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Global existence of mild solutions to semilinear differential equations in Banach spaces

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Introduction

Let X be a Banach space over the real field $\boldsymbol{R}$ with norm $|\cdot|$. Let $\{S(t); t\geqq 0\}$ be a linear contraction semigroup on X of class (C_0) and let A be the infinitesimal generator of $\{S(t); t\geqq 0\}$. Let Ω be a subset of $[a, b)\times X(a<b\leqq+\infty)$ and let B be a nonlinear continuous operator from Ω into X.

In this paper we are concerned with the existence and uniqueness of global mild solutions to the initial-value problem for a semilinear differential equation in X

$$u'(t) = Au(t) + B(t, u(t)), \quad \tau < t < b, \quad u(\tau) = z, \tag{0.1}$$

where (τ, z) is given in Ω. Here by a mild solution is meant an X-valued continuous function u on the interval $[\tau, b)$ satisfying the following Volterra integral equation:

$$u(t) = S(t-\tau)z + \int_{\tau}^{t} S(t-s)B(s, u(s))ds, \quad \tau \leqq t < b. \tag{0.2}$$

In general, a mild solution may not be differentiable and hence need not be an exact solution to (0.1). But this notion is known as the most natural one of the generalized notions of solutions to (0.1). For regularity results of mild solutions, see for instance Martin [10].

Semilinear equations of type (0.1) have been studied by many authors and the present paper is related to the works of Iwamiya [1], Kato [2], [3], Kenmochi and Takahashi [4], Lakshmikantham et al [6], Lovelady and Martin [7], Martin [8], [9], Pavel [11], [12], [13], Pavel and Vrabie [14], [15] and Webb [16].

In case Ω is open, various results have been obtained by the analogy with the theory of ordinary differential equations in $\boldsymbol{R}^n$. The case in which Ω is closed has been considered in relation to so-called flow invariant sets.

For the case in which $A=0$ and equation (0.1) is understood to be an ordinary differential equation in a cylindrical domain $\Omega=[a, b)\times D$ in the product space $[a, b)\times X$, Martin established fundamental results. A properly noncylindrical case was studied by Kenmochi and Takahashi [4] and their results have been recently generalized by Iwamiya [1].

The existence and uniqueness of solutions to (0.1) has been treated by Martin [9] who considered equation (0.1) in the cylindrical case as mentioned above under the "quasi-dissipativity" condition for $B(t, x)$ for each t. The results of Martin have been extended by Pavel [12] to the case of noncylindrical domains.

In this paper we establish the global existence and uniqueness of mild solutions under the following conditions:

($\Omega 1$) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t$ in $[a, b)$ and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$.

($\Omega 2$) $\liminf_{h \downarrow 0} h^{-1} d(S(h)x + hB(t, x), \Omega(t+h)) = 0$
for all $(t, x) \in \Omega$, where $\Omega(t) = \{x \in X; (t, x) \in \Omega\}$ for $t \in [a, b)$.

($\Omega 3$) $[x-y, B(t, x) - B(t, y)]_- \leqq g(t, |x-y|)$
for all $(t, x), (t, y) \in \Omega$, where
$[x, y]_- = \lim_{h \uparrow 0} h^{-1}(|x+hy| - |x|)$ for $x, y \in X$
and g is a function from $[a, b) \times \boldsymbol{R}$ into $\boldsymbol{R}$ with the following properties:

(g1) $g(t, w)$ satisfies so-called Caratheodory's condition.

(g2) $g(t, 0) = 0$; and $w(t) \equiv 0$ is the maximal solution to the initial-value problem

$$\begin{cases} w'(t) = g(t, w(t)), \ a < t < b, \\ w(a) = 0. \end{cases}$$

We here make some brief remarks on these conditions; precise meaning of the notation appeared in them are given in Section 1.

Condition ($\Omega 1$) is a closedness condition in a certain sense for the domain. In particular, it implies that each section $\Omega(t)$ is closed. So it is equivalent to the closedness condition for the sections $\Omega(t)$ provided that Ω is cylindrical.

Condition ($\Omega 2$) is a necessary condition for the mild solutions of (0.1) to exist and it is one of the variants of so-called "subtangential" condition. Notice that in case of $A=0$ condition ($\Omega 2$) is identical with the condition

$$\liminf_{h \downarrow 0} h^{-1} d(x + hB(t, x), \Omega(t+h)) = 0 \quad \text{for all} \quad (t, x) \in \Omega.$$

Further, it should be mentioned that if (t, x) is an interior point of Ω, condition ($\Omega 2$) is always satisfied. We shall see that condition ($\Omega 2$) together with condition ($\Omega 1$) ensures the existence of approximate solutions for (0.1).

Condition ($\Omega 3$) is fulfilled if

$$|B(t, x) - B(t, y)| \leqq g(t, |x-y|) \quad \text{for} \quad (t, x), (t, y) \in \Omega$$

since $[x, y]_- \leqq |y|$. This is a familiar condition in the theory of ordinary differential equations and ensures the unicity of solutions. The operator $B(t, \cdot)$ is said to be ω-dissipative if

$$[x - y, B(t, x) - B(t, y)]_- \leqq \omega |x-y| \quad \text{for} \quad (t, x), (t, y) \in \Omega;$$

hence condition ($\Omega 3$) may be regarded as a relaxation of the "dissipativity" condition as employed in the papers cited above. The assumptions ($g1$) and ($g2$) on the function g seem to be very general, although condition ($\Omega 3$) not only guarantees the unicity of mild solutions to (0.1) but also it is fully applied to establish the convergence of the approximate solutions.

So far, the global existence has been discussed by assuming that Ω is cylindrical ([13]), or else by imposing some additional conditions on the operators A and B. In this regard we treated the regular case in which $A=0$ in the previous paper [1] under the conditions ($\Omega 1$)–($\Omega 3$) and obtained an optimal result concerning the global existence. The result is obtained without any additional conditions. In particular, if Ω is either connected or cylindrical, then the solution exists up to b for any initial data. The result states that the maximal interval of existence of solutions is determined by the connected component of the domain Ω in which the initial data lie. In fact, the verification of the result is based on the fact that the maximal interval of existence depends continuously upon initial data; and this continuous dependence implies the global existence (cf. Theorem 2.2). In this paper we shall show that the above idea is applicable to the global existence problems for a much wider class of semilinear differential equations.

The present paper is organized as follows:

1. Main result

Let X be a Banach space over $\boldsymbol{R}=(-\infty, +\infty)$ with norm $|\cdot|$. Given a subset Q of $R\times X$ we denote by $Q(t)$ the section of Q at $t\in\boldsymbol{R}$, i.e. $Q(t)=\{x\in X; (t, x)\in Q\}$. In what follows, let $[a, b)$ be a fixed subinterval of $\boldsymbol{R}$ and Ω a fixed subset of $[a, b)\times X$ such that $\Omega(t)\neq\phi$ for all $t\in[a, b)$. We denote by $\{S(t); t\geqq 0\}$ a contraction semigroup on X of class (C_0) and write A for the infinitesimal generator of $\{S(t); t\geqq 0\}$, i.e. A is the linear operator defined by

$$Ax = \lim_{h\downarrow 0} h^{-1}[S(h)x-x]$$

for $x\in X$ such that the right side exists. Let B be a continuous function from Ω into X.

Given $(\tau, z)\in\Omega$, we consider the initial-value problem

$$(IVP;\tau, z)\qquad \begin{cases} u'(t) = Au(t) + B(t, u(t)), & \tau < t < b, \\ u(\tau) = z. \end{cases}$$

First we list two notions of solutions of initial-value problems of the form $(IVP;\tau, z)$.

DEFINITION 1.1. Let J be a subinterval of $[a, b)$ which is written in the form $[\tau, c]$ or $[\tau, c)$. A continuous function u from J into X is said to be a solution to $(IVP;\tau, z)$ on J if $u(\tau)=z$, $(t, u(t))\in\Omega$ for all $t\in J$, u is continuously differentiable on (τ, c) and if u satisfies $u'(t)=Au(t)+B(t, u(t))$ for all $t\in(\tau, c)$. Moreover, a continuous function u from J into X is said to be a mild solution to $(IVP;\tau, z)$ on J if it satisfies

$$u(t) = S(t-\tau)z + \int_\tau^t S(t-s)B(s, u(s))ds \quad \text{for all} \quad t\in J. \tag{1.1}$$

It is well known that a solution u to $(IVP;\tau, z)$ on J is a mild solution to $(IVP;\tau, z)$ on J.

We next introduce basic notation and terminologies. For $x, y\in X$ we define

$$\begin{aligned}[x, y]_- &= \lim_{h\downarrow 0} h^{-1}(|x+hy|-|x|) \\ &= \sup_{h<0} h^{-1}(|x+hy|-|x|).\end{aligned} \tag{1.2}$$

Note that $|x|\leqq|x-hy|+h[x, y]_-$ for all $h\geqq 0$ and $[x, y+z]_-\leqq[x, y]_-+|z|$ for $x, y, z\in X$. For each $(t, x)\in \boldsymbol{R}\times X$ and $r>0$ we define

$$S_r(t, x) = \{(s, y)\in \boldsymbol{R}\times X;\ |s-t|<r,\ |y-x|<r\}. \tag{1.3}$$

Moreover, if $x\in X$ and D is a subset of X we define the distance between $\{x\}$ and D by

$$d(x, D) = \inf\{|x-y|;\ y\in D\}. \tag{1.4}$$

Let g be a function from $[a, b)\times\boldsymbol{R}$ into $\boldsymbol{R}$. We impose the following two conditions on g.

$(g1)$ $g(t, w)$ is continuous in w for each fixed t and Lebesgue measurable in t for each fixed w; and for each $r>0$, there is a locally integrable function $L_r(t)$ defined on $[a, b)$ such that $|g(t, w)|\leqq L_r(t)$ for $t\in[a, b)$ and for w with $|w|\leqq r$.

$(g2)$ $g(t, 0)=0$; $w(t)\equiv 0$ is the maximal solution to the initial-value problem:

$$\begin{cases} w'(t) = g(t, w(t)), & a < t < b, \\ w(a) = 0. \end{cases} \tag{1.5}$$

REMARK 1.1. Condition ($g1$) is often called Caratheodory's condition.

Given $(\tau, \eta) \in [a, b) \times \boldsymbol{R}$, we denote by $m(t; \tau, \eta)$ the maximal solution of the initial-value problem

$$(1.5) \qquad \begin{cases} w'(t) = g(t, w(t)), & \tau < t, \\ w(\tau) = \eta. \end{cases}$$

Then condition ($g2$) states that for all $\tau \in [a, b)$, the maximal solution $m(t; \tau, 0)$ is defined on all of $[\tau, b)$ and $m(t; \tau, 0) \equiv 0$.

For convenience of future reference in the rest of this paper, we here list our basic assumptions:

(Ω1) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t$ in $[a, b)$ and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$.

(Ω2) $\liminf_{h \downarrow 0} h^{-1} d(S(h)x + hB(t, x), \Omega(t+h)) = 0$ for all $(t, x) \in \Omega$.

(Ω3) There is a function g from $[a, b) \times \boldsymbol{R}$ into $\boldsymbol{R}$ with properties ($g1$) and ($g2$) for which $[x-y, B(t, x) - B(t, y)]_- \leqq g(t, |x-y|)$ holds for all (t, x), $(t, y) \in \Omega$.

REMARK 1.2. It is well known that condition (Ω3) is equivalent to the following condition:

(Ω3)′ There is a function g from $[a, b) \times \boldsymbol{R}$ into $\boldsymbol{R}$ with properties ($g1$) and ($g2$) for which $|x-y| \leqq |x-y-\delta(B(t, x)-B(t, y))| + \delta g(t, |x-y|)$ holds for all (t, x), $(t, y) \in \Omega$ and $\delta > 0$.

Our main result in this paper is now stated as follows:

MAIN THEOREM. *Suppose that conditions* (Ω1), (Ω2) *and* (Ω3) *are fulfilled. If Ω is a connected subset of $[a, b) \times X$ such that $\Omega(t) \neq \phi$ for all $t \in [a, b)$, then for each $(\tau, z) \in \Omega$, $(IVP; \tau, z)$ has a unique mild solution on $[\tau, b)$.*

We here outline the argument to obtain the above theorem.

First the local existence and the uniqueness of mild solutions to $(IVP; \tau, z)$ are established. We next consider for each $(\tau, z) \in \Omega$ the unique mild solution u of $(IVP; \tau, z)$ that is no longer continuable to the right of $T(\tau, z)$, the final time of u. Then we construct a continuous local semiflow $U(t, \tau, z)$ on Ω (in the sense of Definition 2.1 below) by setting $U(t, \tau, z) = u(t)$ for $t \in [\tau, T(\tau, z))$. The problem on the global existence of mild solutions is reduced to the problem of finding sufficient conditions for the final time of each mild solution to be equal to b. This problem can be handled with the aid of a topological method. Hence it is sufficient to establish the local uniformity of intervals of existence of mild solutions as well as the continuous dependence of mild solutions on initial data. More precisely, we proceed with the argument along the following lines:

(i) Let $(\tau, z) \in \Omega$. Then there are numbers $r > 0$ and $h > 0$ such that for each $(t, x) \in \Omega \cap S_r(\tau, z)$, $(IVP; t, x)$ has a mild solution on $[t, t+h]$.

(ii) Let $\{(\tau_n, z_n)\}_{n\geqq 1}$ be a sequence in Ω converging to $(\tau, z)\in\Omega$ such that $T(\tau_n, z_n)>c$ for $n\geqq 1$ and some number $c\in(\tau, b)$. Then $T(\tau, z)>c$ and $U(t, \tau_n, z_n)$ converges to $U(t, \tau, z)$ on $(\tau, c]$.

Secondly let u_1 and u_2 be mild solutions on $[\tau, c]$ for some $a\leqq\tau<c<b$. Under condition (Ω3) we make an estimate for the difference $|u_1(t)-u_2(t)|$ and derive the integral inequality

$$|u_1(t)-u_2(t)| - |u_1(s)-u_2(s)| \leqq \int_s^t g(\xi, |u_1(\xi)-u_2(\xi)|)d\xi$$

for $\tau\leqq s\leqq t\leqq c$. Applying a comparison theorem well-known in the theory of ordinary differential equations we show that $|u_1(t)-u_2(t)|$ is dominated by a maximal solution of (1.6) and hence the uniqueness of mild solutions follows from condition (g2) on the function g. It should be noted that this argument may be viewed as a prototype of the convergence argument of approximate solutions for $(IVP; \tau, z)$.

Thirdly, in order to establish the local existence, we investigate the subtangential condition (Ω2) and show that it holds locally uniformly. By virtue of this local uniformity in subtangential condition, one constructs ε-approximate solutions on an interval independent of ε by way of the method of Cauchy polygons. The approximate solutions to be constructed could be continuous but might lie outside Ω. If the function g is continuous in both arguments t and w one can apply the techniques evolved by Webb and Martin to show the convergence of the approximate solutions. However, these procedures do not work in the present case since g enjoys only a much weaker continuity. To overcome this difficulty, we construct families $\{u_\delta^\varepsilon; 0<\delta\leqq\delta_\varepsilon\}$ of ε-approximate solutions which might lose the strong continuity but remain in the domain Ω at all time. We then choose an appropriate member from each family to make estimates for the difference between two families. Furthermore, in order to discuss the convergence of such approximate solutions, we need to extend usual comparison theorems for ordinary differential equations so that "bounded measurable" approximate solutions with small errors can be handled. Thus we obtain the local uniformity of intervals of existence of mild solutions.

Finally, combining results obtained in the second stage with those of third stage, we see that the continuous local semiflow providing the mild solutions to $(IVP; \tau, z)$ satisfies conditions (i) and (ii) as mentioned above. Thus the main theorem turns out to be proved.

2. Topological results

This section is devoted to investigate sufficient conditions for the global existence of mild solutions. Those conditions are stated in terms of local semiflow defined as follows.

DEFINITION 2.1. Let C be a subset of $[a, b)\times X$. Let T be a function from C into $(a, b]\subset \boldsymbol{R}\cup\{\infty\}$ such that $T(\tau, z)>\tau$ for $(\tau, z)\in C$. Let U be a function from $D(U)=\{(t, \tau, z); (\tau, z)\in C$ and $\tau\leqq t<T(\tau, z)\}$ into X. U is said to be a local semiflow on C if U satisfies the following conditions:

(S1) $U(\tau, \tau, z) = z$ for $(\tau, z)\in C$ and $(t, U(t, \tau, z))\in C$ for $(t, \tau, z)\in D(U)$;
(S2) $T(t, U(t, \tau, z)) = T(\tau, z)$ for $(t, \tau, z)\in D(U)$;
(S3) $U(t, s, U(s, \tau, z)) = U(t, \tau, z)$ for $(\tau, z)\in C$ and $\tau\leqq s\leqq t<T(\tau, z)$.

U is said to be a continuous local semiflow if U is a local semiflow and satisfies the following additional condition:

(S4) $U(t, \tau, z)$ is continuous in $t\in[\tau, T(\tau, z))$.

In what follows, we consider T as a function from the subset C of the uniform topological space $[a, b)\times X$ into the extended real line $\boldsymbol{R}\cup\{\infty\}$ endowed with the usual uniform topology. If for each $(\tau, z)\in\Omega$, $(IVP; \tau, z)$ has a unique mild solution u that is noncontinuable to the right and $T(\tau, z)$ is its final time, then we can define a continuous local semiflow U on Ω by setting

$$U(t, \tau, z) = u(t) \quad \text{for} \quad t\in[\tau, T(\tau, z)). \tag{2.1}$$

The continuous local semiflow defined through (2.1) is called a continuous local semiflow associated with $(IVP; \tau, z)$ in the following.

Let C be a connected subset of $[a, b)\times X$ and set $d=\sup\{t\in\boldsymbol{R}; C(t)\neq\phi\}$. Let U be a local semiflow on C with domain $D(U)=\{(t, \tau, z); (\tau, z)\in C$ and $\tau\leqq t<T(\tau, z)\}$, where T is a function from C into $(a, b]\subset\boldsymbol{R}\cup\{\infty\}$. It is clear that $C(d)=\phi$, $C\subset[a, b)\times X$ and $T(\tau, z)\leqq d$ for all $(\tau, z)\in C$. The local semiflow U is said to be a semiflow on C if $T(\tau, z)=d$ for all $(\tau, z)\in\Omega$.

Let C be a subset of $[a, b)\times X$. A local semiflow U on C is said to be a semiflow on C if U is a semiflow on each connected component of C. Now let U be a continuous local semiflow associated with $(IVP; \tau, z)$. If U becomes a semiflow on Ω, then $(IVP; \tau, z)$ has a global mild solution for each $(\tau, z)\in\Omega$. Hence it comes to be the main problem to investigate as to when the local semiflow U becomes a semiflow. In this regard we obtain the following useful results.

THEOREM 2.1. *Let C be a connected subset of $[a, b)\times X$. Set $d=\sup\{t\in\boldsymbol{R}; C(t)\neq\phi\}$. Let U be a local semiflow on C with domain $D(U)=\{(t, \tau, z); (\tau, z)\in C$ and $\tau\leqq t<T(\tau, z)\}$ where T is a function from C into $(a, b]\subset\boldsymbol{R}\cup\{\infty\}$. Suppose that the function T satisfies the following conditions:*

(1) *T is lower semicontinuous.*
(2) *If $\{(\tau_n, z_n)\}_{n\geqq 1}$ is a sequence in C such that $(\tau_n, z_n)\to(\tau, z)\in C$ as $n\to\infty$ and if $T(\tau, z_n)>c$ for $n\geqq 1$ and some number c, then $T(\tau, z)>c$.*

Then $T(\tau, z)=d$ *for all* $(\tau, z)\in C$.

Proof. Let c be an arbitrary element of $T(C)$, the range of T. Set $C_1=\{(\tau, z)\in C; T(\tau, z)\leqq c\}$ and $C_2=\{(\tau, z)\in C; T(\tau, z)>c\}$. Since T is lower semicontinuous, C_2 is an open subset of C. Let $\{(\tau_n, z_n)\}_{n\geqq 1}$ be a sequence in C_2 converging to (τ, z) in C. Then it follows from condition (2) that $T(\tau, z)>c$. This means that C_2 is a closed subset of C. Since C is connected, $C=C_1\cup C_2$ (disjoint union) and $C_1\neq\phi$, and it is concluded that $C_2=\phi$. Thus $c=d$. It turns out that $T(C)$ is a singleton set $\{d\}$ and the proof is complete.

In case U is a continuous local semiflow, conditions (1) and (2) can be replaced by conditions (1′) and (2′) listed below.

Theorem 2.2. *Let* C *be a connected subset of* $[a, b)\times X$. *Set* $d=\sup\{t\in \boldsymbol{R}; C(t)\neq\phi\}$. *Let* U *be a continuous local semiflow on* C *with domain* $D(U)=\{(t, \tau, z); (\tau, z)\in C$ *and* $\tau\leqq t<T(\tau, z)\}$ *where* T *is a function from* C *into* $(a, b]\subset \boldsymbol{R}\cup\{\infty\}$. *Suppose that* T *satisfies the following conditions*:

(1′) *For* $(\tau, z)\in C$ *there are* $r>0$ *and* $h>0$ *such that* $T(t, x)>t+h$ *for all* $(t, x)\in C\cap S_r(\tau, z)$.

(2′) *If* $\{(\tau_n, z_n)\}_{n\geqq 1}$ *is a sequence in* C *converging to* $(\tau, z)\in C$ *such that* $T(\tau_n, z_n)>c$ *for* $n\geqq 1$ *and some number* $c\in(\tau, b)$, *then* $T(\tau, z)>c$ *and* $U(t, \tau_n, z_n)$ *converges to* $U(t, \tau, z)$ *on* $(\tau, c]$.

Then $T(\tau, z)=d$ *for all* $(\tau, z)\in C$.

Proof. In view of Theorem 2.1, it suffices to show that T is lower semicontinuous. Let $\{(\tau_n, z_n)\}_{n\geqq 1}$ be a sequence in C converging to (τ, z) in C and set $c=\liminf_{n\to\infty} T(\tau_n, z_n)$. Notice that $c>\tau$ by (1′). Assume that $c<T(\tau, z)$. Then there are numbers $r>0$ and $h>0$ such that $T(t, x)>t+h$ for all $(t, x)\in C\cap S_r(c, U(c, \tau, z))$. Let $\eta\in(0, r)$ be such that $\eta<h$, $\eta<c-\tau$ and $|U(c-\eta, \tau, z)-U(c, \tau, z)|<r/2$. Let N be an integer such that $T(\tau_n, z_n)>c-\eta\geqq\tau_n$ and $|U(c-\eta, \tau_n, z_n)-U(c-\eta, \tau, z)|<r/2$ for $n\geqq N$. Then $(c-\eta, U(c-\eta, \tau_n, z_n))\in C\cap S_r(c, U(c, \tau, z))$ for $n\geqq N$. It follows that $T(\tau_n, z_n)=T(c-\eta, U(c-\eta, \tau_n, z_n))>c-\eta+h$ for $n\geqq N$ and hence $\liminf_{n\to\infty} T(\tau_n, z_n)\geqq c+h>c$, which is a contradiction. Thus $\liminf_{n\to\infty} T(\tau_n, z_n)\geqq T(\tau, z)$ and the proof is complete.

Remark 2.1. Since (1) implies (1′), (1) is equivalent to (1′) for continuous local semiflows under condition (2′).

Remark 2.2. If in particular, C is a cylindrical domain, i.e. $C=[a, b)\times D$, D being a subset of X, condition (2) in Theorem 2.1 can be relaxed to the following condition.

(3) If $\{z_n\}_{n\geqq 1}$ is a sequencein D such that $z_n\to z\in D$ as $n\to\infty$ and $T(\tau, z_n)>c$ for some number c, then $T(\tau, z)>c$.

Similarly, condition (2′) in Theorem 2.2 can be replaced by the following:

(3′) If $\{z_n\}_{n\geqq 1}$ is a sequence in D such that $z_n \to z \in D$ as $n \to \infty$ and $T(\tau, z_n) > c$ for some number $c \in (\tau, b)$, then $T(\tau, z) > c$ and $U(t, \tau, z_n) \to U(t, \tau, z)$ for $t \in [\tau, c]$.

In case U is a continuous local semifiow associated with the problems $(IVP; \tau, z)$, $(\tau, z) \in \Omega$, condition (1′) states that the maximal intervals of mild solutions are locally uniform for initial data $(\tau, z) \in \Omega$ and condition (2′) implies that mild solutions depend continuously upon initial data.

3. Comparison theorems

In this section we make an attempt to extend comparison theorems for ordinary differential equations so that they may be applicable to our problem.

Let g be a function with properties ($g1$) and ($g2$) and let $m(t; \tau, \eta)$ denote the maximal solution to the initial-value problem for the ordinary differential equation:

$$\text{(3.1)} \qquad \begin{cases} w'(t) = g(t, w(t)), & \tau < t, \\ w(t) = \eta, \end{cases}$$

where $(\tau, \eta) \in [a, b) \times \boldsymbol{R}$. Given $\varepsilon > 0$ we define a function g_ε by

$$\text{(3.2)} \qquad g_\varepsilon(t, w) = \sup\{g(t, y); 0 \leqq y - w \leqq \varepsilon\}.$$

It is easy to see that g_ε also satisfies Caratheodory's condition, and that $g_\varepsilon(t, w)$ converges to $g(t, w)$ uniformly on compact subsets in w as $\varepsilon \to 0$. Given $(\tau, \eta) \in [a, b) \times \boldsymbol{R}$, we denote by $m_\varepsilon(t; \tau, \eta)$ the maximal solution of the initial-value problem for an ordinary differential equation:

$$\text{(3.3)} \qquad \begin{cases} w'(t) = g_\varepsilon(t, w(t)), & \tau > t, \\ w(\tau) = \eta. \end{cases}$$

Notice that if $\varepsilon = 0$, then (3.3) coincides with (3.1) and $m_0(t; \tau, \eta) = m(t; \tau, \eta)$.

We first state the following fact (for the proof, see e.g. Lakshmikantham and Leela [5]):

LEMMA 3.1. *Let $\tau \in [a, b)$ and let $[\tau, c]$ be a compact subinterval of $[a, b)$. Then there are $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and $\eta \in (0, \eta_0)$, the maximal solution $m_\varepsilon(t; \tau, \eta)$ of* (3.3) *exists on $[\tau, c]$,*

$$\text{(3.4)} \qquad \lim_{\varepsilon \downarrow 0} m_\varepsilon(t; \tau, \eta) = m(t; \tau, \eta) \textit{ uniformly on } [\tau, c] \textit{ and}$$

$$\text{(3.5)} \qquad \lim_{\varepsilon, \eta \downarrow 0} m_\varepsilon(t; \tau, \eta) = 0 \textit{ uniformly on } [\tau, c].$$

We also employ the following result which may be regarded as an extention of ordinary comparison theorems for ordinary differential equations.

LEMMA 3.2. *Let $\eta>0$. Let $[\tau, c]$ be a compact subinterval of $[a, b)$ on which a maximal solution $m(t; \tau, \eta)$ of* (3.1) *exists. Let α be a bounded measurable function from $[\tau, c]$ into $\boldsymbol{R}$ and suppose that α satisfies the integral inequality*

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} g(t, \alpha(t))dt \tag{3.6}$$

for $\tau \leqq t_1 < t_2 \leqq c$. If $\alpha(\tau) \leqq \eta$, then

$$\alpha(t) \leqq m(t; \tau, \eta) \quad \text{for all} \quad t \in [\tau, c]. \tag{3.7}$$

PROOF. We first show that α is of bounded variation. Let $r>0$ be such that $|\alpha(t)| \leqq r$ for all $t \in [\tau, c]$. Then by condition $(g1)$ there is a locally integrable function $L_r(t)$ defined on $[\tau, c]$ such that $|g(t, w)| \leqq L_r(t)$ for $t \in [\tau, c]$ and $w \in \boldsymbol{R}$ with $|w| \leqq r$. Let $\{s_i\}_{0 \leqq i \leqq n}$ be an arbitrary subdivision of the interval $[\tau, c]$. Using $|h| = 2h^+ - h$ for $h \in \boldsymbol{R}$, where $h^+ = \max\{h, 0\}$, and applying (3.6), we have

$$\begin{aligned}\sum_{i=1}^{n} |\alpha(s_i) - \alpha(s_{i-1})| &= 2\sum_{i=1}^{n} (\alpha(s_i) - \alpha(s_{i-1}))^+ + \alpha(s_0) - \alpha(s_n) \\ &\leqq 2\sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} L_r(t)dt + 2r \\ &= 2\int_{\tau}^{c} L_r(t)dt + 2r.\end{aligned}$$

This means that α is of bounded variation. Therefore, (3.6) yields

$$\lim_{s \downarrow t} \alpha(s) \leqq \alpha(t) \leqq \lim_{s \uparrow t} \alpha(s) \tag{3.8}$$

for all $t \in [\tau, c]$.

Define

$$f(t, w) = \begin{cases} g(t, w) & \text{if} \quad \alpha(t) \leqq w \\ g(t, \alpha(t)) & \text{if} \quad \alpha(t) > w. \end{cases} \tag{3.9}$$

Then the function f also satisfies Caratheodory's condition. Let $m^*(t; \tau, \eta)$ be a maximal solution to the initial value problem for the ordinary differential equation

$$\begin{cases} w'(t) = f(t, w(t)), & \tau < t, \\ w(\tau) = \eta. \end{cases}$$

Assume that $m^*(t; \tau, \eta)$ is defined on $[\tau, c^*]$ with $\tau < c^* \leqq c$.

We now claim that $m^*(t; \tau, \eta) \geqq \alpha(t)$ for $t \in [\tau, c^*]$. Assume to the contrary that $m^*(t_1; \tau, \eta) < \alpha(t_1)$ for some $t_1 \in [\tau, c^*]$. Clearly $t_1 > \tau$, and it follows from

(3.8) that there is an interval $(s, t_1]$, $\tau<s<t_1$, on which $m^*(t; \tau, \eta)<\alpha(t)$. Let $d=\inf\{s; m^*(t; \tau, \eta)<\alpha(t) \text{ on } (s, t_1]\}$. If $d>\tau$, then we see from the definition of d and (3.8) that

$$\alpha(d) - m^*(d; \tau, \eta) = \alpha(d) - \lim_{t\uparrow d} m^*(t; \tau, \eta) \leqq \alpha(d) - \lim_{t\uparrow d} \alpha(t) \leqq 0. \tag{3.10}$$

Combining (3.10) with the assumption that $\eta \geqq \alpha(\tau)$, we obtain

$$m^*(d; \tau, \eta) \geqq \alpha(d). \tag{3.11}$$

Since $m^*(t; \tau, \eta)<\alpha(t)$ for $t\in(d, t_1]$, (3.9) and (3.11) together imply

$$\begin{aligned} 0 &> m^*(t_1; \tau, \eta) - \alpha(t_1) \\ &\geqq [m^*(t_1; \tau, \eta) - m^*(d; \tau, \eta)] - [\alpha(t_1)-\alpha(d)] \\ &\geqq \int_d^{t_1} f(t, m^*(t; \tau, \eta))dt - \int_d^{t_1} g(t, \alpha(t))dt = 0. \end{aligned}$$

This is a contradiction. Hence it follows from (3.9) that

$$\begin{aligned} m^*(t; \tau, \eta) &= \eta + \int_\tau^t f(s, m^*(s; \tau, \eta))ds \\ &= \eta + \int_\tau^t g(s, m^*(s; \tau, \eta))ds \end{aligned}$$

which means that $m^*(t; \tau, \eta)$ is a solution of (3.1). Hence $m^*(t; \tau, \eta)$ eventually exists on all of $[\tau, c]$ and $\alpha(t)\leqq m^*(t; \tau, \eta)\leqq m(t; \tau, \eta)$ for $t\in[\tau, c]$. Thus the proof is complete.

We next give two comparison theorems involving integral inequalities with small errors.

PROPOSITION 3.1. *Let* $\varepsilon>0$ *and* $\eta>0$. *Let* $[\tau, c]$ *be a subinterval of* $[a, b)$ *on which a maximal solution* $m_\varepsilon(t; \tau, \eta)$ *of* (3.3) *exists. Let* α *be a bounded measurable function from* $[\tau, c]$ *into* $\boldsymbol{R}$ *and suppose that* α *satisfies*

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} g(t, \alpha(t))dt + \varepsilon \tag{3.12}$$

for $\tau\leqq t_1<t_2\leqq c$. *If* $\alpha(\tau)\leqq\eta$, *then*

$$\alpha(t) \leqq m_\varepsilon(t; \tau, \eta) + \varepsilon \qquad \textit{for all} \quad t\in[\tau, c]. \tag{3.13}$$

PROOF. Set

$$h(t) = \alpha(t) - \alpha(\tau) - \int_\tau^t g(s, \alpha(s))ds \tag{3.14}$$

for $t \in [\tau, c]$. Then we have

$$h(t_2) - h(t_1) = \alpha(t_2) - \alpha(t_1) - \int_{t_1}^{t_2} g(s, \alpha(s))ds \leqq \varepsilon \tag{3.15}$$

for $\tau \leqq t_1 < t_2 \leqq c$. Define

$$h_*(t) = \inf_{\tau \leqq s \leqq t} h(s), \tag{3.16}$$

and

$$\alpha_*(t) = \alpha(t) + h_*(t) - h(t) \tag{3.17}$$

for $t \in [\tau, c]$. It follows from (3.15) through (3.17) that h_* is a nonincreasing function on $[\tau, c]$, $0 \leqq h(t) - h_*(t) \leqq \varepsilon$, $0 \leqq \alpha(t) - \alpha_*(t) \leqq \varepsilon$ and

$$\alpha_*(t) = \alpha(\tau) + \int_{\tau}^{t} g(s, \alpha(s))ds + h_*(t) \tag{3.18}$$

for all $t \in [\tau, c]$. Hence

$$\begin{aligned} \alpha_*(t_2) - \alpha_*(t_1) &= h_*(t_2) - h_*(t_1) + \int_{t_1}^{t_2} g(s, \alpha(s))ds \\ &\leqq \int_{t_1}^{t_2} g(s, \alpha(s))ds \\ &\leqq \int_{t_1}^{t_2} g_\varepsilon(s, \alpha_*(s))ds \end{aligned} \tag{3.19}$$

for $\tau \leqq t_1 < t_2 \leqq c$. Since $\alpha_*(\tau) = \alpha(\tau) \leqq \eta$, it follows from Lemma 3.2 and (3.19) that $\alpha_*(t) \leqq m_\varepsilon(t; \tau, \eta)$ and hence $\alpha(t) \leqq m_\varepsilon(t; \tau, \eta) + \varepsilon$ for $t \in [\tau, c]$, which completes the proof.

Combining Proposition 3.1 and Lemma 3.1, we have the following result.

PROPOSITION 3.2. *Let $\{\varepsilon_n\}_{n \geqq 1}$ and $\{\eta_n\}_{n \geqq 1}$ be null-sequences. Let $[\tau, c]$ be a subinterval of $[a, b)$. Let $\{\alpha_n\}_{n \geqq 1}$ be a sequence of bounded measurable functions from $[\tau, c]$ into $\boldsymbol{R}$. Suppose that for each $n \geqq 1$, α_n satisfies*

$$\alpha_n(\tau) \leqq \eta_n \tag{3.20}$$

and

$$\alpha_n(t_2) - \alpha_n(t_1) \leqq \int_{t_2}^{t_1} g(t, \alpha_n(t))dt + \varepsilon_n \tag{3.21}$$

for $\tau \leqq t_1 < t_2 \leqq c$. Then we have

$$\limsup_{n \to \infty} \alpha_n(t) \leqq 0 \tag{3.22}$$

for all $\tau \leqq t \leqq c$.

4. Uniqueness of mild solutions

Our objective in this section is to discuss the continuous dependence of mild solutions on initial values and establish a uniqueness theorem for mild solutions.

PROPOSITION 4.1. *Suppose that condition* ($\Omega 3$) *holds. Let* $\eta > 0$ *and let* $[\tau, c]$ *be a subinterval of* $[a, b)$ *on which a maximal solution* $m(t; \tau, \eta)$ *of* (3.1) *exists. Let* $(\tau, z_i) \in \Omega$, $i = 1, 2$. *Suppose that mild solutions* u_i *of the problems* $(IVP; \tau, z_i)$ *exist on* $[\tau, c]$, *respectively. If* $|z_1 - z_2| \leqq \eta$, *then*

$$|u_1(t) - u_2(t)| \leqq m(t; \tau, \eta) \quad \text{for all} \quad t \in [\tau, c]. \tag{4.1}$$

In particular, $(IVP; \tau, z)$ *has at most one mild solution for all* $(\tau, z) \in \Omega$.

PROOF. Set $\alpha(t) = |u_1(t) - u_2(t)|$ for $t \in [\tau, c]$. Let $\varepsilon > 0$. Since $u_i(t)$ and $S(h)B(t, u_i(t))$, $i = 1, 2$, are continuous with respect to $t \in [\tau, c]$ and $h \geqq 0$, there is a number $\delta > 0$ such that $|u_i(t) - u_i(s)| \leqq \varepsilon$ and $|S(h)B(t, u_i(t)) - B(s, u_i(s))| \leqq \varepsilon$, for $i = 1, 2$, $h \in [0, \delta)$ and $s, t \in [\tau, c]$ with $|t - s| \leqq \delta$. By ($\Omega 3$), we have

$$\begin{aligned}
&|u_1(t) - u_2(t)| \\
&\leqq |u_1(t) - u_2(t) - \delta(B(t, u_1(t)) - B(t, u_2(t)))| + \delta g(t, |u_1(t) - u_2(t)|) \\
&\leqq |S(\delta)u_1(t-\delta) - S(\delta)u_2(t-\delta)| + |u_1(t) - S(\delta)u_1(t-\delta) - \delta B(t, u_1(t))| \\
&\quad + |u_2(t) - S(\delta)u_2(t-\delta) - \delta B(t, u_2(t))| + \delta g(t, |u_1(t) - u_2(t)|) \\
&\leqq |u_1(t-\delta) - u_2(t-\delta)| \\
&\quad + \int_{t-\delta}^{t} |S(t-s)B(s, u_1(s)) - B(t, u_1(t))| ds \\
&\quad + \int_{t-\delta}^{t} |S(t-s)B(s, u_2(s)) - B(t, u_2(t))| ds + \delta g(t, |u_1(t) - u_2(t)|) \\
&\leqq |u_1(t-\delta) - u_2(t-\delta)| + \delta g(t, |u_1(t) - u_2(t)|) + 2\delta\varepsilon.
\end{aligned} \tag{4.2}$$

for $t \in [\tau + \delta, c]$. Let t_1 and t_2 be such that $\tau \leqq t_1 \leqq t_1 + \delta < t_2 \leqq c$. Integrating both sides of (4.2) from $t_1 + \delta$ to t_2, we obtain

$$\int_{t_2-\delta}^{t_2} \alpha(s) ds - \int_{t_1}^{t_1+\delta} \alpha(s) ds \leqq \delta \int_{t_1+\delta}^{t_2} g(s, \alpha(s)) ds + 2(c - \tau)\delta\varepsilon. \tag{4.3}$$

Since $|\alpha(t) - \alpha(s)| \leqq |u_1(t) - u_1(s)| + |u_2(t) - u_2(s)| \leqq 2\varepsilon$ for $t, s \in [\tau, c]$ with $|t - s| \leqq \delta$, we have

$$\delta\{\alpha(t_2) - \alpha(t_1)\} \leqq \int_{t_2-\delta}^{t_2} \alpha(s) ds - \int_{t_1}^{t_1+\delta} \alpha(s) ds + 4\delta\varepsilon. \tag{4.4}$$

Combining (4.3) with (4.4) gives

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1+\delta}^{t_2} g(s, \alpha(s))ds + C\varepsilon \tag{4.5}$$

for some constant $C>0$. Since $\varepsilon>0$ is arbitrary, (4.5) yields

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} g(s, \alpha(s))ds$$

for $\tau \leqq t_1 < t_2 \leqq c$. The first assertion now follows from Lemma 3.2. The second assertion follows from the first assertion and the assumption that $m(t; \tau, 0)=0$ on $[\tau, c]$. The proof is thereby complete.

5. Local uniformity in subtangential condition

In what follows, we assume conditions (Ω1) and (Ω2).

First we state the following two lemmas which will be often used in the subsequent argument.

LEMMA 5.1. *Let $\{(s_n, y_n)\}_{n\geqq 0}$ be a sequence in Ω such that $s_n \leqq s_{n+1}$. Then we have the relation*

$$\begin{aligned} y_n - S(s_n - s_0)y_0 &= \textstyle\sum_{k=0}^{n-1} S(s_n - s_{k+1})[y_{k+1} - S(s_{k+1}-s_k)y_k - (s_{k+1}-s_k)B(s_k, y_k)] \\ &\quad + \textstyle\sum_{k=0}^{n-1} (s_{k+1}-s_k)S(s_n - s_{k+1})B(s_k, y_k) \end{aligned} \tag{5.1}$$

for $n \geqq 0$.

LEMMA 5.2. *Let $\varepsilon>0$ and $M>0$. Let $\{(s_n, y_n)\}_{n\geqq 0}$ be a sequence in Ω such that $s_n \leqq s_{n+1}$, $|B(s_n, y_n)| \leqq M$ and*

$$|y_{n+1} - S(s_{n+1}-s_n)y_n - (s_{n+1}-s_n)B(s_n, y_n)| \leqq (s_{n+1}-s_n)\varepsilon \tag{5.2}$$

for $n \geqq 0$. If $s_n \uparrow s \in [a, b)$ as $n \to \infty$, then the sequence $\{y_n\}_{n\geqq 0}$ is a Cauchy sequence in X and the limit $(s, y)=\lim_{n\to\infty}(s_n, y_n)$ lies in Ω.

PROOF. Define a sequence $\{\varepsilon_n\}_{n\geqq 0}$ in X by the equation

$$y_{n+1} = S(s_{n+1}-s_n)y_n + (s_{n+1}-s_n)B(s_n, y_n) + (s_{n+1}-s_n)\varepsilon_n. \tag{5.3}$$

Then $|\varepsilon_n| \leqq \varepsilon$ for $n \geqq 0$. Using (5.3), we have

$$y_n = S(s_n - s_0)y_0 + \textstyle\sum_{k=0}^{n-1} (s_{k+1}-s_k)S(s_n - s_{k+1})[B(s_k, y_k) + \varepsilon_k]$$

by induction. Hence

$$(5.4)\quad y_n - y_m = S(s_m - s_0)[S(s_n - s_m)y_0 - y_0]$$
$$+ \sum_{k=0}^{p-1}(s_{k+1}-s_k)S(s_m - s_{k+1})[S(s_n - s_m)[B(s_k, y_k)+\varepsilon_k] - [B(s_k, y_k)+\varepsilon_k]]$$
$$+ \sum_{k=p}^{n-1}(s_{k+1}-s_k)S(s_n - s_{k+1})[B(s_k, y_k)+\varepsilon_k]$$
$$- \sum_{k=p}^{m-1}(s_{k+1}-s_k)S(s_m - s_{k+1})[B(s_k, y_k)+\varepsilon_k]$$

for $p \leqq m \leqq n$. Let $n>0$ and let p be an integer such that $s-s_p<\eta$. Then there exists a positive number μ such that $\mu \leqq \eta$.

$$\sup_{0\leqq\sigma\leqq u}|S(\sigma)[B(s_k, y_k)+\varepsilon_k]-[B(s_k, y_k)+\varepsilon_k]| \leqq \eta \text{ for } 0 \leqq k \leqq p-1$$

and $\sup_{0\leqq\sigma\leqq u}|S(\sigma)y_0 - y_0| \leqq \eta$. If m and n are such that $p \leqq m \leqq n$ and and $s-s_n \leqq s-s_m<\mu$, then (5.4) yields

$$|y_n - y_m| \leqq |S(s_n - s_m)y_0 - y_0|$$
$$+ \sum_{k=0}^{p-1}(s_{k+1}-s_k)|S(s_n - s_m)[B(s_k, y_k)+\varepsilon_k] - [B(s_k, y_k)+\varepsilon_k]|$$
$$+ \sum_{k=p}^{n-1}(s_{k+1}-s_k)|B(s_k, y_k)+\varepsilon_k| + \sum_{k=p}^{m-1}(s_{k+1}-s_k)|B(s_k, y_k)+\varepsilon_k|$$
$$\leqq \eta + (s-s_0)\eta + 2(M+\varepsilon)\eta.$$

Since η is arbitary, this means that $\{y_n\}_{n\geqq 0}$ is a Cauchy sequence in X. It follows from (Ω1) that $(s, y)=\lim_{n\to\infty}(s_n, y_n)\in\Omega$. This completes the proof.

The next result states that the subtangential condition (Ω2) holds locally uniformly in a certain sense.

PROPOSITION 5.1. *Let* $(t, x)\in\Omega$ *and* $\varepsilon\in(0, 1)$. *Let* $r>0$ *be such that* $|B(s, y)-B(t, x)| \leqq \varepsilon/4$ *for* $(s, y)\in\Omega\cap S_r(t, x)$, $\sup_{0\leqq\sigma\leqq r}|S(\sigma)B(t, x)-B(t, x)| \leqq \varepsilon/4$, *and such that* $|B(s, y)| \leqq M$ *for* $(s, y)\in\Omega\cap S_r(t, x)$ *with some constant* $M>0$. *Set* $h_0=\sup\{h\in(0, b-t);\ h(M+1)+\sup_{0\leqq\sigma\leqq h}|S(\sigma)x-x| \leqq r\}$. *Let* $h\in[0, h_0)$ *and* $y\in\Omega(t+h)$ *satisfy* $|y-S(h)x| \leqq h(M+1)$. *(The existence of such pair* (h, y) *is guaranteed by* (Ω2).) *Then for each* $h^*\in(h, h_0)$ *there exists an element* $y^*\in\Omega(t+h^*)$ *such that*

$$(5.5)\qquad |y^* - S(h^*-h)y - (h^*-h)B(t+h, y)| \leqq (h^*-h)\varepsilon.$$

PROOF. Let $h^*\in(h, h_0)$. We shall define inductively a sequence $\{(s_n, y_n)\}_{n\geqq 0}$ in $\Omega\cap S_r(t, x)$ which possesses the following properties:

$$(5.6)\qquad (s_0, y_0)=(t+h, y) \text{ and } t+h \leqq s_n \leqq s_{n+1} \leqq t+h^* \text{ for } n \geqq 0;$$

$$(5.7)\qquad \lim_{n\to\infty} s_n = t+h^*;$$

$$(5.8)\qquad |y_{n+1} - S(s_{n+1}-s_n)y_n - (s_{n+1}-s_n)B(s_n, y_n)| \leqq (s_{n+1}-s)\varepsilon/4$$

for $n \geqq 0$.

Set $(s_0, y_0)=(t+h, y)$. Suppose that (s_n, y_n) is defined in $\Omega \cap S_r(t, x)$ in such a way that $s_n \in [t+h, t+h^*]$ and $|y_n - S(s_n-t)x| \leqq (s_n-t)(M+1)$, and define $\bar{\sigma}_n$ to be the supremum of those $\sigma \geqq 0$ satisfying $s_n+\sigma \leqq t+h^*$ and

$$d(S(\sigma)y_n + \sigma B(s_n, y_n), \quad \Omega(s_n+\sigma)) \leqq \sigma\varepsilon/8. \tag{5.9}$$

Then, choosing a number $\sigma_n \in [\bar{\sigma}_n/2, \bar{\sigma}_n]$, we put $s_{n+1}=s_n+\sigma_n$ and take an element y_{n+1} of $\Omega(s_{n+1})$ such that

$$|y_{n+1} - S(s_{n+1}-s_n)y_n - (s_{n+1}-s_n)B(s_n, y_n)| \leqq (s_{n+1}-s_n)\varepsilon/4. \tag{5.10}$$

Note that, by (5.9) and (Ω2), $s_n < s_{n+1}$ whenever $s_n < t+h^*$. Since

$$|y_{n+1} - S(s_{n+1}-s_n)y_n| \leqq (s_{n+1}-s_n)(M+1)$$

by (5.10), we have

$$\begin{aligned} &|y_{n+1} - S(s_{n+1}-t)x| \\ &\leqq |y_{n+1} - S(s_{n+1}-s_n)y_n| + |S(s_{n+1}-s_n)[y_n - S(s_n-t)x]| \\ &\leqq (s_{n+1}-s_n)(M+1) + (s_n-t)(M+1) = (s_{n+1}-t)(M+1) \end{aligned}$$

and

$$\begin{aligned} |y_{n+1}-x| &\leqq |y_{n+1} - S(s_{n+1}-t)x| + |S(s_{n+1}-t)x - x| \\ &\leqq (s_{n+1}-t)(M+1) + \sup\{|S(\sigma)x - x|;\ 0 \leqq \sigma \leqq h_0\} < r, \end{aligned}$$

which shows that $(s_{n+1}, y_{n+1}) \in \Omega \cap S_r(t, x)$. In this way we obtain a sequence $\{(s_n, y_n)\}_{n \geqq 0}$ in $\Omega \cap S_r(t, x)$ satisfying (5.6) and (5.8). Now it remains to show that $s=\lim_{n\to\infty} s_n = t+h^*$. Suppose to the contrary that $s<t+h^*$. Then Lemma 5.2 and (5.8) together imply that $\{y_n\}_{n\geqq 0}$ is a Cauchy sequence in X and $(s, y^*)=\lim_{n\to\infty}(s_n, y_n) \in \Omega \cap S_r(t, x)$. Moreover we can find a number $\eta>0$ such that $\eta \leqq t+h^*-s$ and

$$d(S(\eta)y^* + \eta B(s, y^*), \Omega(s+\eta)) \leqq \eta\varepsilon/12. \tag{5.11}$$

Choose an integer $N \geqq 1$ such that $s-s_n \leqq \eta/2$ for $n \geqq N$, and set $\eta_n = s-s_n+\eta$ for each $n \geqq N$. Then $s_n+\eta_n = s+\eta \leqq t+h^*$ and $\eta_n > \eta \geqq 2(s-s_n) > 2\sigma_n \geqq \bar{\sigma}_n$ for all $n \geqq N$. Here we have employed the fact that $s_n < s_{n+1} < s$ for all $n \geqq 0$. Hence it follows from the definition of $\bar{\sigma}_n$ that

$$d(S(\eta_n)y_n + \eta_n B(s_n, y_n), \Omega(s_n+\eta_n)) > \eta_n\varepsilon/8$$

for $n \geqq N$. Using the continuity of B and noting that $s_n+\eta_n=s+\eta$, we obtain

$$\begin{aligned} d(S(\eta)y^* + \eta B(s, y^*), \Omega(s+\eta)) &= \lim_{n\to\infty} d(S(\eta_n)y_n + \eta_n B(s_n, y_n), \Omega(s_n+\eta_n)) \\ &\geqq \eta\varepsilon/8, \end{aligned}$$

which contradicts (5.11) and hence $\lim_{n\to\infty} s_n = s = t+h^*$. Thus it has been shown that the sequence $\{(s_n, y_n)\}_{n\geqq 0}$ has all of the desired properties (5.6)–(5.8).

It is now easy to prove the lemma. First by (5.1) we have

$$\begin{aligned}(5.12)\qquad y_n &- S(s_n-t-h)y - (s_n-t-h)B(t+h, y)\\ &= \textstyle\sum_{k=0}^{n-1} S(s_n-s_{k+1})[y_{k+1}-S(s_{k+1}-s_k)y_k - (s_{k+1}-s_k)B(s_k, y_k)]\\ &\quad + \textstyle\sum_{k=0}^{n-1}(s_{k+1}-s_k)S(s_n-s_{k+1})[B(s_k, y_k) - B(t, x)]\\ &\quad + \textstyle\sum_{k=0}^{n-1}(s_{k+1}-s_k)[S(s_n-s_{k+1})B(t, x) - B(t, x)]\\ &\quad - (s_n-t-h)[B(t+h, y) - B(t, x)]\end{aligned}$$

for $n\geqq 0$. Since $(t+h^*, y^*)$ and (s_n, y_n) are contianed in $\Omega\cap S_r(t, x)$ for $n\geqq 0$, we obtain the estimate

$$|y_n - S(s_n-t-h)y - (s_n-t-h)B(t+h, y)| \leqq (s_n-t-h)\varepsilon.$$

Passing to the limit as $n\to\infty$, we finally obtain

$$|y^* - S(h^*-h)y - (h^*-h)B(t+h, y)| \leqq (h^*-h)\varepsilon.$$

This completes the proof.

6. Approximate solutions

This section is devoted to the construction of approximate solutions to the problems $(IVP; \tau, z)$.

We begin by introducing the notion of ε-approximate solutions.

DEFINITION 6.1. Let $(\tau, z)\in\Omega$. Let $\varepsilon>0$ and $T\in(0, b-\tau)$. A strongly measurable function u from $[\tau, \tau+T]$ into X is said to be an ε-approximate solution to $(IVP; \tau, z)$ on $[\tau, \tau+T]$ if it has the following properties:

(ε1) $u(\tau)=z$ and $(t, u(t))\in\Omega$ for all $t\in[\tau, \tau+T]$;

(ε2) $B(s, u(s))$ is integrable in the sense of Bochner and

$$(6.1)\qquad |u(t) - S(t-\tau)z - \int_\tau^t S(t-s)B(s, u(s))ds| \leqq (t-\tau)\varepsilon$$

for all $t\in[\tau, \tau+T]$.

REMARK 6.1. Since B is continuous and $S(h)x$ is continuous with respect to h for each $x\in X$, the strong measurability of $S(t-s)B(s, u(s))$ with respect to $s\in[0, t]$ follows from the strong measurability of u.

The purpose of this section is to prove the following.

PROPOSITION 6.1. *Suppose that conditions* (Ω1) *and* (Ω2) *hold. Let*

$(\tau, z)\in\Omega$. Let R and M be positive numbers such that $\tau+R<b$ and $|B(t, x)|\leqq M$ for $(t, x)\in\Omega\cap S_R(\tau, z)$. Let T be a positive number such that $T(M+1)+\sup_{0\leqq\sigma\leqq T}|S(\sigma)z-z|<R$. Then for each $\varepsilon>0$ there exist a positive number δ_ε and a family of ε-approximate solutions $\{u_\delta^\varepsilon;\ \delta\in(0, \delta_\varepsilon]\}$ to $(IVP;\ \tau, z)$ on $[\tau, \tau+T]$ with the following properties:

(P1) $u_\delta^\varepsilon(\tau)=z$ *and* $(t, u_\delta^\varepsilon(t))\in\Omega\cap S_R(\tau, z)$ *for* $\delta\in(0, \delta_\varepsilon]$ *and* $t\in[\tau, \tau+T]$;

(P2) $|u_\delta^\varepsilon(t)-u_\delta^\varepsilon(s)|\leqq\varepsilon$ *for* $\delta\in(0, \delta_\varepsilon]$ *and* $t, s\in[\tau, \tau+T]$ *with* $|t-s|\leqq\delta$;

(P3) $\int_{\tau+\delta}^{\tau+T}|u_\delta^\varepsilon(s)-S(\delta)u_\delta^\varepsilon(s-\delta)-\delta B(s, u_\delta^\varepsilon(s))|ds\leqq\delta\varepsilon$;

(P4) $|u_\delta^\varepsilon(t)-u_{\delta'}^\varepsilon(t)|\leqq\varepsilon$ *for* $\delta, \delta'\in(0, \delta_\varepsilon]$ *and* $t\in[\tau, \tau+T]$.

REMARK 6.2. Proposition 6.1 is valid even if the semigroup S is not a contraction semigroup of class (C_0). However, for simplicity, we deal with only the case where S is a contraction semigroup of class (C_0) in this paper.

We first need the following lemma.

LEMMA 6.1. *Let $(t, x)\in\Omega$ and $\varepsilon\in(0, 1)$. Let $r>0$ be a number such that $|B(s, y)-B(t, x)|\leqq\varepsilon/4$ for $(s, y)\in\Omega\cap S_r(t, x)$, $\sup_{0\leqq\sigma\leqq r}|S(\sigma)B(t, x)-B(t, x)|\leqq\varepsilon/4$ and such that $|B(s, y)|\leqq M$ for $(s, y)\in\Omega\cap S_r(t, x)$ with some constant $M>0$. Set $h_0=\sup\{h\in(0, b-t);\ h(M+1)+\sup_{0\leqq\sigma\leqq h}|S(\sigma)x-x|\leqq r\}$ and let t^* and δ be such that $t<t+\delta<t^*\leqq t+h_0$. Then there exist a sequence $\{\gamma_n\}_{n\geqq 0}$ of real-valued step functions on $[t, t^*)$ and a sequence $\{v_n\}_{n\geqq 0}$ of X-valued step functions on $[t, t^*)$ with the following properties:*

(6.3) $\gamma_n(t)=t$ *for* $n\geqq 0$.

(6.4) $\gamma_m(\gamma_n(s))=\gamma_n(\gamma_m(s))=\gamma_m(s)$ *for* $0\leqq m\leqq n$ *and* $s\in[t, t^*)$.

(6.5) $\gamma_n(s-\delta)=\gamma_n(s)-\delta$ *for* $n\geqq 0$ *and* $s\in[t+\delta, t^*)$.

(6.6) *The sequence* $\{\gamma_n(s)\}_{n\geqq 0}$ *is monotone nondecreasing and* $\gamma_n(s)\uparrow s$ *as* $n\to\infty$ *for each* $s\in[t, t^*)$.

(6.7) $v_n(t)=x$ *and* $(\gamma_n(s), v_n(s))\in\Omega\cap S_r(t, x)$ *for* $n\geqq 0$ *and* $s\in[t, t^*)$.

(6.8) $|v_n(s)-S(\gamma_n(s)-t)x|\leqq(\gamma_n(s)-t)(M+1)$ *for* $n\geqq 0$ *and* $s\in[t, t^*)$.

(6.9) $|v_0(s)-S(\delta)v_0(s-\delta)-\delta B(\gamma_0(s-\delta), v_0(s-\delta))|\leqq\delta\varepsilon$ *for* $s\in[t+\delta, t^*)$.

(6.10) $|v_n(s)-S(\gamma_n(s)-\gamma_{n-1}(s))v_{n-1}(s)-(\gamma_n(s)-\gamma_{n-1}(s))B(\gamma_{n-1}(s), v_{n-1}(s))|$
$\leqq(\gamma_n(s)-\gamma_{n-1}(s))\varepsilon$ *for* $n\geqq 1$ *and* $s\in[t, t^*)$.

(6.11) $|v_n(s)-S(\gamma_n(s)-\gamma_0(s))v_0(s)-(\gamma_n(s)-\gamma_0(s))B(\gamma_0(s), v_0(s))|$
$\leqq 2(\gamma_n(s)-\gamma_0(s))\varepsilon$ *for* $n\geqq 1$ *and* $s\in[t, t^*)$.

(6.12) $|v_n(s) - S(\gamma_n(s)-t)x - (\gamma_n(s)-t)B(t, x)| \leqq 2(\gamma_n(s) - t)\varepsilon$ *for* $n \geqq 0$ *and* $s \in [t, t^*)$.

PROOF. For each nonnegative integer n, let $N(n)$ be an integer satisfying $t+N(n)\delta/2^n < t^* \leqq t+(N(n)+1)\delta/2^n$ and set $t_k^n = t+k\delta/2^n$ for $0 \leqq k \leqq N(n)$ and $I_n = \{t_k^n;\ 0 \leqq k \leqq N(n)\}$. Then $I_n \subset I_{n+1}$ and $t_k^{n+1} \in I_n$ for k even. For each $n \geqq 0$, we define a step function γ_n on $[t, t^*)$ with values in I_n by

$$\begin{aligned} \gamma_n(s) &= t_k^n \quad \text{for} \quad s \in [t_k^n, t_{k+1}^n) \quad \text{and} \quad 0 \leqq k \leqq N(n) - 1, \\ &= t_{N(n)}^n \quad \text{for} \quad s \in [t_{N(n)}^n, t^*). \end{aligned}$$

Then it is easy to see that the sequence $\{\gamma_n(s)\}_{n \geqq 0}$ has properties (6.2)–(6.6).

We then construct a sequence $\{v_n\}_{n \geqq 0}$ of X-valued step functions on $[t, t^*)$ with properties (6.7)–(6.10). To this end, we begin by choosing a sequence $\{v_0(t_k^0)\}_{0 \leqq k \leqq N(0)}$ of elements in X such that

(6.13) $v_0(t_0^0) = x$ and $(t_k^0, v_0(t_k^0)) \in \Omega \cap S_r(t, x)$ for $0 \leqq k \leqq N(0)$,

(6.14) $|v_0(t_k^0) - S(t_k^0 - t)x| \leqq (t_k^0 - t)(M+1)$ for $1 \leqq k \leqq N(0)$,

(6.15) $|v_0(t_k^0) - S(\delta)v_0(t_{k-1}^0) - \delta B(t_{k-1}^0, v_0(t_{k-1}^0))| \leqq \delta\varepsilon$ for $1 \leqq k \leqq N(0)$.

This is accomplished by induction on k. In fact, set $v_0(t_0^0) = x$. Suppose that $v_0(t_k^0)$ is chosen so that (6.13) and (6.14) hold. Then we can apply Proposition 5.1 with $h = t_k^0 - t$ and $h^* = t_{k+1}^0 - t = h + \delta$ to select an element $v_0(t_{k+1}^0) \in X$ such that $(t_{k+1}^0, v_0(t_{k+1}^0)) \in \Omega \cap S_r(t, x)$ and $|v_0(t_{k+1}^0) - S(\delta)v_0(t_k^0) - \delta B(t_k^0, v_0(t_k^0))| \leqq \delta\varepsilon$. Hence (6.13) and (6.15) hold for k replaced by $k+1$. From this and (6.14) it follows that $|v_0(t_{k+1}^0) - S(t_{k+1}^0 - x)| \leqq (t_{k+1}^0 - t)(M+1)$. Thus the desired sequence is constructed. We now set

(6.16) $v_0(s) = v_0(\gamma_0(s))$

for $s \in [t, t^*)$. Then it follows from (6.13)–(6.15) that v_0 satisfies (6.7)–(6.9) with $n = 0$. This completes the first stage of our construction.

Next we find the sequence $\{v_n\}_{n \geqq 1}$ by induction. Assume that v_n has been defined in such a way that (6.7)–(6.10) hold for $s \in [t, t^*)$. To construct v_{n+1} on $[t, t^*)$, we first specify the values of v_{n+1} on the set I_{n+1}. Let $s \in I_{n+1}$. If $s \in I_n$, we set $v_{n+1}(s) = v_n(s)$. If $s \in I_{n+1} - I_n$, we can choose with the aid of Proposition 5.1 with $h = \gamma_n(s) - t$, $y = v_n(s)$ and $h^* = s - t$, an element, say $v_{n+1}(s)$, so that $(s, v_{n+1}(s)) \in \Omega \cap S_r(t, x)$ and

$$|v_{n+1}(s) - S(s-\gamma_n(s))v_n(s) - (s-\gamma_n(s))B(\gamma_n(s), v_n(s))| \leqq (s-\gamma_n(s))\varepsilon.$$

Set

$$(6.17) \qquad v_{n+1}(s) = v_{n+1}(\gamma_{n+1}(s))$$

for $s \in [t, t^*)$. Then we infer from the definition of v_{n+1} that

$(\gamma_{n+1}(s), v_{n+1}(s)) \in \Omega \cap S_r(t, x)$,

$|v_{n+1}(s) - S(\gamma_{n+1}(s) - \gamma_n(s))v_n(s) - (\gamma_{n+1}(s) - \gamma_n(s))B(\gamma_n(s), v_n(s))| \leqq (\gamma_{n+1}(s) - \gamma_n(s))\varepsilon$

and

$$|v_{n+1}(s) - S(\gamma_{n+1}(s) - t)x| \leqq (\gamma_{n+1}(s) - t)(M+1).$$

Thus a sequence $\{v_n\}_{n \geqq 0}$ of functions satisfying (6.7)–(6.10) has been constructed.

We now show that the sequences $\{\gamma_n\}_{n \geqq 0}$ and $\{v_n\}_{n \geqq 0}$ constructed above enjoy properties (6.11) and (6.12). Let $s \in [t, t^*)$. Let k be such that $\gamma_0(s) = t_k^0$. First, in view of Lemma 5.1, we observe that $v_n(s)$ can be written as

$$\begin{aligned} v_n(s) &= v_n(\gamma_n(s)) \\ &= \textstyle\sum_{j=0}^{n-1} S(\gamma_n(s) - \gamma_{j+1}(s))\,[v_{j+1}(s) - S(\gamma_{j+1}(s) - \gamma_j(s))v_j(s) \\ &\quad - (\gamma_{j+1}(s) - \gamma_j(s))B(\gamma_j(s), v_j(s))] \\ &\quad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))S(\gamma_n(s) - \gamma_{j+1}(s))B(\gamma_j(s), v_j(s)) \\ &\quad + \textstyle\sum_{j=0}^{k-1} S(\gamma_n(s) - t_{j+1}^0)\,[v_0(t_{j+1}^0) - S(\delta)v_0(t_j^0) - \delta B(t_j^0, v_0(t_j^0))] \\ &\quad + \textstyle\sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)S(\gamma_n(s) - t_{j+1}^0)B(t_j^0, v_0(t_j^0)) + S(\gamma_n(s) - t)x \end{aligned}$$

for $n \geqq 0$. Since $(\gamma_j(s), v_j(s)) \in \Omega \cap S_r(t, x)$ for $0 \leqq j \leqq n$, we have

$$\begin{aligned} &|v_n(s) - S(\gamma_n(s) - t)x - (\gamma_n(s) - t)B(t, x)| \\ &\leqq \textstyle\sum_{j=0}^{n-1} |v_{j+1}(s) - S(\gamma_{j+1}(s) - \gamma_j(s))v_j(s) - (\gamma_{j+1}(s) - \gamma_j(s))B(\gamma_j(s), v_j(s))| \\ &\quad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))|B(\gamma_j(s), v_j(s)) - B(t, x)| \\ &\quad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))|S(\gamma_n(s) - \gamma_{j+1}(s))B(t, x) - B(t, x)| \\ &\quad + \textstyle\sum_{j=0}^{k-1} |v_0(t_{j+1}^0) - S(\delta)v_0(t_j^0) - \delta B(t_j^0, v_0(t_j^0))| \\ &\quad + \textstyle\sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)|B(t_j^0, v_0(t_j^0)) - B(t, x)| \\ &\quad + \textstyle\sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)|S(\gamma_n(s) - t_{j+1}^0)B(t, x) - B(t, x)| \\ &\leqq \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))\varepsilon + \sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))\varepsilon/4 \\ &\quad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s) - \gamma_j(s))\varepsilon/4 \\ &\quad + \textstyle\sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)\varepsilon + \sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)\varepsilon/4 + \sum_{j=0}^{k-1} (t_{j+1}^0 - t_j^0)\varepsilon/4 \\ &\leqq 2(\gamma_n(s) - t)\varepsilon \end{aligned}$$

for $n \geqq 0$. Similarly, we have

$$
\begin{aligned}
&|v_n(s) - S(\gamma_n(s)-\gamma_0(s))v_0(s) - (\gamma_n(s)-\gamma_0(s))B(\gamma_0(s), v_0(s))| \\
&\quad \leqq \textstyle\sum_{j=0}^{n-1} |v_{j+1}(s) - S(\gamma_{j+1}(s)-\gamma_j(s))v_j(s) - (\gamma_{j+1}(s)-\gamma_j(s))B(\gamma_j(s), v_j(s))| \\
&\qquad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s)-\gamma_j(s))|B(\gamma_j(s), v_j(s)) - B(t, x)| \\
&\qquad + \textstyle\sum_{j=0}^{n-1} (\gamma_{j+1}(s)-\gamma_j(s))|S(\gamma_n(s) - \gamma_{j+1}(s))B(t, x) - B(t, x)| \\
&\qquad + (\gamma_n(s)-\gamma_0(s))|B(\gamma_0(s), v_0(s)) - B(t, x)| \\
&\quad \leqq 2(\gamma_n(s)-\gamma_0(s))\varepsilon
\end{aligned}
$$

for $n \geqq 0$ and $s \in [t, t^*)$. This completes the proof.

LEMMA 6.2. *Let (t, x), ε, r, t^* and δ be as in Lemma 6.1. Then there exists an X-valued, strongly measurable function v on $[t, t^*)$ with the following properties*:

(6.18) $\quad v(t) = x$ *and* $(s, v(s)) \in \Omega \cap S_r(t, x)$ *for* $s \in [t, t^*)$.

(6.19) $\quad |v(s) - S(s-t)x - (s-t)B(t, v(t))| \leqq 2(s-t)\varepsilon$ *for* $s \in [t, t^*)$.

(6.20) $\quad |v(s) - S(\delta)v(s-\delta) - \delta B(s, v(s))| \leqq 7\delta\varepsilon$ *for* $s \in [t+\delta, t^*)$.

PROOF. By Lemma 6.1 one finds a sequence $\{\gamma_n\}_{n\geqq 0}$ of real-valued step functions on $[t, t^*)$ and a sequence $\{v_n\}_{n\geqq 0}$ of X-valued step functions on $[t, t^*)$ satisfying (6.2) through (6.12). Since $\lim_{n\to\infty} \gamma_n(s) = s$ and $v_n(s) = v_n(\gamma_n(s))$, it follows from (6.10) and Lemma 5.2 that the sequence $\{v_n(s)\}_{n\geqq 0}$ is a Cauchy sequence in X for each $s \in [t, t^*)$. We then define a function v on $[t, t^*)$ by $v(s) = \lim_{n\to\infty} v_n(s)$. Clearly, v is strongly measurable; and (6.7) and (Ω1) together imply that $(s, v(s)) \in \Omega \cap S_r(t, x)$. Moreover, by use of (6.11) and (6.12), we obtain

$$|v(s) - S(s-t)x - (s-t)B(t, x)| \leqq 2(s-t)\varepsilon$$

and

$$|v(s) - S(s-\gamma_0(s))v_0(s) - (s-\gamma_0(s))B(\gamma_0(s), v_0(s))| \leqq 2(s-\gamma_0(s))\varepsilon$$

for $s \in [t, t^*)$. Using the terms $\pm S(s-\gamma_0(s))v_0(s)$, $\mp(s-\gamma_0(s))B(\gamma_0(s), v_0(s))$, $\pm S(\delta)S(s-\gamma_0(s))v_0(s-\delta)$, $\pm S(\delta)(s-\gamma_0(s))B(\gamma_0(s-\delta), v_0(s-\delta))$, etc., we have

$$
\begin{aligned}
&v(s) - S(\delta)v(s-\delta) - \delta B(s, v(s)) \\
&\quad = v(s) - S(s-\gamma_0(s))v_0(s) - (s-\gamma_0(s))B(\gamma_0(s), v_0(s)) \\
&\qquad - S(\delta)[v(s-\delta) - S(s-\gamma_0(s))v_0(s-\delta) - (s-\gamma_0(s))B(\gamma_0(s-\delta), v_0(s-\delta))] \\
&\qquad + S(s-\gamma_0(s))[v_0(s) - S(\delta)v_0(s-\delta) - \delta B(\gamma_0(s-\delta), v_0(s-\delta))] \\
&\qquad + (s-\gamma_0(s))[B(\gamma_0(s), v_0(s)) - B(t, x)]
\end{aligned}
$$

$$
\begin{aligned}
&-(s-\gamma_0(s))S(\delta)[B(\gamma_0(s-\delta), v_0(s-\delta))-B(t, x)]\\
&+\delta S(s-\gamma_0(s))[B(\gamma_0(s-\delta), v_0(s-\delta))-B(t, x)]-\delta[B(s, v(s))-B(t, x)]\\
&-(s-\gamma_0(s))[S(\delta)B(t, x)-B(t, x)]+\delta[S(s-\gamma_0(s))B(t, x)-B(t, x)].
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
&|v(s)-S(\delta)v(s-\delta)-\delta B(s, v(s))|\\
&\quad\leqq 2(s-\gamma_0(s))\varepsilon+2(s-\gamma_0(s))\varepsilon+\delta\varepsilon+(s-\gamma_0(s))\varepsilon/4+(s-\gamma_0(s))\varepsilon/4\\
&\qquad+\delta\varepsilon/4+\delta\varepsilon/4+(s-\gamma_0(s))\varepsilon/4+\delta\varepsilon/4\leqq 7\delta\varepsilon
\end{aligned}
$$

for all $s\in[t, t^*)$. This completes the proof of Lemma 6.2.

LEMMA 6.3. *Let $(\tau, z)\in\Omega$. Let $R>0$ and $M>0$ be such that $\tau+R<b$ and $|B(t, x)|\leqq M$ for $(t, x)\in\Omega\cap S_R(\tau, z)$. Let $T>0$ be small enough to satisfy $T(M+1)+\sup_{0\leqq\sigma\leqq T}|S(\sigma)z-z|<R$. Then for each $\varepsilon\in(0, 1)$ there exists a sequence $\{(t_i, x_i)\}_{0\leqq i\leqq N}$ in Ω with the properties listed below.*

(i) $(t_0, x_0)=(\tau, z)$ *and* $t_N=\tau+T$.
(ii) $0<t_{i+1}-t_i\leqq\varepsilon$ *for* $0\leqq i\leqq N-1$.
(iii) $(t_i, x_i)\in\Omega\cap S_R(\tau, z)$ *for* $0\leqq i\leqq N$.
(iv) $|x_{i+1}-S(t_{i+1}-t_i)x_i-(t_{i+1}-t_i)B(t_i, x_i)|\leqq(t_{i+1}-t_i)\varepsilon$ *for* $0\leqq i\leqq N-1$.
(v) *For each i with $0\leqq i\leqq N-1$ there eixsts a number $r(i)\in(0, \varepsilon]$ such*

that

$$S_{r(i)}(t_i, x_i)\subset S_R(\tau, z), \tag{6.21}$$

$$|B(s, y)-B(t_i, x_i)|\leqq\varepsilon/4 \text{ for all } (s, y)\in\Omega\cap S_{r(i)}(t_i, x_i), \tag{6.22}$$

$$\sup_{0\leqq\sigma\leqq r(i)}|S(\sigma)B(t_i, x_i)-B(t_i, x_i)|\leqq\varepsilon/4, \tag{6.23}$$

and

$$(t_{i+1}-t_i)(M+1)+\sup\{|S(\sigma)x_i-x_i|;\ 0\leqq\sigma\leqq t_{i+1}-t_i\}\leqq r(i). \tag{6.24}$$

PROOF. The proof can be given in a way similar to that of Proposition 5.1. Let $\varepsilon\in(0, 1)$. Set $(t_0, x_0)=(\tau, z)$. We define a sequence $\{(t_i, x_i)\}_{0\leqq i\leqq N}$ in $\Omega\cap S_R(\tau, z)$ in the following manner: Suppose that (t_i, x_i) is defined in $\Omega\cap S_R(\tau, z)$ in such a way that $t_i\in[\tau, \tau+T]$ and $|x_i-S(t_i-\tau)z|\leqq(t_i-\tau)(M+1)$. First we take the supremum $r(i)$ of all $r\in(0, \varepsilon]$ such that

$$S_r(t_i, x_i)\subset S_R(\tau, z), \tag{6.25}$$

$$|B(s, y)-B(t_i, x_i)|\leqq\varepsilon/4 \quad \text{for all} \quad (s, y)\in\Omega\cap S_r(t_i, x_i) \tag{6.26}$$

and

$$\sup_{0 \leq \sigma \leq r} |S(\sigma)B(t_i, x_i) - B(t_i, x_i)| \leq \varepsilon/4. \tag{6.27}$$

Since $r(i)>0$, we set $h_i = \sup \{h \in (0, b-t);\ h(M+1) + \sup_{0 \leq \sigma \leq h} |S(\sigma)x_i - x_i| \leq r(i)\}$ and define $t_{i+1} = \min\{t_i + h_i, \tau + T\}$. Observe that $t_i < t_{i+1}$ whenever $t_i < \tau + T$. Next, by using Proposition 5.1, one finds an element x_{i+1} of $\Omega(t_{i+1})$ such that

$$|x_{i+1} - S(t_{i+1}-t_i)x_i - (t_{i+1}-t_i)B(t_i, x_i)| \leq (t_{i+1}-t_i)\varepsilon.$$

From this and the hypothesis on x_i we infer that

$$\begin{aligned} &|x_{i+1} - S(t_{i+1}-\tau)z| \\ &\quad \leq |x_{i+1} - S(t_{i+1}-t_i)x_i| + |S(t_{i+1}-t_i)x_i - S(t_{i+1}-\tau)z| \\ &\quad \leq (t_{i+1}-t_i)(M+1) + (t_i-\tau)(M+1) \\ &\quad = (t_{i+1}-\tau)(M+1) \end{aligned}$$

and hence $|x_{i+1}-z| \leq |x_{i+1} - S(t_{i+1}-\tau)z| + |S(t_{i+1}-\tau)z - z| < R$. This shows that $(t_{i+1}, x_{i+1}) \in \Omega \cap S_R(\tau, z)$. We continue this induction argument.

We now claim that $t_N = \tau + T$ for some integer $N \geq 1$. Assume to the contrary that $t_i < \tau + T$ for all $i \geq 0$. Then, by Lemma 5.2, $(t_\infty, x_\infty) = \lim_{i\to\infty}(t_i, x_i)$ exists and $(t_\infty, x_\infty) \in \Omega \cap S_R(\tau, z)$. Since the set $\{(t_i, x_i);\ 0 \leq i \leq \infty\}$ is compact in Ω, there is a number $r \in (0, \varepsilon]$ such that $S_r(t_i, x_i) \subset S_R(\tau, z)$, $|B(s, y) - B(t_i, x_i)| \leq \varepsilon/4$ and $\sup_{0 \leq \sigma \leq r} |S(\sigma)B(t_i, x_i) - B(t_i, x_i)| \leq \varepsilon/4$ for $(s, y) \in \Omega \cap S_r(t_i, x_i)$ and $0 \leq i \leq \infty$. Further, there is a number $h>0$ such that $h(M+1) + \sup_{0 \leq \sigma \leq h} |S(\sigma)x_i - x_i| \leq r$. But in virtue of the definition of $r(i)$ and h_i, we would have $h_i \geq h$ for $i \geq 0$. This contradicts the fact that $h_i = t_{i+1} - t_i \to 0$ as $i \to \infty$. Hence we conclude that there is an integer $N \geq 1$ such that $t_{N-1} < t_N = \tau + T$. It is now easy to see that the sequence $\{(t_i, x_i)\}_{0 \leq i \leq N}$ has the properties (i)–(iv). This completes the proof.

Proposition 6.1 is a direct consequence of the following lemma.

LEMMA 6.4. *Let $(\tau, z) \in \Omega$. Let $R>0$ and $M>0$ be such that $\tau + R < b$ and $|B(t, x)| \leq M$ for $(t, x) \in \Omega \cap S_R(\tau, z)$. Let $T>0$ be small enough to satisfy $T(M+1) + \sup_{0 \leq \sigma \leq T} |S(\sigma)z - z| < R$. Let $\varepsilon \in (0, 1)$ and let $\{(t_i, x_i)\}_{0 \leq i \leq N}$ be a sequence in Ω as in Lemma 6.3. Then there exist a number δ_ε and a family $\{u_\delta;\ \delta \in (0, \delta_\varepsilon]\}$ of X-valued strongly measurable functions on $[\tau, \tau+T]$ such that*

(a) $u_\delta(t_i) = x_i$ *for* $0 \leq i \leq N$ *and* $(t, u_\delta(t)) \in \Omega \cap S_R(\tau, z)$ *for* $t \in [\tau, \tau+T]$,

(b) $|u_\delta(t) - S(t-\tau)z - \int_\tau^t S(t-s)B(s, u_\delta(s))ds| \leq 3(t-\tau)\varepsilon$ *for* $t \in [\tau, \tau+T]$,

(c) $|u_\delta(t) - u_\delta(s)| \leqq 7\varepsilon$ *for* $t, s \in [\tau, \tau+T]$ *with* $|t - s| \leqq \delta$,

(d) $\int_{\tau+\delta}^{\tau+T} |u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))|ds \leqq 14T\delta\varepsilon$,

(e) $|u_\delta(t) - u_{\delta'}(t)| \leqq 4\varepsilon^2$ *for* $t \in [\tau, \tau+T]$ *and* $\delta, \delta' \in (0, \delta_\varepsilon]$.

PROOF. Let $\varepsilon \in (0, 1)$. Set $\delta_\varepsilon = \min\{(t_{i+1} - t_i)/2\colon 0 \leqq i \leqq N-1\}$ and fix any $\delta \in (0, \delta_\varepsilon]$. Applying Lemma 6.2 with $t = t_i$ and $t^* = t_{i+1}$, one finds an X-valued strongly measurable function v_i on $[t_i, t_{i+1})$ satisfying

(6.28) $v_i(t_i) = x_i$ and $(t, v_i(t)) \in \Omega \cap S_R(\tau, z)$ for $t \in [t_i, t_{i+1})$;

(6.29) $|v_i(t) - S(t-t_i)x_i - (t-t_i)B(t_i, v_i(t_i)| \leqq 2(t-t_i)\varepsilon$ for $t \in [t_i, t_{i+1})$;

(6.30) $|v_i(t) - S(\delta)v_i(t-\delta) - \delta B(t, v(t))| \leqq 7\delta\varepsilon$ for $t \in [t_i+\delta, t_{i+1})$.

For each $t \in [\tau, \tau+T]$, set

$$u_\delta(t) = v_i(t) \quad \text{if} \quad t \in [t_i, t_{i+1}), \quad \text{and} \quad u_\delta(\tau+T) = x_N,$$

Then it is clear that the function u_δ is strongly measurable and satisfies conditon (a). Let $t \in [\tau, \tau+T]$ and let i be the integer such that $t \in [t_i, t_{i+1})$. Then we have the following relation:

$$\begin{aligned}
(6.31)\quad & u_\delta(t) - S(t-\tau)z - \int_\tau^t S(t-s)B(s, u_\delta(s))ds \\
&= u_\delta(t) - S(t-t_i)x_i - \int_{t_i}^t S(t-s)B(s, u_\delta(s))ds \\
&\quad + \textstyle\sum_{j=0}^{i-1} S(t-t_{j+1})[x_{j+1} - S(t_{j+1}-t_j)x_j \\
&\qquad\qquad - \displaystyle\int_{t_j}^{t_{j+1}} S(t_{j+1}-s)B(s, u_\delta(s))ds] \\
&= u_\delta(t) - S(t-t_i)x_i - (t-t_i)B(t_i, x_i) \\
&\quad - \int_{t_i}^t S(t-s)[B(s, v_i(s)) - B(t_i, x_i)]ds \\
&\quad - \int_{t_i}^t [S(t-s)B(t_i, x_i) - B(t_i, x_i)]ds \\
&\quad + \textstyle\sum_{j=0}^{i-1} S(t-t_{j+1})[x_{j+1} - S(t_{j+1}-t_j)x_j - (t_{j+1}-t_j)B(t_j, x_j)] \\
&\quad - \textstyle\sum_{j=0}^{i-1} S(t-t_{j+1})\displaystyle\int_{t_j}^{t_{j+1}} S(t_{j+1}-s)[B(s, v_j(s)) - B(t_j, x_j)]ds \\
&\quad - \textstyle\sum_{j=0}^{i-1} S(t-t_{j+1})\displaystyle\int_{t_j}^{t_{j+1}} [S(t_{j+1}-s)B(t_j, x_j) - B(t_j, x_j)]ds.
\end{aligned}$$

It follows from (6.25)–(6.29) and (6.31) that

$$|u_\delta(t) - S(t-\delta)z - \int_\tau^t S(t-s)B(s, u_\delta(s))ds| \leqq 3(t-\tau)\varepsilon$$

which is nothing but condition (b).

Let s, $t \in [\tau, \tau+T]$ with $s<t<s+\delta$. Since $\delta<t_{i+1}-t_i$, we have

$$\begin{aligned}&(t-s)M + |S(t-s)x_i - x_i| \\ &\quad < (t_{i+1}-t_i)(M+1) + \sup\{|S(\sigma)x_i - x_i|;\ 0 \leqq \sigma \leqq t_{i+1}-t_i\} < \varepsilon.\end{aligned}$$

If $t_i \leqq s<t<t_{i+1}$ for some $i \geqq 0$, then we have

$$\begin{aligned}|u_\delta(t) - u_\delta(s)| &\leqq |v_i(t) - S(t-t_i)x_i - (t-t_i)B(t_i, x_i)| \\ &\quad + |v_i(s) - S(s-t_i)x_i - (s-t_i)B(t_i, x_i)| \\ &\quad + (t-s)|B(t_i, x_i)| + |S(t-s)x_i - x_i| \\ &\leqq 2(t-t_i)\varepsilon + 2(s-t_i)\varepsilon + \varepsilon \leqq 5\varepsilon,\end{aligned}$$

since $t_{i+1}-t_i \leqq \varepsilon < 1$. If $t_{i-1} \leqq s < t_i \leqq t < t_{i+1}$ for some $i \geqq 1$,

$$\begin{aligned}&|u_\delta(t) - u_\delta(s)| \\ &\leqq |v_i(t) - S(t-t_i)x_i - (t-t_i)B(t_i, x_i)| \\ &\quad + |v_{i-1}(s) - S(s-t_{i-1})x_i - (s-t_{i-1})B(t_{i-1}, x_{i-1})| \\ &\quad + |S(t-t_i)[x_i - S(t_i-t_{i-1})x_{i-1} - (t_i-t_{i-1})B(t_{i-1}, x_{i-1})]| \\ &\quad + (t_i-t_{i-1})|S(t-t_i)B(t_{i-1}, x_{i-1}) - B(t_{i-1}, x_{i-1})| \\ &\quad + (t-t_i)|B(t_i, x_i)| + (t_i-s)|B(t_{i-1}, x_{i-1})| \\ &\quad + |S(t-t_{i+1})[S(t-s)x_{i-1} - x_{i-1}]| \\ &\leqq 2(t-t_i)\varepsilon + 2(s-t_{i-1})\varepsilon + (t_i-t_{i-1})\varepsilon + (t_i-t_{i-1})\varepsilon/4 + \varepsilon \leqq 7\varepsilon.\end{aligned}$$

Thus conditon (c) is satisfied.

To see that u_δ satisfies condition (d), we estimate the norm of

$$|u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))|$$

for $s \in [\tau+\delta, \tau+T]$. If $t_i+\delta \leqq s < t_{i+1}$ for some i, then (6.30) yields

$$|u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))| \leqq 7\delta\varepsilon. \tag{6.32}$$

If $t_i \leqq s < t_i+\delta$ for some i, then we have

$$\begin{aligned}&|u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))| \\ &\leqq |v_i(s) - S(s-t_i)x_i - (s-t_i)B(t_i, x_i)| \\ &\quad + |S(\delta)[v_{i-1}(s-\delta) - S(s-\delta-t_{i-1})x_{i-1} - (s-\delta-t_{i-1})B(t_{i-1}, x_{i-1})]|\end{aligned} \tag{6.33}$$

$$
\begin{aligned}
&+ |S(s-t_i)[x_i - S(t_i - t_{i-1})x_{i-1} - (t_i - t_{i-1})B(t_{i-1}, x_{i-1})]| \\
&+ (t_i + \delta - s)|B(t_i, x_i) - B(t_{i-1}, x_{i-1})| \\
&+ \delta|B(s, v_i(s)) - B(t_i, x_i)| \\
&+ (t_i - t_{i-1})|S(s-t_i)B(t_i, x_i) - B(t_i, x_i)| \\
&+ (s - \delta - t_{i-1})|S(\delta)B(t_{i-1}, x_{i-1}) - B(t_{i-1}, x_{i-1})| \\
\leqq\ & 2(s-t_i)\varepsilon + 2(s-\delta-t_{i-1})\varepsilon + (t_i - t_{i-1})\varepsilon \\
&+ (t_i + \delta - s)\varepsilon/4 + \delta\varepsilon/4 + (t_i - t_{i-1})\varepsilon/4 + (s-\delta-t_{i-1})\varepsilon/4 \\
\leqq\ & 7(t_i - t_{i-1})\varepsilon.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
&\int_{\tau+\delta}^{\tau+T} |u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))|ds \\
&= \sum_{i=0}^{N-1} \int_{t_i+\delta}^{t_{i+1}} |u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))|ds \\
&\quad + \sum_{i=1}^{N-1} \int_{t_i+\delta}^{t_{i+1}} |u_\delta(s) - S(\delta)u_\delta(s-\delta) - \delta B(s, u_\delta(s))|ds.
\end{aligned}
$$

Applying (6.32) and (6.33) respectively to the first and the second sums on the right side, we see that the left integral is not greater than

$$\sum_{i=1}^{N-1} 7(t_{i+1} - t_i)\delta\varepsilon + \sum_{i=0}^{N-1} 7(t_i - t_{i-1})\delta\varepsilon \leqq 14T\delta\varepsilon.$$

Thus we conclude that u_δ satisfies conditon (d).

To complete the proof, take any pair $\delta, \delta' \in (0, \delta_\varepsilon]$. Let u_δ and $u_{\delta'}$ be the strongly measurable functions on $[\tau, \tau+T]$ constructed for δ and δ', respectively. Let $t \in [\tau, \tau+T]$ and i be such that $t \in [t_i, t_{i+1})$. Then (6.29) implies

$$
\begin{aligned}
&|u_\delta(t) - u_{\delta'}(t)| \\
&\leqq |u_\delta(t) - S(t-t_i)x_i - (t-t_i)B(t_i, x_i)| \\
&\quad + |u_{\delta'}(t) - S(t-t_i)x_i - (t-t_i)B(t_i, x_i)| \\
&\leqq 4(t-t_i)\varepsilon \leqq 4\varepsilon^2,
\end{aligned}
$$

which shows that condition (e) holds. Thus the proof is complete.

7. Local existence

In this section we give a result on the local existence of mild solutions to the problems $(IVP; \tau, z)$.

THEOREM 7.1. *Suppose that conditions* $(\Omega 1)$–$(\Omega 3)$ *are satisfied. Let* $(\tau, z)\in\Omega$. *Let* $R>0$ *and* $M>0$ *satisfy* $\tau+R<b$ *and* $|B(t, x)|\leqq M$ *for* $(t, x)\in\Omega\cap S_R(\tau, z)$. *Let* T *be a positive number such that* $T(M+1)+\sup_{0\leqq\sigma\leqq T}|S(\sigma)z-z|<R$. *Then the problem* $(IVP;\tau, z)$ *has a unique mild solution* u *on* $[\tau, \tau+T]$.

PROOF. Set $r=2(R+|z|)$. Let L_r be an integrable function on $[\tau, \tau+T]$ such that $|g(t, w)|\leqq L_r(t)$ for $t\in[\tau, \tau+T]$ and for w with $|w|\leqq r$; the existence of such function L_r is guaranteed by condition $(g1)$. Let $\{\varepsilon_n\}_{n\geqq 1}$ be a null-sequence in $(0, 1)$. Then Proposition 6.1 implies that for each $n\geqq 1$ there exist a number δ_n and a family $\{u_\delta^n;\ \delta\in(0, \delta_n]\}$ of ε_n-approximate solutions for $(IVP;\tau, z)$ on $[\tau, \tau+T]$ with properties $(P1)$ through $(P4)$. Let m and n be positive integers. Let $\delta>0$ be such that $\delta\leqq\delta_m$ and $\delta\leqq\delta_n$ and such that

$$\int_s^t L_r(\xi)d\xi \leqq \varepsilon_m+\varepsilon_n \tag{7.1}$$

for $\tau\leqq s\leqq t\leqq\tau+T$ with $|t-s|\leqq\delta$. Set $U_\delta^{m,n}(s)=|u_\delta^m(s)-u_\delta^n(s)|$ for $s\in[\tau, \tau+T]$. By $(\Omega 3)$, we have

$$\begin{aligned}
&|u_\delta^m(s)-u_\delta^n(s)|\\
&\quad\leqq |u_\delta^m(s)-u_\delta^n(s)-\delta(B(s, u_\delta^m(s))-B(s, u_\delta^n(s)))|+\delta g(s, |u_\delta^m(s)-u_\delta^n(s)|)\\
&\quad\leqq |u_\delta^m(s-\delta)-u_\delta^n(s-\delta)|+|u_\delta^m(s)-S(\delta)u_\delta^m(s-\delta)-\delta B(s, u_\delta^m(s))|\\
&\qquad+|u_\delta^n(s)-S(\delta)u_\delta^n(s-\delta)-\delta B(s, u_\delta^n(s))|+\delta g(s, |u_\delta^m(s)-u_\delta^n(s)|)
\end{aligned}$$

for $s\in[\tau+\delta, \tau+T]$. Let t_1 and t_2 be such that $\tau\leqq t_1<t_1+\delta<t_2\leqq\tau+T$. Integrating both sides of (7.2) from $t_1+\delta$ to t_2 and using $(P3)$, we obtain

$$\begin{aligned}
&\int_{t_2-\delta}^{t_2} U_\delta^{m,n}(s)\,ds-\int_{t_1}^{t_1+\delta} U_\delta^{m,n}(s)\,ds\\
&\quad\leqq \delta\int_{t_1+\delta}^{t_2} g(s, U_\delta^{m,n}(s))ds+\delta(\varepsilon_m+\varepsilon_n)
\end{aligned}\tag{7.3}$$

Since $|U_\delta^{m,n}(t)-U_\delta^{m,n}(s)|\leqq|u_\delta^m(t)-u_\delta^m(s)|+|u_\delta^n(t)-u_\delta^n(s)|\leqq\varepsilon_m+\varepsilon_n$ for $t, s\in[\tau, \tau+T]$ with $|t-s|\leqq\delta$ by $(P2)$, we obtain

$$\delta U_\delta^{m,n}(t_2)\leqq\int_{t_2-\delta}^{t_2} U_\delta^{m,n}(s)ds+\delta(\varepsilon_m+\varepsilon_n)$$

and

$$\delta U_\delta^{m,n}(t_1)\geqq\int_{t_1}^{t_1+\delta} U_\delta^{m,n}(s)ds-\delta(\varepsilon_m+\varepsilon_n).$$

Hence

$$(7.4) \qquad \delta\{U_\delta^{m,n}(t_2)-U_\delta^{m,n}(t_1)\} \leqq \int_{t_2-\delta}^{t_2} U_\delta^{m,n}(s)ds - \int_{t_1}^{t_1+\delta} U_\delta^{m,n}(s)ds + 2\delta(\varepsilon_m+\varepsilon_n).$$

From (7.1), (7.3) and (7.4) it follows that

$$(7.5) \qquad U_\delta^{m,n}(t_2) - U_\delta^{m,n}(t_1) \leqq \int_{t_1}^{t_2} g(s, U_\delta^{m,n}(s))ds + 4(\varepsilon_m+\varepsilon_n).$$

For simplicity in notation we write u^m for u_δ^m with $\delta=\delta_m$. Since

$$|U_\delta^{m,n}(t) - |u^m(t) - u^n(t)|| \leqq |u_\delta^m(t) - u^m(t)| + |u_\delta^n(t) - u^n(t)| \leqq \varepsilon_m + \varepsilon_n$$

for $t\in[\tau, \tau+T]$ by ($P4$), combining (7.5) with Proposition 3.2, we see that

$$(7.6) \qquad \lim_{m,n\to\infty} |u^m(t) - u^n(t)| = 0$$

holds for $t\in[\tau, \tau+T]$ and the convergence is uniform on $[\tau, \tau+T]$. This means that $\{u^n\}_{n\geqq 1}$ is uniformly Cauchy on $[\tau, \tau+T]$.

We now define $u(t)=\lim_{n\to\infty} u^n(t)$ for each $t\in[\tau, \tau+T]$. It is clear that $u(\tau)=z$ and $(t, u(t))\in\Omega$ for $t\in[\tau, \tau+T]$ by ($P1$) and (Ω1). Also, the continuity of u is deduced from ($P2$) and the uniform convergence of $\{u^n\}_{n\geqq 1}$. Since u^n is an ε_n-approximate solution for (IVP; τ, z), the application of the Lebesgue convergence theorem yields

$$u(t) = S(t-\tau)z + \int_\tau^t S(t-s)B(s, u(s))ds$$

for all $t\in[\tau, \tau+T]$. This show that u is a mild solution to (IVP; τ, z) on $[\tau, \tau+T]$. Since the uniqueness of u follows from Proposition 4.1, the proof is complete.

8. Existence in the large

This section is devoted to the verification of our main result on the global existence we mentioned in Section 1.

In the previous section we established a result on the uniqueness and local existence of mild solutions of the problems (IVP; τ, z), $(\tau, z)\in\Omega$. By virtue of this result, we may think of a family of mild solutions $u(t; \tau, z)$ of (IVP; τ, z), $(\tau, z)\in\Omega$, which are not continuable to the right. Using this family, we may construct a continuous local semiflow by

$$(8.1) \qquad U(t, \tau, z) = u(t, \tau, z), \quad t\in[\tau, T(\tau, z)).$$

In view of Theorem 2.2, the following proposition plays an important role in proving the global existence of mild solutions of $(IVP; \tau, z)$.

PROPOSITION 8.1. *Suppose that conditions* $(\Omega 1)$–$(\Omega 3)$ *are satisfied. Let* $(\tau, z) \in \Omega$. *Then there is a number* c *with the following properties*:

(i) $\tau < c < b$ *and* $(IVP; \tau, \ )$ *has a unique mild solution* u *on* $[\tau, c]$.

(ii) *Let* $\varepsilon > 0$. *Then there is a number* $r > 0$ *such that* $\tau + r < c$ *and for every* $(t, x) \in \Omega \cap S_r(\tau, z)$, $(IVP; t, x)$ *has a unique mild solution* v *on* $[t, c]$ *and* v *satisfies* $|v(s) - u(s)| \leqq \varepsilon$ *for all* $s \in [\max\{\tau, t\}, c]$.

PROOF. Let $R > 0$ and $M > 0$ be such that $\tau + R < b$ and $|B(t, x)| \leqq M$ for $(t, x) \in \Omega \cap S_R(\tau, z)$. Let $T > 0$ be such that

$$T(M+1) + \sup_{0 \leqq \sigma \leqq T} |S(\sigma)z - z| < R.$$

We shall see that any number c in the interval $(\tau, \tau + T)$ is the desired one. The first property follows from Theorem 7.1. To show that c has the second property, let $\varepsilon > 0$. By Lemma 3.1, one can find an $\eta > 0$ such that a maximal solution $m(t; \tau, \eta)$ exists on $[\tau, c]$ and $m(s; \tau, \eta) \leqq \varepsilon$ for all $s \in [\tau, c]$. Choose an $r > 0$ so that $\tau + r \leqq c$, $r \leqq \tau + T - c$, $(c - \tau)(M+1) + r(M+4) + \sup_{0 \leqq \sigma \leqq T} |S(\sigma)z - z| < R$ and $\sup_{0 \leqq \sigma \leqq r} |S(\sigma)z - z| + r(M+3) \leqq \inf_{0 \leqq \sigma \leqq r} m(\tau + \sigma; \tau, \eta)$. This is possible since $\tau < c < \tau + T$ and $\inf_{0 \leqq \sigma \leqq r} m(\tau + \sigma; \tau, \eta) > 0$ for sufficiently small $r > 0$. Take any $(t, x) \in \Omega \cap S_r(\tau, z)$ and set $R^* = R - r$. Note that $c - t < c - \tau + r \leqq T$ and $|S(\sigma)x - x| \leqq |S(\sigma)z - z| + 2r$ for all $\sigma \geqq 0$. Since $|B(s, y)| \leqq M$ for all $(s, y) \in \Omega \cap S_{R^*}(t, x)$ and since

$$\begin{aligned}
&(c - t)(M+1) + \sup_{0 \leqq \sigma \leqq c-t} |S(\sigma)x - x| \\
&\leqq (c - t)(M+1) + \sup_{0 \leqq \sigma \leqq T} |S(\sigma)z - z| + 2r \\
&\leqq (c - \tau)(M+1) + r(M+3) + \sup_{0 \leqq \sigma \leqq T} |S(\sigma)z - z| \\
&< R^* \text{ by the choice of } r > 0,
\end{aligned}$$

Theorem 7.1 implies that the problem $(IVP; t, x)$ has a unique mild solution v on $[t, c]$ such that $(s, v(s)) \in \Omega \cap S_R(\tau, z)$ for $s \in [t, c]$. If $t \leqq \tau$, then

$$\begin{aligned}
|v(\tau) - u(\tau)| &\leqq |v(\tau) - x| + |x - z| \\
&\leqq |S(\tau - t)x - x + \int_t^\tau S(\tau - \xi)B(\xi, v(\xi))d\xi| + r \\
&\leqq \sup_{0 \leqq \sigma \leqq r} |S(\sigma)z - z| + r(M+3) \\
&\leqq m(\tau; \tau, \eta).
\end{aligned}$$

Hence $|v(s) - u(s)| \leqq m(s; \tau, \eta)$ for $s \in [\tau, c]$ by Proposition 4.1. If $t > \tau$,

$$\begin{aligned}|v(t)-u(t)| &\leqq |x-z| + |u(t)-z| \\ &\leqq r + |S(t-\tau)z - z + \int_\tau^t S(t-\xi)B(\xi, u(\xi))d\xi| \\ &\leqq \sup_{0\leqq\sigma\leqq r}|S(\sigma)z-z| + r(M+1) \\ &\leqq m(t; \tau, \eta).\end{aligned}$$

Hence $|v(s)-u(s)|\leqq m(s; \tau, \eta)$ for $s\in[t, c]$ by Proposition 4.1. Thus the proof is complete.

We are now in a position to prove our global existence theorem.

THEOREM 8.1. *Suppose that conditons* (Ω1)-(Ω3) *are satisfied. Let C be a connected component of Ω and set $d=\sup\{t\in[a, b); C(t)\neq\phi\}$. Then for each $(\tau, z)\in C$, $(IVP; \tau, z)$ has a unique mild solution on $[\tau, d)$. In particular, if Ω itself is connected, then for each $(\tau, z)\in\Omega$. $(IVP; \tau, z)$ has a unique mild solution on $[\tau, b)$.*

PROOF. By virtue of Theorem 2.2, it suffices to show that the continuous local semiflow U defined by (8.1) satisfies conditions (1′) and (2′) stated in Theorem 2.2. (1′) follows from Proposition 8.1. Let $\{(\tau_n, z_n)\}_{n\geqq1}$ be a sequence in C such that $(\tau_n, z_n)\to(\tau, z)\in C$ as $n\to\infty$, and suppose that $T(\tau_n, z_n)>c$ for $n\geqq1$ and some number $c>\tau$. Then it follows from Proposition 8.1 that there is a number $c'\leqq c$ such that $T(\tau, z)>c'$ and $U(t, \tau_n, z_n)\to U(t, \tau, z)$ uniformly for $t\in(\tau, c']$. Combining this with Proposition 4.1, we see that $U(t, \tau_n, z_n)$ converges uniformly for $t\in(\tau, c]$ as $n\to\infty$. Define $u(t)=\lim_{n\to\infty} U(t, \tau_n, z_n)$ for $t\in(\tau, c]$. Then it follows from the Lebesgue convergence theorem that the limit function u is a mild solution to $(IVP; \tau, z)$ on $[\tau, c]$. This implies that $T(\tau, z)>c$ and $U(t, \tau_n, z_n)\to U(t, \tau, z)$ uniformly for $t\in(\tau, c]$ as $n\to\infty$. Hence (2′) holds. Thus the proof of Theorem 8.1 is complete.

9. Concluding remarks

1) From the point of view of the flow invariance for semilinear evolution equations, it might be useful to summarize our results in the following form.

THEOREM 9.1. *Suppose that Ω is connected and that conditions* (Ω1) *and* (Ω3) *are satisfied. Then the following are equivalent*:

(a) $\lim_{h\downarrow0} h^{-1}d(S(h)x+hB(t, x), \Omega(t+h))=0$ *for all* $(t, x)\in\Omega$.

(b) $\liminf_{h\downarrow0} h^{-1}d(S(h)x+hB(t, x), \Omega(t+h))=0$ *for all* $(t, x)\in\Omega$.

(c) *For $(t, x)\in\Omega$ and $\varepsilon\in(0, 1)$, there is a number $h_0>0$ with the following property: Let $h\in[0, h_0)$ and $y\in(t+h)$ satisfy $|y-S(h)x|\leqq h(|B(t, x)|+1)$. Then for each $h^*\in(h, h_0)$ there exists an element*

$y^* \in \Omega(t+h^*)$ such that
$|y^* - S(h^* - h)y - (h^* - h)B(t+h, y)| \leqq (h^* - h)\varepsilon.$

(d) *For each $(\tau, z) \in \Omega$, there is a mild solution u to $(IVP; \tau, z)$ on $[\tau, b)$.*

2) It should be mentioned that the topological method evolved in Section 2 is discussed in terms of local semiflow and is not affected by the characteristics of semilinear differential equations. Hence it would be applicable to a much broader class of differential equations in order to deduce the global existence from the local existence.

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