha(P)>0$ the function $L^+(z, re^{i\theta})$ converges to the nonnegative solution

$$
(10)\qquad L^+(z;\sigma)=\begin{cases} 2\sum_{n=1}^{\infty} n\,\dfrac{s_n(z;\rho)}{e_n(\rho)}\,\dfrac{\alpha_n(P)}{\alpha(P)} & (\sigma=0)\,,\\ 2\sum_{n=1}^{\infty} \dfrac{s_n(z;\rho)}{e_n(\rho)}\,\dfrac{\sin n\sigma}{\sin\sigma}\,\dfrac{\alpha_n(P)}{\alpha(P)} & (0<\sigma<\pi)\,,\\ 2\sum_{n=1}^{\infty} (-1)^{n+1} n\,\dfrac{s_n(z;\rho)}{e_n(\rho)}\,\dfrac{\alpha_n(P)}{\alpha(P)} & (\sigma=\pi)\\ \qquad\qquad (z\in\Omega^+,\ \rho\in(0,\,|z|)) & \end{cases}
$$

of (1) on Ω^+ as $r\to 0$ and $\theta\to\sigma$ uniformly on every compact subset of Ω^+. The functions $L^+(z;\sigma)$ $(\sigma\in[0,\pi])$ are positive on Ω^+ by Lemma 1. Then the P-Martin kernel $K^+(z,\zeta)$ converges to the positive solution

$$
(11)\qquad k^+(z;\sigma)=L^+(z;\sigma)/L^+(i/2;\sigma)\quad(z\in\Omega^+,\ \sigma\in[0,\pi])
$$

of (1) on Ω^+ as $|\zeta|\to 0$ and $\arg\zeta\to\sigma$ uniformly on every compact subset of Ω^+ if $\alpha(P)>0$. Again by Lemma 1 we have $k^+(z;\sigma)\not\equiv k^+(z;\tau)$ $(\sigma\neq\tau)$.

7. Positive solutions $k^+(z)$ and $k^+(z;\sigma)$ $(\sigma\in[0,\pi])$ of (1) on Ω^+ vanish on $\partial\Omega^+-\{0\}$ by the boundary Harnack principle (cf. e.g. [1]). On the other hand the P-Martin kernel $K^+(z,\zeta)$ converges to a minimal solution of (1) with vanishing boundary values on $\partial\Omega^+-\{\xi\}$ $(\xi\in\partial\Omega^+-\{0\})$ as $\zeta\to\xi$ uniformly on every compact subset of Ω^+ again by the boundary Harnack principle. Therefore, the arguments in the previous no. show that (cf. [7, 4.2 and 4.3]) the homeomorphism

$$
\pi_P^+(z)=(\alpha(P)+(1-\alpha(P))|z|)z/|z|\quad(z\in\Omega^+)
$$

from Ω^+ to $D^+(\alpha(P))=\{\alpha(P)<|z|<1,\ \mathrm{Im}\,z>0\}$ can be extended to a homeomorphism from the Martin compactification $(\Omega^+)_P^*$ of Ω^+ to $\bar{D}^+(\alpha(P))=\{\alpha(P)\leqq |z|\leqq 1,\ \mathrm{Im}\,z\geqq 0\}$, and

$$
\Gamma_P^+(0)=(\pi_P^+)^{-1}(\{|z|=\alpha(P),\ \mathrm{Im}\,z\geqq 0\})\,.
$$

8. Now $k^+(z)$ and $k^+(z;\sigma)$ $(\sigma\in[0,\pi])$ are P-Martin kernels on Ω^+ with poles at ideal boundary points $(\pi_P^+)^{-1}(0)$ and $(\pi_P^+)^{-1}(\alpha(P)e^{i\sigma})$, respectively:

$$
k^+(z)=K^+(z,(\pi_P^+)^{-1}(0)),\quad k^+(z;\sigma)=K^+(z,(\pi_P^+)^{-1}(\alpha(P)e^{i\sigma}))\,.
$$

By the boundary Harnack principle, these functions vanish on $\partial\Omega^+-\{0\}$. We show in nos. 8–10 that these P-Martin kernels are all minimal. Since $K^+(z,\zeta^*)$ vanishes at $z=0$ for $\zeta^*\in\Gamma_P^+-\Gamma_P^+(0)$, again by the boundary Harnack principle, we easily see that the support of the representing measure of $k^+(z)$ consists of a single point $(\pi_P^+)^{-1}(0)$, and hence $k^+(z)$ is minimal.

9. The minimality of $k^+(z;\sigma)$ $(\sigma\in(0,\pi))$ are derived from the minimality of $k(z;\sigma)$. Let $u^+(z)$ be a positive solution of (1) on Ω^+ with

$$u^+(z) \leqq k^+(z;\sigma) \quad (z \in \Omega^+).$$

Since $u^+(z)$ vanishes on $\partial\Omega^+ - \{0\}$, the function

$$u(z) = \begin{cases} u^+(z) & (z \in \Omega^+), \\ -u^+(\bar{z}) & (z \in \Omega - \Omega^+) \end{cases}$$

is a solution of (1) on Ω. By the expansions (8) of $L(z;\sigma)$, $L(z;2\pi-\sigma)$ and (10) of $L^+(z;\sigma)$ we have

$$L^+(z;\sigma) = \frac{1}{\alpha(P)\sin\sigma}(L(z;\sigma) - L(z;2\pi-\sigma))$$

and hence

$$k^+(z;\sigma) = C_1 k(z;\sigma) - C_2 k(z;2\pi-\sigma) \quad (z \in \Omega^+)$$

for positive constants $C_1 = C_1(P,\sigma)$ and $C_2 = C_2(P,\sigma)$. Then the solution

$$v(z) = C_1 k(z;\sigma) - u(z)$$

of (1) on Ω satisfies that

$$v(z) = \begin{cases} C_1 k(z;\sigma) - u^+(z) \geqq k^+(z;\sigma) - u^+(z) \geqq 0 & (z \in \Omega^+), \\ C_1 k(z;\sigma) + u^+(\bar{z}) \geqq 0 & (z \in \Omega - \Omega^+) \end{cases}$$

and

$$v(z) \leqq \begin{cases} C_1 k(z;\sigma) & (z \in \Omega^+), \\ C_1 k(z;\sigma) + C_1 k(\bar{z};\sigma) & (z \in \Omega - \Omega^+). \end{cases}$$

By the equality $G(\bar{z}, re^{i\sigma}) = G(z, re^{i(2\pi-\sigma)})$ we have $L(\bar{z};\sigma) = L(z;2\pi-\sigma)$ and hence

$$k(\bar{z};\sigma) = C_3 k(z;2\pi-\sigma) \quad (z \in \Omega)$$

for a positive constant $C_3 = C_3(P,\sigma)$. Therefore

$$0 \leqq v(z) \leqq C_1 k(z;\sigma) + C_1 C_3 k(z;2\pi-\sigma)$$

on Ω so that the minimality of $k(z;\sigma)$ and $k(z;2\pi-\sigma)$ yiedls the following representation of $v(z)$:

$$v(z) = C_4 k(z;\sigma) + C_5 k(z;2\pi-\sigma)$$

for nonnegative constants C_4 and C_5. Then $u^+(z)$ has a form

$$u^+(z) = (C_1 - C_4)k(z; \sigma) - C_5 k(z; 2\pi - \sigma) \quad (z \in \Omega^+).$$

We remark that $C_1 - C_4 > 0$ since $C_5 \geqq 0$ and $u^+(z) > 0$. Further we have $C_2/C_1 = C_5/(C_1 - C_4)$ since $k(x; \sigma) > 0$, $k(x; 2\pi - \sigma) > 0$, and $k^+(x; \sigma) = u^+(x) = 0$ for any x in the subset $(-1, 1) - \{0\}$ of the real axis. Hence $u^+(z)$ is proportional to $k^+(z; \sigma)$.

10. The minimality of $k^+(z; 0)$ and $k^+(z; \pi)$ are derived from Lemma 1. Let μ_0^+ be the representing measure of $k^+(z; 0)$. Since $k^+(z; 0)$ vanishes on $\partial\Omega^+ - \{0\}$ and $K^+(z, \zeta^*)$ vanishes at $z = 0$ for $\zeta^* \in \Gamma_P^+ - \Gamma_P^+(0)$, the support of μ_0^+ is contained in $(\pi_P^+)^{-1}(\{|z| = \alpha(P), \operatorname{Im} z \geqq 0\})$ as in no. 8. Assume that $k^+(z; 0)$ is nonminimal. Then there exists a measure ν_0^+ on $(0, \pi)$ such that

$$k^+(z; 0) = \int_{(0,\pi)} k^+(z; \sigma) d\nu_0^+(\sigma)$$

and hence by (10) and (11)

$$k^+(z; 0) = \int_{(0,\pi)} \frac{2}{L^+(i/2; \sigma)} \sum_{n=1}^{\infty} \frac{s_n(z; \rho)}{e_n(\rho)} \frac{\sin n\sigma}{\sin \sigma} \frac{\alpha_n(P)}{\alpha(P)} d\nu_0^+(\sigma).$$

The function $L^+(i/2; \sigma)$ of σ is positive continuous on $[0, \pi]$ since the series in (10) is uniformly convergent for $\sigma \in [0, \pi]$ by (9) and (5). Therefore $\sup_{[0,\pi]} L^+(i/2; \sigma)^{-1} < \infty$. In view of this and (9) again $k^+(z; 0)$ has the following expansion:

$$k^+(z; 0) = 2 \sum_{n=1}^{\infty} \frac{s_n(z; \rho)}{e_n(\rho)} \frac{\alpha_n(P)}{\alpha(P)} \int_{(0,\pi)} \frac{1}{L^+(i/2; \sigma)} \frac{\sin n\sigma}{\sin \sigma} d\nu_0^+(\sigma)$$
$$(z \in \Omega^+, \ \rho \in (0, |z|)).$$

Then by (10), (11), and Lemma 1 we have

$$\frac{n}{L^+(i/2; 0)} = \int_{(0,\pi)} \frac{1}{L^+(i/2; \sigma)} \frac{\sin n\sigma}{\sin \sigma} d\nu_0^+(\sigma) \quad (n = 1, 2, \cdots)$$

so that the Lebesgue theorem yields a contradiction that

$$\frac{1}{L^+(i/2; 0)} = \lim_{n\to\infty} \int_{(0,\pi)} \frac{1}{L^+(i/2; \sigma)} \frac{\sin n\sigma}{n \sin \sigma} d\nu_0^+(\sigma)$$
$$= \int_{(0,\pi)} \frac{1}{L^+(i/2; \sigma)} \lim_{n\to\infty} \frac{\sin n\sigma}{n \sin \sigma} d\nu_0^+(\sigma) = 0.$$

Thus $k^+(z; 0)$ is minimal. By symmetry, $k^+(z; \pi)$ is also minimal. Hence we have shown that any point in the ideal boundary $\Gamma_P^+(0)$ over $z = 0$ is minimal.

§3. The Martin compactification $(\Omega_\theta)^*_P$ of Ω_θ

11. We characterize in this section the Martin compactification $(\Omega_\theta)^*_P$ ($\theta \in (0, 2\pi)$) of the region $\Omega_\theta = \{0<|z|<1, 0<\arg z<\theta\}$ with respect to (1) with a rotation free density P on Ω. Consider the conformal mapping

$$\phi(z) = \phi_\theta(z) = z^{\theta/\pi} \quad (z \in \Omega^+)$$

from Ω^+ to Ω_θ. If we set $v(z) = u(\phi(z))$ ($z \in \Omega^+$) for a C^2 function u on Ω_θ, then v satisfies

$$\Delta v(z) = \left| \frac{d}{dz} \phi(z) \right|^2 (\Delta u)(\phi(z)).$$

Let Q be the rotation free density on Ω defined by

$$Q(z) = \left| \frac{d}{dz} \phi(z) \right|^2 P(\phi(z)) = \frac{\theta^2}{\pi^2} |z|^{2\theta/\pi - 2} P(z^{\theta/\pi})$$

on $\bar{\Omega}^+ - \{0\}$ and $Q(z) = Q(|z|i)$ on $\bar{\Omega} - \bar{\Omega}^+$. Then a C^2 function u on Ω_θ is a solution of (1) on Ω_θ if and only if $v = u \circ \phi$ is a solution of $\Delta v = Qv$ on Ω^+ so that ϕ is extended to a homeomorphism from $(\Omega^+)^*_Q$ to $(\Omega_\theta)^*_P$.

12. For a positive real number λ, consider the rotation free density $P_\lambda(z) = P(z) + \lambda^2 |z|^{-2}$ and denote by $e_\lambda(z)$ the P_λ-unit on Ω. In this no., we show

LEMMA 2. *If* $0 < \lambda \leqq \nu$, *then*

$$\{e_\lambda(r)/e_0(r)\}^{(\nu/\lambda)^2} \leqq e_\nu(r)/e_0(r) \leqq e_\lambda(r)/e_0(r), \quad 0 < r < 1.$$

PROOF. By the usual maximum principle (cf. e.g. [7, 1.1]), we see that $e_\nu(r) \leqq e_\lambda(r)$, which implies the second inequality.

In order to prove the first inequality, let $\kappa = (\nu/\lambda)^2 \geqq 1$ and put $F(r) = e_\lambda(r)^\kappa \cdot e_0(r)^{1-\kappa}$. Then F is a C^2 function on (0, 1) with $0 < F(r) \leqq 1$ and $F(1) = 1$. Further, we have

$$\frac{F'(r)}{F(r)} = \kappa \frac{e'_\lambda(r)}{e_\lambda(r)} - (\kappa - 1) \frac{e'_0(r)}{e_0(r)},$$

$$\begin{aligned}
\frac{F''(r)}{F(r)} &= \left\{ \frac{F'(r)}{F(r)} \right\}^2 + \left\{ \frac{F'(r)}{F(r)} \right\}' \\
&= \kappa \frac{e''_\lambda(r)}{e_\lambda(r)} - (\kappa - 1) \frac{e''_0(r)}{e_0(r)} - \kappa \frac{e'_\lambda(r)^2}{e_\lambda(r)^2} + (\kappa - 1) \frac{e'_0(r)^2}{e_0(r)^2} \\
&\quad + \left\{ \kappa \frac{e'_\lambda(r)}{e_\lambda(r)} - (\kappa - 1) \frac{e'_0(r)}{e_0(r)} \right\}^2,
\end{aligned}$$

and hence

$$\frac{F''(r)+r^{-1}F'(r)}{F(r)} = \kappa\left(P(r)+\frac{\lambda^2}{r^2}\right)-(\kappa-1)P(r)$$

$$+\kappa(\kappa-1)\left\{\frac{e'_\lambda(r)}{e_\lambda(r)}-\frac{e'_0(r)}{e_0(r)}\right\}^2 \geqq P_\nu(r)\,.$$

Therefore $F(r)\leqq e_\nu(r)$, which means that $\{e_\lambda(r)/e_0(r)\}^\kappa \leqq e_\nu(r)/e_0(r)$.

13. If we denote by $f_n(z)$ the Q_n-unit on Ω, then the observation in no. 11 shows that $f_0(r)=e_0(r^{\theta/\pi})$ and $f_1(r)=e_{\pi/\theta}(r^{\theta/\pi})$. Hence

$$\alpha(Q)=\lim_{r\to 0}\frac{f_1(r)}{f_0(r)}=\lim_{r\to 0}\frac{e_{\pi/\theta}(r)}{e_0(r)}\,.$$

Therefore, in view of Lemma 2, we see that $\alpha(Q)=0$ if and only if $\alpha(P)=0$. Thus, we have shown

COROLLARY. *Let P be a rotation free density on Ω. Then the homeomorphism $\pi_{\theta P}$ $(\theta\in(0,2\pi))$ from Ω_θ to $\{\alpha(P)<|z|<1,\ 0<\arg z<\theta\}$ defined by $\pi_{\theta P}(z)=(\alpha(P)+(1-\alpha(P))|z|)z/|z|$ can be extended to a homeomorphism from $(\Omega_\theta)^*_P$ to $\{\alpha(P)\leqq|z|\leqq 1,\ 0\leqq\arg z\leqq\theta\}$ and every point in the ideal boundary $\Gamma_{\theta P}(0)=(\pi_{\theta P})^{-1}(\{|z|=\alpha(P),\ 0\leqq\arg z\leqq\theta\})$ over $z=0$ is minimal.*

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