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Liapunov functions and boundedness for differential and delay equations

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1. Introduction

In the theory of Liapunov's direct method for a system of ordinary differential equations

$$x' = H(t, x) \tag{1}$$

where H: $[0, \infty) \times R^n \to R^n$ is continuous, in order to prove that all solutions tend to zero as $t \to \infty$ it suffices to find a continuous function V: $[0, \infty) \times R^n \to [0, \infty)$ and continuous functions W_i: $[0, \infty) \to [0, \infty)$, $W_i(0)=0$, $W_i(r)$ increasing, such that

(i) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$,
(ii) $V'_{(1)}(t, x) \leq -W_3(|x|)$,

and

(iii) $W_1(r) \to \infty$ as $r \to \infty$.

Corduneanu [2] showed that if

(iv) $V'_{(1)}(t, x) \leq -h(t, V)$

and if the solutions of $\{r'=h(t, r), r(t_0)=V(t_0, x_0)\}$ tend to zero, then $V(t, x(t)) \to 0$ as $t \to \infty$.

Three problems have persisted in fundamental applications: (ii), (iii), and (iv) fail. The typical example is the scalar system

$$x' = y$$
$$y' = -q(x, y)y - g(x)$$

where $q(x, y) \geq 0$, $xg(x) > 0$ if $x \neq 0$. The function

$$V(x, y) = y^2 + 2\int_0^x g(s)ds$$

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satisfies

$$V'(x, y) = -2q(x, y)y^2.$$

It is clear that (ii) fails since $V'(x, 0)=0$. Also, if $\int_0^x g(s)ds$ is bounded for $x>0$ or for $x<0$, then (iii) fails. And there seems to be no way of writing (iv) so that $r'=h(t, r)$ has all solutions tending to zero.

In this paper we show that if (ii) and (iii) fail, then this may be used to construct a V for which (iv) holds. We first formulate an abstract result to that effect and then provide detailed examples from ordinary and functional differential equations.

The difficulties displayed in this simple example have been attacked with vigor by many investigators. A far from complete list would include Krasovskii [6; p. 67 and 153], Haddock [3], Hatvani [4], Kato [5], La Salle [8], Murakami [10], and the author [1]. Those references contain extensive lists of related investigations.

General theory of ordinary and functional differential equations is found in Yoshizawa [14], for example.

2. The main result

Suppose that $V_1, P, U: [0, \infty)\times R^n\to[0, \infty)$ and $Q: [0, \infty)\times[0, \infty)\times R^n\to[0, \infty)$ are continuous functions with

$$V_1(t, x) = P(t, x) + U(t, z) \tag{2}$$

where $x=(x_1,\ldots, x_{n-1}, z)$ and that the derivative of V_1 along a solution of (1) satisfies

$$V_1'(t, x) \leq -Q(P(t, x), t, x) \tag{3}$$

with $Q(P, t, x)>0$ if $P>0$.

THEOREM 1. *Let* (2) *and* (3) *hold and suppose there is an* $L>0$ *such that if* $t_n\to\infty$ *and* $z_n\to\infty$ *then*

$$U(t_n, z_n)\to L, \quad U(t, z) < L \quad \textit{for all} \quad (t, z) \quad \textit{with} \quad z > 0. \tag{4}$$

If $V(t, x)=V_1(t, x)-L$ *and if* $x(t)$ *is a solution of* (1) *on* $[t_0, \infty)$ *with* $\overline{\lim}_{t\to\infty} z(t) =\infty$, *then*

$$V'(t, x) \leq -Q(V(t, x)/2, t, x) \qquad \textit{for} \quad z(t) > 0. \tag{5}$$

PROOF. Notice that for the solution described, $P(t, x(t))\geq L-U(t, z(t))$.

If this were false, then there would be a t_1 with $P(t_1, x(t_1)) < L - U(t_1, z(t_1))$.

(a) It is surely true that $P(t_1, x(t_1)) > 0$, otherwise

$$\begin{aligned} V(t_1, x(t_1)) &= P(t_1, x(t_1)) + U(t_1, z(t_1)) - L \\ &= U(t_1, z(t_1)) - L = -\varepsilon_1 < 0. \end{aligned}$$

Since $V'(t, x(t)) \leq 0$, we would then have $V(t, x(t)) \leq -\varepsilon_1$ for $t \geq t_1$ and this would combine with (4) to produce a contradiction, since for large z we have

$$-\varepsilon_1 \geq V(t, x(t)) = P(t, x(t)) + U(t, z) - L$$

and $P(t, x(t)) \geq 0$.

(b) Since $P(t_1, x(t_1)) > 0$ it follows that for $P(t_1, x(t_1)) < L - U(t_1, z(t_1))$ we have $V'(t_1, x(t_1)) < 0$ and there is an $\varepsilon_2 > 0$ with $V(t, x(t)) < -\varepsilon_2$ for $t \geq t_1 + 1$; this again contradicts $P \geq 0$ when $z(t)$ becomes large.

Thus, (a) and (b) prove that our opening statement is true and we then have

$$\begin{aligned} P(t, x(t)) &\geq [P(t, x(t)) + L - U(t, z(t))]/2 \\ &\geq [P(t, x(t)) + U(t, z(t)) - L]/2 \\ &= V(t, x(t))/2. \end{aligned}$$

Then by (3) we have (5) and the proof is complete.

NOTE. We point out that (5) requires $z(t) > 0$ because (4) requires $U(t, z) < L$ for $z > 0$. If $U(t, z) < L$ for all t and z, then (5) is true for any solution with $\overline{\lim}_{t\to\infty} z(t) = \infty$ and for all $t \in [t_0, \infty)$.

REMARKS.

I. The theorem has an obvious counterpart for $\underline{\lim}_{t\to\infty} z(t) = -\infty$. Moreover, (4) need not hold along a coordinate, but could be phrased along any unbounded curve.

II. The theorem is fully valid for functional differential equations; V, P, and U can all be functionals.

III. Frequently an equation is given an integrable perturbation $e(t)$. The standard response of the investigator is to change V_1 to

$$V_1(t, x) = [P(t, x) + 1 + U(t, z)] \exp\left(-\int_0^t |e(s)| ds\right)$$

and, frequently, the analysis proceeds just as before. It is easy to verify that if (2) is replaced by

$$V_1(t, x) = [P(t, x) + 1 + U(t, z)] B(t) \tag{2)*$$

where $A \geq B(t) \geq C > 0$, $B'(t) \leq 0$, and (3) is replaced by

$$V_1'(t, x) \leq -Q(P(t, x)C, t, x) \tag{3)*}$$

then Theorem 1 remains valid and the proof is changed only in letting $U \to U+1$.

IV. In the proof we see that if there is an unbounded solution then $P+U \geq L$. For typical problems La Salle [8] points out that the set $V_1(t, x) < L$ is contained in the domain of attraction of the zero solution. Thus, for $V_1(t, x) \geq L$ we have $V'(t, x) \leq -Q(V/2, t, x)$. And in all of the examples we obtain $V' \leq -Q(V)$ where Q is a positive definite function. Under reasonable conditions the set $V=0$ is uniformly asymptotically stable.

V. In the examples that follow we speak of boundedness. But the results can also be used to obtain continuation of solutions. Instead of asking in the theorem that $x(t)$ be a solution on $[t_0, \infty)$ with $\overline{\lim}_{t\to\infty} z(t) = +\infty$, one may ask instead that all conditions hold for $0 \leq t \leq T$, where T is fixed but arbitrary. Then the assumption that $\overline{\lim}_{t\to T^-} z(t) = +\infty$ yields (5) when $z(t) > 0$ and $t_0 \leq t < T$. As seen in the examples, (5) can be used to show that, in fact, $z(t)$ is bounded on $[t_0, T)$ and the solution is continuable to $+\infty$.

3. Examples

EXAMPLE A. The scalar equation

$$x'' + h(t, x, x')|x'|^\alpha x' + f(x) + g(t, x, x') + e(t, x, x') = 0 \tag{A1}$$

with all functions continuous, $\alpha \geq 0$,

$$h(t, x, y) \geq 0, \quad yg(t, x, y) \geq 0, \quad xf(x) > 0 \quad \text{if} \quad x \neq 0, \tag{A2}$$

$$|e(t, x, y)| \leq \beta(t), \quad \int_0^\infty \beta(t)dt < \infty, \tag{A3}$$

has been considered in some form by many authors. A recent discussion is given by Murakami [10] and one by Yoshizawa [16] when $g=e=0$. These authors are interested in limit sets and both ask that

$$F(x) = \int_0^x f(s)ds \to \infty \quad \text{as} \quad |x| \to \infty;$$

and that settles the question of boundedness of solutions since the system

$$\begin{cases} x' = y \\ y' = -h(t, x, y)|y|^\alpha y - f(x) - g(t, x, y) - e(t, x, y) \end{cases} \tag{A4}$$

has the Liapunov function

$$V(t, x, y) = (y^2 + 2F(x) + 1)B(t)$$

(where $B(t)=\exp\left(-\int_0^t \beta(s)ds\right)$) which is then radially unbounded and satisfies $V'\le 0$. In that case, the question of the size of α is of little importance. But when $F(\infty)$ or $F(-\infty)$ is finite then the size of α is crucial, as may be seen in Thurston and Wong [12] when $g(t, x, y)=0$.

As mentioned in the remarks, we may prove the following result for $e(t, x, y)$ as in (A3), but the illustration when $e\equiv 0$ makes the desired point.

PROPOSITION A. *Let* (A2) *hold*, $e(t, x, y)=0$, *and* $0\le\alpha\le 1$. *Suppose that for each* $M>0$ *there is a continuous function* $q\colon(-\infty, \infty)\to[0, \infty)$ *with*

(A5) $$h(t, x, y) \ge q(x) \quad \text{when} \quad |y| \le M \quad \text{and} \quad |x| \ge M$$

such that

(A6) $$\int_0^{\pm\infty} [q(x)+|f(x)|]dx = \pm\infty$$

and let

(A7) $$h(t, x, y) + |g(t, x, y)| > 0 \quad \text{if} \quad y \ne 0.$$

Then all solutions of (A4) *are bounded.*

PROOF. Define

$$V_1(x, y) = y^2 + 2F(x)$$

and obtain

$$V_1'(x, y) = -2h(t, x, y)|y|^{2+\alpha} - 2g(t, x, y)y$$

which is negative by (A7) for $y\ne 0$. Note that this means that for each solution $(x(t), y(t))$ there is an M with $|y(t)|\le M$ which, from $x'=y$, yields $|x(t)|\le|x(t_0)|+ +Mt$; thus, all solutions are defined for all future time.

With reference to Theorem 1,

$$P = y^2 \quad \text{and} \quad U = 2F(x).$$

To be definite, let us suppose that there is an unbounded solution $(x(t), y(t))$ with $|y(t)|\le M$ and $\overline{\lim}_{t\to\infty} x(t)=+\infty$. This means that $F(\infty)<\infty$ and that $\int_0^\infty q(x)dx =\infty$. When $x(t)>0$ we have $2F(x)<2F(\infty)=L$.

Consider any interval $[t_0, t_1]$ with $x(t)\ge M$. Then $V(x, y)=V_1(x, y)-2F(\infty)$ and by Theorem 1

$$V'(x(t), y(t)) \le -h(t, x, y)|y|V^\gamma(x(t), y(t))$$

where $\gamma=(1+\alpha)/2\le 1$. Let $V=V(t)=V(x(t), y(t))$. When $0\le\alpha<1$, then

$$V^{-\gamma}V' \leq -q(x)|x'|$$

and so

$$V^{1-\gamma}(t) - V^{1-\gamma}(t_0) \leq -(1-\gamma)\left|\int_{x(t_0)}^{x(t)} q(s)ds\right|$$

which yields $x(t)$ bounded above because $\gamma<1$. If $\alpha=1$, then

$$V(t) \leq V(t_0)\exp\left|\int_{x(t_0)}^{x(t)} q(s)ds\right|$$

with the same conclusion. This completes the proof.

EXAMPLE B. Krasovskii [6] and more recently Yoshizawa [15] have considered the delay equation

$$x'' + q(t, x, x')x' + g(x(t-r(t))) = 0 \tag{B1}$$

in which all functions are continuous and

$$G(x) = \int_0^x g(s)ds \geq 0. \tag{B2}$$

Krasovskii asks that $q=q(t, x')$ with q periodic in t and $q \geq c_0 > 0$, while Yoshizawa asks that $q(t, x, y) \geq b\beta(t)$, $b>0$, $0 \leq r(t) \leq \beta(t)$, $\beta'(t) \leq \beta_0 < 1$,

$$|g^*(x)| \leq N, \quad g^*(x) = (d/dx)g(x), \tag{B3}$$

and $N^2 < b^2(1-\beta_0)$; thus, Yoshizawa allows $\beta(t)\to 0$ if $r(t)\to 0$ and we are unable to handle that case for we obtain, via Theorem 1, the relation $V' \leq -\beta(t)|x'|V^{1/2}$ from which we are unable to obtain $x(t)$ bounded. Thus, our work here is more in the line of Krasovskii's for we ask that

$$0 \leq r(t) \leq r, \quad q(t, x, y) \geq b > 0, \tag{B4}$$

and

$$r < b/N, \tag{B5}$$

where r, b, and N are positive constants. Both Krasovskii and Yoshizawa obtain boundedness of solutions by asking that $G(x)\to\infty$ as $|x|\to\infty$. Here, we prove the boundedness from (B4) instead.

PROPOSITION B. *Let* (B2)–(B5) *hold. Then all solutions of* (B1) *are bounded.*

PROOF. Write (B1) as the system

(B6)
$$\begin{cases} x' = y \\ y' = -q(t, x, y)y - g(x) + \int_{-r(t)}^{0} g^*(x(t+s))y(t+s)ds \end{cases}$$

and define

$$V_1(x_t, y_t) = 2G(x) + y^2 + [b/r]\int_{-r}^{0}\int_{t+u}^{t} y^2(s)dsdu$$

where $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$. After a calculation we obtain

$$V_1'(x_t, y_t) \leq -\gamma\left[y^2(t) + \int_{t-r}^{t} y^2(u)du\right]$$

for some $\gamma > 0$. Denote the integrals in V_1 and V_1' by I_1 and I_2, respectively.

Suppose that $(x(t), y(t))$ is an unbounded solution with $|y(t)| \leq M$ and $\overline{\lim}_{t\to\infty} x(t) = +\infty$ so that $G(\infty) < \infty$. In Theorem 1 let

$$P = y^2 + I_1 \quad \text{and} \quad U = 2G(x).$$

Then $L = 2G(\infty)$ and for $x(t) > 0$ we have $2G(x) < L$. Define $V(x_t, y_t) = V_1 - 2G(\infty)$ and notice that there is a $J > 0$ with

$$\int_{t-r}^{t} y^2(u)du \geq J[b/r]\int_{-r}^{0}\int_{t+u}^{t} y^2(s)dsdu.$$

By Theorem 1, if $x(t) > 0$ on an interval $[t_0, t_1]$, then for $V = V(t) = V(x_t, y_t)$ we have

$$V' \leq -\bar{\gamma}V \leq -\bar{\gamma}|y|V^{1/2}$$

for some $\bar{\gamma} > 0$. Hence,

$$V'V^{-1/2} \leq -\bar{\gamma}|x'|$$

and

$$V^{1/2}(t) - V^{1/2}(t_0) \leq -(1/2)\bar{\gamma}|x(t) - x(t_0)|$$

so that $x(t)$ is bounded. This completes the proof.

EXAMPLE C. Somolinos [11; pp. 196–9] considers the control system

(C1)
$$\begin{cases} x' = p(t, \sigma, x_t) + bf(\sigma) \\ \sigma' = q(t, x_t) - rf(\sigma) \end{cases}$$

where p: $[0, \infty) \times R \times C \to R^n$, q: $[0, \infty) \times C \to R^n$, f: $R \to R$, all functions are continuous, C is the space of continuous functions φ: $[-h, 0] \to R^n$, and h is a positive constant. It is assumed that

$$\text{(C2)} \qquad |p(t, \sigma_1, \Psi) - p(t, \sigma_2, \Psi)| \leq L(\sigma_1, \sigma_2) \|\Psi\| \, |\sigma_1 - \sigma_2|$$

and the solutions of

$$\text{(C3)} \qquad y' = p(t, \sigma, y_t)$$

satisfy

$$\text{(C4)} \qquad |y(t, t_0, \sigma, \Psi)| \leq K(\sigma) \|\Psi\| \exp\{-\beta(\sigma)(t-t_0)\}$$

with L, K, and β being appropriate functions, while

$$\text{(C5)} \qquad |q(t, \Psi)| \leq c\|\Psi\|, \quad c > 0.$$

Here, $\|\cdot\|$ is the supremum norm and $|\cdot|$ is any convenient norm on R^n.

Using (C2) and (C4), Somolinos applies a converse theorem to conclude that there exists a functional $V(t, \sigma, \Psi)$ for (C3) satisfying

$$\|\Psi\| \leq V(t, \sigma, \Psi)$$

and the derivative of V along a solution of (C3) satisfies

$$V'(t, \sigma, \Psi) \leq -\gamma(\sigma)V(t, \sigma, \Psi)$$

where $\gamma(\sigma) > 0$. He then constructs a Liapunov functional for (C1) of the form $W(t, \sigma, \Psi) = V^2(t, \sigma, \Psi) + F(\sigma)$ where

$$F(\sigma) = \int_0^\sigma f(s)ds$$

and shows that

$$\text{(C6)} \qquad W'(t, \sigma, \Psi) \leq -\beta V^2 + (K|\sigma| + c)V|f(\sigma)| - rf^2(\sigma)$$

with constants such that if either

$$\text{(C7)} \qquad K(\sigma) = 1, \quad \beta(\sigma) \geq \beta > 0 \quad \text{and} \quad 4r\beta > (|b| + c)^2$$

or

$$\text{(C8)} \qquad L(\sigma) = 0 \quad \text{and} \quad 4r\beta > (K|b| + c)^2,$$

then the right-hand side of (C6) is a negative definite quadratic form in V and $f(\sigma)$.

Under the foregoing conditions, Somolinos proves that if

$$\text{(C9)} \qquad F(\sigma) \to \infty \quad \text{as} \quad |\sigma| \to \infty,$$

then all solutions of (C1) are bounded and tend to zero as $t \to \infty$. In the next proposition we show that (C9) is not needed for the boundedness.

To put this result in perspective, we remark that (C1) is a generalization of the classical problem of Lurie, extensively discussed in Lefschetz [9]. It is the ordinary differential equation counterpart of (C1) with p and q linear functions of x alone. The classical results called for (C9) in order to show boundedness. LaSalle [7] showed that (C9) was not needed. Thus, our proposition may be viewed as an extension of LaSalle's result to the functional differential system (C1).

Proposition C. *Let* (C2)–(C8) *hold. Then every solution of* (C1) *is bounded.*

Proof. Conditions on p and q imply that (C1) takes bounded sets into bounded sets; thus, bounded solutions are continuable for all future time. Because $\|\Psi\| \leq V(t, \sigma, \Psi)$ and $W' \leq 0$, for any solution $(x(t), \sigma(t))$ there is an $M > 0$ with $|x(t)| \leq M$. Then using (C5) in the second equation in (C1) we see that any solution starting at t_0 can be defined for all future time.

Suppose that $(x(t), \sigma(t))$ is an unbounded solution with $\overline{\lim}_{t\to\infty} \sigma(t) = \infty$. Clearly, $F(\infty) < \infty$ and, with regard to Theorem 1, we let

$$P = V^2(t, \sigma, \Psi) \quad \text{and} \quad U = F(\sigma).$$

Define

$$\overline{W} = V^2 + F(\sigma) - F(\infty)$$

and note from the fact that Somolinos has proved that the right-hand side of (C6) is a negative definite quadratic form in V and $f(\sigma)$, there is a $\bar{\beta} > 0$ with

$$\overline{W}' \leq -\bar{\beta} V V.$$

By Theorem 1 we have

$$\overline{W}' \leq -(\bar{\beta}/2) V \overline{W}^{1/2}$$

so that

$$\overline{W}^{-1/2} \overline{W}' \leq -(\bar{\beta}/2) V$$

and

$$\overline{W}^{1/2}(t) - \overline{W}^{1/2}(t_0) \leq -(\bar{\beta}/4) \int_{t_0}^{t} V(s) ds.$$

Thus, $V(t) \in L^1[t_0, \infty)$. But $V(t) \geq \|x_t\|$ and $|q(t, x_t)| \leq c\|x_t\|$, so $q \in L^1[0, \infty)$.

In $\sigma' = q(t, x_t) - rf(\sigma)$, define a Liapunov function $H(\sigma) = |\sigma|$ so that $H'(\sigma) \leq |q(t, x_t)|$. Hence, $H(\sigma)$ is bounded and the proof is complete.

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