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Modularity conditions in Lie algebras

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1. Introduction

For several years some authors have been studying the properties of algebras by considering the subalgebra lattice structure. Throughout this paper lattice conditions will be defined in an algebra context. It will become clear that much of the earlier work will hold true for any lattice.

A subalgebra U of an algebra A is called *modular* in A if it is a modular element in the lattice of subalgebras of A; that is

$$\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C$$

and

$$\langle U, B \rangle \cap C = \langle B \cap C, U \rangle \quad \text{for all subalgebras } U \subseteq C.$$

(Here $\langle X, Y \rangle$ denotes the subalgebra of A generated by X and Y.) We say A is *completely modular* if every subalgebra is modular in A.

Modular subalgebras were studied by Amayo and Schwarz in [1]. A natural question to ask is, "Can the hypothesis of modularity be weakened in such a way that useful information about the algebra can still be obtained?" The answer to this question is "yes", but it is unclear as to what is the "best" weakened hypothesis as there are several sensible versions and their relationships to one another are still unclear. It is to a better clarification of this situation and a deeper understanding of different types of modularity that this paper is directed.

Throughout this section U will denote a subalgebra of a general algebra A. We shall denote that U is maximal in B by $U < B$.

We now give some of the key definitions: We say that U is *upper semi-modular* in A or *u.s.m.* in A if for every subalgebra B of A such that $U \cap B < U, B$ then $U, B < \langle U, B \rangle$. We say that A is *completely upper semi-modular* or *completely u.s.m* if every subalgebra of A is u.s.m. in A. We say that U is *lower semi-modular* in A or *l.s.m.* in A if for every subalgebra B of A such that $U, B < \langle U, B \rangle$ then $U \cap B < U, B$. We say that A is *completely lower semi-modular* or *completely l.s.m.* if every subalgebra of A is l.s.m. in A. (Note that completely u.s.m. and completely l.s.m. algebras were called upper semi-modular and lower semi-modular respectively by Kolman, Gein and Varea.)

In [9] different conditions were defined as upper and lower semi-modular. We shall see in §3 that these are indeed different and so we define these conditions again but give them slightly different names: We say that U is *upper modular* in A or *u.m.* in A if $U \lessdot \langle U, B \rangle$ for every subalgebra B of A such that $U \cap B \lessdot B$. We say that A is *completely upper modular* or *completely u.m.* if every subalgebra of A is u.m. in A. We say that U is *lower modular* in A or *l.m.* in A if $U \cap B \lessdot B$ for every subalgebra B of A such that $U \lessdot \langle U, B \rangle$. We say that A is *completely lower modular* or *completely l.m.* if every subalgebra of A is l.m. in A. If U is both u.m. in A and l.m in A then we say that U is *semi-modular* in A or *s.m.* in A; and if every subalgebra of A is s.m. in A then we say that A is *completely semi-modular* or *completely s.m.*

It is a surprising fact that under certain conditions modularity and semi-modularity are equivalent and we shall use this fact several times. Thus we give:

THEOREM 1.1. *Let L be a finite dimensional Lie algebra over a field of characteristic zero and let U be a subalgebra of L. Then U is s.m. in L if and only if U is modular in L.*

PROOF. Theorem 3.6 of [9].

Upon studying the definition of U being modular in A, it can be seen that it may be dualised to give a subalgebra U is *modular** in A if

$$\langle U, B\rangle \cap C = \langle B, U \cap C\rangle \quad \text{for all subalgebras } B \subseteq C$$

and

$$\langle U \cap B, C\rangle = \langle B, C\rangle \cap U \quad \text{for all subalgebras } C \subseteq U.$$

We say that A is *completely modular** if every subalgebra of A is modular* in A.

Similarly we can define U to be u*.m., l*.m. and s*.m. in A and also A to be completely u*.m., l*.m. and s*.m. Notice that upper semi-modularity and lower semi-modularity are dual concepts.

If $x_1, \dots, x_n$ belong to A we shall denote by $((x_1, \dots, x_n))$ the subspace spanned by $x_1, \dots, x_n$. The symbol $\oplus$ will denote an algebra direct sum, whereas $\dotplus$ will indicate a direct sum of the vector space structure alone. All algebras will be finite dimensional unless otherwise stated.

2. Basic properties

Firstly we note that if we compare the definitions of modularity and modularity* one of the identities occurs in both. This immediately gives us the following from [1].

LEMMA 2.1. *Let M be modular* in A and let B be any subalgebra of A. Then $M \cap B$ is modular* in B.*

PROOF. We need to show $\langle M \cap B, B^* \rangle \cap C^* = \langle B^*, M \cap B \cap C^* \rangle$ for all $B^* \subset C^*$ that are subalgebras of B; and also that $\langle M \cap B \cap B^*, C^* \rangle = \langle B^*, C^* \rangle \cap M \cap B$ for all $C^* \subset M \cap B$ and $B^* \subset B$. Now $\langle M \cap B \cap B^*, C^* \rangle = \langle M \cap B^*, C^* \rangle = \langle B^*, C^* \rangle \cap M = \langle B^*, C^* \rangle \cap M \cap B$. Also we have $\langle M \cap B, B^* \rangle \cap C^* = \langle M, B^* \rangle \cap B \cap C^* = \langle B^*, M \cap B \cap C^* \rangle$.

LEMMA 2.2. *Let M be modular* in A and let I be an ideal of A with $I \subseteq M$. Then M/I is modular* in A/I.*

PROOF. Similar to Proposition 1.2 of [1].

LEMMA 2.3. *Let M and N be modular* in A. Then $M \cap N$ is modular* in A.*

PROOF. Since M and N are modular* in A we have:

(i)	$\langle M, B^* \rangle \cap C^* = \langle B^*, M \cap C^* \rangle$	for all $B^* \subseteq C^*$,
(ii)	$\langle N, B^* \rangle \cap C^* = \langle B^*, N \cap C^* \rangle$	for all $B^* \subseteq C^*$,
(iii)	$\langle M \cap B^*, C^* \rangle = \langle B^*, C^* \rangle \cap M$	for all $C^* \subseteq M$,
(iv)	$\langle N \cap B^*, C^* \rangle = \langle B^*, C^* \rangle \cap N$	for all $C^* \subseteq N$.

We need to show that

(a)	$\langle M \cap N, B \rangle \cap C = \langle B, M \cap N \cap C \rangle$	for all $B \subseteq C$,
(b)	$\langle M \cap N \cap B, C \rangle = \langle B, C \rangle \cap M \cap N$	for all $C \subseteq M \cap N$.

$\langle M \cap N \cap B, C \rangle = \langle N \cap B, C \rangle \cap M$ by (iii) since $C \subseteq M \cap N \subseteq M$ $= \langle B, C \rangle \cap M \cap N$ by (iv). Also

$$
\begin{aligned}
\langle B, M \cap N \cap C \rangle &= \langle \langle M \cap N \cap C, N \cap B \rangle, B \rangle \\
&= \langle \langle M, N \cap B \rangle \cap N \cap C, B \rangle \quad \text{from (i) putting} \\
&\qquad\qquad M = M;\ B^* = N \cap B;\ C^* = N \cap C \\
&= \langle \langle N \cap M, N \cap B \rangle \cap C, B \rangle \quad \text{from (iv) putting} \\
&\qquad\qquad N = N;\ B^* = M;\ C^* = N \cap B \\
&= \langle \langle B, N \cap M \rangle \cap N \cap C, B \rangle \quad \text{from (iv) putting} \\
&\qquad\qquad N = N;\ B^* = B;\ C^* = N \cap M \\
&= \langle N, B \rangle \cap C \cap \langle B, N \cap M \rangle \quad \text{from (ii) putting} \\
&\qquad\qquad N = N;\ B^* = B;\ C^* = C \cap \langle B, N \cap M \rangle \\
&= \langle M \cap N, B \rangle \cap C \text{ since } \langle B, N \cap M \rangle \subseteq \langle N, B \rangle
\end{aligned}
$$

and thus the lemma is proved.

LEMMA 2.4. *Let M be modular* in a Lie algebra L and let U be a subalgebra of L with $\langle M, U \rangle = M + U$. Then:*

(i) *M is permutable with all quasi-ideals W of U.*

(ii) *Every subalgebra V of L with $U \cap M \subseteq V \subseteq U$ is permutable with M.*

(iii) *If in addition, $U \cap M$ is a quasi-ideal of U, then M is a quasi-ideal of $M+U$.*

PROOF. Lemma 1.6 of [1].

LEMMA 2.5. *Let M be a modular* and maximal subalgebra of a Lie algebra L. Then* $\dim V \leq 1$ *for every subalgebra V of L with $V \cap M = 0$.*

PROOF. Lemma 1.9 of [1].

LEMMA 2.6. *Let M be modular* and maximal in A. Then $M \cap U$ is modular* and maximal in U for every subalgebra U of A with $U \nsubseteq M$.*

PROOF. We have $M \cap U$ is modular* in U by Lemma 2.1. If $x \in U \setminus (M \cap U)$ then $\langle x, M \rangle = A$ holds and since M is modular* in A, $U = U \cap A = U \cap \langle x, M \rangle = \langle x, U \cap M \rangle$.

The following lemmas are straightforward and so some of the proofs are left as simple exercises.

LEMMA 2.7. (i) *If U is modular* in A then U is u*.m., l*.m., l.m. and l.s.m. in A.*

(ii) *If U is modular in A then U is u.m., l.m., l*.m. and u.s.m. in A.*

PROOF. (i) Let B, C be subalgebras of A such that $B < \langle U, B \rangle$ and that $U \cap B \subsetneq C \subseteq U$. Then $C = \langle U \cap B, C \rangle = \langle B, C \rangle \cap U$ since $C \subseteq U$ and U is modular* in A. Now $B \subseteq \langle C, B \rangle \subseteq \langle U, B \rangle$ and so $\langle C, B \rangle = B$ or $\langle C, B \rangle = \langle U, B \rangle$. If the former holds then $C \subseteq B$ and so $C \subseteq U \cap B$, contrary to our assumption. Hence $\langle C, B \rangle = \langle U, B \rangle$. Thus $U = U \cap \langle U, B \rangle = U \cap \langle C, B \rangle = C$. It follows that $U \cap B$ is maximal in U. Thus U is u*.m. in A. Similar arguments can be used to show that U is l*.m. and l.m. in A. To see that U is also l.s.m. in A simply note that if we pick B to be a subalgebra of A such that $U, B < \langle U, B \rangle$ then since U is u*.m. in A we have that $U \cap B < U$ and since U is l.m. in A, $U \cap B < B$. Hence U is l.s.m. in A as required.

(ii) See [9] and similar to part (i).

LEMMA 2.8. *Let U be a subalgebra of A and let I be an ideal of A contained in U.*

(i) *If U is u*.m. in A then U/I is u*.m. in A/I.*

(ii) *If U is l*.m. in A then U/I is l*.m. in A/I.*

PROOF. Straightforward.

LEMMA 2.9. (i) *All minimal subalgebras of A are u*.m. in A.*

(ii) *All maximal subalgebras of A are u.m. in A.*

PROOF. (i) Let M be any minimal subalgebra of A and let B be a subalgebra of A such that B is maximal in $\langle M, B\rangle$. Then clearly $M \cap B = 0$ which is maximal in M.

(ii) Lemma 1.3 of [9].

LEMMA 2.10. *Let U be $u^*.m.$ in A. Then $U \gtrdot U \cap M$ for all maximal subalgebras M of A such that $U \nsubseteq M$.*

Recall that the *Frattini subalgebra* of an algebra A, denoted by $F(A)$, is the intersection of all the maximal subalgebras of A and also that the *Frattini ideal* of A, denoted by $\phi(A)$, is the largest ideal of A contained in $F(A)$.

COROLLARY 2.11. *Let A be any algebra such that $F(A) = \phi(A)$ and suppose that U is u*.m. in A. Then $\phi(U) \subseteq \phi(A)$.*

PROOF. $U \cap \phi(A) = U \cap (\bigcap_{M \lessdot A} M) = U \cap (\bigcap_{U \nsubseteq M} M) = \bigcap_{U \nsubseteq M} (U \cap M) \supset \phi(U)$.

Notice that if L is a Lie algebra that is completely u*.m. and also satisfies the hypothesis of Corollary 2.11 then L is an E-algebra (see [10] for results on E-algebras). It may be hoped for that a generalization of Stitzinger's Theorem 2 of [7] (if L^2 is nilpotent then L is an E-algebra) could be obtained but we show in §5 that if L^2 is nilpotent and L is completely u*.m. then L is supersolvable.

LEMMA 2.12. *Let U be a minimal subalgebra of A. Then U is modular* in A if and only if $B \lessdot \langle U, B\rangle$ for all subalgebras B of A such that $U \nsubseteq B$.*

PROOF. ($\Rightarrow$) Let $B \subseteq M \subseteq \langle U, B \rangle$. Then $\langle U, B\rangle \cap M = \langle B, U \cap M\rangle$. Now if $U \subseteq M$; this implies that $\langle U, B\rangle \subseteq M \subseteq \langle U, B\rangle$ and so $M = \langle U, B\rangle$. So suppose $U \nsubseteq M$. Then $U \cap M = 0$ so $B = \langle B, U \cap M\rangle = \langle U, B\rangle \cap M = M$. Thus B is maximal in $\langle U, B\rangle$.

($\Leftarrow$) Assume $B \subseteq C$. *If* $U \subseteq B$, then $\langle U, B\rangle \cap C = B \cap C = B$ and $\langle B, U \cap C\rangle = \langle B, U\rangle = B$. So suppose that $U \nsubseteq B$. Then B is maximal in $\langle U, B\rangle$ by the hypothesis. Since $B \subseteq C$ we have that $B \subseteq \langle U, B\rangle \cap C \subseteq \langle U, B\rangle$. If $\langle U, B\rangle \cap C = B$ then $\langle U, B\rangle \cap C \subseteq \langle B, U \cap C\rangle$. Clearly $\langle B, U \cap C\rangle \subseteq \langle U, B\rangle \cap C$. If $\langle U, B\rangle \cap C = \langle U, B\rangle$ then $\langle U, B\rangle \subseteq C$ which implies that $U \subseteq C$. Hence $\langle B, U \cap C\rangle = \langle B, U\rangle$. Assume that $C \subseteq U$. If $C = U$ then $\langle U \cap B, C\rangle = \langle U \cap B, U\rangle = U$. Also $\langle B, C\rangle \cap C = \langle B, U\rangle \cap U = U$. If $C = 0$ then $\langle U \cap B, C\rangle = \langle U \cap B\rangle = U \cap B$. Also $\langle B, C\rangle \cap U = U \cap B$. Thus the result is proved.

We can now prove the dual to Theorem 2.3 of [9], namely

THEOREM 2.13. *Let U be a minimal subalgebra of A. Then the following are equivalent*:

(i) *U is l^*.m. in A.*

(ii) *$B < \langle U, B \rangle$ for all subalgebras B of A such that $U \nsubseteq B$.*

(iii) *U is modular* in A.*

PROOF. (i) $\Rightarrow$ (ii): Clear since U is minimal and l*.m. in A. (ii) $\Rightarrow$ (iii): Lemma 2.12. (iii) $\Rightarrow$ (i): Lemma 2.7(i).

3. Inter-relationships

In this section we study the question, "Under what conditions , if any, are the various modularities equivalent or weaker versions of each other?" We start by showing that in the case of Lie algebras they can be different conditions.

Let L be a Lie algebra such that $L = A \dot{+} ((x))$ where A is a minimum abelian ideal of L and dim $A \geq 3$. Pick any non-zero $a \in A$. Then clearly $((a))$ is not maximal in A. Now it is easily seen that $\langle a, x \rangle = L$ and also that $((x))$ is a maximal subalgebra of L. Thus we have that $((x))$ is u.m. and u*.m. in L but not u.s.m. or l*.m. in L.

For our next example we shall need the following definition. We say a Lie algebra L is *semiabelian if every proper subalgebra of L* is abelian.

LEMMA 3.1. *Let L be a simple semiabelian Lie algebra and let U be a proper subalgebra of L such that* $\dim U \geq 2$. *Then U is u.s.m. in L.*

PROOF. Suppose that B is a subalgebra of L such that $U \cap B$ is maximal in both U and B. Then two possibilities can occur:

(i) $\langle U, B \rangle = L$. Then there exist maximal subalgebras M and N of L with $M \neq N$ and $U \subseteq M$, $B \subseteq N$. Then $U \cap B \subseteq M \cap N = 0$ since $M \cap N$ is an ideal of $\langle M, N \rangle = L$. Therefore $\dim U = 1$ which is a contradiction.

(ii) $\langle U, B \rangle$ is abelian. Then $\langle U, B \rangle = U + B$. Thus $(U + B)/B \simeq U/(U \cap B)$ and also $(U + B)/U \simeq B/(U \cap B)$. Hence U and B are maximal in $\langle U, B \rangle$. Thus U is u.s.m. in L as required.

We can now give our next example.

COROLLARY 3.2. *Let L be a simple semiabelian Lie algebra of rank $\geqslant 3$ over a field of characteristic zero. Let U be a proper subalgebra of a Cartan subalgebra of L and* $\dim U \geqslant 2$. *Then U is u.s.m. in L but not u.m. in L.*

PROOF. The fact that U is u.s.m. in L is given by Lemma 3.1. Moreover U is not maximal in L and so by Theorem 1.6 of [9], U is not u.m. in L.

LEMMA 3.3. *Let A be any algebra and let U be a subalgebra of A. Then*:

(i) *If U is u.m. and l^*.m. in A then U is u.s.m. in A.*

(ii) *If U is u*.m. and l.m. in A then U is l.s.m. in A.*

PROOF. This is straightforward.

We now consider what can be said about maximal and minimal subalgebras of a general algebra A.

LEMMA 3.4. *Let M denote any maximal subalgebra of A. Then:*
(i) *M is u.m. in A.*
(ii) *M is u.s.m. in A if and only if M is l*.m. in A.*
(iii) *M is l.s.m. in A implies that M is u*.m. in A.*

PROOF. (i) Lemma 2.9 (ii).
(ii) ($\Rightarrow$) Choose B to be a subalgebra of A such that $M \cap B$ is maximal in M. Let C be a subalgebra of A such that $C \subseteq B$ and $M \cap B$ is maximal in C. Then $M \cap C = M \cap B$ which is maximal in M and C. So M and C are maximal in $\langle M, C \rangle = \langle M, B \rangle = A$. Clearly then $C = B$ and B is maximal in $\langle M, B \rangle$; that is M is l*.m. in A.
($\Leftarrow$) Part (i) and Lemma 3.3 (i).
(iii) Suppose that B is maximal in $\langle M, B \rangle = A$. Then M and B are maximal in $\langle M, B \rangle$ and so $M \cap B$ is maximal in M and B. Hence M is u*.m. in A.

The proof of the above lemma uses simple lattice arguments and so the dual result is immediately true, namely

LEMMA 3.5. *Let M denote any minimal subalgebra of A. Then:*
(i) *M is u*.m. in A.*
(ii) *M is l.s.m. in A if and only if M is l.m. in A.*
(iii) *M is u.s.m. in A implies that M is u.m. in A.*

To get further relationships we shall (not surprisingly) restrict our attention to certain types of algebra. We therefore introduce the following classes of algebras. If A and B are subalgebras of an algebra L with $B \subsetneq A$ then we say a *J-series* (or *Jordan-Dedekind series*) for (A, B) is a series

$$B = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

of subalgebras such that A_{i-1} is maximal in A_i for $1 \leq i \leq n$. This series has *length*, $l(A, B)$, equal to n. We call L a *J-algebra* if whenever A and B are subalgebras of L with $B \subsetneq A$ then all J-series for (A, B) have the same finite length. J-algebras were studied by Gein in [3]. A *special J-algebra* is a J-algebra in which $l(A, B) = \dim A - \dim B$ for all subalgebras A and B of L such that $B \subsetneq A$.

LEMMA 3.6. *Let J be a J-algebra and let U be a subalgebra of J. Then:*
(i) *If U is u.m. in J then U is u.s.m. in J.*

(ii) *If U is u*.m. in J then U is l.s.m. in J.*
(iii) *If U is l.m. in J then U is l.s.m. in J.*
(iv) *If U is l*.m. in J then U is u.s.m. in J.*

PROOF. (i) Let U be u.m. in J and suppose B is a subalgebra of J such that $U \cap B$ is maximal in U and B. Then since U is u.m. in J we have that U is maximal in $\langle U, B \rangle$. Hence $l(\langle U, B \rangle, U \cap B) = 2$. Since $U \cap B \subseteq B \subseteq \langle U, B \rangle$ is a chain from $U \cap B$ to $\langle U, B \rangle$ it follows that it has length 2. Thus we have B is maximal in $\langle U, B \rangle$ and so U is u.s.m. in J.

(ii), (iii) and (iv) follow by similar arguments.

Putting together the previous results we have

THEOREM 3.7. *Let J be a J-algebra. Then:*

(i) *All maximals of J are u.m., l*.m. and u.s.m. in J. Furthermore if M is a maximal subalgebra of J then M is l.s.m. in J if and only if M is u*.m. in J.*

(ii) *All minimals of J are u*.m., l.m. and l.s.m. in J. Furthermore if M is a minimal subalgebra of J then M is u.s.m. in J if and only if M is u.m. in J.*

4. Modularity conditions on the whole subalgebra lattice

In this section we study the effect of imposing different types of modularity on the whole subalgebra lattice. We are able, rather surprisingly, still to obtain important results by considering a general algebra A.

We start by showing

THEOREM 4.1. *Let A be any algebra and let U be a subalgebra of A.*

(i) *If U and all its subalgebras are u.s.m. in A then U is u.m. in A.*

(ii) *If U and all subalgebras of A containing U are u.s.m. in A then U is l*.m. in A.*

PROOF. (i) Let U and all its subalgebras be u.s.m. in A and suppose that B is a subalgebra of A such that $U \cap B$ is maximal in B. Clearly $B \nsubseteq U$. Let $U \cap B = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = U$ be a Jordan-Dedekind chain from $U \cap B$ to U. Set $B_1 = B$ and put $B_{i+1} = \langle M_i, B_i \rangle$ for $0 \leqslant i \leqslant k$. Now $M_1 \cap B_1 < B_1$, M_1. Thus we have that $B_2 = \langle M_1, B_1 \rangle > M_1, B_1$ since M_1 is u.s.m. in A. Now consider $M_2 \cap B_2$. So $M_1 \subseteq M_2 \cap B_2 \subseteq B_2$. But since M_1 is maximal in B_2 we have that $M_2 \cap B_2 = B_2$ or $M_2 \cap B_2 = M_1$. If the former holds then $B_2 \subseteq M_2$ and so $B \subseteq U$, a contradiction. Hence $M_2 \cap B_2 = M_1$ and so $B_3 = \langle M_2, B_2 \rangle > M_2, B_2$. Continuing in this way we see that $B_{k+1} = \langle M_k, B_k \rangle > M_k, B_k$; that is U is maximal in $\langle U, B \rangle$ and so U is u.m. in A.

(ii) Let U and all subalgebras of A containing U be u.s.m. in A. Pick B to be a subalgebra of A such that $U \cap B < U$. Clearly $U \nsubseteq B$. Now let $U \cap B = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_k = B$ be a Jordan-Dedekind chain from $U \cap B$ to

B. Set $U_1 = U$ and put $U_{i+1} = \langle U_i, B_i \rangle$ for $0 \leqslant i \leqslant k$. Then by an argument similar to (i) we see that $U_{k+1} = \langle U_k, B_k \rangle > U_k, B_k$; that is B is maximal in $\langle U, B \rangle$ and so U is 1*.m. in A as required.

This immediately gives us the following corollary, namely

COROLLARY 4.2. *Let A be any algebra, then the following are equivalent*:
(i) *A is completely u.s.m.*
(ii) *A is completely u.m.*
(iii) *A is completely l*.m.*

THEOREM 4.3. *Let A be any algebra and let U be a subalgebra of A.*
(i) *If U and all its subalgebras are l.s.m. in A then U is l.m. in A.*
(ii) *If U and all subalgebras of A containing U are l.s.m. in A then U is u*.m. in A.*

PROOF. Similar to Theorem 4.1.

COROLLARY 4.4. *Let A be any algebra, then the following are equivalent*:
(i) *A is completely l.s.m.*
(ii) *A is completely l.m.*
(iii) *A is completely u*.m.*

Comparing these results with that of Kolman [4] and Gein [3] we have the following classifying theorem for the Lie algebra case.

THEOREM 4.5. *Let L be a Lie algebra over a field of characteristic zero.*
(a) *The following four conditions are equivalent*:
- (i) *L is completely u.m.*
- (ii) *L is completely u.s.m.*
- (iii) *L is completely l*.m.*
- (iv) *L is abelian, almost abelian or* 3-*dimensional non-split simple.*

(b) *The following four conditions are equivalent*:
- (i) *L is completely l.m.*
- (ii) *L is completely l.s.m.*
- (iii) *L is completely u*.m.*
- (iv) *$L = R \oplus S_1 \oplus \cdots \oplus S_n$ where R is supersolvable and the S_i's are non-isomorphic* 3-*dimensional simple algebras.*

PROOF. (a) Corollary 4.2 and Theorem 2.4 of [4].
(b) Corollary 4.4 and Theorem 3 of [3].

5. The Lie algebra case

In this section we concentrate on Lie algebras. We impose conditions on only certain subalgebras and study the overall effect on the algebra. Let A be any algebra, then in [11] Varea defined the following: A is an *$M(1)$-algebra* if all maximals of A are modular in A. So we define A is an *$M(0)$-algebra* if all minimals of A are modular in A. We can also define A to be an $M^*(1)$, $M^*(0)$, u.m.(1)-algebra etc. in the natural way.

Suppose A is an $M(0)$-algebra, then by Lemma 2.7 (ii) and Theorem 2.13 A is an $M^*(0)$-algebra. The natural question to ask is, "If A is an $M^*(0)$-algebra does this imply A is an $M(0)$-algebra?" Under certain conditions we can answer this question, but first of all we require the following

LEMMA 5.1. *Let A be any algebra. Then A is an $M^*(0)$-algebra if and only if A is completely u.s.m.*

PROOF. Let A be an $M^*(0)$-algebra. Pick subalgebras H and K of A such that $H \cap K$ is maximal in H and in K. Clearly $H, K \neq A$ and the result is straightforward if H or K are minimal subalgebras of A. So we can assume that H and K are not minimal subalgebras of A. Thus there exists a minimal subalgebra M of A such that $M \nsubseteq H$ and $M \subsetneq K$. For if every minimal subalgebra of A contained in K were also contained in H then $K \subseteq H$ which is a contradiction to the fact that $H \cap K$ is maximal in K. Now since M is modular* in A, we have that M is 1*.m. in A by Lemma 2.7 (i). Hence since $M \cap H = 0$ then H is maximal in $\langle H, M \rangle$. Consider $\langle H \cap K, M \rangle$; this equals K since $M \nsubseteq H \cap K$ and $H \cap K$ is maximal in K. Then $\langle H, K \rangle = \langle H, \langle H \cap K, M \rangle \rangle$, but $\langle H, \langle H \cap K, M \rangle \rangle = \langle H, M \rangle$. Hence $\langle H, K \rangle = \langle H, M \rangle$. Thus H is maximal in $\langle H, K \rangle$. Similarly we can show that K is maximal in $\langle H, K \rangle$. Thus A is completely u.s.m.

Conversely let A be completely u.s.m. Then by Corollary 4.2, all minimals are 1*.m. in A. Thus by Lemma 2.13, all minimals are modular* in A. Hence A is an $M^*(0)$-algebra.

THEOREM 5.2. *Let L be a Lie algebra over a field of characteristic zero. Then the following are equivalent*:

(i) *L is an $M(0)$-algebra.*

(ii) *L is an $M^*(0)$-algebra.*

PROOF. (i) $\Rightarrow$ (ii): This is shown by the remarks preceding Lemma 5.1.

(ii) $\Rightarrow$ (i): Let L be an $M^*(0)$-algebra. Then, by Lemma 2.7 (i), all minimals are in particular l.m. in L. Moreover, by Lemma 5.1, L is completely u.s.m. and so, by Corollary 4.2, L is completely u.m. Thus all minimals are s.m. in L. So, by Theorem 1.1, all minimals are modular in L; that is L is an $M(0)$-algebra.

We now study the effect of imposing one type of modularity on all the minimal subalgebras. From this point onwards *L will denote a finite-dimensional Lie algebra over a field of characteristic zero* unless otherwise stated.

LEMMA 5.3. *Let L be semisimple. If L has a minimal subalgebra that is u.m. in L then L is 3-dimensional non-split simple.*

PROOF. Suppose that $((x))$ is u.m. in L. Then $((x))$ cannot be an ideal of L, since this would contradict the semisimplicity of L. So by Lemma 1.5 of [9], $((x))$ is self-idealizing. Thus $((x))$ is a Cartan subalgebra of L and hence L is 3-dimensional non-split simple.

COROLLARY 5.4. *Let L be semisimple and a u.m.(0)-algebra. Then L is completely u.m.*

PROOF. Apply Lemma 5.3 and Theorem 4.5 (a).

Suppose we consider L to be a simple infinite-dimensional Lie algebra in which all 1-dimensional subalgebras are maximal in L; then clearly L would be a u.s.m.(0)-algebra. Such an algebra is discussed by Lashi in [5] and is denoted by P_∞. However, as he states, it is still an open question as to whether such an algebra exists. Such an algebra is the analogue of the Tarski monster group and its existence is one of the most interesting problems in the general theory of Lie algebras.

We continue our examination of finite-dimensional Lie algebras with

THEOREM 5.5. *Let L be a Lie algebra over any field F such that L^2 is nilpotent.*

(i) *If L is a u.s.m.(0)-algebra then L is supersolvable.*

(ii) *If L is a u.m.(0)-algebra then L is supersolvable.*

PROOF. (i) Let L be a minimal counterexample. Then L is not supersolvable but every proper subalgebra of L is supersolvable. Thus L is minimal non-supersolvable and is described by Theorem 1.1 of [1]. In this case $L = N \dotplus ((x))$, where N is the nil radical of L, $N = N^2 \dotplus ((e_1, \cdots, e_r))$, $e_1 x = e_2, \cdots, e_{r-1}x = e_r$, $e_r x = c_0 e_1 + \cdots + c_{r-1}e_r$, $\mathrm{ad}x|_{N^2}$ is split, the polynomial $p(Y) = Y^r - c_{r-1}Y^{r-1} - \cdots - c_1 Y - c_0$ is irreducible in $F[Y]$ and $r > 1$. Now $((e_1)) \not\lessdot L$ since $((e_1)) \subsetneq N \subsetneq L$. Moreover $\langle e_1, x \rangle$ is not supersolvable and so is equal to L. But $((e_1)) \cap ((x)) = 0 \lessdot ((e_1)), ((x))$ and since L is a u.s.m.(0)-algebra we have that $((e_1)), ((x)) \lessdot L$, a contradiction. Thus no such counterexample exists.

(ii) Similar to part (i).

COROLLARY 5.6. *Let L be a u.m.(0)-algebra. Then L is solvable if and only if L is supersolvable.*

Here we introduce a piece of terminology. If L is abelian or almost abelian we say that L is *quasi-abelian*. We now give the key theorem to the behaviour of u.m.(0)-algebras.

THEOREM 5.7. *Let L be such that all 2-generator subalgebras are at most 2-dimensional or 3-dimensional non-split simple. Then L is quasi abelian or 3-dimensional non-split simple. In particular L is completely u.m.*

PROOF. Let L be a minimal counterexample. Clearly L cannot be semisimple since it could be generated by two elements and thus would be of the required form. Suppose then that $L = R$ where R is solvable. Then all proper subalgebras of R are completely u.m. Let M be a maximal ideal of R. Then $R = M + ((x))$ and by the minimality of R we know that M is quasi-abelian. Now if $m \in M$ then $xm \in M$ and $xm \in \langle x, m \rangle = ((x, m))$ by the hypothesis. So $xm \in M \cap ((x, m))$. Thus $xm = \lambda_{(x,m)} m$ where $\lambda_{(x,m)}$ belongs to the underlying field of R and is dependent on x and m. So suppose that M is almost abelian. Then there exist non-zero m_1 and m_2 belonging to M such that $m_1 m_2 = m_2$. Now $\lambda_{(x,m_2)} m_2 = xm_2 = x(m_1 m_2) = -m_1(m_2 x) - m_2(xm_1)$. But this is the same as $\lambda_{(x,m_2)} m_1 m_2 + \lambda_{(x,m_1)} m_1 m_2 = \lambda_{(x,m_2)} m_2 + \lambda_{(x,m_1)} m_2$. Thus $\lambda_{(x,m_1)} m_2 = 0$ so $\lambda_{(x,m_1)} = 0$. Since $\lambda_{(x,m_1)} = 0$ we have that $xm_1 = 0$. Now consider $m_1(x + m_2) = \mathrm{m}_1 x + m_1 m_2 = m_2$. But $m_2 \in \langle m_1, x + m_2 \rangle = ((m_1, x + m_2))$ and so $m_2 = \lambda m_1 + \mu(x + m_2)$. But this implies that $\mu = 0$ and 1 simultaneously which is impossible. Thus M is abelian. Now consider $x(m_1 + m_2) = xm_1 + xm_2$ for any m_1 and m_2 in M. Thus $\lambda_{(x,m_1+m_2)}(m_1 + m_2) = \lambda_{(x,m_1)} m_1 + \lambda_{(x,m_2)} m_2$. So we have $\lambda_{(x,m_1+m_2)} m_1 + \lambda_{(x,m_1+m_2)} m_2 = \lambda_{(x,m_1)} m_1 + \lambda_{(x,m_2)} m_2$ and it follows that $\lambda_{(x,m_1)} = \lambda_{(x,m_2)}$ for all m_1 and m_2 in M. So $xm = \lambda_{(x)} m$ for all $m \in M$ where $\lambda_{(x)}$ is independent of m. Since $R = M + ((x))$ we have that R, and so L, is quasi-abelian. Suppose now that L is neither solvable nor semisimple. Then $L = R \dotplus S$ where S is 3-dimensional non-split simple. Pick any non-zero $s \in S$, then $R \dotplus ((s))$ is a proper solvable subalgebra of L and hence it is quasi-abelian; in particular R is abelian. Now as above we have $sr = \lambda_s r$ for all $s \in S$ and all $r \in R$. Choose basis elements s_1, s_2 and s_3 of S such that $s_1 s_2 = s_3$. Then $\lambda_{s_3} r = s_3 r = (s_1 s_2)\mathrm{r} = -(s_2 r)s_1 - (rs_1)s_2 = -\lambda_{s_2} rs_1 + \lambda_{s_1} rs_2 = \lambda_{s_2}\lambda_{s_1} r - \lambda_{s_1}\lambda_{s_2} r = 0$. Thus $\lambda_{s_3} = 0$ and so $s_3 r = 0$ for all $r \in R$. Similarly we can show that λ_{s_1}, $\lambda_{s_2} = 0$ and so $s_1 r = s_2 r = 0$ for all $r \in R$. Hence $L = R \oplus S$ with $R^2 = 0$. Now consider the subalgebra $\langle s_1, s_2 + r \rangle$. This is not two dimensional since S is contained in it. Moreover it is not 3-dimensional non-split simple. But this is a contradiction, so L cannot be of this form and thus the theorem is established.

LEMMA 5.8. *If all maximal subalgebras of L are 1-dimensional then L is 2-dimensional quasi-abelian or 3-dimensional non-split simple.*

PROOF. Suppose L is simple. Then the Cartan subalgebras are 1-dimensional so L has rank 1 and hence is 3-dimensional. If L were split simple then it would have a 2-dimensional subalgebra which contradicts the hypothesis. If L is semisimple then L must be simple and so is of the required form. Let $L = R \dotplus S$ where $R \neq 0$ and solvable then $S = 0$ otherwise not all maximal subalgebras would be 1-dimensional. So if L is solvable then it is 2-dimensional since it has a maximal subalgebra of codimension one in L. Hence L is quasi-abelian. The result follows.

We are now able to give

COROLLARY 5.9. *L is a u.m.(0)-algebra if and only if L is completely u.m.*

PROOF. Let L be a u.m.(0)-algebra and let x be any non-zero element of L. Choose a non-zero $y \in L$ such that y is linearly independent of x. Then $0 = ((x)) \cap ((y))$ which is maximal in $((y))$ and so $((x))$ is maximal in $\langle x, y \rangle$. Clearly all maximals of $\langle x, y \rangle$ are 1-dimensional. Now apply Lemma 5.8 and then Theorem 5.7 which gives that L is completely u.m. The converse is obvious and thus the result is proved.

THEOREM 5.10. *L is a u.s.m.(0)-algebra if and only if L is completely u.s.m.*

PROOF. If L is a u.s.m.(0)-algebra then L is a u.m.(0)-algebra by Lemma 3.5, and L is completely u.m. by Corollary 5.9. Hence L is completely u.s.m. by Corollary 4.2. The converse is obvious and the result is proved.

Collecting together many of our previous results we can show

THEOREM 5.11. *Let L be a finite-dimensional Lie algebra over a field of characteristic zero, then the following conditions are equivalent:*

(i) *L is completely u.s.m.*
(ii) *L is a u.s.m.(0)-algebra.*
(iii) *L is an $M(0)$-algebra.*
(iv) *L is an $M^*(0)$-algebra.*
(v) *L is a u.m.(0)-algebra.*
(vi) *L is an l^*.m.(0)-algebra.*

PROOF. (i)$\Leftrightarrow$(ii): Theorem 5.10. (iii)$\Leftrightarrow$(iv): Theorem 5.2. (i)$\Leftrightarrow$(iv): Lemma 5.1. (i)$\Leftrightarrow$(v): Corollary 5.9 and Corollary 4.2. (iv)$\Leftrightarrow$(vi): Theorem 2.13.

For our final result we give the following

LEMMA 5.12. *L is an l.s.m.(0)-algebra if and only if L is an l.m.(0)-algebra.*

PROOF. Let L be an l.s.m.(0)-algebra and suppose that $\langle x \rangle$ is maximal in $\langle x, B \rangle$ for some subalgebra B of L. We must show that $\langle x \rangle \cap B$ is maximal in B, so we can clearly assume that $\langle x \rangle \nsubseteq B$. So suppose $B \subseteq B^*$ which is maximal in $\langle x, B \rangle$. Then since $\langle x \rangle$ is l.s.m. in L, B^* must be 1-dimensional and so $B = B^*$. Hence $\langle x \rangle$ is l.m. in L. Conversely let L be an l.m.(0)-algebra with $\langle x \rangle$ and B maximal in $\langle x, B \rangle$ for some subalgebra B of L. Since $\langle x \rangle$ is l.m. in L, $0 = \langle x \rangle \cap B$ is maximal in B. Hence B is 1-

dimensional. Thus L is an l.s.m.(0)-algebra which completes the proof.

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