HIROSHIMA MATH. J.
19 (1989), 37–76

Relations between several Adams spectral sequences

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(Received January 18, 1988)

Introduction

In the stable homotopy theory, the G-Adams spectral sequence

(1) $$E(G) = \{E(G)_r^{s,t}, d_r\colon E(G)_r^{s,t} \to E(G)_r^{s+r,t+r-1}\} \quad \text{abutting to } \pi_{t-s}(X)$$

(cf. [4, III, §15]) is useful, where X is a CW spectrum, $\pi_*(X)$ is its homotopy group and G is a ring spectrum. For X and $G = E$, F with some conditions, H. R. Miller [10] introduced the May and Mahowald spectral sequences

(2) $$E^{\mathrm{May}} = \{E_{u,r}^{s,t}, d_r^{\mathrm{May}}\colon E_{u,r}^{s,t} \to E_{u+r,r}^{s+1,t+r}\} \quad \text{abutting to } E(E)_2^{s,u-t} \text{ and}$$
$$E^{\mathrm{Mah}} = \{\tilde{E}_{u,r}^{s,t}, d_r^{\mathrm{Mah}}\colon \tilde{E}_{u,r}^{s,t} \to \tilde{E}_{u,r}^{s+r,t-r+1}\} \quad \text{converging to } E(F)_2^{s+t,u}$$

for $E(G)_2$ in (1), which satisfy the following

(o) $E_{u,1}^{s,t} = \tilde{E}_{u,2}^{s,t} = A_u^{s,t}$; *and for any* $x \in A_u^{s,t}$,
(ii) *if* x *converges to* x^F *in* E^{Mah}, *then so does* $d_1^{\mathrm{May}}x$ *to* $(-1)^t d_2^F x^F$.

Especially, he defined these algebraically in case when
(3) $X = S^0$, $E = BP$ at a prime p, and $F = HZ_p$ (BP is the Brown-Peterson spectrum, and HZ_p is the spectrum of the ordinary homology $H_*(\ \ ; Z_p)$); and calculated some differential $d_2^{HZ_p}$ in (1) for $X = S^0$.

The purpose of this paper is to argue the existence and relations of these spectral sequences. Let $\bar{G}$ denote the mapping cone of the unit $S^0 \to G$ of a ring spectrum G, and $\bar{G}^n$ the smash product of n copies of $\bar{G}$. Then the main result in this paper, stated in Theorem 7.2, implies the following

THEOREM. *For a CW spectrum X and ring spectra E, F, assume that*
(4) *there is a unit-preserving map* $\lambda\colon E \to F$, *and*
(5) *the F-Adams spectral sequence abutting to* $\pi_*(E \wedge \bar{E}^n \wedge X)$ *in* (1) *converges and collapses for any* $n \geqq 0$.
Then we have the spectral sequences E^{May} *and* E^{Mah} *in* (2) *satisfying* (o), (ii),
(i) $d_1^{\mathrm{May}} d_2^{\mathrm{Mah}} x = d_2^{\mathrm{Mah}} d_1^{\mathrm{May}} x$ *for any* $x \in A_u^{s,t}$,
(iii) *if* x *converges to* x^E *in* E^{May}, *then so does* $d_2^{\mathrm{Mah}}x$ *to* $d_2^E x^E$, *and*
(iv) *if the assumptions in* (ii)–(iii) *hold, then some* $y \in A_{u+1}^{s+2,t}$ *converges to* $d_2^E x^E$ *in* E^{May} *and to* $(-1)^t d_2^F x^F$ *in* E^{Mah}.

Especially, in case (3), we see (4)–(5) by the Thom map $BP \to HZ_p$, and

$$A_u^{s,t} = \mathrm{Ext}_{P_*}^{s,u}(Z_p, \mathrm{Ext}_{A_*}^{t,*}(Z_p, P_*)) \quad \text{in (o)}$$

$(A_* = (HZ_p)_*(HZ_p)$, $P_* = (HZ_p)_*(BP)$, and $\mathrm{Ext}_{A_*}^{*,*}(Z_p, P_*) = Z_p[a_0, a_1, a_2, \dots]$ $(a_n \in \mathrm{Ext}^{1,2p^n-1})$), and we obtain Examples 8.3-4 on the differentials $d_2^{Z_p}$ and d_{2p-1}^{BP} in (1) for $X = S^0$.

For our purpose, we argue in §§1–3 the construction of the Adams spectral sequences. We introduce the notion of an *E_2-group* $B = \{B_t^s\}$ related to a given homology theory h_* in Definition 1.8, so that we have in Theorem 1.9 the spectral sequence of Adams type

(6) $\{E(B)_r^{s,t},\ d_r^B\}$ abutting to $h_{t-s}(X)$ and satisfying $E(B)_2^{s,t} = B_t^s(X)$.

Then for any ring spectrum G, we have the E_2-group $GA = \{GA_t^s\}$ in (2.1.1-4) related to π_* and define the G-Adams spectral sequence $E(G)$ in (1) by

$$E(G) = E(GA),\ \text{i.e.,}\ E(G)_2^{s,t} = GA_t^s(X) \quad \text{(see Theorem 2.3)}.$$

We note that the E_2-term may be seen by the definition of GA even if $G_*(G)$ is not flat over $G_*(S^0)$; e.g., we have Example 2.5 for the connective K-theory spectrum bu or the corresponding one buQ_2 with coefficients in Q_2.

We define an *E_2-functor* $B = \{B_t^s\}$ to be an E_2-group satisfying the functoriality on the category of cofiberings in Definition 3.2, so that we can compare $E(B)$ in (6) for $B = C$, D (see Theorems 3.4-5)). Then GA is an E_2-functor by definition, $\lambda: E \to F$ in (4) induces the homomorphism $\bar{\lambda}_*$: $E(E)_r^{s,t} \to E(F)_r^{s,t}$ between G-Adams spectral sequences, and we have Theorem 3.8 on the conditions that $\bar{\lambda}_*$ is isomorphic, monomorphic or epimorphic. Examples 3.9-10 hold when λ is the Thom map $BP \to HZ_p$, etc.; in particular, we see $E(MO) \cong E(HZ_2)$ for the Thom spectrum MO of the bordism theory.

Moreover, we introduce in §§4–5 the notion of a *double E_2-functor* $A = \{A_u^{s,t}\}$ related to an E_2-functor D or indirectly related to C (see (Definitions 4.3 and 5.3), so that we have the Mahowald or May spectral sequence

(7) $\{\tilde{E}_{u,r}^{s,t}, d_r^{\mathrm{Mah}}\}$ converging to $D_u^{s+t}(X)$ with $\tilde{E}_{u,2}^{s,t} = A_u^{s,t}(X)$, or
$\{E_{u,r}^{s,t}, d_r^{\mathrm{May}}\}$ abutting to $C_{u-t}^s(X)$ with $E_{u,1}^{s,t} = A_u^{s,t}(X)$

(see Theorem 4.4 and Corollary 5.6)). In particular, for some ring spectra E and F (e.g., satisfying (4)–(5)), we have the double E_2-functor $EFA = \{EFA_u^{s,t}\}$ in (4.6.8) and the spectral sequences in (7) by taking $A = EFA$, $D = FA$ and $C = EA$ (see Theorems 4.7 and 5.8), which are taken to be E^{Mah} and E^{May} in (2). Example 4.8 gives a note on E^{Mah} for $E = BP$ at p and $F = KQ_p$ (the K-theory spectrum with coefficients in Q_p) when p is an odd prime.

Now, we prepare in §6 some lemmas on commutative diagrams of coficberings. Then we can consider the case stated in Definition 7.1 that for a CW spectrum X, a homology theory h_*, E_2-functors $B = C$, D and a double E_2-functor A, the spectral sequences of Adams type in (6) and the Mahowald and May ones in (7) are all defined (see (7.1.8)); and we prove in Theorem 7.2 some relations between them. By taking $h_* = \pi_*$, $C = EA$, $D = FA$ and $A = EFA$, Theorem 7.2 implies the above theorem and Examples 8.3-4.

Here, we notice that the cohomology version of E_2-functors can be obtained by the dual consideration, by which we may argue several spectral sequences, e.g., the Adams universal coefficient one or the one of Bousfield-Kan type; the details will be discussed in a forthcoming paper.

The author is deeply in debt to Professors M. Sugawara, T. Kobayasi and T. Matumoto for their valuable suggestions and discussions. He also thanks Professor J. F. Adams for the kind letter of Nov. 24, 1981 on $KQ_pA(S^0)$ in Example 4.8.

§1. Spectral sequences and E_2-groups

Throughout this paper, we work in the category $\mathscr{C}$ of CW spectra (cf. [4] or [16] for the definition and the basic properties of CW spectra and the related notions).

Let h_* be a homology theory on $\mathscr{C}$, and for a given $X_0 \in \mathscr{C}$, assume that

(1.1.1) there are cofiberings α_n: $X_n \xrightarrow{f_n} W_n \xrightarrow{g_{n+1}} X_{n+1}$ $(n = 0, 1, 2, \dots)$ in $\mathscr{C}$ (i.e., X_{n+1} is the mapping cone $W_n \cup_{f_n} CX_n$ of f_n and g_{n+1} is the inclusion map, up to homotopy equivalence).

Then, we have the induced exact sequences

(1.1.2) $$\cdots \longrightarrow h_t(X_s) \xrightarrow{f_*} h_t(W_s) \xrightarrow{g_*} h_t(X_{s+1}) \xrightarrow{\partial} h_{t-1}(X_s) \longrightarrow \cdots \quad (f_* = f_{s*}, g_* = g_{s+1*})$$

for any t and any $s \geqq 0$; and the standard argument on exact couples defines the spectral sequence given by (1.1.3), where $\partial^r = \partial \circ \cdots \circ \partial$: $h_{t+r}(X_{s+r}) \to h_t(X_s)$:

(1.1.3)

$$Z_r^{s,t} = g_*^{-1} \operatorname{Im} [\partial^{r-1}: h_{t+r-1}(X_{s+r}) \to h_t(X_{s+1})] \subset h_t(W_s), \quad Z_\infty^{s,t} = \textstyle\bigcap_{r \geqq 1} Z_r^{s,t},$$

$$B_r^{s,t} = f_* \operatorname{Ker} [\partial^{r-1}: h_t(X_s) \to h_{t-r+1}(X_{s-r+1})] (r \leqq s+1),$$

$$= B_{s+1}^{s,t} = B_\infty^{s,t} \quad (r \geqq s+1)$$

$$E_r^{s,t} = Z_r^{s,t}/B_r^{s,t}, \quad d_r: E_r^{s,t} \overset{\mathrm{pr}}{\twoheadrightarrow} Z_r^{s,t}/Z_{r+1}^{s,t} \cong B_{r+1}^{s+r,t+r-1}/B_r^{s+r,t+r-1} \subset E_r^{s+r,t+r-1},$$

$$E_\infty^{s,t} = Z_\infty^{s,t}/B_\infty^{s,t}, \quad F^{s,t} = \operatorname{Im} [\partial^s: h_t(X_s) \to h_{t-s}(X_0)], \quad \bar{Z}_\infty^{s,t} = \operatorname{Ker} g_* = \operatorname{Im} f_* \subset Z_\infty^{s,t},$$

$$A^{s,t} = \operatorname{Im} g_* \cap \textstyle\bigcap_{r \geqq 1} \operatorname{Im} [\partial^r: h_{t+r}(X_{s+r+1}) \to h_t(X_{s+1})] \subset h_t(X_{s+1}).$$

PROPOSITION 1.2. *For a homology theory* h_* *on* $\mathscr{C}$, $X_0 \in \mathscr{C}$ *and cofiberings* α_n *in* (1.1.1), *the exact sequences* (1.1.2) *associate the spectral sequence* $\{E_r^{s,t}, d_r\}$ *in* (1.1.3) *such that*

(1.2.1) $E_1^{s,t} = h_t(W_s)$, $\quad d_1 = f_* \circ g_*: E_1^{s,t} = h_t(W_s) \to h_t(X_{s+1}) \to h_t(W_{s+1}) = E_1^{s+1,t}$, *and*

(1.2.2) *by the filtration* $h_{t-s}(X_0) = F^{0,t-s} \supset \cdots \supset F^{s,t} \supset F^{s+1,t+1} \supset \cdots$, *we have the exact sequence*

$$0 \to F^{s,t}/F^{s+1,t+1}(\cong \bar{Z}_\infty^{s,t}/B_\infty^{s,t}) \to E_\infty^{s,t} \to A^{s,t}(\cong Z_\infty^{s,t}/\bar{Z}_\infty^{s,t}) \to 0\,.$$

In this paper, we present such a case by the following

(1.2.3) $\{E_r^{s,t}\}$ *abuts to* $h_{t-s}(X_0)$: $E_1^{s,t} = h_t(W_s) \Rightarrow h_{t-s}(X_0)$ *(abut).*

To represent the E_2-term of this spectral sequence, we consider the following

DEFINITION 1.3. Let $C = \{C_t^s | s, t \in Z\}$ be a collection of covariant functors

$C_t^s: \mathscr{C} \to \mathscr{A}$ (the category of abelian groups) with $C_t^s = 0$ for $s < 0$.

Then, we say that C *is related to a homology theory* h_* *at* X_0 *by a natural transformation* $\phi: h_t \to C_t^0$ $(t \in Z)$ *and cofiberings* α_n *in* (1.1.1), if

(1.3.1) $\phi: h_t(W_n) \cong C_t^0(W_n)$, $C_t^s(W_n) = 0$ for $s > 0$, and there are homomorphisms $\bar{\delta}$ so that the following sequences are exact:

$$\cdots \longrightarrow C_t^s(X_n) \xrightarrow{f_{n*}} C_t^s(W_n) \xrightarrow{g_{n+1*}} C_t^s(X_{n+1}) \xrightarrow{\bar{\delta}} C_t^{s+1}(X_n) \longrightarrow \cdots.$$

(1.3.2) Then, we have $\bar{\delta}: C_t^s(X_{n+1}) \cong C_t^{s+1}(X_n) (s > 0)$ and the exact sequence

$$0 \longrightarrow C_t^0(X_n) \xrightarrow{f_{n*}} C_t^0(W_n) \xrightarrow{g_{n+1*}} C_t^0(X_{n+1}) \xrightarrow{\bar{\delta}} C_t^1(X_n) \longrightarrow 0\,.$$

Furthermore, for $d_1^{s,t} = d_1 = f_* \circ g_*$ in (1.2.1), we have the commutative diagram

$$\begin{array}{ccccccccc}
E_1^{s-1,t} & & \xrightarrow{d_1^{s-1,t}} & & E_1^{s,t} & & \xrightarrow{d_1^{s,t}} & & E_1^{s+1,t} \\
\| & & & & \| & & & & \| \\
h_t(W_{s-1}) & \xrightarrow{g_*} & h_t(X_s) & \xrightarrow{f_*} & h_t(W_s) & \xrightarrow{g_*} & h_t(X_{s+1}) & \xrightarrow{f_*} & h_t(W_{s+1}) \\
\cong \downarrow \phi & & \downarrow \phi & & \cong \downarrow \phi & & \downarrow \phi & & \cong \downarrow \phi \\
C_t^0(W_{s-1}) & \xrightarrow{g_{s*}} & C_t^0(X_s) & \xrightarrow{f_{s*}} & C_t^0(W_s) & \xrightarrow{g_{s+1*}} & C_t^0(X_{s+1}) & \xrightarrow{f_{s+1*}} & C_t^0(W_{s+1})\,.
\end{array} \tag{1.3.3}$$

Then, (1.3.2) implies that f_{s*} is monomorphic and we have the isomorphisms $f_{s*}^{-1} \circ \phi$: Ker $d_1^{s,t} \cong C_t^0(X_s)$, Im $d_1^{s-1,t} \cong$ Im g_{s*} and

(1.3.4) $\quad \bar{\phi} = \bar{\delta}^{s-1} \circ \bar{\delta} \circ (f_{s*}^{-1} \circ \phi): E_2^{s,t} \cong C_t^0(X_s)/\text{Im}\, g_{s*} \cong C_t^1(X_{s-1}) \cong C_t^s(X_0)$

$(\bar{\delta}^{s-1} = \bar{\delta} \cdots \circ \bar{\delta})$. Thus, we see the following

THEOREM 1.4. *In case of Definition* 1.3, *we have the associated spectral sequence* $\{E_r^{s,t}\}$ *in Proposition* 1.2, *which abuts to* $h_*(X_0)$ *and whose* E_2*-term* $E_2^{s,t}$ *is isomorphic to* $C_t^s(X_0)$ *by* $\bar{\phi}$ *in* (1.3.4):

$$E_2^{s,t} = C_t^s(X_0) \Rightarrow h_{t-s}(X_0) \quad (abut)\,.$$

COROLLARY 1.5. *In Theorem* 1.4, *the following* (1.5.1-3) *are equivalent*:
(1.5.1) $E_2^{s,t} = C_t^s(X_0) = 0$ *for* $s > 0$ *and* $\bar{\phi} = \phi$: $E_2^{0,t} = h_t(X_0) \cong C_t^0(X_0)$.
(1.5.2) $0 \to h_t(X_n) \xrightarrow{f_*} h_t(W_n) \xrightarrow{g_n} h_t(X_{n+1}) \to 0$ *is exact in* (1.1.2) *for all* $n \geqq 0$.
(1.5.3) ϕ: $h_t(X_n) \cong C_t^0(X_n)$ *and* $C_t^s(X_n) = 0$ *for all* $s > 0$ *and* $n \geqq 0$.

PROOF. (1.5.2) implies $\partial = 0$ and so (1.5.1) by (1.1.3). (1.5.1) means (1.5.3) for $n = 0$; and (1.5.3) for n implies (1.5.2) for n and (1.5.3) for $n+1$ by (1.3.2) and 5-Lemma. Thus, (1.5.1-3) are equivalent by induction. q.e.d.

We use the following terminologies for $\{E_r, d_r\}$ in Proposition 1.2:
(1.6.1) $d_r x = x'$ for $x \in E_u^{s,t}$, $x' \in E_u^{s',t'}$ with $u \leqq r$, if $s' = s + r$, $t' = t + r - 1$, $x \in Z_r/B_u$, $x' \in Z_r'/B_u'$ and the equality holds for their images $x \in E_r$, $x' \in E_r'$ $(G_* = G_*^{s,t}, G_*' = G_*^{s',t'})$.
(1.6.2) $\bar{Z}E_r^{s,t} = \bar{Z}_\infty^{s,t}/B_r^{s,t}$ is the subgroup of all *permanent cycles* in $E_r^{s,t}$; and $x \in E_r^{s,t}$ *converges to* $y \in h_{t-s}(X_0)$ if $x \in \bar{Z}E_r^{s,t}$, $y \in F^{s,t}$ and they coincide in $\bar{Z}E_\infty^{s,t} = F^{s,t}/F^{s+1,t+1}$.
(1.6.3) $\{E_r, d_r\}$ *converges*: $E_r^{s,t} \Rightarrow h_{t-s}(X_0)$ (*conv*), if $\bar{Z}_\infty = Z_\infty$ (or $A^{s,t} = 0$) and $\bigcap_{n \geqq 0} F^{n,t+n} = 0$; and it *collapses* (for $r \geqq 2$) if $d_r = 0$ or $E_r = E_\infty$ for $r \geqq 2$.

COROLLARY 1.7. *In Theorem* 1.4, *consider*
(1.7.1) $\bar{Z}C_t^s(X_0) = \bar{Z}E_2^{s,t} \subset E_2^{s,t} = C_t^s(X_0)$ (*by regarding* $\bar{\phi} = \mathrm{id}$), *and*
(1.7.2) $\bar{\phi} = \bar{\delta}^s \circ \phi$: $h_t(X_s) \to C_t^0(X_s) \to C_t^1(X_{s-1}) \cong C_t^s(X_0)$.

(i) *Then*, $\bar{Z}C_t^s(X_0) = \mathrm{Im}\, \bar{\phi}$; *and* $x \in C_t^s(X_0) = E_2^{s,t}$ *converges to* $y \in h_{t-s}(X_0)$ *if and only if* $x = \bar{\phi} y_s$ *and* $\partial^s y_s = y$ *for some* $y_s \in h_t(X_s)$. *Also* $d_r x = x'$ *holds for* $x \in C_t^s(X_0)$, $x' \in C_{t'}^{s'}(X_0)$ *if and only if* $s' = s + r$, $t' = t + r - 1$ *and* $x = \bar{\delta}^s x_{s'}, f_{s*} x_s = \phi w$, $g_{s+1*} w = \partial^{r-1} y$ *and* $\bar{\phi} y = x'$ *for some* $x_s \in C_t^0(X_s)$, $w \in h_t(w_s)$ *and* $y \in h_{t'}(X_{s'})$.

(ii) $\{E_r\}$ *converges and collapses if and only if* (1.7.3) *and one of* (1.7.4-6) *hold*:
(1.7.3) inv $\lim_n \{h_{t+n}(X_n), \partial: h_{t+n+1}(X_{n+1}) \to h_{t+n}(X_n)\} = 0$ for any t.
(1.7.4) $\{E_r^{s,t}\}$ *converges weakly* (*i.e.*, $\bar{Z}_\infty = Z_\infty$ *or* $A^{s,t} = 0$) *and collapses*.
(1.7.5) ϕ: $h_t(X_s) \to C_t^0(X_s)$ *is epimorphic for any* s, t.
(1.7.6) Ker ∂^n = Ker ∂ *for* ∂^n: $h_t(X_s) \to h_{t-n}(X_{s-n})$, *for any* n $(1 \leqq n \leqq s)$ *and* s, t.

PROOF. (i) follows immediately from (1.1.3) and (1.3.1-4) and (1.6.1-2).

(ii) Assume (1.7.6), and take any $x \in C_t^0(X_s)$. Then by (1.3.2-3), we see $f_{s*} x = \phi w$ for some $w \in h_t(W_s)$, and so $\phi f_* g_* w = 0$ and $g_* w = \partial y$ for some

$y \in h_{t+1}(X_{s+2})$. Hence $\partial^2 y = 0$, $g_* w = \partial y = 0$ by (1.7.6), and $w = f_* y'$ for some $y' \in h_t(X_s)$. Thus $x = \phi y'$ since f_{s*} is monomorphic; and (1.7.5) holds. (1.7.5) implies $\bar{Z}_\infty = Z_\infty = Z_r = Z_2$ ($r \geqq 2$) by (i), and so $d_r = 0$ and (1.7.4).

Conversely, assume (1.7.4), and take any $y \in \operatorname{Ker} \partial^n$ ($n \geqq 2$) in (1.7.6). Then by (1.1.3) and (1.7.4), we have $f_* y \in B_{n+1} = B_2$, $f_* y = f_* y'$, $y - y' = \partial z$ and $\partial y = \partial^2 z$ for some $y' \in \operatorname{Ker} \partial$ and $z \in h_{t+1}(X_{s+1})$. Hence $\partial y \in \bigcap_r \operatorname{Im} \partial^r$ by induction. Therefore, $\partial^{n-1} y \in \operatorname{Ker} \partial \cap \bigcap_r \operatorname{Im} \partial^r = A^{s-n,t-n+1} = 0$ by (1.7.4); and $\partial y = 0$ by induction, which shows (1.7.6). Thus (1.7.4-6) are equivalent.

Now, consider $p: \bar{h}_t = \operatorname{inv\,lim}_n h_{t+n}(X_n) \to \bar{F}_t = \bigcap_n F^{n,t+n}$ given by $p\{y_n\} = y_0$ for $y_n \in h_{t+n}(X_n)$ with $\partial y_{n+1} = y_n$ ($n \geqq 0$); and assume (1.7.6). If $p\{y_n\} = y_0 = 0$, then $y_{n+1} \in \operatorname{Ker} \partial^{n+1} = \operatorname{Ker} \partial$ by (1.7.6), and $y_n = 0$. If $y_0 \in \bar{F}_t$, then we have $y_n \in h_{t+n}(X_n)$ with $\partial^n y_n = y_0$. Thus $\partial y_{n+2} - y_{n+1} \in \operatorname{Ker} \partial$ by (1.7.6), and $\{\partial y_{n+1}\} \in \bar{h}_t$ with $p\{\partial y_{n+1}\} = y_0$. Therefore p is isomorphic, and we see (ii).

q.e.d.

The exact sequence in the assumption (1.3.1) is given by the following

DEFINITION 1.8. (1) For convariant functors $C_t^s: \mathscr{C} \to \mathscr{A}$ ($s, t \in Z$) with $C_t^s = 0$ for $s < 0$, assume the following (1.8.1):

(1.8.1) For any cofibering $\alpha: X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$ in $\mathscr{C}$, there are given abelian groups $KC_t^s(\alpha; i)$ ($s, t \in Z$; $i = 0, 1, 2$) and exact sequences

$$\cdots \longrightarrow KC_t^s(\alpha; i) \xrightarrow{\iota} C_t^s(X_i) \xrightarrow{\kappa} KC_t^s(\alpha; i+1) \xrightarrow{\delta} KC_t^{s+1}(\alpha; i) \longrightarrow \cdots$$

($\rho = \rho_i$ for $\rho = \iota, \kappa, \delta$) with $KC_t^s(\alpha; 3) = KC_{t-1}^s(\alpha; 0)$, $KC_t^s(\alpha; i) = 0$ for $s < 0$ and

$$(1.8.2) \quad \iota_{i+1} \circ \kappa_i = f_{i*}: C_t^s(X_i) \longrightarrow KC_t^s(\alpha; i+1) \longrightarrow C_t^s(X_{i+1}) \quad \text{for } i = 0, 1\,.$$

Then, we call a collection $C = \{C_t^s, KC_t^s(\ ;i)\}$ an *E_2-group*. In this case, we call $X \in \mathscr{C}$ *C-injective* if $C_t^s(X) = 0$ for $s > 0$; and $\alpha: X_0 \to X_1 \to X_2$ a *C-cofibering* if $KC_t^s(\alpha; 0) = 0$, and a *C-injective cofibering* if X_1 is C-injective in addition.

(2) Furthermore, we say that C *has enough injective objects* if

(1.8.3) any $X \in \mathscr{C}$ is in a C-injective cofibering $\omega(X): X \xrightarrow{f} W(X) \xrightarrow{g} \bar{W}(X)$.

By this definition, we see the following (1.8.4-6):

(1.8.4) For any C-cofibering $\alpha: X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$, we have the exact sequence

$$\cdots \longrightarrow C_t^s(X_0) \xrightarrow{f_{0*}} C_t^s(X_1) \xrightarrow{f_{1*}} C_t^s(X_2) \xrightarrow{\bar{\delta}} C_t^{s+1}(X_0) \longrightarrow \cdots$$

by taking $\bar{\delta} = \kappa_0^{-1} \circ \delta_1 \circ \iota_2^{-1}: C_t^s(X_2) \cong KC_t^s(\alpha; 2) \to KC_t^{s+1}(\alpha; 1) \cong C_t^{s+1}(X_0)$ in (1.8.1). In fact, the exact sequences in (1.8.1) show that ι_2 and κ_0 are isomorphic by $KC_t^s(\alpha; 0) = 0$, and then the desired one is exact by (1.8.2).

(1.8.5) If α_n's in (1.1.1) are C-cofiberings, then exact sequences in (1.3.1) are given by (1.8.4); and if they are C-injective cofiberings, then (1.3.2) holds.

(1.8.6) We note that $C_t^s(*) = 0$ for any s, t, where $*$ is the one point spectrum. In fact, consider ι, κ in (1.8.1) for $\alpha: * \to * \to *$ and $i = 1$. Then ι_1 is epimorphic and κ_1 is monomorphic by (1.8.2); hence $C_t^s(*) = 0$ by exactness.

Therefore, Theorem 1.4 imlies the following

THEOREM 1.9. *Let h_* be a homology theory, $C = \{C_t^s, KC_t^s\}$ be an E_2-group, and $\phi: h_t \to C_t^0$ be a natural transformation. For $X_0 \in \mathscr{C}$, let be given*
(1.9.1) *C-injective cofiberings $\alpha_n: X_n \to W_n \to X_{n+1}$ with $\phi: h_t(W_n) \cong C_t^0(W_n)$.*
Then, $C = \{C_t^s\}$ is related to h_* at X_0 by ϕ and $\{\alpha_n\}$, and we have the spectral sequence $\{E_r^{s,t}\}$ in Theorem 1.4 with $E_2^{s,t} = C_t^s(X_0) \Rightarrow h_{t-s}(X_0)$ (abut).

When C has enough injective objects by $\omega(X)$ in (1.8.3) with $\phi: h_t(W(X)) \cong C_t^0(W(X))$, this is obtained for any X_0 by taking
(1.9.2) $\alpha_n = \omega(X_n): X_n \to W_n = W(X_n) \to X_{n+1} = \overline{W}(X_n)$, *inductively.*

§ 2. Adams spectral sequences

We recall the Adams spectral sequence for a given ring spectrum E with unit $\iota = \iota_E: S^0 \to E$ and product $\mu = \mu_E: E \wedge E \to E$.

For any $X \in \mathscr{C}$, consider the homotopy and homology groups

$$\pi_t(X) = [\textstyle\sum^t S^0, X] \quad \text{and} \quad E_t(X) = \pi_t(E \wedge X).$$

Then, we obtain the cochain complex

$$\text{(2.1.1)} \qquad E_t^*(X) = \{E_t^s(X) = \pi_t(E^{s+1} \wedge X) \quad (s \geqq 0), = 0 \quad (s < 0)\}$$

with coboundary $\delta^s = \sum_{i=0}^{s+1} (-1)^i \delta_{i*}^s$, where $E^n = E \wedge \cdots \wedge E$ (n copies) and

$$\begin{aligned} \delta_i^s = 1 \wedge \iota \wedge 1: E^{s+1} \wedge X &= E^{s+1-i} \wedge S^0 \wedge E^i \wedge X \\ &\to E^{s+1-i} \wedge E \wedge E^i \wedge X = E^{s+2} \wedge X . \end{aligned}$$

(2.1.2) We note that if a map $\lambda: E \to F$ between ring spectra E and F *preserves units* (i.e., $\iota_F \sim \lambda \circ \iota_E: S^0 \to F$), then $\lambda^{s+1} \wedge 1: E^{s+1} \wedge X \to F^{s+1} \wedge X$ induces the cochain map $\lambda_* = \{(\lambda^{s+1} \wedge 1)_*\}: E_t^*(X) \to F_t^*(X)$.

Furthermore, for any cofibering $\alpha: X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$,

(2.1.3) we have the homotopy exact sequences

$$\cdots \longrightarrow E_t^s(X_0) \xrightarrow{f_{0*}} E_t^s(X_1) \xrightarrow{f_{1*}} E_t^s(X_2) \xrightarrow{f_{2*}} E_{t-1}^s(X_0) \longrightarrow \cdots \quad (f_{2*} = \partial),$$

the subcomplexes $KE_t^*(\alpha; i) = \{\operatorname{Ker} f_{i*}\}$ of $E_t^*(X_i)$ and the exact sequences

$$0 \to KE_t^*(\alpha; i) \to E_t^*(X_i) \to KE_t^*(\alpha; i+1) \to 0 \quad (KE_t^*(\alpha; 3) = KE_{t-1}^*(\alpha; 0))$$

of cochain complexes. Thus, taking their cohomologies,

(2.1.4) *we obtain the* E_2*-group* $EA = \{EA_t^s, KEA_t^s(\ ;i)\}$ *given by*

$$EA_t^s(X) = H^s(E_t^*(X)) \quad (X \in \mathscr{C}), \quad KEA_t^s(\alpha; i) = H^s(KE_t^*(\alpha; i)) \quad (\alpha \in \mathscr{C}\mathscr{F}).$$

Now, the Hurewicz map $(\iota \wedge 1)_*: \pi_t(X) \to E_t^0(X)$ induced from $\iota \wedge 1: X = S^0 \wedge X \to E \wedge X$ satisfies $\delta^0 \circ (\iota \wedge 1)_* = 0$ for δ^0 in (2.1.1). Thus we have the natural Hurewicz map

(2.1.5) $\phi^E = (\iota_E \wedge 1)_*: \pi_t(X) \to EA_t^0(X) = H^0(E_t^*(X)) = \operatorname{Ker} \delta^0$ for $X \in \mathscr{C}$.

Furthermore, we have the induced cofiberings

(2.1.6) $\omega^E: S^0 \xrightarrow{\iota} E \xrightarrow{j} \bar{E} = C_\iota, \quad \omega^E X: X \xrightarrow{\iota \wedge 1} E \wedge X \xrightarrow{j \wedge 1} \bar{E} \wedge X$ and

$$\alpha_n^E = \omega^E \wedge X_n: X_n \to E \wedge X_n \to X_{n+1} \quad \text{with} \quad X_n = \bar{E}^n \wedge X_0 \quad (n \geqq 0).$$

LEMMA 2.2. $(1 \wedge \mu \wedge 1)_* \circ (\iota \wedge 1)_* = \mathrm{id}: E_t^s(X) \to E_t^s(E \wedge X) \to E_t^s(X)$ *for* $1 \wedge \mu \wedge 1: E^s \wedge E^2 \wedge X \to E^s \wedge E \wedge X$; *and* $KE_t^s(\omega^E \wedge X; 0) = 0$. *Moreover,* $\phi^E: \pi_t(E \wedge X) \cong EA_t^0(E \wedge X)$ *and* $EA_t^s(E \wedge X) = 0$ $(s > 0)$. *Thus* $\omega^E \wedge X$ *is an* EA*-injective cofibering, and* EA *has enough injective objects.*

PROOF. The first part holds since $\mu \circ (1 \wedge \iota) \sim 1: E \to E$. Consider δ_{i*}^s, $\delta^s: \pi_t(E^{s+1} \wedge W) \to \pi_t(E^{s+2} \wedge W)$ in (2.1.1) for $W = E \wedge X$ when $s \geqq 0$, and $\delta_0^{-1} = \iota \wedge 1$, $\delta^{-1} = \delta_{0*}^{-1}$ when $s = -1$; and

$$\sigma^s = \textstyle\sum_{i=0}^s (-1)^i \sigma_{i*}^s: \pi_t(E^{s+1} \wedge W) \to \pi_t(E^s \wedge W) \quad \text{for} \quad s \geqq 0,$$

where $\sigma_i^s = 1 \wedge \mu \wedge 1: E^{s-i} \wedge E^2 \wedge E^i \wedge X \to E^{s-i} \wedge E \wedge E^i \wedge X$. Then, $\sigma_{i*}^{s+1} \circ \delta_{j*}^s$ is $\delta_{j-1*}^{s-1} \circ \sigma_{i*}^s$ if $i < j$, id if $i = j$, $j+1$, and $\delta_{j*}^{s-1} \circ \sigma_{i-1*}^s$ if $i > j+1$; hence $\sigma^0 \circ \delta^{-1} = \mathrm{id}: \pi_t(W) \to \pi_t(W)$ and

$$\sigma^{s+1} \circ \delta^s + \delta^{s-1} \circ \sigma^s = \mathrm{id}: \pi_t(E^{s+1} \wedge W) \to \pi_t(E^{s+1} \wedge W) \quad \text{when} \quad s \geqq 0.$$

Since $\phi^E = \delta^{-1}$ by (2.1.5), these imply the second part. q.e.d.

By this lemma and Theorem 1.9, we see the following

THEOREM 2.3. *For the homotopy theory* π_* *on* $\mathscr{C}$ *and any ring spectrum* E, *we have the* E_2*-group* EA *in* (2.1.4) *with the Hurewicz map* ϕ *in* (2.1.5). *Thus, we have the E-Adams spectral sequence* $\{E_r^{s,t}\}$ *for any CW spectrum* X_0, *given in Theorem* 1.9 *by* $\{\alpha_n^F\}$ *in* (2.1.6), *with*

(2.3.1) $$E_2^{s,t} = EA_t^s(X_0) = H^s(E_t^*(X_0)) \Rightarrow \pi_{t-s}(X_0) \quad (abut).$$

Moreover, it satisfies

(2.3.2) $E_2^{s,t} = EA_t^s(X_0) = \mathrm{Ext}_{E_*(E)}^{s,t}(E_*(S^0), E_*(X_0))$ *when*

(2.3.3) *the* $E_*(S^0)$*-module* $E_*(E)$ *is flat.*

PROOF. The cofibering $\{\alpha_n^E\}$ in (2.1.6) induces the one $E \wedge X_n \to X_{n+1} \to \Sigma X_n$ (the subspension of X_n), and we have the filtration $X_0 \leftarrow \Sigma^{-1}X_1 \leftarrow \Sigma^{-2}X_2 \leftarrow \cdots$ of X_0, which is the Adams filtration. Thus we have the Adams spectral sequence $\{E_r^{s,t}\}$ given by Proposition 1.2 for $h_* = \pi_*$ and $\{\alpha_n^E\}$. The latter half holds by the following:

(2.3.4) If (2.3.3) holds, then $E_*^s(X)$ in (2.1.1) is $E_*(E) \otimes \cdots \otimes E_*(E) \otimes E_*(X)$ (the tensor product over $E_*(S^0)$ of s copies of $E_*(E)$ and $E_*(X)$) (cf. [16, 13.75]) and $E_*^*(X_0) = \{E_*^s(X_0), \delta^s\}$ is just the cobar complex for $E_*(X_0)$. q.e.d.

In this paper, we consider the following ring spectra as examples:
(2.4.1) For a ring R, HR is the *Eilenberg-MacLane spectrum* of the ordinary homology theory $H_*(\;\; ; R)$, SR is the *Moore spectrum of type* R, and for any ring spectrum E, $ER = E \wedge SR$ is the corresponding one *with coefficients in* R. KO or K is the spectrum of *real* or *complex K-theory*, and bu is the one of the *connective K-theory*. For $G = O$, U or SU, MG is the *Thom spectrum* of the G-bordism theory. For a prime p, BP is the *Brown-Peterson spectrum at p*.

(2.4.2) ([4, III, 15.1]) (2.3.2-4) hold for $E = HR$ or SR when R is a field, KO, K, MO, MU or BP.

(2.4.3) In this case, $EA_*^0(X) = PE_*(X)$, the group of all primitive elements in $E_*(X)$, by (2.3.2) and definition.

When $E_*(E)$ is not flat, we have to calculate $EA_t^s(X_0) = H^s(E_t^*(X_0))$ in (2.3.1) directly by definition. As examples, we have the following

EXAMPLE 2.5. *Consider bu or* buQ_2 *in* (2.4.1) *for* $Q_2 = \{a/b \in Q | b\colon odd\}$. *Then*:

(i) $EA_t^0(S^0) = Z$ *(resp.* Q_2*) if* $t = 0$, $= 0$ *if* $t \neq 0$, *for* $E = bu$ *(resp.* buQ_2*).*

(ii) $buQ_2A_*^1(S^0)$ *is the direct sum of the groups* $Z_2\langle h_n\rangle$ *in degree* $n = 2^\nu \geqq 2$ *and* $Z_{a(n)}\langle \alpha_n\rangle$ *in degree* $2n \geqq 2$, *where the generators* h_n *and* α_n *are given in* (2.5.4,7) *below, and* $a(n) = 2^{\nu+2}$ *if n is even* $\geqq 4 = 2^{\nu+1}$ *otherwise, for* $n = 2^\nu q$ *(q: odd).*

PROOF. We use the following (2.5.1-3) given by Adams [4, III, §§16–17]:

(2.5.1) There is a map j $(= f^0 j$ in [4]): $bu \to HZ_2$ preserving units such that

$$j_*\colon (HZ_2)_*(bu) \to (HZ_2)_*(HZ_2) = A_* = Z_2[\xi_1, \xi_2, \xi_3, \ldots]$$

is monomorphic and $\operatorname{Im} j_* = Z_2[\xi_1^2, \xi_2^2, \xi_3, \ldots]$. Also, the HZ_2-Adams spectral sequence $\{E_r^{s,t}\}$ in Theorem 2.3 for $X_0 = bu^2$ with (2.3.2) satisfies

$$E_2^{s,t} = \operatorname{Ext}_{B_*}^{s,t}(Z_2, (HZ_2)_*(bu)) \Rightarrow \pi_{t-s}(bu^2) \quad \text{for} \quad B_* = A_*/(\xi_1^2, \xi_2^2, \xi_3, \ldots)$$

by the change-of-rings theorem, which converges weakly and collapses for $r \geqq 2$;

and $(j \wedge 1)_* = \mathrm{in} \circ \phi\colon \pi_*(bu^2) \to E_2^{0,*} \subset (HZ_2)_*(bu)$ for its edge homomorphism ϕ. Moreover, there is a homomorphism $E_r^{s,t} \to E_r^{s+1,t+1}$ which for $r = 2$ is multiplication by ξ_1 and for $r = \infty$ is obtained by passing to quotients from multiplication by 2.

(2.5.2) $HZ_*(bu^n)$ is a direct sum of groups Z_p (p: prime) and groups Z in even degree; hence so is $HZ_*(BU^n) = HZ_*(bu^n) \otimes Q_2$ of Z_2 and Q_2, where $BU = buQ_2$ in this proof. Also, $\pi_*(BU^n) = \pi_*(bu^n) \otimes Q_2$, the Hurewicz homomorphism $h\colon \pi_*(BU^2) \to HZ_*(BU^2)$ is monomorphic, and it induces the monomorphism $h\colon \tilde{F}^{s,t} \otimes Q_2 \to \tilde{G}^{s,t}(\tilde{H}^{s,t} = H^{s,t}/H^{s+1,t+1})$ for the filtrations $\{F^{s,t}\}$ of $\pi_*(bu^2)$ corresponding to $\{E_r^{s,t}\}$ in (2.5.1) and $\{G^{s,t} = 2^s HZ_{t-s}(BU^2)\}$. Moreover, the torsion subgroup T_*^n of $\pi_*(BU^{n+1})$ is a direct sum of groups Z_2, and $(j \wedge 1)_*$ $(= (j \wedge 1)_* \otimes 1)\colon \pi_*(BU^{n+1}) \to (HZ_2)_*(BU^n)$ is monomorphic on T_*^n.

(2.5.3) $\pi_*(bu) = Z[t]$ $(\deg t = 2)$ and $\pi_*(bu^2) \otimes Q = Q[u, v]$ for $u = (1 \wedge \iota)_* t$ and $v = (\iota \wedge 1)_* t$. Moreover, a polynomial $f(u, v) \in Q[u, v]$ lies in $\mathrm{Im}\,[\pi_*(BU^2) = \pi_*(bu^2) \otimes Q_2 \to \pi_*(bu^2) \otimes Q = Q[u, v]]$ if and only if
(*) $f(kx, lx) \in Q_2[x, x^{-1}]$ for any odd integers k and l, and $f(u, v) \in Q_2[u/2, v/2]$.

Proof of (i): The coboundary $\delta^0\colon bu_*^0(S^0) = Z[t] \to bu_*^1(S^0) = \pi_*(bu^2)$ in (2.1.1) satisfies $\delta^0 t^n = u^n - v^n$ $(\neq 0$ for $n \geqq 1)$ by definition and (2.5.3). Hence we see (i) for bu, and for $BU = buQ_2$ in the same way.

In (2.5.1), $\Delta\xi_1 = \xi_1 \otimes 1 + 1 \otimes \xi_1$ for the coproduct $\Delta\colon A_* \to A_* \otimes A_* \to B_* \otimes A_*$; hence for $n = 2^\nu \geqq 2$, $j_*^{-1}\xi_1^n \in (HZ_2)_*(bu)$ lies in $E_2^{0,n}$ since $\Delta\xi_1^n = 1 \otimes \xi_1^n$, and we have $x_n \in \pi_n(bu^2)$ with $(j \wedge 1)_* x_n = \phi x_n = j_*^{-1}\xi_1^n$ since ϕ is epimorphic by (1.7.5). Also, $\xi_1 \cdot j_*^{-1}\xi_1^n = 0$ in $E_2^{1,n+1}$ since $\Delta\xi_1^{n+1} = \xi_1 \otimes \xi_1^n + 1 \otimes \xi_1^{n+1}$, and so $2x_n = 0$ in $\tilde{F}^{1,n+1} \subset E_\infty$. Therefore, in (2.5.2), $hx_n \in \tilde{G}^{0,n}$ for $x_n \in \pi_n(BU^2)$ is mapped to 0 by $\times 2\colon \tilde{G}^{0,n} \to \tilde{G}^{1,n+1}$, whose kernel is a direct sum of groups Z_2; and so $x_n \in h_n + F^{1,n+1} \otimes Q_2$ for some $h_n \in T_n^1$. Moreover, $(j^s)_*\colon \pi_*(BU^s) \to \pi_*((HZ_2)^s)$ is monomorphic on T_*^{s-1}, and is a cochain map by (2.1.2). Now, $HZ_2 A_*^1(S^0) = \mathrm{Ext}_{A_*}^{1,*}(HZ_2(S^0), HZ_2(S^0))$ is generated by $\{\xi_1^n\colon n = 2^\nu \geqq 1\}$ (cf. [16, p. 477]). Thus:

(2.5.4) For any $n = 2^\nu \geqq 2$, there exists $h_n \in T_n^1 \subset \pi_n(BU^2) = BU_n^1(S^0)$ $(BU = buQ_2)$ such that $(j \wedge j)_* h_n = \xi_1^n$ in A_*, h_n is a cocycle and its class h_n in $BUA_n^1(S^0)$ generates Z_2. Moreover, if a cocycle $y \in T_n^1$ is not 0 in $BUA_n^1(S^0)$, then $n = 2^\nu \geqq 2$ and $y = h_n$.

On the other hand, let $t_u'\colon BP \to BU = buQ_2$ be the map for BP at 2 induced from the Atiyah-Bott-Shapiro map $t_u\colon MU \to K$ (cf. [5]). Then:

(2.5.5) $t_{u*}' v_1 = t \in \pi_2(bu) \otimes Q_2 = \pi_2(BU)$ for $v_1 = [CP^1] \in \pi_2(BP)$.

(2.5.6) ([11, Cor. 4.23] or [12, Th. 5.5 (b)]) $\alpha_n' = ((1 \wedge \iota_{BP})_* - (\iota_{BP} \wedge 1)_*) v_1^n \in \pi_{2n}(BP^2)$ is divisible by $a(n)$ given in (ii) of the example, and $\alpha_n'/a(n) \in \pi_2(BP^2) = BP_2^1(S^0)$ is a cocycle.

(2.5.7) We have the cocycle $\alpha_n = (t'_u \wedge t'_u)_*(\alpha'_n/a(n)) \in \pi_{2n}(BU^2) = BU^1_{2n}(S^0)$ $(BU = buQ_2)$ with $a(n)\alpha_n = u^n - v^n$ in Im [ ] in (2.5.3), and $\alpha_n \in BUA^1_{2n}(S^0)$ generates $Z_{a(n)}$.

(2.5.8) $f_i(u, v) = (u^n - v^n)/2^i \notin$ Im [ ] in (2.5.3) for any $2^i > a(n)$.
In fact, the first part of (*) in (2.5.3) for $f = f_i$ implies $i \leqq \nu + 2$ if $\nu \geqq 1$ and $i \leqq 1$ if $\nu = 0$ where $n = 2^\nu q$, q: odd (cf. [16, 19.21, 25]), and the second one implies $i \leqq n$. Thus we see (2.5.8).

Proof of (ii): Take any $x \in \pi_*(BU^2) = \pi_*(bu^2) \otimes Q_2$ with $\delta^1 x = 0$. Then, for its image $\bar{x}$ in $\pi_*(bu^2) \otimes Q$, we have $\delta^1 \bar{x} = 0$ and so $\bar{x} = a(u^n - v^n)$ $(a \in Q)$ by [16, 19.20]. Hence, $a = b/a(n)$ for $b \in Q_2$ and $x = b\alpha_n + y$ for $y \in T^1_*$ with $\delta^1 y = 0$ by (2.5.7-8); and we see (ii) by (2.5.4) and (2.5.7). q.e.d.

Here, we notice the following notions due to Miller [10]:
(2.6) $f: X \to Y$ *splits* if $g \circ f \sim 1: X \to X$ for some $g: Y \to X$, X is *E-injective* if $\iota_E \wedge 1: X \to E \wedge X$ splits, and $f: X \to Y$ is *E-monic* if $1 \wedge f: E \wedge X \to E \wedge Y$ splits.

LEMMA 2.7. (i) *For a ring spectrum E, X is EA-injective if X is E-injective; and* $\alpha: X_0 \xrightarrow{f_0} X_1 \to X_2$ *is an EA-cofibering if* f_0 *is E-monic.*

(ii) *K is HZA-injective but not HZ-injective; and* $\alpha^{HZ}: S^0 \xrightarrow{\iota} HZ \to C_\iota$ *is a KA-cofibering, but* ι *is not K-monic.*

PROOF. (i) is seen by Lemma 2.2 and its proof.

(ii) By [16, 13.92, 16.25, 17.21],

(2.7.1) $\pi_*(K) = Z[t, t^{-1}]$ $(\deg t = 2)$, $HZ_*(K) = Q[u, u^{-1}]$ $(\deg u = 2)$ and $K_*(K)$ is torsion free.
Thus, $HZ^s_*(K) = \pi_*(HZ) \otimes \cdots \otimes \pi_*(HZ) \otimes HZ_*(K)$ by [16, 17.9], which is $Q[u, u^{-1}]$ for any s with $\delta^s_{i*} = \mathrm{id}$ in (2.1.1). Hence, $HZA^s_*(K) = 0$ for $s \geqq 1$, and K is HZA-injective. Since $K^s_t(S^0)$ is torsion free by (2.3.4) and (2.7.1), $\iota_*: K^s_t(S^0) \to K^s_t(HZ) = K^s_t(S^0) \otimes Q[u, u^{-1}]$ is monomorphic. Hence α^{HZ} is a KA-cofibering. Since $(\iota \wedge 1)_*: \pi_2(K) = Z \to \pi_2(HZ \wedge K) = Q$ does not split as groups, we see that K is not HZ-injective and ι is not K-monic. q.e.d.

§3. E_2-functors and comparison of spectral sequences

Let denote by $\mathscr{CF}$ the *category of cofiberings* in $\mathscr{C}$, where
(3.1) a *mah* $\psi: \alpha_1 \to \alpha_2$ between cofiberings $\alpha_j: X_{j0} \xrightarrow{f_{j0}} X_{j1} \xrightarrow{f_{j1}} X_{j2}$ $(j = 1, 2)$ consists of maps $\psi_i: X_{1i} \to X_{2i}$ $(i = 0, 1, 2)$ which make the homotopy commutative diagram

$$\begin{array}{ccccccc} \alpha_1: X_{10} & \xrightarrow{f_{10}} & X_{11} & \xrightarrow{f_{11}} & X_{12} & \xrightarrow{f_{12}} & \Sigma X_{10} \\ \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \Sigma\psi_0} \\ \alpha_2: X_{20} & \xrightarrow{f_{20}} & X_{21} & \xrightarrow{f_{21}} & X_{22} & \xrightarrow{f_{22}} & \Sigma X_{20} \end{array}$$

of the induced cofiber sequences of α_j for the suspension functor Σ.

DEFINITION 3.2. We define an *E_2-functor* on $\mathscr{C}$ to be an E_2-group $C = \{C_t^s, KC_t^s(\ \ ; i)\}$ in Definition 1.8 with the following (3.2.1) in addition:
(3.2.1) $C_t^s\colon \mathscr{C} \to \mathscr{A}$ is a homotopy functor, $KC_t^s(\ \ ; i)\colon \mathscr{C}\mathscr{F} \to \mathscr{A}$ is a covariant functor and the exact sequences in (1.8.1) are *natural*, i.e., ι, κ and δ commute with the induced homomorphism ψ_* and ψ_{i*} for any map $\psi = \{\psi_i\}\colon \alpha_1 \to \alpha_2$ in (3.1); hence so are the ones in (1.8.4) for C-cofiberings.

By definition, we see immediately the following
(3.2.2) *For a ring spectrum E, the E_2-group EA in* (2.1.4) *is an E_2-functor.*

Now, for $X_0 \in \mathscr{C}$, a homology theory h_* and E_2-functors $B = C$ and D, let be given
(3.3.1) B-injective cofiberings $\alpha_n^B\colon X_n^B \to W_n^B \to X_{n+1}^B$ and maps $\bar{\lambda} = \{\bar{\lambda}_n, \tilde{\lambda}_n\}\colon \alpha_n^C \to \alpha_n^D$ in $\mathscr{C}\mathscr{F}$ $(n = 0, 1, 2, \dots)$ in homotopy commutative diagrams

$$\begin{array}{ccccccc} \alpha_n^C\colon X_n^C & \xrightarrow{f_n^C} & W_n^C & \xrightarrow{g_{n+1}^C} & X_{n+1}^C & \longrightarrow & \Sigma X_n^C \\ \downarrow{\scriptstyle \bar{\lambda}_n} & & \downarrow{\scriptstyle \tilde{\lambda}_n} & & \downarrow{\scriptstyle \bar{\lambda}_{n+1}} & & \downarrow{\scriptstyle \Sigma\bar{\lambda}_n} \\ \alpha_n^D\colon X_n^D & \xrightarrow{f_n^D} & W_n^D & \xrightarrow{g_{n+1}^D} & X_{n+1}^D & \longrightarrow & \Sigma X_n^D \end{array} \qquad \bar{\lambda}_0 = \mathrm{id} \text{ on } X_0^B = X_0\,,$$

(3.3.2) natural transformations $\phi^B\colon h_t \to B_t^0$ with $\phi^B\colon h_t(W_n^B) \cong B_t^0(W_n^B)$, and
(3.3.3) an *E_2-map* $\lambda\colon C \to D$, consisting of natural transformations $\lambda\colon C_t^s \to D_t^s$, $KC_t^s \to KD_t^s$ compatible with ι, κ and δ in (1.8.1), such that $\phi^D = \lambda \circ \phi^C\colon h_t \to C_t^0 \to D_t^0$.

Then, ϕ^B and $\{\alpha^B\}$ in (3.3.1-2) give us the spectral sequences

(3.3.4) $\{E(B)_r^{s,t}\}$ in Theorem 1.9 with $E(B)_2^{s,t} = B_t^s(X_0) \Rightarrow h_{t-s}(X_0)$ (abut).

Furthermore, the maps $\bar{\lambda}$ in (3.3.1) induce the commutative diagrams

$$(3.3.5)\quad \begin{array}{ccccccccc} \cdots \longrightarrow & h_t(X_s^C) & \xrightarrow{f_*^C} & h_t(W_s^C) & \xrightarrow{g_*^C} & h_t(X_{s+1}^C) & \xrightarrow{\partial} & h_{t-1}(X_s^C) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \bar{\lambda}_{s*}} & & \downarrow{\scriptstyle \tilde{\lambda}_{s*}} & & \downarrow{\scriptstyle \bar{\lambda}_{s+1*}} & & \downarrow{\scriptstyle \bar{\lambda}_{s*}} & \\ \cdots \longrightarrow & h_t(X_s^D) & \xrightarrow{f_*^D} & h_t(W_s^D) & \xrightarrow{g_*^D} & h_t(X_{s+1}^D) & \xrightarrow{\partial} & h_{t-1}(X_s^D) & \longrightarrow \cdots \end{array} \quad (\bar{\lambda}_{0*} = \mathrm{id})$$

of the exact sequences in (1.1.2). Therefore, by Proposition 1.2, we have the induced map
(3.3.6) $\bar{\lambda}_*\colon \{E(C)_r^{s,t}\} \to \{E(D)_r^{s,t}\}$ between the spectral sequences in (3.3.4) with

$$\bar{\lambda}_* = \tilde{\lambda}_{s*}\colon E(C)_1^{s,t} = h_t(W_s^C) \to E(D)_1^{s,t} = h_t(W_s^D) \Rightarrow \mathrm{id} \text{ on } h_{t-s}(X_0) \quad \text{(abut)}\,.$$

We see that this is represented on the E_2-terms by λ in (3.3.3):

(3.3.7) $\bar{\lambda}_* = \lambda\colon E(C)_2^{s,t} = C_t^s(X_0) \to E(D)_2^{s,t} = D_t^s(X_0)$, more precisely,

$$\bar{\phi}^D \circ \bar{\lambda}_* = \lambda \circ \bar{\phi}^C \quad \text{for} \quad \bar{\phi}^B = (\bar{\delta}^B)^s \circ (f_{s*}^B)^{-1} \circ \phi^B\colon E(B)_2^{s,t} \cong B_t^s(X_0) \quad \text{in (1.3.4)}.$$

In fact, we see that $(f_{s*}^D)^{-1} \circ \phi^D \circ \tilde{\lambda}_{s*} = \bar{\lambda}_{s*} \circ \lambda \circ (f_{s*}^C)^{-1} \circ \phi^C$ and the diagram

$$\begin{array}{ccccc} C_t^s(X_{n+1}^C) & \xrightarrow{\lambda} & D_t^s(X_{n+1}^C) & \xrightarrow{\bar{\lambda}_{n+1*}} & D_t^s(X_{n+1}^D) \\ \downarrow \bar{\delta}^D & & & & \downarrow \bar{\delta}^D \\ C_t^{s+1}(X_n^C) & \xrightarrow{\lambda} & D_t^{s+1}(X_n^C) & \xrightarrow{\bar{\lambda}_{n*}} & D_t^{s+1}(X_n^D) \end{array} \qquad (\bar{\delta}^B = \kappa^{-1} \circ \delta \circ \iota^{-1} \text{ in (1.8.4)})$$

is commutative by (3.3.1-3) and (3.2.1); and these imply the desired equality

$$\bar{\phi}^D \circ \bar{\lambda}_* = (\bar{\delta}^D)^s \circ (f_{s*}^D)^{-1} \circ \phi^D \circ \tilde{\lambda}_{s*} = \bar{\lambda}_{0*} \circ \lambda \circ (\bar{\delta}^C)^s \circ (f_{s*}^C)^{-1} \circ \phi^C = \lambda \circ \bar{\phi}^C.$$

For this induced map $\bar{\lambda}_*$, we have the following

THEOREM 3.4. *In addition to* (3.3.1-3), *assume that*
(3.4.1) *each α_n^C is also a D-injective cofibering and $\phi^D\colon h_t(W_n^C) \cong D_t^0(W_n^C)$.*
Then, the spectral sequences $\{E(B)_r^{s,t}\}$ $(B = C, D)$ in (3.3.4) *are isomorphic for $r \geqq 2$ by the induced map $\bar{\lambda}_*$ in* (3.3.6), *and $\lambda\colon C_t^s(X_0) \cong D_t^s(X_0)$ for any s and t.*

PROOF. By (3.4.1), Theorem 1.9 for ϕ^D and $\{\alpha_n^C\}$ shows that

$$\lambda\colon C_t^s(X_0) \cong D_t^s(X_0), \quad \bar{\lambda}_*\colon E(C)_r^{s,t} \cong E(D)_r^{s,t} \quad \text{for} \quad r = 2;$$

hence the latter is isomorphic also for any $r \geqq 2$. q.e.d.

By weakening the assumption (3.4.1), we can prove the following

THEOREM 3.5. *In addition to* (3.3.1-3), *assume the following* (3.5.1-3) *for some integers $a \geqq 0$ and b:*
(3.5.1) *α_n^C is a D-cofibering if $n \leqq a$.*
(3.5.2) *$D_t^s(W_n^C) = 0$ if $n < t - b - 1 = s + n < a$ (when $a \geqq 2$).*
(3.5.3) *$\phi^D\colon h_t(W_s^C) \to D_t^0(W_s^C)$ is*
(*) *monomorphic if $t - b = s \leqq a$ and epimorphic if $t - b - 1 = s < a$.*

(i) *Then, $\bar{\lambda}_* = \lambda\colon E(C)_2^{s,t} = C_t^s(X_0) \to E(D)_2^{s,t} = D_t^s(X_0)$ in* (3.3.7) *is* (*).

(ii) *Furthermore, for the subgroups $\bar{Z}E$ in Corollary* 1.7 (i), *the restriction $\lambda | \bar{Z}C_t^s(X_0)\colon \bar{Z}C_t^s(X_0) \to \bar{Z}D_t^s(X_0)$ for $t = b + s$ is epimorphic if $s \leqq a + 1$; hence it is isomorphic if $s \leqq a$ by* (i).

PROOF. (i) By (3.5.1) and (3.3.3), we have the commutative diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_t^s(X_n^C) & \longrightarrow & C_t^s(W_n^C) & \longrightarrow & C_t^s(X_{n+1}^C) & \xrightarrow{\bar{\delta}^C} & C_t^{s+1}(X_n^C) & \longrightarrow \cdots \\
(3.5.4) \quad & \downarrow \lambda & & \downarrow \lambda & & \downarrow \lambda & & \downarrow \lambda & \quad (n \leqq a) \\
\cdots \longrightarrow & D_t^s(X_n^C) & \longrightarrow & D_t^s(W_n^C) & \longrightarrow & D_t^s(X_{n+1}^C) & \xrightarrow{\bar{\delta}^D} & D_t^{s+1}(X_n^C) & \longrightarrow \cdots
\end{array}
$$

of the exact sequences in (1.8.4) for $n \leqq a$, where

(3.5.5) $C_t^* = D_t^* = 0$ if $* < 0$, $C_t^*(W_n^C) = 0$ if $* \geqq 1$, and $\bar{\delta}^D\colon D_t^n(X_{s-n}^C) \to D_t^{n+1}(X_{s-n-1}^C)$ $(0 < n < s)$ and $\lambda = \phi^D \circ (\phi^C)^{-1}\colon C_t^0(W_s^C) \to D_t^0(W_s^C)$ are $(*)$ in (3.5.3),

because $\phi^D = \lambda \circ \phi^C$ and ϕ^C is isomorphic for W_s^C by (3.3.2-3). Thus, by 5-Lemma and by induction, we see that

(3.5.6) $\lambda\colon C_t^n(X_{s-n}^C) \to D_t^n(X_{s-n}^C)$ $(0 \leqq n \leqq s)$ is also $(*)$; and (i) holds.

(ii) By (3.3.5-6) and the definition of $\bar{Z}B_t^s(X_0)$, $\lambda(\bar{Z}C_t^s(X_0)) \subset \bar{Z}D_t^s(X_0)$ holds and (ii) is proved by showing the following in the commutative diagram (3.3.5):

(3.6.1) Let $t = b + s \leqq a + b + 1$. Then, for any $y \in h_t(X_s^D)$, there exist $x_n \in h_{b+n}(X_n^C)$ $(0 \leqq n \leqq s)$ with $x_0 = y_0$ and $\partial\bar{\lambda}_{n*}x_n = y_{n-1}$ for $n > 0$, where $y_n = \partial^{s-n}y$.

In fact, (3.6.1) shows that $\partial\bar{\lambda}_{s*}x_s = \partial y$; hence $y - \bar{\lambda}_{s*}x_s \in \operatorname{Ker}\partial = \operatorname{Im} g_*^D$, and so $f_*^D y - \tilde{\lambda}_{s*}f_*^C x_s \in \operatorname{Im} d_1^D$ $(d_1^B = f_*^B \circ g_*^B)$ for any $y \in h_t(X_s^D)$ and some $x_s \in h_t(X_s^C)$. Thus, $\tilde{\lambda}_{s*}\colon \operatorname{Im} f_*^C/\operatorname{Im} d_1^C \to \operatorname{Im} f_*^D/\operatorname{Im} d_1^D$ is epimorphic, which means (ii).

Now, assume inductively that there exists x_n in (3.6.1) for $n < s$. Then, $\bar{\lambda}_*x_n - y_n \in \operatorname{Ker}\partial = \operatorname{Im} g_*^D$ and so $\tilde{\lambda}_{n*}f_*^C x_n = f_*^D(\bar{\lambda}_*x_n - \partial y_{n+1}) \in \operatorname{Im} d_1^D$ $(\bar{\lambda} = \bar{\lambda}_n)$. Thus $\tilde{\lambda}_n f_*^C x_n = 0$ in $E(D)_2^{n,m}$ $(m = b + n)$; hence $f_*^C x_n = 0$ in $E(C)_2^{n,m}$ by (i), and $f_*^C x_n = d_1^C w = f_*^C g_*^C w$, $x_n - g_*^C w = \partial x$ for some $w \in h_m(W_{n-1}^C)$, $x \in h_{m+1}(X_{n+1}^C)$. Therefore, $\partial^2\bar{\lambda}'_*x = \partial\bar{\lambda}_*x_n = \partial y_n$ $(\bar{\lambda}' = \bar{\lambda}_{n+1})$; hence

(3.6.2) $g_*^D z = \partial\bar{\lambda}'_*x - y_n = \partial(\bar{\lambda}'_*x - y_{n+1})$ for some $z \in h_m(W_{n-1}^D)$.

This implies $d_1^D z = f_*^D g_*^D z = 0$, and so we see by (i) that

(3.6.3) $z - \tilde{\lambda}_{n-1*}w' \in \operatorname{Im} d_1^D$ for some $w' \in h_m(W_{n-1}^C)$ with $d_1^C w' = f_*^C g_*^C w' = 0$; hence $g_*^C w' = \partial x'$ and so $\partial\bar{\lambda}'_*x' = \bar{\lambda}_*g_*^C w' = g_*^D z$ for some $x' \in h_{m+1}(X_{n+1}^C)$. Thus $\partial\bar{\lambda}'_*x_{n+1} = y_n$ for $x_{n+1} = x - x'$; and (3.6.1) is proved by induction. q.e.d.

As applications to Theorems 3.4-5, we compare the Adams spectral sequences $\{E(G)_r^{s,t}\}$ given in Theorem 2.3 for

(3.7.1) ring spectra $G = E$ and F with a unit-preserving map $\lambda\colon E \to F$ $(\iota_F \sim \lambda \circ \iota_E\colon S^0 \to F)$.

In this case, λ induces $\bar{\lambda}\colon \bar{E} = C_{\iota_E} \to \bar{F} = C_{\iota_F}$ (cf. [16, 8.31]) and the maps

(3.7.2) $\bar{\lambda} = \{\bar{\lambda}_n, \tilde{\lambda}_n\}\colon \alpha_n^E \to \alpha_n^F$ between the cofiberings of (2.1.6) in

$$
\begin{array}{ccccccc}
\alpha_n^E\colon X_n^E & \xrightarrow{\iota_E \wedge 1} & E \wedge X_n^E & \longrightarrow & X_{n+1}^E & \longrightarrow & \Sigma X_n^E \\
\downarrow \bar{\lambda}_n & & \downarrow \tilde{\lambda}_n & & \downarrow \bar{\lambda}_{n+1} & & \downarrow \Sigma\bar{\lambda}_n \quad (X_0^E = X_0 = X_0^F) \\
\alpha_n^F\colon X_n^F & \xrightarrow{\iota_F \wedge 1} & F \wedge X_n^F & \longrightarrow & X_{n+1}^F & \longrightarrow & \Sigma X_n^F
\end{array}
$$

given by $\bar{\lambda}_n = \bar{\lambda}^n \wedge 1: X_n^E = \bar{E}^n \wedge X_0 \to X_n^F = \bar{F}^n \wedge X_0$ and $\tilde{\lambda}_n = \lambda \wedge \bar{\lambda}_n$ $(n \geqq 0)$. Furthermore, $\lambda^{s+1} \wedge 1: E^{s+1} \wedge X_0 \to F^{s+1} \wedge X_0$ $(s \geqq 0)$ induce the cochain maps $\lambda_*: E_t^*(X) \to F_t^*(X)$ $(X \in \mathscr{C})$ and $\lambda_*: KE_t^*(\alpha; i) \to KF_t^*(\alpha; i)$ $(\alpha \in \mathscr{C}\mathscr{F})$, which induce the E_2-map

$$(3.7.3) \qquad \lambda_*: EA = \{EA_t^s, KEA_t^s(\ \ ; i)\} \to FA = \{FA_t^s, KFA_t^s(\ \ ; i)\}$$

between the E_2-functors GA given in (2.1.4) (see (3.2.2)). This satisfies

(3.7.4) $\phi^F = \lambda_* \circ \phi^E: \pi_t(X) \to EA_t^0(X) \to FA_t^0(X)$ for ϕ^G in (2.1.5).

Thus, by Theorem 2.3 and (3.3.6-7), we have the map

(3.7.5) $\bar{\lambda}_*: \{E(E)_r^{s,t}\} \to \{E(F)_r^{s,t}\}$ between the Adams spectral sequences with

$$\bar{\lambda}_* = \lambda_*: E(E)_2^{s,t} = EA_t^s(X_0) \to E(F)_2^{s,t} = FA_t^s(X_0) \Rightarrow \text{id on } \pi_{t-s}(X_0) \quad \text{(abut)}.$$

Now, $(\iota_F \wedge 1)_* = (\lambda \wedge 1)_* \circ (\iota_E \wedge 1)_*: F_t^s(X) \to F_t^s(E \wedge X) \to F_t^s(F \wedge X)$ is monomorphic by Lemma 2.2, and so is $(\iota_E \wedge 1)_*$. Hence:

(3.7.6) $KF_t^s(\omega^E \wedge X; 0) = 0$ and $\alpha_n^E = \omega^E \wedge X_n^E$ is also an FA-cofibering, by definition. Therefore, Theorems 3.4-5 imply the following

THEOREM 3.8. *Let $\lambda: E \to F$ be a unit-preserving map between ring spectra, and consider $W_n^E = E \wedge X_n^E = E \wedge \bar{E}^n \wedge X_0$ $(n \geqq 0)$ in (3.7.2) for $X_0 \in \mathscr{C}$. Then:*

(i) *$\bar{\lambda}_*: E(E)_r^{s,t} \to E(F)_r^{s,t}$ in (3.7.5) is isomorphic for $r \geqq 2$, if*

(3.8.1) *each W_n^E is FA-injective and ϕ^F (or λ_*) in (3.7.4) for $X = W_n^E$ is isomorphic.*

(ii) *Assume that there are integers $a \geqq 0$ and b such that*

(3.8.2) *$FA_t^s(W_n^E) = 0$ if $n < t - b - 1 = n + s < a$ (when $a \geqq 2$), and*

(3.8.3) *ϕ^F (or λ_*) in (3.7.4) for $X = W_s^E$ is*

(∗) *monomorphic if $t - b = s \leqq a$ and epimorphic if $t - b - 1 = s < a$.*

Then, $\bar{\lambda}_*: E(E)_2^{s,t} \to E(F)_2^{s,t}$ in (3.7.5) is also (∗). Furthermore the restriction

$$(3.8.4) \qquad \bar{\lambda}_* = \lambda_*: \bar{Z}E(E)_2^{s,t} = \bar{Z}EA_t^s(X_0) \to \bar{Z}E(F)_2^{s,t} = \bar{Z}FA_t^s(X_0)$$

for $t = b + s$ is isomorphic if $s \leqq a$ and epimorphic if $s = a + 1$.

(iii) *(ii) holds for $a = 1$ (resp. 0) and any b, if*

(3.8.5) *$\phi^F: \pi_*(E) \to FA_*^0(E)$ is isomorphic (resp. monomorphic), and*

(3.8.6) *$E_*(E)$ and $E_*(X_0)$ (resp. $E_*(X_0)$) are the flat $E_*(S^0)$-modules.*

PROOF OF (iii). We see inductively that

(3.8.7) if $E_*(E)$ and $E_*(X_0)$ are flat, then so is $E_*(X_n^E)$ for any n, because then $E_*(W_n^E) = E_*(E) \otimes E_*(X_n^E)$ by [16, 13.75], and

(3.8.8) the split exact sequence $0 \to E_*(X_n^E) \to E_*(W_n^E) \to E_*(X_{n+1}^E) \to 0$ holds, by Lemma 2.2. Then, by [16, Note after 13.75], we see that

(3.8.9) if (3.8.6) holds, then for $n \leqq a = 1$ (resp. 0), $F_*^t(W_n^E) = \pi_*(F^{t+1} \wedge E) \otimes E_*(X_n^E)$, and so $FA_*^t(W_n^E) = FA_*^t(E) \otimes E_*(X_n^E)$ and $\phi^F = \phi^F \otimes \text{id}: \pi_*(W_n^E) = \pi_*(E) \otimes E_*(X_n^E) \to FA_t^0(W_n^E) = FA_*^0(E) \otimes E_*(X_n^E)$.

Thus (3.8.5-6) imply (3.8.3) for $a = 1$ (resp. 0). q.e.d.

EXAMPLE 3.9. *In Theorem* 3.8, (i) *is valid when* $E = F$ *for any unit-preserving map* $\lambda\colon E \to E$, *or when* λ *is the Thom map* $\Phi\colon MO \to HZ_2$.
Also, under the assumption that $E_*(X_0)$ *is flat,* (iii) *is valid for* $a = 1$ *when* λ *is the Atiyah-Bott-Shapiro map* $t_u\colon MU \to K$ *or* $t_u^{BP}\colon BP \to KQ_p$ *at a prime p induced from* t_u; *and for* $a = 0$ *when* λ *is the Conner-Floyd map* $t_{su}\colon MSU \to KO$ (*cf.* [15, 7.10]).

PROOF. When $F = E$, (3.8.1) holds by Lemma 2.2. $MO \simeq \bigvee_i \Sigma^{n_i} HZ_2$ (homotopy equivalent) by [4, p. 207], and so $W_n^{MO} \simeq (\bigvee_i \Sigma^{n_i} HZ_2) \wedge X_n^{MO}$. Hence, we see that $HZ_2 A_t^s(W_n^{MO}) = \mathrm{Ext}_{A_*}^{s,t}(Z_2, (HZ_2)_*(W_n^{MO}))$ ($A_* = (HZ_2)_*(HZ_2)$) in (2.3.2) is isomorphic to $\pi_t(W_n^{MO})$ by ϕ^{HZ_2} if $s = 0$ and is 0 if $s > 0$; and (3.8.1) holds.

(3.8.6) holds in each case by (2.4.2). $\phi^K\colon \pi_*(MU) \cong PK_*(MU)$ by the Hattori-Stong theorem (cf. [4, II, 14.1]). By [4, II, §16], BP is the direct summand of MUQ_p, and so the isomorphism ϕ^K induces $\phi^{K'}\colon \pi_*(BP) \cong PK'_*(BP)$ ($K' = KQ_p$). Also, $\phi^{KO}\colon \pi_*(MSU) \to PKO_*(MSU)$ is monomorphic by [15, 7.10]. Since $PF_*(X) = FA_*^0(X)$ by (2.4.3), these show the latter half.
q.e.d.

EXAMPLE 3.10. *Theorem* 3.8 (ii) *is valid for the Thom map* $\Phi^{BP}\colon BP \to HZ_p$ *at a prime p,* $X_0 = S^0$, $a = q - r - 1$ *and* $b = kq + r$ *with* $0 < r < q$, *where* $q = 2(p-1)$; *and* $BPA_t^s(S^0)$ $(q \nmid t)$, $HZ_pA_{nq+t}^s(S^0)$ $(s+1 < t < q)$, $\bar{Z}HZ_pA_{nq+t}^s(S^0)$ $(s < t < q)$ *are* 0, *and* $\Phi_*^{BP}\colon \bar{Z}BPA_{nq}^s(S^0) \to \bar{Z}HZ_pA_{nq}^s(S^0)$ $(s < q)$ *is epimorphic.*

PROOF. We use the following (3.10.1) (cf. [4, II, §16]):

(3.10.1) If $q \nmid t$, then $\pi_t(BP)$, $BP_t(BP)$ and $HZ_pA_{t+s}^s(BP)$ are all 0, (for the last one, we see that $\mathrm{Ext}_{A_*}^{*,*}(Z_p, (HZ_p)_*(BP))$ ($A_* = (HZ_p)_*(HZ_p)$) in (2.3.2) is $Z_p[a_0, a_1, \dots]$ ($a_i \in \mathrm{Ext}^{1,t_i}$, $t_i = 2(p^i - 1) + 1$) by the structure of $(HZ_p)_*(BP)$ in [7] and by the same argument as in [16, pp. 500–503].)
Then, according to (2.4.2) and (3.8.7-9), we see the following

(3.10.2) If $q \nmid t$, then $BP_t^s(S^0)$, $BP_t(X_n^{BP})$ and $HZ_pA_{t+s}^s(W_n^{BP})$ are 0, where $X_0 = S^0$; which implies (3.8.2) and the desired results. q.e.d.

§4. Mahowald spectral sequences and double E_2-functors

Let $D = \{D_u^t, KD_u^t\}$ be an E_2-functor, and for a given X_0, assume that
(4.1.1) there exist D-cofiberings $\omega_s\colon X_s \xrightarrow{i_s} W_s \xrightarrow{j_{s+1}} X_{s+1}$ for $s \geqq 0$. Then, by (1.8.4), we have the exact sequences

(4.1.2)
$$\cdots \longrightarrow D_u^t(X_s) \xrightarrow{i_*} D_u^t(W_s) \xrightarrow{j_*} D_u^t(X_{s+1}) \xrightarrow{\delta} D_u^{t+1}(X_s) \longrightarrow \cdots (i_* = i_{s*}, j_* = j_{s+1*})\,;$$

and the same argument as Proposition 1.2 and $D_u^t = 0$ $(t < 0)$ imply the following

PROPOSITION 4.2. *For an E_2-functor D and X_0 with* (4.1.1), *we have the spectral sequence* $\{\tilde{E}_{u,r}^{s,t}, d_r\colon \tilde{E}_{u,r}^{s,t} \to \tilde{E}_{u,r}^{s+r,t-r+1}\}$ *associated to* (4.1.2) *such that*

(4.2.1) $d_1 = i_* \circ j_*\colon \tilde{E}_{u,1}^{s,t} = D_u^t(W_s) \to \tilde{E}_{u,1}^{s+1,t} = D_u^t(W_{s+1})$, *and*

(4.2.2) $\{\tilde{E}_{u,r}^{s,t}\}$ *converges to* $D_u^{s+t}(X_0)$, $\tilde{E}_{u,\infty}^{s,t} \cong F_u^{s,t}/F_u^{s+1,t-1}$, *in the sense of* (1.6.2), *by the finite filtration* $D_u^{s+t}(X_0) = F_u^{0,s+t} \supset \cdots \supset F_u^{s,t} = \operatorname{Im}[\bar{\delta}^s\colon D_u^t(X_s) \to D_u^{t+s}(X_0)] \supset F_u^{s+1,t-1} \supset \cdots \supset F_u^{s+t+1,-1} = 0$.

We now represent the E_2-term of this spectral sequence in a similar way to Theorem 1.9.

DEFINITION 4.3. Let be given a collection of covariant functors

$$A = \{A_u^{s,t}\colon \mathscr{C} \to \mathscr{A};\ KA_u^{s,t}(\ \ ; i),\ LA_u^{s,t}(\ \ ; i, j)\colon \mathscr{CF} \to \mathscr{A} \mid s, t, u \in Z;\ i, j = 0, 1, 2\}$$

with $A_u^{s,t} = KA_u^{s,t}(\ \ ; i) = LA_u^{s,t}(\ \ ; i, j) = 0$ for $s < 0$ or $t < 0$.

(1) We say that A is a *double E_2-functor* on $\mathscr{C}$, if

(4.3.1) for any $\alpha\colon X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$ in $\mathscr{CF}$, there hold *natural* exact sequences

$$\cdots \longrightarrow LA_u^{s,t}(\alpha; i, j) \xrightarrow{\iota} A_{u,j}^{s,t}(\alpha; i) \xrightarrow{\kappa} LA_u^{s,t}(\alpha; i + j, j + 1)$$
$$\xrightarrow{\delta} LA_u^{s+1,t}(\alpha; i, j) \longrightarrow \cdots,$$

$\rho = \rho_{i,j}$ for $\rho = \iota,\ \kappa,\ \delta$, where $A_{u,j}^{s,t}(\alpha; i) = KA_u^{s,t}(\alpha; i)$ $(j = 0, 2)$, $A_{u,1}^{s,t}(\alpha; i) = A_u^{s,t}(X_i)$ and $LA_u^{s,t}(\alpha; a, b) = LA_{u-1}^{s,t}(\alpha; a - 3, b)$ if $a \geqq 3$, $= LA_{u+1}^{s,t+1}(\alpha; a, b - 3)$ if $b \geqq 3$; and these satisfy the equalities

$$(4.3.2)\quad f_{i*} = \iota_{i+1,1} \circ \kappa_{i+1,0} \circ \iota_{i+1,2} \circ \kappa_{i,1}\colon A_u^{s,t}(X_i) \to A_u^{s,t}(X_{i+1}) \quad \text{for} \quad i = 0, 1\ .$$

(2) We call $\alpha\colon X_0 \to X_1 \to X_2$ in $\mathscr{CF}$ an *$A(1)$-injective cofibering* if it is an *$A(1)$-cofibering*, i.e., $KA_u^{s,t}(\alpha; 0) = 0 = LA_u^{s,t}(\alpha; i, j)$ for $j = 0$ (hence for $i = 0$ by (4.3.1)), and X_1 is *$A(1)$-injective*, i.e., $A_u^{s,t}(X_1) = 0$ for $s \neq 0$.

(3) We say that A *is related to an E_2-functor D at X_0 by ψ^D and* $\{\omega_s\}$, if (4.3.3) each $\omega_s\colon X_s \xrightarrow{i_s} W_s \xrightarrow{j_{s+1}} X_{s+1}$ is a D-cofibering and $A(1)$-injective cofibering and $\psi^D\colon D_u^t \to A_u^{0,t}$ is a natural transformation with $\psi^D\colon D_u^t(W_s) \cong A_u^{0,t}(W_s)$.

By this definition, the exact sequences in (4.3.1-2) imply the following:

(4.3.4) Any $A(1)$-cofibering $\alpha\colon X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$ induces the exact sequence

$$\cdots \longrightarrow A_u^{s,t}(X_0) \xrightarrow{f_{0*}} A_u^{s,t}(X_1) \xrightarrow{f_{1*}} A_u^{s,t}(X_2) \xrightarrow{\bar{\delta}} A_u^{s+1,t}(X_0) \longrightarrow \cdots,$$

where $\bar{\delta} = (\kappa_{1,0} \circ \iota_{1,2} \circ \kappa_{0,1})^{-1} \circ \delta_{1,1} \circ (\iota_{2,1} \circ \kappa_{2,0} \circ \iota_{2,2})$ by the isomorphisms κ and ι in it.

Hence, for ω_s and ψ in (4.3.3), the following (4.3.5-6) hold:

(4.3.5) $\bar{\delta}\colon A_u^{n,t}(X_{s+1}) \cong A_u^{n+1,t}(X_s)$ for $n \geqq 1$, and we have the exact sequence

$$0 \longrightarrow A_u^{0,t}(X_s) \xrightarrow{i_{s*}} A_u^{0,t}(W_s) \xrightarrow{j_{s+1*}} A_u^{0,t}(X_{s+1}) \xrightarrow{\bar{\delta}} A_u^{1,t}(X_s) \longrightarrow 0\,.$$

(4.3.6) $\bar{\psi} = \bar{\delta}^s \circ (i_{s*}^{-1} \circ \psi)\colon \tilde{E}_{u,2}^{s,t} \cong A_u^{0,t}(X_s)/\mathrm{Im}\, j_{s*} \cong A_u^{s,t}(X_0)$ for $\{\tilde{E}_{u,r}^{s,t}\}$ in Proposition 4.2. Moreover, $\psi\colon D_u^0(X_s) \cong A_u^{0,0}(X_s)$.

In fact, the first isomorphism is seen in the same way as (1.3.4) by the exact sequences in (4.1.2) and (4.3.5) with ψ in (4.3.3); and the second one by those for $t = 0$, $D_u^{-1} = 0$ and 5-Lemma. Thus, we have proved the following

THEOREM 4.4 (Mahowald spectral sequence). *In case of Definition* 4.3(3), *we have the spectral sequence* $\{\tilde{E}_{u,r}^{s,t}\}$ *in Proposition* 4.2 *which converges to* $D_u^{s+t}(X_0)$ *and whose* E_2*-term* $\tilde{E}_{u,2}^{s,t}$ *is isomorphic to* $A_u^{s,t}(X_0)$ *by* $\bar{\psi}$ *in* (4.3.6): $\tilde{E}_{u,2}^{s,t} = A_u^{s,t}(X_0) \Rightarrow D_u^{s+t}(X_0)$ *(conv).*

The same proof as Corollary 1.7 and the last half of (4.3.6) give us the following

COROLLARY 4.5. (i) *In Theorem* 4.4. $\bar{Z}A_u^{s,t}(X_0) = \mathrm{Im}\,[\bar{\psi} = \bar{\delta}^s \circ \psi\colon D_u^t(X_s) \to A_u^{s,t}(X_0)]$ *for* $\bar{Z}A_u^{s,t}(X_0) = \bar{\psi}(\tilde{Z}_{u,\infty}^{s,t}/\tilde{B}_{u,2}^{s,t}) = \bar{\psi}(\mathrm{Im}\, i_*/i_*\, \mathrm{Ker}\, \bar{\delta})$; *and* $\bar{Z}A_u^{s,0}(X_0) = A_u^{s,0}(X_0)$.

(ii) *When* $\{\tilde{E}_{u,r}^{s,t}\}$ *collapses, the similar results to Corollary* 1.7 (ii) *hold.*

By Theorem 4.4, we can construct a spectral sequence which converges to a given E_2-functor, or to the E_2-term of a spectral sequence in Theorem 1.9, by finding a double E_2-functor related to it. We call a spectral sequence of this theorem a *Mahowald* one according to Miller [10].

For a ring spectrum E and an E_2-functor $D = \{D_u^t, KD_u^t(\ \ ;i)\}$, we obtain a double E_2-functor ED in the same way as (2.1.1-4), as follows: For $X \in \mathscr{C}$, let

(4.6.1) $$DE_u^{*,t}(X) = \{DE_u^{s,t}(X) = D_u^t(E^{s+1} \wedge X)\ (s \geqq 0),\ = 0\ (s < 0)\}$$

be the cochain complex with $\delta^s = \sum_{i=0}^{s+1}(-1)^i \delta_{i*}^s$ for $\delta_i^s\colon E^{s+1} \wedge X \to E^{s+2} \wedge X$ in (2.1.1). Also, for $\alpha\colon X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$ in $\mathscr{C}\mathscr{F}$, consider $E^s \wedge \alpha\colon E^s \wedge X_0 \xrightarrow{1 \wedge f_0} E^s \wedge X_1 \xrightarrow{1 \wedge f_1} E^s \wedge X_2$ and $\delta_i^s = \delta_i^s \wedge 1\colon E^{s+1} \wedge \alpha \to E^{s+2} \wedge \alpha$ in $\mathscr{C}\mathscr{F}$. Then, according to (3.2.1),

(4.6.2) $$KDE_u^{*,t}(\alpha; i) = \{KDE_u^{s,t}(\alpha; i) = KD_u^t(E^{s+1} \wedge \alpha; i)\ (s \geqq 0),\ = 0\ (s < 0)\}$$

is the cochain complex with $\delta^s = \sum_{i=0}^{s+1}(-1)^i \delta_{i*}^s$, and by the exact sequences

(*) $$\cdots \longrightarrow KDE_u^{s,t}(\alpha; i) \xrightarrow{\iota} DE_u^{s,t}(X_i) \xrightarrow{\kappa} KDE_u^{s,t}(\alpha; i+1) \xrightarrow{\delta} KDE_u^{s,t+1}(\alpha; i) \longrightarrow \cdots$$

in (1.8.1) for $E^{s+1} \wedge \alpha$, $\iota_{i,0} = \iota$, $\iota_{i,1} = \kappa$ and $\iota_{i+1,2} = \delta$ give us

(4.6.3) the subcomplexes $LDE_u^{*,t}(\alpha; i,j) = \{\mathrm{Ker}\, \iota_{i,j}\}$ with the exact sequences

$$0 \to LDE_u^{s,t}(\alpha; i, j) \to DE_{u,j}^{*,t}(\alpha; i) \to LDE_u^{*,t}(\alpha; i+j, j+1) \to 0$$

of cochain complexes ($DE_{u,0}^{*,t} = DE_{u,2}^{*,t} = KDE_u^{*,t}$, $DE_{u,1}^{*,t}(\alpha; i) = DE_u^{*,t}(X_i)$, and $LDE_u^{*,t}(\alpha; a, b) = LDE_{u-1}^{*,t}(\alpha; a-3, b)$ $(a \geqq 3)$, $= LDE_{u+1}^{*,t+1}(\alpha; a, b-3)$ $(b \geqq 3)$).

(4.6.4) Thus *we have the double E_2-functor ED*, where $ED_u^{s,t}(X)$, $KED_u^{s,t}(\alpha; i)$ and $LED_u^{s,t}(\alpha; i, j)$ are the cohomologies H^s of the cochain complexes in (4.6.1-3). Moreover, in the same way as ϕ^E in (2.1.5), we have

(4.6.5) $$\psi^D = (\iota_E \wedge 1)_*: D_u^t(X) \to \operatorname{Ker} \delta^0 = H^0(DE_u^{*,t}(X)) = ED_u^{0,t}(X);$$

and by the same proof as Lemma 2.2, we see that

(4.6.6) $$\psi^D: D_u^t(E \wedge X) \cong ED_u^{0,t}(E \wedge X), \quad \text{and} \quad ED_u^{s,t}(E \wedge X) = 0 \quad \text{if} \quad s > 0.$$

Now, consider the case that

(4.6.7) each $E^s \wedge \alpha_n^E$ for $\alpha_n^E: X_n \xrightarrow{\iota_E \wedge 1} E \wedge X_n \to X_{n+1}$ in (2.1.6) is a D-cofibering.

Then $KDE_u^{s,t}(\alpha_n^E; 0) = 0$ by definition. Hence $\operatorname{Ker} \iota_{0,0} = 0$ and $\operatorname{Ker} \iota_{2,0} = \operatorname{Im} \iota_{0,2} = 0$ in (*). Also, $\iota_{1,2} = 0$, $\operatorname{Ker} \iota_{1,2} = \operatorname{Im} \iota_{0,1}$ and $\iota_{1,0} \circ \iota_{0,1} = \iota \circ \kappa = (\iota_E \wedge 1)_*$ by (1.8.2), which show that $\iota_{1,0}$ is monomorphic since so is $(\iota_E \wedge 1)_*$ and $\iota_{0,1}$ is epimorphic. Thus $LDE_u^{s,t}(\alpha_n^E; i, 0) = \operatorname{Ker} \iota_{i,0} = 0$; and we see the following:

(4.6.8) *If* (4.6.7) *holds, then α_n^E is an $ED(1)$-cofibering, and ED is related to D at X_0 by ψ^D and $\{\alpha_n^E\}$. In particular, when $D = FA$ in* (2.1.4) *for a ring spectrum F,* (4.6.7) *holds if*
(4.6.9) $(1 \wedge \iota_E \wedge 1)_*: F_*(F^t \wedge X_n) \to F_*(F^t \wedge E \wedge X_n)$ *is monomorphic, e.g., there is a unit-preserving map $\lambda: E \to F$.*

Therefore, we have proved the following

THEOREM 4.7. *Let E be a ring spectrum and $D = \{D_u^t, KD_u^t\}$ an E_2-functor.*

(i) *If* (4.6.7) *holds, then we have the Mahowald spectral sequence $\{\tilde{E}_{u,r}^{s,t}\}$ in Theorem* 4.4 *for $A = ED$ in* (4.6.4):

(4.7.1) $$\tilde{E}_{u,2}^{s,t} = ED_u^{s,t}(X_0) \Rightarrow D_u^{s,t}(X_0) \quad (conv).$$

(ii) (Miller [10]) *If* (4.6.9) *holds for another ring spectrum F, then we have the one $\{\tilde{E}_{u,r}^{s,t}\}$ in* (i) *for $D = FA$ in* (2.1.4):

(4.7.2) $$\tilde{E}_{u,2}^{s,t} = EFA_u^{s,t}(X_0) \Rightarrow FA_u^{s+t}(X_0) \quad (conv).$$

If $G_*(G)$ is flat over $G_*(S^0)$ for $G = E, F$, in addition, then

(4.7.3) $$EFA_u^{s,t}(X_0) = \mathrm{Ext}_{E_*(E)}^{s,u}(E_*(S^0), FA_*^t(E \wedge X_0)),$$
$$FA_u^t(X) = \mathrm{Ext}_{F_*(F)}^{t,u}(F_*(S^0), F_*(X)) \quad (X = E \wedge X_0, X_0).$$

In fact, (4.7.3) is seen in the same way as the proof of (2.3.2).

EXAMPLE 4.8. *Let p be an odd prime. Then, on the groups in* (4.7.3) *for* $E = BP$ *at p and* $F = KQ_p$ ($Q_p = \{a/b \in Q | (b, p) = 1\}$), *we have the following*

(i) (Adams–Baird) $KQ_pA_u^t(S^0)$ *is* Q_p *if* $t = u = 0$, Z_{p^v} *if* $t = 1$, $u = 2(p-1)bp^{v-1}$ *with* $(b, p) = 1$, Q/Q_p *if* $t = 2$, $u = 0$, *and* 0 *otherwise.*

(ii) $KQ_pA_u^t(BP) = 0$ *for* $t \geqq 2$.

(iii) $$BPA_u^s(S^0) \cong KQ_pA_u^s(S^0) \qquad (\text{if } s = 0, 1)$$
$$\cong BPKQ_pA_u^{s-2,1}(S^0) \qquad (\text{if } s \geqq 4 \text{ or } s = 2, 3, u \neq 0).$$

PROOF. Denote simply by $K = KQ_p$ in this proof. Then, by [16, §17],

(4.8.1) $K_*(K)$ is flat over $\pi_*(K) = Q_p[t, t^{-1}]$ ($\deg t = 2$) and is identified with the subring of all finite Laurent series $f(u, v) \in K_*(K) \otimes Q = Q[u, v, u^{-1}, v^{-1}]$ ($u = (1 \wedge \iota)_* t$, $v = (\iota \wedge 1)_* t$) satisfying

(*) $f(\lambda t, \mu t) \in Q_p[t, t^{-1}]$ for any integers λ, μ prime to p.

(i) Let k be a generator for the multiplicative group of reduced residue classes mod p^2 (and so mod p^n for any n). Then, we have the exact sequence

(4.8.2)
$$0 \longrightarrow K_*(S^0) \xrightarrow{\iota} K_*(K) \xrightarrow{\psi} K_*(K) \xrightarrow{c} K_*(SQ) \quad (= Q[t, t^{-1}]) \longrightarrow 0,$$

with $\iota t = u$, $\psi(u^iv^j) = (k^j - 1)u^iv^j$, and $c(u^iv^j) = 0$ $(j \neq 0)$, $= t^i$ $(j = 0)$, by taking $\iota = \iota_*$, $\psi = \psi_*^k - \mathrm{id}$ ($\psi^k \in K^0(K) = \mathrm{Hom}_{\pi_*(K)}(K_*(K), \pi_*(K))$ is the Adams operation given by $\psi^k(u^iv^j) = k^jt^{i+j}$) and $c = \mathrm{ch}_*$ (ch: $K \to SQ$ is the Chern charactor).

In fact, the equalities are seen by definition; and $\psi \circ \iota = 0 = c \circ \psi$. Let $f = \sum f_{ij}u^iv^j \in K_*(K)$. If $\psi f = 0$, then $f_{ij} = 0$ $(j \neq 0)$, $f_{i0} \in Q_p$ (by (*) in (4.8.1)) and $f = \sum f_{i0}u^i \in \mathrm{Im}\,\iota$. If $cf = 0$, then $f_{i0} = 0$ and we have $g = \sum_{j \neq 0} f_{ij}u^i(v^j - u^j)/(k^j - 1)$ with $\psi g = f$ and $g(\lambda t, \lambda t) = 0$. Thus $g(\lambda t, k\mu t) = g(\lambda t, \mu t) + f(\lambda t, \mu t)$ by $\psi g(u, v) = g(u, kv) - g(u, v)$, and $g(\lambda t, k^n\lambda t) \in Q_p[t, t^{-1}]$ for λ prime to p and any n by (*) for f and induction; hence $g(\lambda t, \mu t) \in Q_p[t, t^{-1}]$ for any λ, μ prime to p, and $g \in K_*(K)$. Finally, $q_n = \{\prod_{i=1}^{n-1}(v - iu)\}/n!v^{n-1} \in K_*(K)$ (cf. [16, 17.31]) and $cq_n = 1/n!$; hence c is epimorphic. Therefore, the sequence (4.8.2) is exact.

Now, consider $I = \mathrm{Im}\,\psi$ in (4.8.2). Then, we have the exact sequences

$$0 \longrightarrow K_*(S^0) \longrightarrow K_*(K) \xrightarrow{\psi} I \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow I \longrightarrow K_*(K) \xrightarrow{c} K_*(SQ) \longrightarrow 0;$$

and these induce the long exact sequences for $\mathrm{Ext}^{s,*}(-) = \mathrm{Ext}_{K_*(K)}^{s,*}(K_*(S^0), -)$, where $\mathrm{Ext}^{s,*}(K_*(X)) = KA_*^s(X)$ and $KA_*^s(X) = 0$ $(s > 0)$, $= \pi_*(X)$ $(s = 0)$ for $X = K$ by Lemma 2.2 and for $X = SQ$ by taking $\sigma_0^s = 1 \wedge \mathrm{ch}\colon K^s \wedge K \wedge SQ \to K^s \wedge SQ$, $W = SQ$ in the proof of Lemma 2.2. Thus, we see that

(4.8.3) $KA_*^s(S^0)$ is isomorphic to $\mathrm{Ext}^{s-1,*}(I)$ if $s \geqq 2$, which is 0 if $s \geqq 3$; and to $\mathrm{Coker\,ch}_*$ if $s = 2$, $\mathrm{Ker\,ch}_*/\mathrm{Im}\,(\psi_*^k - \mathrm{id})$ if $s = 1$, $\mathrm{Ker}\,(\psi_*^k - \mathrm{id})$ if $s = 0$, for $\mathrm{ch}\colon \pi_*(K) \to \pi_*(SQ)$ and $\psi_*^k - \mathrm{id}\colon \pi_*(K) \to \pi_*(K) = Q_p[t, t^{-1}]$ with $\mathrm{ch}_* t^i = 0$ $(i \neq 0)$, $= 1$ $(i = 0)$ and $\psi_*^k t^i = k^i t^i$.

Then, the order of $k \in (Z/p^\nu Z)^\times$ is $p^{\nu-1}(p-1)$, and so (i) is seen by (4.8.3).

(ii) By taking the tensor products over $\pi_*(K)$ with the flat module $K_*(BP) = \pi_*(K)[t_i']$, the exact sequence (4.8.2) gives us the one

$$(4.8.4) \quad 0 \longrightarrow K_*(BP) \xrightarrow{(\iota \wedge 1)_*} K_*(K \wedge BP) \xrightarrow{(\psi^k \wedge 1)_* - \mathrm{id}} K_*(K \wedge BP) \xrightarrow{(\mathrm{ch} \wedge 1)_*} K_*(SQ \wedge BP) \longrightarrow 0\,;$$

hence, we see in the same way as (4.8.3) that

(4.8.5) $KA_*^t(BP)$ is the cohomology of the cochain complex $0 \to \pi_*(K \wedge BP) \xrightarrow{\psi'} \pi_*(K \wedge BP) \xrightarrow{c'} \pi_*(SQ \wedge BP) \to 0 \to \cdots$ for $\psi' = (\psi^k \wedge 1)_* - \mathrm{id}$ and $c' = (\mathrm{ch} \wedge 1)_*$. Here,

$$K_*(BP) = \pi_*(K)[t_i'] \xrightarrow{c'} SQ_*(BP) = \pi_*(BP) \otimes Q = Q[l_i] \xrightarrow{\phi} K_*(BP) \otimes Q$$

$(\pi_*(K) = Q_p[t, t^{-1}]$, $t_0' = 1$, $\phi = \phi^K \otimes 1)$ satisfy by [4, II, 16.1, pp. 63–64] that

$$c'(t^n \phi\alpha) = \alpha (n = 0)\,, \; = 0 \; (n \neq 0)\,, \quad \text{and} \quad \phi l_i = \textstyle\sum_{j=0}^i t^{-1}(tt'_{i-j})^{p^j}/p^j.$$

Now, for any $\alpha = \prod l_i^{\alpha_i} \neq 1$ and $n \geqq 1$, consider the elements

$$x = p^{a-1}\phi\alpha - (t^b/p)\,, \quad x_n = (-t^b)^{1-n} x^n \quad (a = \textstyle\sum i\alpha_i,\, b = \sum (p^i - 1)\alpha_i)\,.$$

Then, by the above equalities for ϕ and c', we see that x is in $K_*(BP)$, so is x_n for any n, and $c' x_n = n\alpha/p^{n-a}$, $c'(t^{-b}x_n) = -1/p^n$. Thus c' is epimorphic; and (ii) is proved.

(iii) $\{\tilde{E}_{u,r}^{s,t}\}$ in Theorem 4.7 (ii) for $E = BP$, $F = K$ $(= KQ_p)$ and $X_0 = S^0$ satisfies

(4.8.6) $\tilde{E}_{u,2}^{s,t} = BPKA_u^{s,t}(S^0) = 0$ if $t \geqq 2$ and $\tilde{E}_{u,\infty}^{s,t-s} = 0$ if $t \geqq 3$ or $t = 2$, $u \neq 0$,

by (4.7.2-3) and (i)–(ii). Thus, the differential $d_r\colon \tilde{E}_{u,r}^{s,t} \to \tilde{E}_{u,r}^{s+r,t-r+1}$ is 0 except for $r = 2$, $t = 1$; and $d_2\colon \tilde{E}_{u,2}^{s,1} \cong \tilde{E}_{u,2}^{s+2,0}$ for $s \geqq 2$ or $s = 0, 1$, $u \neq 0$. Since $BPKA_u^{s,0}(S^0) = BPA_u^s(S^0)$ by the Hattori-Stong theorem (cf. [4, II, 14.1]), the above isomorphism d_2 implies (iii). q.e.d.

In the rest of this section, we note on the differential of $\{\tilde{E}_{u,r}^{s,t}\}$ in Theorem 4.7 (ii) for ring spectra E and F with (4.6.9). For $X \in \mathscr{C}$, we consider

(4.9.1) $FE_u^{s,t}(X) = \pi_u(F^{t+1} \wedge E^{s+1} \wedge X)$ $(s, t \geqq 0)$, $= 0$ (otherwise), with coboundary $\delta^G = \sum_{i=0}^{*+1} (-1)^i \delta_{i*}^G\colon FE_u^{s,t}(X) \to FE_u^{s+1,t}(X)$ or $FE_u^{s,t+1}(X)$ for $G = E$

or F, respectively, $(* = s$ or t, $\delta_i^G = 1 \wedge \iota_G \wedge 1 \colon Y \wedge S^0 \wedge Z \to Y \wedge G \wedge Z$, $Z = E^i \wedge X$ or $F^i \wedge E^{s+1} \wedge X)$; i.e., $\{FE_u^{s,*}(X); \delta^F\} = F_u^*(E^{s+1} \wedge X)$ with $H^t(FE_u^{s,*}(X)) = FAE_u^{s,t}(X)$ and $\{FAE_u^{*,t}(X); \delta_*^E\}$ with $H^s(FAE_u^{*,t}(X)) = EFA_u^{s,t}(X)$ are the ones in (2.1.1-4) and (4.6.1-4).

According to the assumption (4.6.9), the cofibering

$$\alpha_n^E \colon X_n \xrightarrow{i} E \wedge X_n \xrightarrow{j} X_{n+1} = \bar{E} \wedge X_n$$
$$(i = \iota_E \wedge 1, j = j \wedge 1 \text{ for } \omega^E \colon S^0 \xrightarrow{\iota_E} E \xrightarrow{j} \bar{E})$$

in (2.1.6) induces the short exact sequence of the cochain complexes $\{F_u^*; \delta^F\}$:

(4.9.2)
$$0 \longrightarrow F_u^*(E^m \wedge X_n) \xrightarrow{i_*} F_u^*(E^{m+1} \wedge X_n) \xrightarrow{j_*} F_u^*(E^m \wedge X_{n+1}) \longrightarrow 0 \quad (k = 1 \wedge k):$$

and by the definition of δ^G in (4.9.1), we see the equalities

(4.9.3) $\delta^F \circ j_* = j_* \circ \delta^F$, $\delta^F \circ j^s = j^s \circ \delta^F$ and $i_* \circ j_* \circ j^s = (-1)^{s+1} j^{s+1} \circ \delta^E$, for the compositions $j^s = (j_*)^s \colon FE_u^{s,*}(X_0) \to FE_u^{0,*}(X_s)$ and $i_* \circ j_* \colon FE_u^{0,*}(X_s) \to F_u^*(X_{s+1}) \to FE_u^{0,*}(X_{s+1})$, where $i_* \circ j_* = (\iota_E \wedge j)_* \colon F_u^*(S^0 \wedge E \wedge X_s) \to F_u^*(E \wedge X_{s+1})$.

Moreover, (4.9.2) induces the cohomology exact sequence

$$\text{(4.9.4)} \quad \cdots \longrightarrow FA_u^t(E^m \wedge X_n) \xrightarrow{i_*} FA_u^t(E^{m+1} \wedge X_n) \xrightarrow{j_*} FA_u^t(E^m \wedge X_{n+1})$$
$$\xrightarrow{\delta_*} FA_u^{t+1}(E^m \wedge X_n) \longrightarrow \cdots \quad (k_* = (k_*)_*, \delta_* = (i_*^{-1} \circ \delta^F \circ j_*^{-1})_*);$$

and by the definition of δ_* and the equalities in (4.9.3), we see the following:

(4.9.5) If $\delta^F y = (-1)^{s+1} \delta^E x$ for $x \in FE_u^{s,t+1}(X_0)$ and $y \in FE_u^{s+1,t}(X_0)$, then $\delta^F j_* j^{s+1} y = 0$ and $\delta_*[j_* j^{s+1} y] = [j_* j^s x]$ in $FA_u^{t+1}(X_{s+1})$ for the cohomology classes [ ].

On the other hand, by (4.6.9) and the definition of FA in (2.1.1-4), we see that

(4.9.6) (4.9.4) is the one in (1.8.4) for the FA-cofibering $E^m \wedge \alpha_n^E$ (i.e. $\delta_* = \bar{\delta}$).

(4.9.7) Thus, $\{\tilde{E}_{u,r}^{s,t}, d_r\}$ in Theorem 4.7 (ii) is the one in Proposition 4.2 associated to (4.9.4) for $m = 0$. So $\tilde{E}_{u,1}^{s,t} = FA_u^t(E \wedge X_s)$, $d_1 = i_* \circ j_*$, and we have

$$J_* \colon EFA_u^{s,t}(X_0) \to \tilde{E}_{u,2}^{s,t} \text{ induced by } J = (j^s)_* \colon FAE_u^{s,*}(X_0) \to FAE_u^{0,*}(X_s) = \tilde{E}_{u,1}^{s,*},$$

where j^s is the composition in (4.9.3).

Therefore, we see the following

LEMMA 4.10. (i) *Assume that* $x_i \in FE_u^{s+i,t-i}(X_0)$ $(0 \leqq i \leqq n)$ *satisfy* $\delta^F x_0 = 0$ *and* $\delta^F x_{i+1} = (-1)^{s+i+1} \delta^E x_i$ *for* $i < n$. *Then*, for the cohomology classes $[x_0] \in FAE_u^{s,t}(X_0)$, $[\delta^E x_n] \in FAE_u^{s+n,t-n}(X_0)$ *and the differential* d_r *in* (4.9.7), *there hold*

$$d_r J[x_0] = 0\ (1 \leqq r \leqq n) \quad \textit{and} \quad d_{n+1} J[x_0] = (-1)^{s+n+1} J[\delta^E x_n]\,.$$

(ii) *Assume that* $x_i \in FE_u^{s-i-1,t+i}(X_0)$ $(0 \leqq i \leqq s-1)$ *and* $x_s \in F_u^{s+t}(X_0)$ *satisfy* $\delta^F \delta^E x_0 = 0$, $\delta^F x_i = (-1)^{s-i-1} \delta^E x_{i+1}$ *for* $i < s-1$, $\delta^F x_{s-1} = i_* x_s$ (i.e., $\delta_* j_* x_{s-1} = x_s$). *Then, for* $[\delta^E x_0] \in FAE_u^{s,t}(X_0)$ *and* $[x_s] \in FA_u^{s+t}(X_0)$ *in* (4.9.7), $(-1)^s J[\delta^E x_0]$ *converges to* $[x_s]$.

(iii) *Assume that we have a unit-preserving map* $\lambda\colon E \to F$ *and* $\delta^{FE} x = 0$, $\delta^F \delta^E x = 0$ *for* $x \in FE_u^{s-1,t}(X_0)$, *where* $\delta^{FE} = \delta^E \circ \lambda_* + (-1)^s \delta^F\colon FE_u^{s-1,t}(X_0) \to FE_u^{s-1,t+1}(X_0)$ $(\lambda_* = (\lambda \wedge 1)_*\colon F_u^n(E \wedge E^m \wedge X_0) \to F_u^n(F \wedge E^m \wedge X_0))$. *Then, for* $[\delta^E x] \in FAE_u^{s,t}(X_0)$ *and* $[\lambda_* \lambda^{s-1} x] \in FA_u^{s+t}(X_0)$ $(\lambda^i = (\lambda_*)^i\colon FE_u^{s-1,t}(X_0) \to FE_u^{s-i-1,t+i}(X_0))$, $(-1)^s J[\delta^E x]$ *converges to* $[\lambda_* \lambda^{s-1} x]$.

(iv) $J_*\colon EFA_u^{s,t}(X_0) \to \tilde{E}_{u,2}^{s,t}$ *is isomorphic.*

Proof. By (1.6.1-2), (1.1.3) and (4.9.3), (4.9.5) implies (i)–(ii).

(iii) By the definition of δ^{FE}, $\delta^{FE} \circ \lambda_* = \delta^E \circ \lambda^{i+1} + (-1)^{s-i} \delta^F \circ \lambda^i$; and so $\delta^F \lambda^i x = (-1)^{s-i-1} \delta^E \lambda^{i+1} x$ $(0 \leqq i < s-1)$ and $\delta^F \lambda^{s-1} x = i_* \lambda_* \lambda^{s-1} x$. By (ii), these imply (iii).

(iv) We consider the cochain complexes $M(r)_u^{*,t} = \{M(r)_u^{s,t}, \delta(r)_M^s\}$ and $K(r)_u^{*,t} = \{K(r)_u^{s,t}, \delta(r)_K^s\}$ for $r \geqq 0$ given as follows:

$$M(r)_u^{s,t} = \tilde{E}_{u,1}^{s,*} \text{ in (4.9.7) if } s \leqq r\,,\ = FA_u^t(E^{s-r+1} \wedge X_r) \text{ if } s > r\,, \text{ and}$$
$$\delta(r)_M^s = d_1 = (\iota_E \wedge j)_* \text{ in (4.9.7) if } 0 \leqq s < r\,,\ = \delta^{s-r} \text{ in (4.6.1)}$$
$$(D = FA,\ X = X_r) \text{ if } s \geqq r\,,$$
$$K(r)_u^{s,t} = 0 \text{ if } s \leqq r\,,\ = FA_u^t(E^{s-r} \wedge X_r) \text{ if } s > r\,, \text{ and}$$
$$\delta(r)_K^s = 0 \text{ if } s \leqq r\,,\ = (\iota_E \wedge 1)_* \text{ if } s = r+1\,,\ = \delta^{s-r-2} \text{ in (4.6.1)}$$
$$(D = FA,\ X = E \wedge X_r) \text{ if } s \geqq r+2\,.$$

Furthermore, we have the cochain maps $i(r) = \{i(r)^*\}\colon K(r)_u^{*,t} \to M(r)_u^{*,t}$ and $j(r) = \{j(r)^*\}\colon M(r)_u^{*,t} \to M(r+1)_u^{*,t}$ by taking

$$i(r)^s = 0 \quad \text{if } s \leqq r\,, \qquad = (1 \wedge i)_* \qquad \text{if } s > r\,, \text{ and}$$
$$j(r)^s = \mathrm{id} \quad \text{if } s \leqq r\,, \qquad = (-1)^{s-r}(1 \wedge j)_* \quad \text{if } s > r\,.$$

Then, we have the short exact sequence

$$0 \longrightarrow K(r)_u^{*,t} \xrightarrow{i(r)} M(r)_u^{*,t} \xrightarrow{j(r)} M(r+1)_u^{*,t} \longrightarrow 0\,;$$

because i_* in (4.9.4) is monomorphic for $m \geqq 1$. By (4.6.6) $(D = FA, X = X_r)$, $H^s(K(r)_u^{*,t}) = 0$ for any s; hence $j(r)_*$ is isomorphic on the cohomology groups. Thus, by $M(0)_u^{*,t} = FAE_u^{*,t}(X_0)$ and $J = (-1)^\varepsilon j(s)^s \circ \cdots \circ j(0)^s\colon M(0)_u^{s,t} \to M(s+1)_u^{s,t} = \tilde{E}_{u,1}^{s,t}$ $(\varepsilon = s(s+1)/2)$, this implies (iv). q.e.d.

§5. May spectral sequences

In this section, we construct another spectral sequence which abuts to an E_2-functor and whose E_1-term is a double E_2-functor.

Let $C = \{C_u^s, KC_u^s\}$ be an E_2-functor, and assume that

$$
(5.1.1)\qquad
\begin{array}{cccccc}
 & \underset{\cdot\cdot}{\omega_{s,t}} & & \underset{\cdot\cdot}{\eta_{s,t}} & & \underset{\cdot\cdot}{\omega_{s,t+1}} \\
\alpha_{s,t}: & X_{s,t} & \xrightarrow{f} & V_{s,t} & \xrightarrow{g} & X_{s,t+1} \\
 & \downarrow i & & \downarrow i & & \downarrow i \\
\beta_{s,t}: & W_{s,t} & \xrightarrow{f} & Y_{s,t} & \xrightarrow{g} & W_{s,t+1} \\
 & \downarrow j & & \downarrow j & & \downarrow j \\
\alpha_{s+1,t}: & X_{s+1,t} & \xrightarrow{f} & V_{s+1,t} & \xrightarrow{g} & X_{s+1,t+1}
\end{array}
\qquad (s, t \geqq 0)
$$

are diagrams of cofiberings $\xi_{s,t}$ ($\xi = \alpha, \beta, \omega, \eta$) with the following (5.1.2-4):

(5.1.2) $\{k\}$ ($k = i, j, f, g$) are maps in $\mathscr{CF}$ (see (3.1)).

(5.1.3) Each $\omega_{s,0}$ is a C-injective cofibering.

(5.1.4) Each $\beta = \beta_{s,t}$ is C^0-*homological*, i.e., we have the exact sequence

$$\cdots \longrightarrow C_u^0(W) \xrightarrow{f_*} C_u^0(Y) \xrightarrow{g_*} C_u^0(W_2) \xrightarrow{\partial} C_{u-1}^0(W) \longrightarrow \cdots$$
$$(Z = Z_{s,t},\ Z_2 = Z_{s,t+1})$$

by the composition $\partial = \iota \circ \kappa$: $C_u^0(W_2) \to KC_{u-1}^0(\beta; 0) \to C_{u-1}^0(W)$ in (1.8.1).

(5.1.5) When W, Y and W_2 are C-injective, (5.1.4) holds if $KC_u^*(\beta; i) = 0$ ($* \neq 0$) for some (or any) i, which is seen by (1.8.1-2).

(5.1.6) For ϕ: $h_u \to C_u^0$ in (1.3.1), assume that ∂ in the exact sequence

$$\cdots \longrightarrow h_u(W) \xrightarrow{f_*} h_u(Y) \xrightarrow{g_*} h_u(W_2) \xrightarrow{\partial} h_{u-1}(W) \longrightarrow \cdots$$

and ∂ in (5.1.4) satisfy $\phi \circ \partial = \partial \circ \phi$ (then ϕ is called *natural for* β), and that ϕ is isomorphic for W and Y. Then, (5.1.4) is equivalent to ϕ: $h_u(W_2) \cong C_u^0(W_2)$.

Then, the same construction as Proposition 1.2 gives us the following:

(5.2.1) For any $s \geqq 0$, the spectral sequence $\{E(s)_r^{t,u},\ d_r\colon E(s)_r^{t,u} \to E(s)_r^{t+r,u+r-1}\}$ is associated to the exact sequences in (5.1.4) such that

$$E(s)_1^{t,u} = C_u^0(Y_{s,t}) \Rightarrow C_{u-t}^0(W_{s,0}) = G_{u-t}^{s,0} \quad \text{(abut)}, \text{ i.e.},$$

(5.2.2) $G_{u-t}^{s,0} \supset G_u^{s,t} \supset G_{u+1}^{s,t+1}$ and $G_u^{s,t}/G_{u+1}^{s,t+1} \cong \bar{Z}(s)_\infty^{t,u}/B(s)_\infty^{t,u} \subset E(s)_\infty^{t,u}$ for $\bar{Z}(s)_\infty^{t,u} = \operatorname{Im} f_*$, $B(s)_\infty^{t,u} = f_*(\operatorname{Ker} \partial^t)$ and $G_u^{s,t} = \operatorname{Im} \partial^t$ where $\partial^t\colon C_u^0(W_{s,t}) \to C_{u-t}^0(W_{s,0})$.

On the other hand, $\omega_{s,0}$ in (5.1.3) induces the exact sequence

$$(5.2.3)\quad 0 \longrightarrow C_u^0(X_{s,0}) \xrightarrow{i_*} C_u^0(W_{s,0}) \xrightarrow{j_*} C_u^0(X_{s+1,0}) \xrightarrow{\delta^{s+1}} C_u^{s+1}(X_{0,0}) \longrightarrow 0$$

by (1.8.4). Also, by (5.1.2) and (3.2.1), we see the following:

(5.2.4) $\delta = (i \circ j)_*: C_u^0(Z_{s,t}) \to C_u^0(Z_{s+1,t})$ for $Z = W$, Y satisfy $\delta \circ \delta = 0$, $\delta \circ k_* = k_* \circ \delta$ and $\delta \circ \partial = \partial \circ \delta$ for $k_* = f_*, g_*$ and $\partial = \iota \circ \kappa$ in (5.1.4).

Thus, in (5.2.1-2), we have the cochain complexes

(5.2.5) $\{E_u^{s,t} = E(s)_2^{t,u}\} \supset \{\bar{Z}_u^{s,t} = \bar{Z}_u(s)_\infty^{t,u}/B\} \supset \{B_u^{s,t} = B(s)_\infty^{t,u}/B\}$ $(B = B(s)_2^{t,u} = f_*(\mathrm{Ker}\ \partial))$ and $\{G_u^{s,t}\}$ with coboundary $\delta = (i \circ j)_*$.

Taking their cohomologies, we see the following by (5.2.2-3):

(5.2.6) $\bar{\delta}^s \circ i_*^{-1}: H^s(G_{u-t}^{*,0}) \cong C_{u-t}^s(X_{0,0})$. Furthermore, the exact sequence

$$\cdots \to H^s(G_u^{*,t}) \to H^s(G_u^{*,t}/G_{u+1}^{*,t+1}) \to H^{s+1}(G_{u+1}^{*,t+1}) \to H^{s+1}(G_u^{*,t}) \to \cdots$$

associates the spectral sequence $\{E_{u,r}^{s,t}, d_r: E_{u,r}^{s,t} \to E_{u+r,r}^{s+1,t+r}\}$ with

$$E_{u,1}^{s,t} = H^s(G_u^{*,t}/G_{u+1}^{*,t+1}) \cong H^s(\bar{Z}_u^{*,t}/B_u^{*,t}) \Rightarrow H^s(G_{u-t}^{*,0}) \cong C_{u-t}^s(X_{0,0}) \quad \text{(abut)},$$

i.e., $F_u^{s,t}/F_{u+1}^{s,t+1} \subset E_{u,\infty}^{s,t}$ for $F_u^{s,t} = \mathrm{Im}\,[H^s(G_u^{*,t}) \to H^s(G_{u-t}^{*,0})]$.

To represent $H^s(E_u^{*,t})$ of $\{E_u^{s,t}\}$ in (5.2.5), we use the following

Definition 5.3. Let $A = \{A_u^{s,t}, KA_u^{s,t}, LA_u^{s,t}\}$ be a double E_2-functor in Definition 4.3 (1).

(1) We call $X \in \mathscr{C}$ *$A(2)$-injective* if $A_u^{s,t}(X) = 0$ for $t \neq 0$, and $\alpha \in \mathscr{C}\mathscr{F}$ an *$A(2)$-cofibering* if $KA_u^{s,t}(\alpha; 0) = 0 = LA_u^{s,t}(\alpha; i, 0)$ for $i = 0, 2$ and $LA_u^{s,t}(\alpha; 1, 1) = 0$ for $t \neq 0$.

(2) We say that *A is indirectly related to an E_2-functor C at $X_{0,0}$ by a natural transformation $\psi: C_u^s \to A_u^{s,0}$ and cofiberings in* (5.1.1) with (5.1.2-4), if

(5.3.1) each $\omega_{s,0}$ is an $A(1)$-cofibering, $W_{s,0}$ is $A(1)$-injective, $\beta_{s,t}$ is an $A(2)$-cofibering, $Y_{s,t}$ is $A(1)$- and $A(2)$-injective, and

(5.3.2) $\psi: C_u^0(Y_{s,t}) \cong A_u^{0,0}(Y_{s,t})$ for any $s, t = 0, 1, 2, \ldots$.

In (1) of this definition, the exact sequences in (4.3.1-2) imply the following:

(5.3.3) Let $\alpha: X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2$ be an $A(2)$-cofibering. Then, $\iota_{2,1} \circ \kappa_{2,0}: KA_u^{s,t}(\alpha; 2) \cong A_u^{s,t}(X_2)$ and $\iota_{1,2} \circ \kappa_{0,1}: A_u^{s,t}(X_0) \cong KA_u^{s,t}(\alpha; 1)$ by $LA_u^{s,t}(\alpha; 0, 1) \cong 0 \cong LA_u^{s,t}(\alpha; 0, 2)$, $\kappa_{1,0}: KA_u^{s,0}(\alpha; 1) \cong LA_u^{s,0}(\alpha; 1, 1)$ by $LA_u^{s,0}(\alpha; 1, 0) = LA_u^{s,-1}(\alpha; 4, 3) = 0$; $\iota_{1,0}: LA_u^{s,t}(\alpha; 1, 0) \cong KA_u^{s,t}(\alpha; 1)$, $\kappa_{1,1}: A_u^{s,t}(X_1) \cong LA_u^{s,t}(\alpha; 2, 2)$ $(t > 0)$; and we have the exact sequences

$$\cdots \longrightarrow A_u^{s,0}(X_0) \xrightarrow{f_{0*}} A_u^{s,0}(X_1) \xrightarrow{\kappa_{1,1}} LA_u^{s,0}(\alpha; 2, 2) \longrightarrow A_u^{s+1,0}(X_0) \longrightarrow \cdots,$$

$$\cdots \longrightarrow LA_u^{s,t}(\alpha; 2, 2) \xrightarrow{\bar{\iota}} A_u^{s,t}(X_2) \xrightarrow{\bar{\kappa}} A_u^{s,t+1}(X_0) \longrightarrow LA_u^{s+1,t}(\alpha; 2, 2) \longrightarrow \ldots,$$

where $\bar{\iota} = \iota_{2,1} \circ \kappa_{2,0} \circ \iota_{2,2}$ (hence $\bar{\iota} \circ \kappa_{1,1} = f_{1*}$) and $\bar{\kappa} = (\iota_{1,2} \circ \kappa_{0,1})^{-1} \circ \iota_{1,0} \circ \kappa_{2,2} \circ (\iota_{2,1} \circ \kappa_{2,0})^{-1}$.

(5.3.4) In (5.3.3), if X_1 is $A(2)$-injective, then

$$\bar{\kappa}\colon A_u^{s,t}(X_2) \cong A_u^{s,t+1}(X_0) \quad \text{for} \quad t>0\,.$$

If X_0 and X_1 are $A(1)$-injective, then so is X_2. Furthermore, if both of these hold, then we have the exact sequence

$$0 \longrightarrow A_u^{0,0}(X_0) \xrightarrow{f_{0*}} A_u^{0,0}(X_1) \xrightarrow{f_{1*}} A_u^{0,0}(X_2) \xrightarrow{\bar{\kappa}} A_u^{0,1}(X_0) \longrightarrow 0\,.$$

Now, consider the case of Definition 5.3 (2). Then, for $\beta_{s,t}$ with (5.3.1),

$$(5.4.1)\quad 0 \longrightarrow A_u^{0,0}(W_{s,t}) \xrightarrow{f_*} A_u^{0,0}(Y_{s,t}) \xrightarrow{g_*} A_u^{0,0}(W_{s,t+1})) \xrightarrow{\bar{\kappa}^{t+1}} A_u^{0,t+1}(W_{s,0}) \longrightarrow 0$$

is exact by (5.3.3-4), since $W_{s,t}$ is $A(1)$-injective by induction on t. Thus, in the same way as Theorem 1.4, (5.4.1) and a natural transformation $\psi\colon C_u^s \to A_u^{s,0}$ with (5.3.2) imply the isomorphism

$$(5.4.2)\quad \bar{\psi} = \bar{\kappa}^t \circ (f_*^{-1} \circ \psi)\colon E(s)_2^{t,u} \cong A_u^{0,0}(W_{s,t})/\mathrm{Im}\, g_* \cong A_u^{0,t}(W_{s,0})$$

for the spectral sequence $\{E(s)_r^{t,u}\}$ in (5.2.1).

On the other hand, (4.3.4) for the $A(1)$-cofibering $\omega_{s,0}$ in (5.3.1) implies the exact sequence

$$(5.4.3)\quad 0 \longrightarrow A_u^{0,t}(X_{s,0}) \xrightarrow{i_*} A_u^{0,t}(W_{s,0}) \xrightarrow{j_*} A_u^{0,t}(X_{s+1,0}) \xrightarrow{\bar{\delta}^{s+1}} A_u^{s+1,t}(X_{0,0}) \longrightarrow 0\,,$$

and i_* and j_* commute with $\bar{\kappa}$ in (5.4.1) (see (5.3.3)) by (5.1.2). Thus:

(5.4.4) The cochain complex $\{A_u^{0,t}(W_{s,0}), \delta = (i \circ j)_*\}$ is isomorphic to $\{E_u^{s,t} = E(s)_2^{t,u}, \delta = (i \circ j)_*\}$ in (5.2.5) by $\bar{\psi}$ in (5.4.2), and $\bar{\delta}^s \circ i_*^{-1}\colon H^s(A_u^{0,t}(W_{*,0})) \cong A_u^{s,t}(X_{0,0})$.

Therefore, we have proved the following

THEOREM 5.5 (May spectral sequence). *If a double E_2-functor $A = \{A_u^{s,t}, KA_u^{s,t}, LA_u^{s,t}\}$ is indirectly related to an E_2-functor $C = \{C_u^s, KC_u^s\}$ at $X_{0,0}$, then we have the spectral sequence $\{E_{u,r}^{s,t}\}$ in* (5.2.6) *such that it abuts to $C_{u-t}^s(X_{0,0})$ and*

$$E_{u,1}^{s,t} = H^s(\bar{Z}_u^{*,t}/B_u^{*,t})\,, \quad H^s(E_u^{*,t}) = A_u^{s,t}(X_{0,0})$$

for the cochain complexes $E_u^{*,t} \supset \bar{Z}_u^{*,t} \supset B_u^{*,t}$ in (5.2.5).

COROLLARY 5.6 (i) *If each $\{E(s)_r^{t,u}\}$ in* (5.2.1) *converges and collapses, then $E_u^{*,t} = \bar{Z}_u^{*,t} \supset B_u^{*,t} = 0$ and $E_{u,1}^{s,t} = A_u^{s,t}(X_{0,0})$ in Theorem* 5.5.

(ii) *The assumption in* (i) *is equivalent to* (5.6.1) *and one of* (5.6.2-3):

(5.6.1) $\mathrm{inv\,lim}_n\, \{C_{t+n}^0(W_{s,n}), \partial\} = 0$ (*for ∂ in* (5.1.4)).

(5.6.2) $\psi\colon C_u^0 \to A_u^{0,0}$ *is epimorphic for $W_{s,t}$*.

(5.6.3) $\mathrm{Ker}\,[\partial^n\colon C_u^0(W_{s,t}) \to C_{u-n}^0(W_{s,t-n})] = \mathrm{Ker}\,\partial$ *for* $1 \leqq n \leqq t$.

In fact, (ii) is the same as Corollary 1.7 (ii).

For given ring spectra E and F, and $X_0 \in \mathscr{C}$, we take
(5.7.1) the commutative diagram (5.1.1) defined by $X_{s,0} = \bar{E}^s \wedge X_0$ and

$$X_{s,t} = \bar{F}^t \wedge X_{s,0}, \quad V_{s,t} = F \wedge X_{s,t}, \quad W_{s,t} = \bar{F}^t \wedge E \wedge X_{s,0}, \quad Y_{s,t} = F \wedge W_{s,t},$$
$$\alpha_{s,t} = \omega^F \wedge X_{s,t}, \quad \beta_{s,t} = \omega^F \wedge W_{s,t}, \quad \omega_{s,t} = \bar{F}^t \wedge \omega^E \wedge X_{s,0}, \quad \eta_{s,t} = F \wedge \omega_{s,t},$$

where $\omega^G \wedge X: X \xrightarrow{\iota_G \wedge 1} G \wedge X \to \bar{G} \wedge X$ is the cofibering in (2.1.6). Then, by Lemma 2.2 and (5.1.5-6), we see the following:

(5.7.2) The above diagram satisfies (5.1.2-4) for $C = EA$ in (2.1.4), where the exact sequence in (5.1.4) is isomorphic to the homotopy one

$$\cdots \longrightarrow \pi_u(W_{s,t}) \longrightarrow \pi_u(Y_{s,t}) \longrightarrow \pi_u(W_{s,t+1}) \xrightarrow{\partial} \pi_{u-1}(W_{s,t}) \longrightarrow \cdots$$

by $\phi^E: \pi_u(E \wedge X) \cong EA_u^0(E \wedge X)$. Thus *the spectral sequence* $\{E(s)_r^{t,u}\}$ *in* (5.2.1) *is* (*isomorphic to*) *the F-Adams one*: $E(s)_2^{t,u} = FA_u^t(W_{s,0}) \Rightarrow \pi_{u-t}(W_{s,0})$.

On the other hand, by (4.6.1-5) for $D = FA$, we have
(5.7.3) the double E_2-functor EFA, with the natural transformations ψ^{FA}: $FA_u^t(X) \to EFA_u^{0,t}(X)$ and $\psi^E: EA_u^s(X) \to EFA_u^{s,0}(X)$ induced from $\phi^F: \pi_*(Y) \to FA_*^0(Y)$ $(Y = E^{s+1} \wedge X)$, satisfying $\psi^E \circ \phi^E = \psi^{FA} \circ \phi^F: \pi_*(X) \to EFA_*^{0,0}(X)$.
Then, by Lemma 2.2 for F, (4.6.6-9) for $D = FA$ and definition, we see that

(5.7.4) $Y_{s,t}$ is $EFA(i)$-injective for $i = 1, 2$, so is $W_{s,0}$ for $i = 1$, ψ^E: $EA_u^0(Y_{s,t}) \cong EFA_u^{0,0}(Y_{s,t})$ and $\beta_{s,t}$ is an $EFA(2)$-cofibering. *If* (4.6.9) *holds for* $X_n = X_{n,0}$ *then* $\omega_{s,0}$ *is an* $EFA(1)$-*cofibering so that* EFA *is indirectly related to* EA *at* $X_{0,0}$ *by* ψ^E *in* (5.7.3) *and the cofiberings in* (5.7.1).

Therefore, Theorem 5.5 and Corollary 5.6 imply the following

THEOREM 5.8. *Let* $X_0 \in \mathscr{C}$ *and* E *and* F *are ring spectra satisfying* (4.6.9) *for* $X_n = X_{n,0}$. *Then, we have the May spectral sequence* $\{E_{u,r}^{s,t}\}$ *in Theorem* 5.5 *abutting to* $EA_{u-t}^s(X_0)$ *in* (2.1.4). *Moreover, if the F-Adams spectral sequence* $\{E(s)_r^{t,u}\}$ *in* (5.7.2) *converges and collapses for any* $s \geqq 0$, *then we have* $E_{u,1}^{s,t} = EFA_u^{s,t}(X_0)$ (*in* (4.9.1)) $\Rightarrow EA_{u-t}^s(X_0)$ (*abut*).

§6. Some preliminary lemmas

For the main result in the next section, we prepare some lemmas.

LEMMA 6.1. *If the compositions of maps* $X' \xrightarrow{i} W' \xrightarrow{f} Y'$ *and* $X' \xrightarrow{f'} V' \xrightarrow{i'} Y'$ *in* $\mathscr{C}$ *are homotopic to each other, then these are homotopy equivalent to inclusions*
(6.1.1) $X \subset W \subset Y$ *and* $X \subset V \subset Y$ *with* $X = W \cap V$.

PROOF. The double mapping cylinder $\bar{X} = W' \cup_i X' \wedge [0, 1]^+ \cup_{f'} V'$ of i and f' is the union of the mapping cylinders $W = W' \cup_i X' \wedge [0, 1/2]^+$ and

$V = V' \cup_{f'} X' \wedge [1/2, 1]^+$ and $X = X' \wedge \{1/2\}^+ = W \cap V$. Furthermore, i, f' and a homotopy h': $X' \wedge [0, 1]^+ \to Y'$ of $f \circ i$ to $i' \circ f'$ define the map h: $\bar{X} \to Y'$ and $Y = Y' \cup_h \bar{X} \wedge [0, 1]^+ \supset \bar{X}$, as desired. q.e.d.

According to this lemma, we may assume the following:
(6.1.2) In (5.1.1), denoting by $Z_{s,t} = Z$, $Z_{s+1,t} = Z_1$ and $Z_{s,t+1} = Z_2$, we have

$$X = W \cap V \subset \bar{X} = W \cup V \subset Y, \quad X_1 = W/X = \bar{X}/V \subset Y/V = V_1 ,$$
$$X_2 = V/X = \bar{X}/W \subset Y/W = W_2 , \quad X_{s+1,t+1} = Y/\bar{X} = V_1/X_1 = W_2/X_2 ,$$

and the horizontal and vertical sequences α, β, ω, and η are the cofiberings ξ: $A \overset{a}{\subset} B \overset{b}{\to} B/A$ with the inclusions $a = f, i$ and the collapsing maps $b = g, j$.

(6.1.3) Hence, $\{i\}$, $\{j\}$, $\{f\}$ and $\{g\}$ are maps in $\mathscr{CF}$, and (5.1.2) holds.

LEMMA 6.2 *For a homology theory h_*, consider the induced exact sequences*

(6.2.1) $$\cdots \longrightarrow h_u(A) \xrightarrow{a_*} h_u(B) \xrightarrow{b_*} h_u(B/A) \xrightarrow{\partial_\xi} h_{u-1}(A) \longrightarrow \cdots$$

of the above cofiberings ξ, and the diagram formed by them. Then:

(6.2.2) $$\partial_\omega \circ \partial_{\alpha'} = -\partial_\alpha \circ \partial_{\omega'}: h_{u+1}(X_{s+1,t+1}) \to h_{u-1}(X_{s,t})$$

for $\xi = \xi_{s,t}$, $\alpha' = \alpha_{s+1,t}$ and $\omega' = \omega_{s,t+1}$; and the other squares are commutative.

(6.2.3) *For $y \in h_u(Y_{s,t})$ with $j_* g_* y = 0$, there are $x_k \in h_u(X_k)$ ($X_k = X_{s+2-k,t-1+k}$, $k = 1, 2$) with $\partial_\omega x_1 = -\partial_\alpha x_2$, $f_* x_1 = j_* y$ and $i_* x_2 = g_* y$. Conversely, for x_k with the first equality, there is y with the last two ones. In particular, if each i_*: $h_u(V_{s,t}) \to h_u(Y_{s,t})$ is monomorphic, then for any $x_1 \in h_u(X_1)$, there is $x_2 \in h_u(X_2)$ with $\partial_\omega x_1 = \partial_\alpha x_2$.*

(6.2.4) *For $z \in h_{u+1}(X_{s+1,t+1})$ with $\partial_\omega \partial_{\alpha'} z = 0$, there are $w \in h_u(W_{s,t})$ and $v \in h_u(V_{s,t})$ with $j_* w = \partial_{\alpha'} z$, $g_* v = \partial_{\omega'} z$ and $f_* w = -i_* v$. Here, if w or v is given, then there is v or w.*

PROOF. In addition to ξ with the maps in (6.1.2-3), we have also
(6.2.5) the cofiberings γ: $X \xrightarrow{i'} \bar{X} \xrightarrow{j'} \bar{X}/X = X_1 \vee X_2$, ρ: $\bar{X} \overset{f'}{\subset} Y \xrightarrow{g'} Y/\bar{X}$ and $X_k \xrightarrow{i_k} X_1 \vee X_2 \xrightarrow{j_l} X_l$ ($l = 3 - k$) with the maps $\{1, f_1$: $W \subset \bar{X}, i_1\}$: $\omega \to \gamma$, $\{1, f_2$: $V \subset \bar{X}, i_2\}$: $\alpha \to \gamma$, $\{j'_1, j, 1\}$: $\rho \to \alpha'$, $\{j'_2, g, 1\}$: $\rho \to \omega'$ ($j'_k = j_k \circ j'$) for ξ, α' and ω' in (6.2.2), so that

(6.2.6) $\partial_\omega = \partial_\gamma \circ i_{1*}$, $\partial_\alpha = \partial_\gamma \circ i_{2*}$, $\partial_{\alpha'} = j'_{1*} \circ \partial_\rho$, $\partial_{\omega'} = j'_{2*} \circ \partial_\rho$, and

(6.2.7) (j_{1*}, j_{2*}): $h_*(X_1 \vee X_2) \cong h_*(X_1) \oplus h_*(X_2)$ with $(j_{1*}, j_{2*})^{-1} = i_{1*} + i_{2*}$.

(6.2.2): $\partial_\omega \circ \partial_{\alpha'} + \partial_\alpha \circ \partial_{\omega'} = \partial_\gamma \circ (i_{1*} \circ j'_{1*} + i_{2*} \circ j'_{2*}) \circ \partial_\rho = \partial_\gamma \circ j'_* \circ \partial_\rho = 0$ by (6.2.6-7); and the other squares are commutative by (6.1.3).

(6.2.3): If $j_* g_* y = 0$, then $g'_* y = 0$ and $y = f'_* \bar{x}$ for some $\bar{x} \in h_u(\bar{X})$; hence $x_k = j'_k \bar{x}$ are the desired ones, since $\partial_\omega x_1 + \partial_\alpha x_2 = \partial_\gamma j'_* \bar{x} = 0$. Conversely, if $\partial_\omega x_1 = -\partial_\alpha x_2$, then $\partial_\gamma \tilde{x} = 0$ for $\tilde{x} = i_{1*} x_1 + i_{2*} x_2$, and $\tilde{x} = j'_* \bar{x}$ for some

$\bar{x} \in h_u(\bar{X})$; hence $y = f'_* \bar{x}$ is the desired one. The last holds, since $f_* \partial_\omega x_1 = 0$ by $i_* f_* \partial_\omega x_1 = f_* i_* \partial_\omega x_1 = 0$ and assumption.

(6.2.4): If $\partial_\omega \partial_{\alpha'} z = 0$, then $\partial_\alpha \partial_{\omega'} z = 0$ by (6.2.2), and there are w and v with the first two equalities. Hence $j'_* \bar{x} = 0$ for $\bar{x} = f_{1*} w + f_{2*} v - \partial_\rho z$, and $\bar{x} = i'_* x$ for some $x \in h_u(X)$. Thus $f_* w + i_* v = f'_*(\bar{x} + \partial_\rho z) = i_* f_* x = f_* i x$; and (6.2.4) holds for w and $v - f_* x$, or $w - i_* x$ and v. q.e.d.

According to (3.2.1), (6.1.3) and (6.2.5-7), the same proof gives us the following

LEMMA 6.3. *For an E_2-functor $D = \{D^t_u, KD^t_u\}$, we assume that*
(6.3) *ξ in* (6.1.2) *and γ, ρ in* (6.2.5) *are all D-cofiberings, and D splits with wedge sum, i.e., for i_k and j_k in* (6.2.5), *there holds the isomorphism*

$$(j_{1*}, j_{2*})\colon D^t_u(X_1 \vee X_2) \cong D^t_u(X_1) \oplus D^t_u(X_2) \text{ with } (j_{1*}, j_{2*})^{-1} = i_{1*} + i_{2*}\,.$$

(6.3.1) *Then, ξ induces the exact sequence in* (1.8.4):

$$\cdots \longrightarrow D^r_u(A) \xrightarrow{a_*} D^r_u(B) \xrightarrow{b_*} D^r_u(B/A) \xrightarrow{\delta_\xi} D^{r+1}_u(A) \longrightarrow \cdots \quad (\delta_\xi = \bar{\delta})\,.$$

(6.3.2) *These sequences form the diagram, which is commutative except for*

$$\delta_\omega \circ \delta_{\alpha'} = -\delta_\alpha \circ \delta_{\omega'}\colon D^{r-1}_u(X_{s+1,t+1}) \to D^{r+1}_u(X_{s,t}) \quad (\textit{by the notations in } (6.2.2))\,.$$

(6.3.3) *For $y^D \in D^r_u(Y_{s,t})$ with $j_* g_* y^D = 0$, there are $x^D_k \in D^r_u(X_k)$ (for X_k in* (6.2.3)) *satisfying the equalities $\delta_\omega x^D_1 = -\delta_\alpha x^D_2$, $f_* x^D_1 = j_* y^D$ and $i_* x^D_2 = g_* y^D$. Conversely, for x^D_k with the first equality, there is y^D with the last two ones.*

(6.3.4) *For $z^D \in D^{r-1}_u(X_{s+1,t+1})$ with $\delta_\omega \delta_{\alpha'} z^D = 0$, there are $w^D \in D^r_u(W_{s,t})$ and $v^D \in D^r_u(V_{s,t})$ with $j_* w^D = \delta_{\alpha'} z^D$, $g_* v^D = \delta_{\omega'} z^D$ and $f_* w^D = -i_* v^D$. Here, if w^D or v^D is given, then there is v^D or w^D.*

LEMMA 6.4. *Furthermore, let $\phi^D\colon h_u \to D^0_u$ be a natural transformation. Then:*

(6.4.1) *i_* and f_* for D^0_u are monomorphic, and $\phi^D \circ \partial_\xi = 0$ for ∂_ξ in* (6.2.1).

(6.4.2) *For $x_k \in h_u(X_k)$ with $\partial_\omega x_1 = -\partial_\alpha x_2$ (cf.* (6.2.3)), *$\delta_\omega \phi^D x_1 = -\delta_\alpha \phi^D x_2$ holds.*

(6.4.3) *In* (6.3.3) *for $r = 0$, the last two equalities imply the first one.*

(6.4.4) *For z, w and v in* (6.2.4), *there is $x^D \in D^0_u(X_{s,t})$ with $i_* x^D = \phi^D w$ and $f_* x^D = -\phi^D v$.*

PROOF. (6.4.1): We see the first half by (6.3.1) and $D^{-1}_u = 0$, and so the second half since $a_* \circ \phi \circ \partial_\xi = \phi \circ a_* \circ \partial_\xi = 0$ $(a = i, f)$, where $\phi = \phi^D$.

(6.4.2): $j_* g_* \phi y = 0$ for y in (6.2.3), and there are x^D_k in (6.3.3) for $y^D = \phi y$ and $r = 0$. Then $f_* x^D_1 = \phi j_* y = f_* \phi x_1$ and $x^D_1 = \phi x_1$ by (6.4.1); and $x^D_2 = \phi x_2$ similarly. Thus (6.4.2) holds.

(6.4.3) holds, since the last two equalities determine x_k^D uniquely by (6.4.1).

(6.4.4): $f_* j_* \phi w = \phi f_* \partial_{\alpha'} z = 0$, and $j_* \phi w = 0$ by (6.4.1); hence $\phi w = i_* x^D$ for some x^D. Then $i_* f_* x^D = f_* \phi w = -i_* \phi v$, and $f_* x^D = -\phi v$ by (6.4.1).

q.e.d.

§ 7. Comparison of spectral sequences by a double E_2-functor

Under Definitions 1.8, 4.3 and 5.3, we consider the following

DEFINITION 7.1. We say that a double E_2-functor $A = \{A_u^{s,t}, KA_u^{s,t}, LA_u^{s,t}\}$ *is related to a homology theory* h_* *at* $X_0 = X_{0,0}$ by

(7.1.1) E_2-functors $B = \{B_u^s, KB_u^s\}$ $(B = C, D)$, natural transformations ϕ^B: $h_u \to B_u^0$, ψ^C: $C_u^s \to A_u^{s,0}$, ψ^D: $D_u^t \to A_u^{0,t}$ with $\psi^C \circ \phi^C = \psi^D \circ \phi^D$, and cofiberings

$$(7.1.2) \qquad \alpha_{s,t}: X_{s,t} \xrightarrow{f} V_{s,t} \xrightarrow{g} X_{s,t+1}, \quad \omega_{s,t}: X_{s,t} \xrightarrow{i} W_{s,t} \xrightarrow{j} X_{s+1,t},$$
$$\beta_{s,t}: W_{s,t} \xrightarrow{f} Y_{s,t} \xrightarrow{g} W_{s,t+1}, \quad \eta_{s,t}: V_{s,t} \xrightarrow{i} Y_{s,t} \xrightarrow{j} V_{s+1,t},$$

in (5.1.1) with (6.1.2), if these satisfy the following (7.1.3-5):

(7.1.3) For each $\eta_{s,t}$, $0 \to h_u(V_{s,t}) \xrightarrow{i_*} h_u(Y_{s,t}) \xrightarrow{j_*} h_u(V_{s+1,t}) \to 0$ is exact.

(7.1.4) $\xi_{s,t}(\xi = \alpha, \beta, \omega, \eta)$ and γ, ρ in (6.2.5) are all D-cofiberings, and D splits with wedge sum (cf. (6.3)).

(7.1.5) Each $\beta_{s,t}$ is also a C^0-homological $A(2)$-cofibering, $\{E(s)_r^{t,u}\}$ in (5.2.1) converges and collapses, ϕ^C is natural for $\beta_{s,t}$ (cf. (5.1.4-6)), and $\omega_{s,0}$ is a C- and $A(1)$-injective cofibering; $Y_{s,t}$ is D- and $A(i)$-injective $(i = 1, 2)$; ϕ^C, ϕ^D and ψ^C: $C_u^0 \to A_u^{0,0}$ are isomorphic for $Y_{s,t}$, and so are ϕ^C and ψ^D for $W_{s,0}$.

Under this definition, we see the following:

(7.1.6) Lemmas 6.2-4 hold by (7.1.4). ϕ^D is isomorphic also for $V_{s,t}$ which is D-injective, by Corollary 1.5 for $\eta_{s,t}$.

(7.1.7) For $W_{s,t}$, ϕ^C and ψ^D are isomorphic since so are for $Y_{s,t}$ and $W_{s,0}$, and ϕ^D is epimorphic; and $\operatorname{Ker} \partial_\beta^n = \operatorname{Ker} \partial_\beta$ for $t \geqq n \geqq 1$ and ∂_β^n: $h_u(W_{s,t}) \to h_{u-n}(W_{s,t-n})$ in (6.2.1), by (5.1.6) and (5.6.2-3).

(7.1.8) Moreover, A is indirectly related to C at X_0 by ψ^C and (7.1.2); and A (resp. C, D) is related to D (resp. h_*, h_*) by ψ^D and $\{\omega_{s,0}\}$ (resp. ϕ^C, ϕ^D and $\{\omega_{s,0}\}$, $\{\alpha_{0,t}\}$). Thus, *Theorem* 1.9, 4.4 *and Corollary* 5.6 *give us the following spectral sequences*:

the May one $\quad E^{\text{May}} = \{E_{u,r}^{s,t}, d_r^{\text{May}}: E_{u,r}^{s,t} \to E_{u+r,r}^{s+1,t+r}\}$,

the Mahowald one $E^{\text{Mah}} = \{\tilde{E}_{u,r}^{s,t}, d_r^{\text{Mah}}: \tilde{E}_{u,r}^{s,t} \to \tilde{E}_{u,r}^{s+r,t-r+1}\}$ *and*

$E(B) = \{E(B)_r^{s,t}, d_r^B: E(B)_r^{s,t} \to E(B)_r^{s+r,t+r-1}\}$, *with*

$$\begin{array}{ccc} A_u^{s,t}(X_0) = E_{u,1}^{s,t} \underset{\text{abut}}{\Rightarrow} C_{u-t}^s(X_0) = E(C)_2^{s,u-t} & \underset{\text{abut}}{\Rightarrow} & h_{u-s-t}(X_0) \\ \| & & \| \\ A_u^{s,t}(X_0) = \tilde{E}_{u,2}^{s,t} \underset{\text{conv}}{\Rightarrow} D_u^{s+t}(X_0) = E(D)_2^{s+t,u} & \underset{\text{abut}}{\Rightarrow} & h_{u-s-t}(X_0) \, . \end{array}$$

(7.1.9) If E^{Mah} collapses, then ψ^D is epimorphic also for $X_{s,0}$ and $\operatorname{Ker} \delta_\omega^n = \operatorname{Ker} \delta_\omega$ for $s \geqq n \geqq 1$ and $\delta_\omega^n \colon D_u^t(X_{s,0}) \to D_u^{t+n}(X_{s-n,0})$ in (6.3.1), by Corollary 4.5 (ii).

The purpose of this section is to argue some relations betwen these spectral sequences by the following main result.

THEOREM 7.2. *In case of Definition* 7.1, *consider the condition*

$\mathrm{C}(a, b, n)$: $h_{b-a+i}(W_{i,0}) = 0$ *for* $a \leqq i < a + n$ *(this is nothing when* $n = 0$*)* .

Then, the spectral sequences in (7.1.8) *satisfy the following* (i)–(iv) *for* $x \in A_u^{s,t}(X_0) = \tilde{E}_{u,2}^{s,t} = E_{u,1}^{s,t}$:

(i) $d_1^{\text{May}} d_2^{\text{Mah}} x = d_2^{\text{Mah}} d_1^{\text{May}} x$ *in* $A_{u+1}^{s+3,t}(X_0)$. *More generally, if* $\mathrm{C}(a, b, n)$ *for* $a = s+2$, $s+3$ *and* $b = u - t + 1$ *hold for an integer* $n \geqq 0$, *then* $d_r^{\text{Mah}} x = 0 = d_r^{\text{Mah}} d_1^{\text{May}} x$ *for* $r \leqq \min\{n+1, t\}$; *and* $d_1^{\text{May}} d_{n+2}^{\text{Mah}} x = d_{n+2}^{\text{Mah}} d_1^{\text{May}} x$ *when* $n < t$, *and* x *converges in* E^{Mah} *when* $n \geqq t$.

(ii) *If* x *converges to* $x^D \in D_u^{s+t}(X_0)$ *in* E^{Mah}, *then so does* $d_1^{\text{May}} x \in A_{u+1}^{s+1,t+1}(X_0)$ *to* $(-1)^t d_2^D x^D \in D_{u+1}^{s+t+2}(X_0)$. *If* E^{Mah} *collapses and* $d_2^D x^D = 0$ *in addition, then so does* $d_2^{\text{May}} x \in A_{u+2}^{s+1,t+2}(X_0)$ *to* $(-1)^t d_3^D x^D \in D_{u+2}^{s+t+3}(X_0)$.

(iii) *If* x *converges to* $x^C \in C_{u-t}^s(X_0)$ *in* E^{May}, *then so does* $d_2^{\text{Mah}} x \in A_u^{s+2,t-1}(X_0)$ *to* $d_2^C x^C \in C_{u-t+1}^{s+2}(X_0)$. *If* $\mathrm{C}(s+2, u-t+1, n)$ *holds in addition, then* $d_r^{\text{Mah}} x = 0 = d_r^C x^C$ *for* $r \leqq \min\{n+1, t\}$; *and* $d_{n+2}^{\text{Mah}} x \in A_u^{s+n+2,t-n-1}(X_0)$ *converges to* $d_{n+2}^C x^C \in C_{u-t+n+1}^{s+n+2}(X_0)$ *in* E^{May} *when* $n < t$, *and* x *converges in* E^{Mah} *when* $n \geqq t$.

(iv) *If* x *converges to* x^C *in* E^{May} *and to* x^D *in* E^{Mah}, *then there is* $y \in A_{u+1}^{s+2,t}(X_0)$ *converging to* $d_2^C x^C$ *in* E^{May} *and to* $(-1)^t d_2^D x^D$ *in* E^{Mah}. *If* $\mathrm{C}(s+2, u-t+1, n)$ *holds in addition, then* $d_r^C x^C = 0$ *for* $r \leqq n+1$, $d_r^D x^D = 0$ *for* $r \leqq n - t + 1$, *and there is* $y' \in A_{u+1+b}^{s+n+2,a}(X_0)$ *converging to* $d_{n+2}^C x^C$ *in* E^{May} *and to* $(-1)^t d_{b+2}^D x^D$ *in* E^{Mah}, *where* $a = \max\{t-n, 0\}$ *and* $b = \max\{n-t, 0\}$.

Here, '*converge*' is used in the sense of (1.6.2). Thus, in the same way as Corollary 1.7 (i), we see the following by the definitions of E^{Mah} and E^{May} in §§4–5:

(7.3.1) $x \in A_u^{s,t}(X_0)$ converges to $x^D \in D_u^{s+t}(X_0)$ in E^{Mah} if and only if $x = \bar{\psi}_\omega \bar{x}^D$ and $\delta_\omega^s \bar{x}^D = x^D$ for some $\bar{x}^D \in D_u^t(X_{s,0})$, where $\bar{\psi}_\omega = (\delta_\omega^A)^s \circ \psi^D$: $D_u^t(X_{s,0}) \to A_u^{0,t}(X_{s,0}) \to A_u^{s,t}(X_0)$, $\delta_\omega^A = \bar{\delta}$, is in Corollary 4.5, and δ_ω in (7.1.9).

(7.3.2) For x in (7.3.1) and $x_r \in A_u^{s',t'}(X_0)$, $d_r^{\text{Mah}} x = x_r$ in E^{Mah} (cf. (1.6.1)) if and only if $s' = s + r$, $t' = t - r + 1$ and $x = (\delta_\omega^A)^s \bar{x}$, $i_* \bar{x} = \psi^D \bar{w}^D$, $j_* \bar{w}^D =$

$\delta_\omega^{r-1}\bar{x}_r^D$ and $\bar{\psi}_\omega \bar{x}_r^D = x_r$ for some $\bar{x} \in A_u^{0,t}(X_{s,0})$, $\bar{w}^D \in D_u^t(W_{s,0})$ and $\bar{x}_r^D \in D_u^{t'}(X_{s',0})$.

(7.3.3) x in (7.3.1) converges to $x^C \in C_{u-t}^s(X_0)$ in E^{May} if and only if $x = (\delta_\omega^A)^s \bar{x}$, $i_* \bar{x} = \psi^D \bar{\phi}_\beta w$, $\phi^C \partial_\beta^t w = i_* \bar{x}^C$ and $(\delta_\omega^C)^s \bar{x}^C = x^C$ for some $\bar{x}$ in (7.3.2), $w \in h_u(W_{s,t})$ and $\bar{x}^C \in C_{u-t}^0(X_{s,0})$, where $\bar{\phi}_\beta = \delta_\beta^t \circ \phi^D$: $h_u(W_{s,t}) \to D_u^0(W_{s,t}) \to D_u^t(W_{s,0})$, $\delta_\beta = \bar{\delta}$, is in Corollary 1.7 (i) for $\beta = \beta_{s,t}$, ∂_β in (7.1.7), and $\delta_\omega^C = \bar{\delta}$: $C_u^t(X_{s,0}) \to C_u^{t+1}(X_{s-1,0})$ in (1.8.4).

(7.3.4) For x in (7.3.1) and $y_r \in A_{u'}^{s',t'}(X_0)$, $d_r^{\mathrm{May}} x = y_r$ in E^{May} if and only if $s' = s+1$, $t' = t+r$, $u' = u+r$ and $x = (\delta_\omega^A)^s \bar{x}$, $i_* \bar{x} = \psi^D \bar{\phi}_\beta w$, $i_* j_* \partial_\beta^t w = \partial_\beta^{t'} w_r$, $\psi^D \bar{\phi}_\beta w_r = i_* \bar{y}_r$ and $(\delta_\omega^A)^{s'} \bar{y}_r = y_r$ for some $\bar{x}$, w in (7.3.3), $w_r \in h_{u'}(W_{s',t'})$ and $\bar{y}_r \in A_{u'}^{0,t'}(X_{s',0})$.

Also, (6.2.3) and (7.1.3) imply inductively the following:

(7.4.1) For any $z \in h_u(X_{s,t})$, there are $z_i \in h_u(X_{i,j})$ $(j = s+t-i)$ for $s \geqq i \geqq 0$ with $z_s = z$ and $\partial_\alpha z_i = \partial_\omega z_{i+1}$; hence

(7.4.2) $\delta_\omega^{s-i} \bar{\phi}_\alpha z = (-1)^{\varepsilon(j,t)} \bar{\phi}_\alpha z_i$ $(\varepsilon(j,t) = \sum_{k=t}^{j-1} k)$

by (6.3-4.2) for $\bar{\phi}_\alpha = \delta_\alpha^* \circ \phi^D$: $h_u(X_{i,*}) \to D_u^0(X_{i,*}) \to D_u^*(X_{i,0})$.

Moreover, (6.2.3-4), (7.1.3) and (6.4.4) imply the following:

(7.4.3) For $\tilde{x}^D \in D_u^0(X_{s,t})$, $w \in h_u(W_{s,t})$ and $z \in h_{u+1}(X_{s+1,t+1})$ with $i_* \tilde{x}^D = \phi^D w$ and $j_* w = \partial_\alpha z$, there are $z_i \in h_{u+1}(X_{i,j+2})$, $x_i^D \in D_u^0(X_{i,j})$, $v_i \in h_u(V_{i,j})$, $w_i \in h_u(W_{i-1,j+1})$ and $y_i \in h_u(Y_{i-1,j})$ $(j = s+t-i)$ for $s \geqq i \geqq 0$ with $\partial_\alpha z_i = \partial_\omega z_{i+1} = g_* v_i$ $(z_{s+1} = z)$, $i_* v_i = -f_* w_{i+1}$ $(w_{s+1} = w)$, $i_* x_i^D = \phi^D w_{i+1}$, $f_* x_i^D = -\phi^D v_i$, $v_i = j_* y_i$, $w_i = g_* y_i$ and so $j_* w_i = \partial_\alpha z_i$; hence $i_* \tilde{x}^D = i_* x_s^D$ and so $\tilde{x}^D = x_s^D$ by (6.4.1); and $\delta_\omega x_i^D = \delta_\alpha x_{i-1}^D$ by (6.4.3). Thus,

(7.4.4) $\delta_\omega^{s-i} \delta_\alpha^t \tilde{x}^D = (-1)^{\varepsilon(j,t)} \delta_\alpha^j x_i^D (\varepsilon(j,t)$ is in (7.4.2)) by (6.3.2).

On the other hand, $\mathrm{C}(a,b,n)$ implies $\partial_\beta^m = 0$: $h_k(W_{i,j}) \to h_{k-m}(W_{i,j-m})$ $(k = b+i+j-a)$ for $a \leqq i < a+n$, $j \geqq m \geqq 1$, by (7.1.7); hence for any $z \in h_k(X_{i,j})$, there is $z' \in h_k(X_{i+1,j-1})$ with $\partial_\omega z' = \partial_\alpha z$ when $j \geqq 1$, and $z' \in h_{k+1}(X_{i+1,0})$ with $\partial_\omega z' = z$ when $j = 0$. Thus:

(7.4.5) Assume $\mathrm{C}(a,b,n)$. Then for any $z \in h_u(X_{a,c})$ $(u = b+c)$, there are $z_i \in h_u(X_{i,j})$ $(j = a+c-i)$ for $a \leqq i \leqq a + \min\{n,c\}$ with $z_a = z$ and $\partial_\omega z_i = \partial_\alpha z_{i-1}$, hence $\delta_\omega^{i-a} \bar{\phi}_\alpha z_i = (-1)^{\varepsilon(c,j)} \bar{\phi}_\alpha z$ in the same way as (7.4.2); and moreover when $n > c$, we have $z_i \in h_{b+i-a}(X_{i,0})$ for $c < i-a \leqq n$ with $\partial_\omega z_i = z_{i-1}$.

Also, by (6.2.2,4) and (6.4.4), we see the following:

(7.4.6) Assume $\mathrm{C}(a,b,n)$ and $\mathrm{C}(a+1,b,n)$. Then for $\tilde{x}^D$, w and z in (7.4.3) with $s = a$, $t = c$ and $u = b+c$, there are $z_i \in h_{u+1}(X_{i+1,j+1})$, $y_i \in h_u(Y_{i-1,j})$, $v_i = j_* y_i \in h_u(V_{i,j})$, $w_i \in h_u(W_{i,j})$ and $x_i^D \in D_u^0(X_{i,j})$ $(j = a+c-i)$ for $a < i \leqq a + \min\{n,c\}$ with $\partial_\omega z_i = \partial_\alpha z_{i-1} = g_* v_i$ $(z_a = z)$, $g_* y_i = w_{i-1}$ $(w_a = w)$, $j_* w_i = \partial_\alpha z_i$, $f_* w_i = -i_* v_i$, $i_* x_i^D = \phi^D w_i$, and $f_* x_i^D = -\phi^D v_i$; hence $\delta_\omega^{i-a} \delta_\alpha^j x_i^D = (-1)^{\varepsilon(c,j)} \delta_\alpha^c \tilde{x}^D$ by the same way as (7.4.4); and moreover $\delta_\alpha^c \tilde{x}^D = 0$ when $n > c$, since $w_{a+c} \in h_u(W_{a+c,0}) = 0$ and so $x_{a+c}^D = 0$.

PROOF OF THEOREM 7.2. (i) For x,
(1) put $y_1 = d_1^{\mathrm{May}}x$, $a' = a + 1$, and take $\bar{x}$, w, w_1 and $\bar{y}_1$ in (7.3.4) for $r = 1$.
Then, $\partial_\beta^t w' = 0$ for $w' = i_* j_* w - \partial_\beta w_1$, and $\partial_\beta w' = 0$ by (7.1.7). Thus, $w' = g_* y$ for some $y \in h_u(Y_{s',t-1})$, and $j_* g_* \phi^D y = -j_* \phi^D \partial_\beta w_1 = 0$ by (6.4.1) (in (7.1.6)). Hence, there are $x_k^D \in D_u^0(X_{s+3-k,t-2+k})$ in (6.3.3) with $\delta_\omega x_1^D = -\delta_\alpha x_2^D$, $f_* x_1^D = j_* \phi^D y$ and $i_* x_2^D = g_* \phi^D y$. Therefore, $i_* x_2^D = i_* j_* \phi^D w$ and $x_2^D = j_* \phi^D w$ by (6.4.1). Thus, by (6.3.2), (1) and (7.3.2),

(2) $j_* \bar{\phi}_\beta w = \delta_\alpha^t x_2^D = \delta_\omega \bar{x}_1^D$ for $\bar{x}_1^D = (-1)^t \delta_\alpha^{t-1} x_1^D$, and so $\bar{\psi}_\omega \bar{x}_1^D = d_2^{\mathrm{Mah}} x$.

Also, $\partial_\beta^2 i_* j_* w_1 = -i_* j_* \partial_\beta w' = 0$ and $\partial_\beta i_* j_* w_1 = 0$ by (7.1.7); hence $\partial_\alpha j_* w_1 = \partial_\omega z$ for some $z \in h_{u'}(X_{s+3,t})$, and $\delta_\alpha \phi^D j_* w_1 = \delta_\omega \phi^D z$ by (6.4.2). Thus, in the same way,

(3) $j_* \bar{\phi}_\beta w_1 = \bar{\phi}_\alpha j_* w_1 = (-1)^t \delta_\omega \bar{\phi}_\alpha z$, and so $(-1)^t \bar{\psi}_\omega \bar{\phi}_\alpha z = d_2^{\mathrm{Mah}} y_1 (\bar{\phi}_\alpha = \delta_\alpha^t \circ \phi^D)$.

Moreover, $\partial_\omega z = j_* \partial_\beta w_1 = -g_* j_* y$. Hence, (6.2.4) and (6.4.4) for $v = -j_* y$ give us $w_{s+2} \in h_u(W_{s+2,t-1})$ and $x^D \in D_u^0(X_{s+2,t-1})$ with $j_* w_{s+2} = \partial_\alpha z$, $f_* w_{s+2} = -i_* v = i_* j_* y$, $i_* x^D = \phi^D w_{s+2}$ and $f_* x^D = \phi^D j_* y = f_* x_1^D$. Thus $x^D = x_1^D$ by (6.4.1), and

(4) $i_* \psi^D \bar{x}_1^D = (-1)^t \psi^D \bar{\phi}_\beta w_{s+2}$ and so $d_1^{\mathrm{May}}(\bar{\psi}_\omega \bar{x}_1^D) = (-1)^t \bar{\psi}_\omega \bar{\phi}_\alpha z$ for x_1^D in (2),

by (7.3.4). Now, (1)–(4) show the desired first equality in (i). (Note that w', z, w_{s+2} and x^D are all 0 when $t = 0$.)

Assume $C(a, b, n)$ and $C(a + 1, b, n)$ for $a = s + 2$ and $b = u - t + 1$. Then, by (7.4.6) for x^D, z and w_a $(a = s + 2, c = t - 1)$ of above, we have elements x_i^D, z_i, w_i $(a \leqq i \leqq a + \min\{n, c\})$ in (7.4.6) with $x_a^D = x^D$, $z_a = z$, $i_* x_i^D = \phi^D w_i$ and $j_* w_i = \partial_\alpha z_i$. Thus, by (7.3.2-4) and (1)–(4),

(5) $\bar{\psi}_\omega \bar{x}_i^D = d_{i-s}^{\mathrm{Mah}} x$ and $d_1^{\mathrm{May}} \bar{\psi}_\omega \bar{x}_i^D = (-1)^{\varepsilon(c,j)+t} \bar{\psi}_\omega \bar{\phi}_\alpha z_i = d_{i-s}^{\mathrm{Mah}} y_1$ for $\bar{x}_i^D = (-1)^{\varepsilon(c,j)+t} \delta_\alpha^j x_i^D$ (these are 0 when $i < a + \min\{n, c\}$); and when $n \geqq t$, $d_r^{\mathrm{Mah}} x = 0$ for any $r \geqq 2$ by taking $\bar{x}_r^D = 0$ in (7.3.2), and so x converges in E^{Mah}.

These imply the last half of (i).

(ii) Assume that x converges to x^D in E^{Mah}. Then, by (7.3.1), (7.1.4, 6-7) and (1.3.2),

(1) we have $\bar{x}^D \in D_u^t(X_{s,0})$, $\tilde{x}^D \in D_u^0(X_{s,t})$, $w \in h_u(W_{s,t})$ and $z \in h_{u'}(X_{s',t'})$ $(a' = a + 1)$ with $x = \bar{\psi}_\omega \bar{x}^D$, $x^D = \delta_\omega^s \bar{x}^D$, $\bar{x}^D = \delta_\alpha^t \tilde{x}^D$, $i_* \tilde{x}^D = \phi^D w$ and $j_* w = \partial_\alpha z$;

because the fourth equality implies $\phi^D f_* j_* w = f_* j_* i_* \tilde{x}^D = 0$ and so $f_* j_* w = 0$. Hence,

(2) $d_1^{\mathrm{May}} x = \bar{\psi}_\omega \bar{\phi}_\alpha z$ by (7.3.4), and this converges to $\delta_\omega^{s'} \bar{\phi}_\alpha z$ in E^{Mah} by (7.3.1).

Now, by $i_* \tilde{x}^D = \phi^D w$ and $j_* w = \partial_\alpha z$, we have elements z_i, x_i^D, v_i, y_i and w_i $(s \geqq i \geqq 0)$ in (7.4.3). Then,

(3) $d_2^D(\delta_\alpha^{t+s}x_0^D) = -\bar{\phi}_\alpha z_0$ by the last part of Corollary 1.7 (i) for $E(D)$, $x^D = \delta_\omega^s \bar{x}^D = \delta_\omega^s \delta_\alpha^t \tilde{x}^D = (-1)^{\varepsilon(s+t,t)}\delta_\alpha^{s+t}x_0^D$ by (1) and (7.4.4), and $\delta_\omega^{s'}\bar{\phi}_\alpha z = (-1)^{\varepsilon(s'+t',t')}\bar{\phi}_\alpha z_0$ by (7.4.2).
By (7.3.1), (2)–(3) imply the first half of (ii).

Assume in addition that E^{Mah}collapses and $d_2^D x^D = 0$. Then, $\delta_\omega^{s'}\bar{\phi}_\alpha z = 0$ by (3), and so $\delta_\omega \bar{\phi}_\alpha z = 0$ by Corollary 4.5 (ii). Hence, (7.1.4), (7.1.6-7) and (1.3.2) imply that $\bar{\phi}_\alpha z = j_* \phi_\beta w'$, $\phi^D z - j_* \phi^D w' = g_* \phi^D v'$ and $z - j_* w' - g_* v' = \partial_\alpha z'$ for some $w' \in h_{u'}(W_{s,t'})$, $v' \in h_{u'}(V_{s',t})$ and $z' \in h_{u'+1}(X_{s',t'+1})$; and $\partial_\alpha^2 z' = j_* w''$ and $\phi^D w'' = i_* \tilde{x}^D$ for $w'' = w - \partial_\beta w'$. Therefore,

(4) $d_2^{\mathrm{May}} x = \bar{\psi}_\omega \bar{\phi}_\alpha z'$ by (7.3.4).

Then, we have $z'_i \in h_{u'+1}(X_{i,j})$ $(j = s' + t' - i + 1)$ for $s' \geqq i \geqq 0$ in (7.4.1) with $z = z'$. Also, we have v'_i, y'_i, w'_i and $x_i'^D$ $(s \geqq i \geqq 0)$ in (7.4.3) for $\tilde{x}^D$, w'' and $\partial_\alpha z'$ with $\partial_\alpha^2 z'_i = -\partial_\omega \partial_\alpha z'_{i+1} = (-1)^{s'-i} g_* v'_i$ and the equalities in (7.4.3). Thus, in the same way as (3),

(5) $d_3^D(\delta_\alpha^{t+s}x_0^D) = (-1)^s \bar{\phi}_\alpha z'_0$, $x^D = (-1)^\varepsilon \delta_\alpha^{s+t} x_0^D$ $(\varepsilon = \varepsilon(s+t,t))$, and $\delta_\omega^{s'}\bar{\phi}_\alpha z' = (-1)^{\varepsilon'}\bar{\phi}_\alpha z'_0$ $(\varepsilon' = \varepsilon(s'+t'+1, t'+1))$.

(4)–(5) imply the last half of (ii) by $\varepsilon' - \varepsilon - s = t + 2s + 2$.

(iii) Assume that x converges to x^C in E^{May}. Then,

(1) we have $\bar{x}$, w and $\bar{x}^C$ in (7.3.3), and so $z \in h_u(X_{s+2,t-1})$ with $\partial_\omega z = \partial_\alpha j_* w$;

because $\phi^C \partial_\beta^t i_* j_* w = 0$ by the third equality in (7.3.3), and $i_* \partial_\alpha j_* w = \partial_\beta i_* j_* w = 0$ by (7.1.7). Therefore, $j_* \partial_\beta^t w = (-1)^{t-1}\partial_\omega \partial_\alpha^{t-1} z$ by (6.2.2), and $\delta_\omega \phi^D z = \delta_\alpha \phi^D j_* w$ and $\delta_\omega \bar{\phi}_\alpha z = (-1)^{t-1}\bar{\phi}_\alpha j_* w$ by (6.4.2), (6.3.2). Thus, by Corollary 1.7 (i) and (7.3.2),

(2) $d_2^C x^C = (-1)^{t-1}\bar{\phi}_\omega^C \partial_\alpha^{t-1} z$ $(\bar{\phi}_\omega^C = (\delta_\omega^C)^* \circ \phi^C)$ and $d_2^{\mathrm{Mah}} x = (-1)^{t-1}\bar{\psi}_\omega \bar{\phi}_\alpha z$.

Hence $d_2^{\mathrm{Mah}} x$ converges to $d_2^C x^C$ by (7.3.3).

Assume in addition $\mathrm{C}(a, b, n)$ for $a = s + 2$ and $b = u - t + 1$. Then,

(3) we have z_i $(a \leqq i \leqq a + \min\{n, c\})$; and when $n > c$, z_{a+c+1} in (7.4.5), for z and $c = t - 1$.

Then, $\partial_\omega^{i-a+1}\partial_\alpha^j z_i = (-1)^{\varepsilon+c} j_* \partial_\beta^{c+1} w$ and $\delta_\omega^{i-a+1}\bar{\phi}_\alpha z_i = (-1)^{\varepsilon+c}\bar{\phi}_\alpha j_* w$ where $\varepsilon = \varepsilon(c, j)$; and when $n \geqq t$, $\phi^D z_{a+c} = 0$ by (6.4.1), and $\bar{\phi}_\alpha j_* w = 0$. Therefore, by Corollary (1.7) (i) and (7.3.2),

(4) $d_r^C x^C = (-1)^{\varepsilon+c}\bar{\phi}_\omega^C \partial_\alpha^j z_i$ and $d_r^{\mathrm{Mah}} x = (-1)^{\varepsilon+t-1}\bar{\psi}_\omega \bar{\phi}_\alpha z_i$ $(r = i - a + 2$, $\varepsilon = \varepsilon(c, j))$ for $a \leqq i \leqq a + \min\{n, c\}$; and when $n > c$, $d_r^{\mathrm{Mah}} x = 0$ for any $r \geqq 2$ and so x converges in E^{Mah}.

These imply the last half of (iii).

(iv) Assume that x converges to x^D in E^{Mah} and to x^C in E^{May}. Then, we have $\bar{x}^D$, $\tilde{x}^D$, w and z in (1) (in the proof) of (ii), and $\bar{x}$, w' (this is w in (7.3.3)),

$\bar{x}^C$ in (7.3.3). Now, $\psi^D\bar{x}^D - \bar{x} = j_*\psi^D\bar{\phi}_\beta w_1$ for some $w_1 \in h_u(W_{s-1,t})$ ($w_1 = 0$ if $s = 0$) by $\bar{\psi}_\omega\bar{x}^D = x = (\delta_\omega^A)^s\bar{x}$, (7.1.5) and (7.1.7), and so $\psi^D\bar{\phi}_\beta i_* j_* w_1 = \psi^D i_*\bar{x}^D - i_*\bar{x} = \psi^D\delta_\beta^t i_*\tilde{x}^D - \psi^D\bar{\phi}_\beta w' = \psi^D\bar{\phi}_\beta(w - w')$ by (1) of (ii) and (7.3.3). Hence, $\phi^D(w - w' - i_* j_* w_1) = \phi^D g_* y'$ and $w - w' - i_* j_* w_1 - g_* y' = \partial_\beta w_2$ for some $y' \in h_u(Y_{s,t-1})$ ($y' = 0$ if $t = 0$) and $w_2 \in h_{u'}(W_{s,t'})$ ($a' = a + 1$) by (1.3.2) and (7.1.5-6). Therefore, by taking $w - \partial_\beta w_2$, $z - j_* w_2$, $\psi^D\bar{x}^D$ and $\bar{x}^C + j_*\phi^C\partial_\beta^t w_1$ to be new w, z, $\bar{x}$, $\bar{x}^C$, respectively,

(1) we have $\bar{x}^D$, $\tilde{x}^D$, w, z, $\bar{x}$ and $\bar{x}^C$ with the equalities in (7.3.3) and (1) of (ii).

Then, by the same way as (1) of (iii), we have $z' \in h_{u'}(X_{s'+1,t})$ with $\partial_\omega z' = \partial_\alpha z = j_* w$; and so $(-1)^t\partial_\omega\partial_\alpha^t z' = \partial_\alpha^{t'} z = j_*\partial_\beta^t w$. Therefore, by Corollary 1.7 (i) and (7.3.3),

(2) $y = (-1)^t\bar{\psi}_\omega\bar{\phi}_\alpha z'$ converges to $d_2^C x^C = (-1)^t\bar{\phi}_\omega^C\partial_\alpha^t z'$ in E^{May}.

Also, by the same way as (3) of the proof of (ii), we have z_i ($z_{s'} = z$), x_i^D ($x_s^D = \tilde{x}^D$), v_i, y_i and w_i ($w_{s'} = w$) in (7.4.3) for $s \geqq i \geqq 0$, and

(3) $d_2^D(\delta_\alpha^{t+s}x_0^D) = -\bar{\phi}_\alpha z_0$, $x^D = \delta_\omega^s\bar{x}^D = (-1)^{\varepsilon'}\delta_\alpha^{s+t}x_0^D$ ($\varepsilon' = \varepsilon(s + t, t)$), and $(-1)^t\delta_\omega^{s'+1}\bar{\phi}_\alpha z' = \delta_\omega^{s'}\bar{\phi}_\alpha z = (-1)^{\varepsilon(s'+t',t')}\bar{\phi}_\alpha z_0$.

Therefore,

(4) $y = (-1)^t\bar{\psi}_\omega\bar{\phi}_\alpha z'$ in (2) converges to $(-1)^t d_2^D x^D$ in E^{Mah}.

(2) and (4) imply the first half of (iv).

Assume $C(s + 2, u - t + 1, n)$ in addition. Then, we have z_i ($z_a = z'$) in (7.4.5) for $a = s + 2$, $b = u - t + 1$ and $c = t$. Hence, for $a \leqq i \leqq a + \min\{n, t\}$, $\partial_\omega^{i-s-1}\partial_\alpha^j z_i = (-1)^{\varepsilon(t+1,j)}j_*\partial_\beta^t w$ by (6.2.2) and $\partial_\omega\partial_\alpha^t z' = (-1)^t j_*\partial_\beta^t w$; and for $a + t < i \leqq a + n$, $\partial_\omega^{i-s-1}z_i = \partial_\omega^{t+1}z_{s+t+2} = (-1)^{\varepsilon(t+1,0)}j_*\partial_\beta^t w$. Therefore, by Corollary 1.7 (i) and (7.3.3),

(5) $d_{i-s}^C x^C = 0$ for $s + 2 \leqq i < s + n + 2$, and $y' = (-1)^\varepsilon\bar{\psi}_\omega\bar{\phi}_\alpha z_{s+n+2}$ converges to $d_{n+2}^C x^C$ in E^{May} ($\varepsilon = \varepsilon(t + 1, t - n)$ if $n \leqq t$, $= \varepsilon(t + 1, 0)$ if $n > t$).

Also, when $n \leqq t$, $(-1)^{\varepsilon(t,t-n)}\delta_\omega^n\bar{\phi}_\alpha z_{a+n} = \bar{\phi}_\alpha z'$ by (6.3.2), and so $(-1)^\varepsilon\delta_\omega^{a+n}\bar{\phi}_\alpha z_{a+n} = (-1)^{\varepsilon(s'+t',t')}\bar{\phi}_\alpha z_0$ ($\varepsilon = \varepsilon(t + 1, t - n)$) for z_0 in (3) by $z_a = z'$. Therefore, by (3),

(6) when $n \leqq t$, y' in (5) converges to $(-1)^t d_2^D x^D$ in E^{Mah}.

If $n > t$, then we have z_i' ($z_{a+n}' = z_{a+n}$) for $a + n \geqq i \geqq 0$ in (7.4.1) for $z = z_{a+n}$; and $z_{a+t} = \partial_\omega^{n-t}z_{a+n} = (-1)^{\varepsilon''}\partial_\alpha^{n-t}z_{a+t}'$ ($\varepsilon'' = \varepsilon(n - t, 0)$) by (6.2.2), and $\partial_\alpha z_i = (-1)^{\varepsilon''+e}\partial_\alpha^{n-t+1}z_i'$ ($e = (a + t - i)(n - t)$) for $a + t > i \geqq 0$ by induction. In fact, $z_i - (-1)^{\varepsilon''+e}\partial_\alpha^{n-t}z_i' = g_* v'$ for some $v' \in h_{u+1}(V_{i,j-1})$, ($g_* v' = 0$ for $i = a + t$, $j = 0$) by the assumption of induction, and so $\partial_\alpha z_{i-1} = \partial_\omega z_i = (-1)^{\varepsilon''+e}\partial_\omega\partial_\alpha^{n-t}z_i' = (-1)^{\varepsilon''+e+n-t}\partial_\alpha^{n-t+1}z_{i-1}'$ by (6.2.2) and (7.1.3). Especially, $(-1)^{\varepsilon''+e}\partial_\alpha^{n-t+1}z_0' = \partial_\alpha z_0 = g_* v_0$ ($e = (s + t + 2)(n - t)$) for z_0 and v_0 in (3). Therefore,

(7) $d_r^D(\delta_\alpha^{t+s}x_0^D) = 0$ for $r < n - t + 2$, $d_{n-t+2}^D(\delta_\alpha^{t+s}x_0^D) = (-1)^{\varepsilon''+e+1}\bar{\phi}_\alpha z_0'$, and $\delta_\omega^{a+n}\phi^D z_{a+n} = (-1)^{\varepsilon'''}\bar{\phi}_\alpha z_0'$ ($\varepsilon''' = \varepsilon(a + n, 0)$). Thus, by (3) and $\varepsilon''' - \varepsilon'' - \varepsilon' + \varepsilon - a - 1 = t^2 + 2t + 2s$,

(8) when $n > t$, $d_r^D x^D = 0$ for $r < n - t + 2$, and y' in (6) converges to $(-1)^t d_{n-t+2}^D x^D$.
(6)–(8) imply the latter half of (iv). q.e.d.

§8. The case $B = GA$ for ring spectra $G = E, F$

For ring spectra $G = E, F$ and a CW spectrum X_0, consider
(8.1.1) the E_2-functors GA with $\phi^G: \pi_* \to GA_*^0$ in (2.1.1-6), the double E_2-functor EFA with $\psi^F = \psi^{FA}: FA_u^t \to EFA_u^{0,t}$, $\psi^E: EA_u^s \to EFA_u^{s,0}$ in (4.6.1-8) for $D = FA$ and in (5.7.3), and the diagram (5.1.1) of the cofiberings given by (5.7.1), by assuming the following (8.1.2):
(8.1.2) (4.6.9) holds for $X_n = X_{n,0}$ (e.g., there is a unit-preserving map $\lambda: E \to F$), and the F-Adam spectral sequence $\{E(s)_r^{t,u}\}$, $E(s)_2^{t,u} = FA_u^t(W_{s,0}) \Rightarrow \pi_{u-t}(W_{s,0})$, in (5.7.2) converges and collapses for any $s \geqq 0$.

(8.1.3) Then, for $A = EFA$, $C = EA$, $D = FA$, $h_* = \pi_*$ and the ones in (8.1.1), (7.1.3-5) hold by (4.6.1-9), (5.7.1-4) and Lemma 2.2; and

(8.1.4) *we have the spectral sequences in* (7.1.8), *which are the G-Adams ones* $E(G) = \{E(G)_r^{s,t}, d_r^G\}$, *the Mahowald and May ones* $E^{\mathrm{Mah}} = \{\tilde{E}_{u,r}^{s,t}\}$ *and* $E^{\mathrm{May}} = \{E_{u,r}^{s,t}\}$ *given in Theorem* 2.3, 4.7 *and* 5.8, *respectively*:

$$\begin{array}{ccc} EFA_u^{s,t}(X_0) = E_{u,1}^{s,t} \overset{\mathrm{May}}{\Rightarrow} EA_{u-t}^s(X_0) = E(E)_2^{s,u-t} & \overset{E-\mathrm{Adams}}{\Rightarrow} & \pi_{u-s-t}(X_0) \\ \| & & \| \\ EFA_u^{s,t}(X_0) = \tilde{E}_{u,2}^{s,t} \overset{\mathrm{Mah}}{\Rightarrow} FA_u^{s+t}(X_0) = E(F)_2^{s+t,u} & \overset{F-\mathrm{Adams}}{\Rightarrow} & \pi_{u-s-t}(X_0)\,. \end{array}$$

(8.1.5) *Moreover, Theorem* 7.2 *holds for the spectral sequences in* (8.1.4).

In the rest of this section, we consider the case that
(8.2.1) $X_0 = S^0$, $E = BP$ at a prime p and $F = HZ_p$ with the Thom map Φ^{BP} $BP \to HZ_p$,
(cf. Example 3.10). We notice that

(8.2.2) the Thom map Φ^{BP} induces a monomorphism $\Phi_*^{BP}: (HZ_p)_*(BP) = P_* = Z_p[t_i] \to (HZ_p)_*(HZ_p) = A_*$, and $\Phi_*^{BP} t_i = \eta_i$ if p is an odd prime, $= \eta_i^2$ if $p = 2$, where η_i is the conjugate of Milnor's ξ_i, and we regard P_* as a subalgebra of A_* by Φ_*^{BP}.
Then $\{E(s)_r^{t,u}\}$ in (5.7.2) satisfies

$$(8.2.3) \qquad \{E(s)_r^{t,u}, d(s)_r\} = \{E(0)_r^{t,u} \otimes BP_*(X_{s,0}), d(0)_r \otimes 1\}\,,$$

because $BP_*(X_{s,0})$ is flat over $BP_*(S^0)$ for $s \geqq 0$ by (3.8.7); and

$$(8.2.4) \qquad E(0)_2^{t,u} = \mathrm{Ext}_{A_*}^{t,u}(Z_p, P_*) = Z_p[a_i]\,,$$

($a_i \in \mathrm{Ext}^{1,*}$, $* = 2(p^i - 1) + 1$), which is 0 if $u - t \not\equiv 0 \mod 2p - 2$, by (3.10.1).

Thus, $d(0)_r = 0$, $d(s)_r = 0$ and $\{E(s)_r^{t,u}\}$ collapses. Also, this converges by [16, 19.12]. Thus:

(8.2.5) In case (8.2.1), the assumption (8.1.2) and so (8.1.4-5) hold.

(8.2.6) Moreover, C(a, b, n) in Theorem 7.2 holds if $b - 1 \equiv 0 \bmod 2p - 2$ and $n = 2p - 3$, by (3.10.2); and E^{Mah} collapses if p is odd, by [10, 8.15].

Therefore, Theorem 7.2 implies the following

EXAMPLE 8.3. *In case* (8.2.1), *the spectral sequences in* (8.1.4) *satisfy the following* (i)–(iv) *for* $x \in EFA_u^{s,t}(S^0)$ $(E = BP,\ F = HZ_p)$:

(i) $d_1^{\mathrm{May}} d_{2p-1}^{\mathrm{Mah}} x = d_{2p-1}^{\mathrm{Mah}} d_1^{\mathrm{May}} x$ *if* $t \geqq 2p - 2$, *and* x *converges in* E^{Mah} *if* $t < 2p - 2$.

(ii) *If* x *converges to* $x^F \in FA_u^{s+t}(S^0)$ *in* E^{Mah}, *then so does* $d_1^{\mathrm{May}} x$ *to* $(-1)^t d_2^F x^F$. *If* p *is odd and* $d_2^F x^F = 0$ *in addition, then so does* $d_2^{\mathrm{May}} x$ *to* $(-1)^t d_3^F x^F$.

(iii) *If* x *converges to* $x^E \in EA_{u-t}^s(S^0)$ *in* E^{May}, *then so does* $d_{2p-1}^{\mathrm{Mah}} x$ *to* $d_{2p-1}^E x^E$ *when* $t \geqq 2p - 2$, *and* x *converges in* E^{Mah} *when* $t < 2p - 2$.

(iv) *If* x *converges to* x^E *in* E^{May} *and to* x^F *in* E^{Mah}, *then there is* $y \in EFA_{u+m}^{s+2p-1,v}(S^0)$ $(v = \max\{t - 2p + 3, 0\},\ m = \max\{1, 2p - t - 2\})$ *which converges to* $d_{2p-1}^E x^E$ *in* E^{May} *and to* $(-1)^t d_{m+1}^F x^F$ *in* E^{Mah}.

Now, by [4, II, 16.1],

(8.3.1) $\pi_*(E) = Q_p[v_i]$ and $E_*(E) = \pi_*(E)[t_i]$ $(E = BP)$ with $\Delta t_1 = 1 \otimes t_1 + t_1 \otimes 1$, $\eta v_1 = v_1 + p t_1$ for the copoduct $\Delta\colon E_*(E) \to E_*(E) \otimes E_*(E)$ and the (right) unit $\eta\colon \pi_*(E) \to E_*(E)$ $(\eta_L x = x$ for the lef unit $\eta_L)$.
Then, for the cochain complex $E_u^*(S^0)$ in (2.1.1), $E_*^s(S^0) = E_*(E) \otimes \cdots \otimes E_*(E)$ (s times), and $\delta^s = \sum_{i=0}^{s+1}(-1)^i \delta_{i*}^s$, $\delta_{i*}^s = 1 \otimes \Delta \otimes 1$: $E_*^{s-i}(S^0) \otimes E_*(E) \otimes E_*^{i-1}(S^0) \to E_*^{s-i}(S^0) \otimes E_*(E) \otimes E_*(E) \otimes E_*^{i-1}(S^0)$ for $0 < i \leqq s$, $\delta_{0*}^s x = x \otimes 1$ and $\delta_{s+1*}^s(x) = 1 \otimes x$.

(8.3.2) Thus, we have the elements

$$\alpha_t^E \in EA_*^1(S^0) \text{ and } \beta_{q/t}^E \in EA_*^2(S^0) \quad \text{for} \quad q = p^n\ (E = BP)\ (\text{cf. } [11]),$$

represented respectively by $\alpha_t^E = (\eta v_1^t - v_1^t)/p$ in $E_*^1(S^0)$ $(\alpha_1 = t_1)$ and $\beta_{q/t}^E = \{\eta v_1^{q-t} \otimes t_1^{pq} - \eta v_1^{pq-t} \otimes t_1^q - v_1^{q-t} \cdot \Delta t_1^{pq} + v_1^{pq-t} \cdot \Delta t_1^q + v_1^{q-t} t_1^{pq} \otimes 1 - v_1^{pq-t} t_1^q \otimes 1\}/p$ in $E_*^2(S^0)$.

(8.3.3) Also, we have the elements

$$a_0^F,\ h_n^F \in FA_*^1(S^0) \quad \text{and} \quad b_n \in FA_*^2(S^0) \quad (F = HZ_p),$$

represented respectively by $a_0^F = e_0$, $h_n^F = \eta_1^q$ $(q = p^n)$ in $F_*^1(S^0) = A_*$ and $b_n^F = \sum_{i=1}^{p-1} c_i \eta_1^{(p-i)q} \otimes \eta_1^{iq}$ $(q = p^n,\ pc_i = \binom{p}{i})$ in $F_*^2(S^0) = A_* \otimes A_*$, where e_i and η_i are the conjugates of Milnor's τ_i and ξ_i, respectively.
Moreover, for $E = BP$, $F = HZ_p$ and $X = S^0$, consider

$$FE_*^{s,t}(S^0) = (A_*)^t \otimes (P_*)^{s+1} \text{ with } \delta^G = \textstyle\sum_{i=0}^{*+1}(-1)^i\delta_{i*}^G \quad (* = s \text{ or } t) \text{ in (4.9.1)},$$

where $(N_*)^t = N_* \otimes \cdots \otimes N_*$ (t times) (cf. (2.3.2)). Then for $x \in (A_*)^t \otimes (P_*)^{s+1}$, $\delta_{i*}^G x = x \otimes 1$ if $G = E$ and $i = 0$, $= 1 \otimes x$ if $G = F$ and $i = t + 1$, and $\delta_{i*}^G = 1 \otimes \Delta \otimes 1$ otherwise, where the coproduct $\Delta: A_* \to A_* \otimes A_*$, $P_* \to A_* \otimes P_*$ or $P_* \to P_* \otimes P_*$ satisfies $\Delta\eta_1 = \eta_1 \otimes 1 + 1 \otimes \eta_1$, $\Delta\eta_2 = \eta_2 \otimes 1 + \eta_1 \otimes \eta_1^p + 1 \otimes \eta_2$, $\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1$. Also, by (4.9.1) (cf. (2.3.2)),

$$C_u^{s,t} = FAE_u^{s,t}(S^0) = H^t(FE_u^{s,*}(S^0); \delta^F) = FA_*^t(E) \otimes (P_*)^s,$$
$$FA_*^t(E) = Z_p[a_i] \text{ in (8.2.4)} \quad \text{and} \quad FEA_u^{s,t}(S^0) = H^s(C_u^{*,t}; \delta_*^E).$$

Here, by (8.2.3-5) and dimensional reason, we take a_i so that

(8.3.4) a_i converges to $v_i \in \pi_*(E)$ ($v_0 = p$) in (8.3.1) in $\{E(0)_r^{t,u}\}$.

(8.3.5) Hence, for $E = BP$ and $F = HZ_p$, we have the elements

$$h_n,\ b_n,\ a_0,\ \alpha_t,\ \alpha_1^s,\ \beta_{q/t},\ \alpha_1^s\beta_{q/q-1}\ (q = p^n) \text{ in } FEA_*^{u,v}(S^0),$$

represented respectively by the elements

$$h_n = 1 \otimes t_1^q, \quad b_n = 1 \otimes \textstyle\sum_{i=1}^{p-1} c_i t_1^{(p-i)q} \otimes t_1^{iq} \qquad \left(q = p^n, pc_i = \binom{p}{i}\right), \quad a_0,$$
$$\alpha_t = \textstyle\sum_{i=0}^{t-1}\binom{t}{i} a_0^{t-i-1}a_1^i \otimes t_1^{t-i}, \quad \alpha_1^s = 1 \otimes t_1 \otimes \cdots \otimes t_1 \quad (s \text{ times}),$$
$$\beta_{q/t} = \alpha_{q-t} \otimes t_1^{pq}, \quad \alpha_1^s\beta_{q/q-1} = \alpha_1^{s+1} \otimes t_1^{pq}\ (q = p^n) \text{ in } C_*^{u,v},$$

where, $(u, v) = (1, 0)$, $(2, 0)$, $(0, 1)$, $(1, t-1)$, $(s, 0)$, $(2, q-t-1)$, $(s+2, 0)$, respectively.

(8.3.6) In particular, when $p = 2$, the following elements $a(n)$ ($n = 0, 1, 2$) in $F_*^2(E) = A_* \otimes A_* \otimes P_*$, represent $a_0^{2-n}a_1^n \in FA_*^2(E)$ ($E = BP$ at 2, $F = HZ_2$):

$$a(0) = \eta_1 \otimes \eta_1 \otimes 1, \quad a(1) = \eta_1 \otimes \eta_2 \otimes 1 + \eta_1 \otimes \eta_1 \otimes t_1,$$
$$a(2) = (\eta_2 \otimes \eta_2 + \eta_1 \otimes \eta_1^2\eta_2 + \eta_1\eta_2 \otimes \eta_1^2) \otimes 1 + \eta_1 \otimes \eta_1 \otimes t_1^2 + \eta_1^2 \otimes \eta_1^2 \otimes t_1 \qquad (t_1 = \eta_1^2).$$

Moreover, for $\Delta: P_* \to P_* \otimes P_*$, $(1 \otimes \Delta)a(n) - a(n) \otimes 1$ is equal to 0 if $n = 0$, $a(0) \otimes t_1$ if $n = 1$, and $a(0) \otimes t_1^2 + \eta_1^2 \otimes \eta_1^2 \otimes 1 \otimes t_1$ if $n = 2$.

Now, by (8.2.3) and (8.3.4),

(8.3.7) $\{G_u^{s,t}\}$ in (5.2.2) satisfies $G_{*+t}^{0,t} = (I^t)_* \subset \pi_*(E)$, $(I^t/I^{t+1})_* = FA_{*+t}^t(E)$, $G_{*+t}^{s,t} = I^t \cdot E_*(\bar{E}^s) \subset E_*(\bar{E}^s)$ ($\bar{E}^s = X_{s,0}$) and $G_*^{s,t}/G_{*+1}^{s,t+1} = FA_*^t(E \wedge \bar{E}^s) = \bar{E}_{*,1}^{s,t}$ for the ideal $I = (v_0 = p, v_1, \ldots)$ of $\pi_*(E)$ ($E = BP$ at p, $F = HZ_p$). Moreover, for $\tilde{G}_{*+t}^{s,t} = I^t \cdot E_*(E^s) \subset E_*^s(S^0)$ with $\tilde{G}_*^{s,t}/\tilde{G}_{*+1}^{s,t+1} = FAE_*^{s,t}(S^0) = C_*^{s,t}$, j^s: $E^s \to \bar{E}^s$ of j: $E \to \bar{E}$ induces the following maps:

$$J^E = (j^s)_*\colon E_*^s(S^0) \to E_*(\bar{E}^s), \text{ the restriction } J^G = J^E|\tilde{G}_*^{s,t}\colon \tilde{G}_*^{s,t} \to G_*^{s,t},$$
$$J\colon C_*^{s,t} \to \tilde{E}_{*,1}^{s,t} \text{ in (4.9.7) and } J' = \mathrm{pr} \circ J^G = J \circ \mathrm{pr}\colon \tilde{G}_*^{s,t} \to \tilde{E}_{*,1}^{s,t}$$

for the projections pr: $G_*^{s,t} \to \tilde{E}_{*,1}^{s,t}$ and $\tilde{G}_*^{s,t} \to C_*^{s,t}$.
Furthermore, for δ^* in (2.1.1) and $(i \circ j)_*\colon E_*(\bar{E}^s) \to \pi_*(\bar{E}^{s+1}) \to E_*(\bar{E}^{s+1})$ in (5.2.5) ($E_*(X) \cong EA_*^0(E \wedge X)$),

(8.3.8) $(i \circ j)_* \circ J^E = (-1)^{s+1} J^E \circ \delta^*$; hence we have the map

$$J_*^E = (J^E)_*\colon EA_u^s(S^0) = H^s(E_u^*(S^0); \delta^*) \to H^s(E_u(\bar{E}^s); (i \circ j)_*) = EA_u^s(S^0).$$

Then, by (8.3.8), (5.2.6) and (1.6.1-2), we see the following:

(8.3.9) Assume that $x \in \tilde{G}_{u+t}^{s,t} \subset E_u^s(S^0)$ satisfies $\delta^s x \in \tilde{G}_{u+t+r}^{s+1,t+r} \subset E_u^{s+1}(S^0)$. Then, $J'x \in \tilde{E}_{u+t,1}^{s,t}$ and $J'\delta^s x \in \tilde{E}_{u+t+r,1}^{s+1,t+r}$ represent the elements in $\tilde{E}_{*,2}^{*,*} = E_{*,1}^{*,*}$ such that $d_r^{\mathrm{May}}[J'x] = (-1)^{s+1}[J'\delta^s x]$ ($[J'x] = J_*[\mathrm{pr}\, x]$ for $J_*\colon \tilde{E}_{*,2}^{s,t} \cong \tilde{E}_{*,2}^{s,t}$ in Lemma 4.10 (iv)). If $\delta^s x = 0$, then $[J'x] = J_*[\mathrm{pr}\, x]$ converges to $J_*^E[x]$ in E^{May}.

EXAMPLE 8.4. *In Example* 8.3 ($E = BP$ at p, $F = HZ_p$ *and* $X_0 = S^0$), *the elements given in* (8.3.2-5) *satisfy the following*:

(i) *In* E^{Mah}, $J_* h_n$ *(resp.* $J_* b_n$, $J_*(a_0 b_n)$*) converges to* h_n^F *(resp.* b_n^F, $a_0^F b_n^F$*). In* E^{May}, $J_* \alpha_t$ *(resp.* $J_* \beta_{q/t}$, $J_*(\alpha_1^s \beta_{q/q-1})$ *for* $q = p^n$*) converges to* $J_*^E \alpha_t^E$ *(resp.* $J_*^E \beta_{q/t}^E$, $J_*^E((\alpha_1^E)^s \beta_{q/q-1}^E)$*).*

(ii) *For* $n \geqq 1$, $d_1^{\mathrm{May}} J_* h_{n+1} = -J_*(a_0 b_n)$; *hence* $d_2^F h_{n+1}^F = -a_0^F b_n^F$.

(iii) *Assume* $p = 2$. *Then* $d_3^{\mathrm{Mah}} J_* \alpha_3 = J_*(\alpha_1^4)$ *and* $d_3^{\mathrm{Mah}} J_* \beta_{q/q-3} = J_*(\alpha_1^3 \beta_{q/q-1})$ *for* $q = 2^n$; *hence* $d_3^E J_*^E \alpha_3 = (J_*^E \alpha_1^E)^4$ (cf. [13]) *and* $d_3^E J_*^E \beta_{q/q-3}^E = J_*^E((\alpha_1^E)^3 \beta_{q/q-1}^E)$ *for* $q = 2^n$.

PROOF. (i) The first half is seen by the equality of Φ_*^{BP} in (8.2.2) and Lemma 4.10 (iii). By (8.3.9) and $\mathrm{pr}\, \alpha_t^E = \alpha_t$ ($\alpha_t^E \in \tilde{G}_*^{1,t-1}$), $J\alpha_t$ converges to $J_*^E \alpha_t^E$. Also, $\beta_{q/t}^E = (\eta v_1^{q-t} - v_1^{q-t})/p \otimes t_1^{pq} + I^{q-t} \cdot E_*(E) \otimes E_*(E) \in G_*^{2,q-t-1}$ and $\mathrm{pr}\, \beta_{q/t}^E = \beta_{q/t}$; hence we see (i) by (8.3.9).

(ii) $t_1^{pq} \in E_*^1(S^0) = G_*^{1,0}$ and $\delta^1 t_1^{pq} \equiv -p \sum_{i=1}^{p-1} c_i t_1^{(p-i)q} \otimes t_1^{iq} \pmod{p^2} \in G_*^{2,1}$, and so $\mathrm{pr}\, t_1^{pq} = h_{n+1}$ and $\mathrm{pr}\,(\delta^1 t_1^{pq}) = -a_0 b_n$. Hence $d_1^{\mathrm{May}} J_* h_{n+1} = -J_* a_0 b_n$ by (8.3.9). Thus, (i) and Example 8.3 (iii) imply (ii).

(iii) By (8.3.5-6), $\bar{\alpha}_3 = \sum_{n=0}^{2} a(n) \otimes t_1^{n+1} \in A_*^2 \otimes P_*^2$ represents α_3. Then, for $x = \eta_1^2 \otimes t_1 \otimes t_1 \otimes t_1 \in A_* \otimes P_*^3$ and $y = t_1 \otimes t_1 \otimes t_1 \otimes t_1 \in P_*^4$, we see that

$$\delta^E \bar{\alpha}_3 = \eta_1^2 \otimes \eta_1^2 \otimes 1 \otimes t_1 \otimes t_1 = \delta^F x, \quad \delta^E x = \delta^F y \text{ and } \delta^E y = \alpha_1^4.$$

Thus, $d_3^{\mathrm{Mah}} J_* \alpha_3 = J_*(\alpha_1)^4$ by Lemma 4.10 (i). Also, $\bar{\alpha}_3 \otimes t_1^{2q}$ ($q = 2^n$) represents $\beta_{q/q-3}$; and the above equalities hold for $\bar{\alpha}_3 \otimes t_1^{2q}$, $x \otimes t_1^{2q}$, $y \otimes t_1^{2q}$ and $\alpha_1^3 \otimes t_1 \otimes t_1^{2q}$ instead of $\bar{\alpha}_3$, x, y and α_1^4, which show the second equality by Lemma 4.10 (i). Thus, Example 8.3 (iii) implies (iii). q.e.d.

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