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Construction of diffusion processes associated with a porous medium equation

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§1. Introduction

Let us consider the following Cauchy problem

$$(1.1a)\qquad u_t = \frac{1}{2}(u^\alpha)_{xx}, \qquad t > 0,\ x \in \mathbf{R},$$

$$(1.1b)\qquad u(0, x) = u_0(x), \qquad x \in \mathbf{R},$$

for given $\alpha > 1$. The equation (1.1a) is called a *porous medium equation.* The equation was studied by Kalashnikov, Oleinik, Yui-lin, Aronson, Kamenomostskaya, Peletier and so on. They studied *weak solutions* of the Cauchy problem (1.1)(= (1.1a) + (1.1b)) that are functions $u = u(t, x)$ satisfying

$$(1.2a)\qquad u \in L^1([0, T] \times \mathbf{R}) \cap L^\infty([0, T] \times \mathbf{R}) \quad \text{for all } T > 0 \text{ and}$$

$$(1.2b)\qquad \int_0^\infty dt \int_{\mathbf{R}} (\varphi_t u + \frac{1}{2}\varphi_{xx} u^\alpha) dx + \int_{\mathbf{R}} \varphi(0, x) u_0(x) dx = 0$$

for all $\varphi \in C_0^\infty([0, \infty) \times \mathbf{R})$.

Our interest is in a diffusion process ($X = \{X(t)\}$, P) such that $P(X(t) \in dx) = u(t, x)dx$ for all $t \geq 0$ and the density function $u = u(t, x)$ is a weak solution of the Cauchy problem (1.1). We call it a *diffusion process associated with* (1.1). But such a diffusion process is not unique. In this paper, we will construct a class of diffusion processes associated with (1.1). Our main result is the following

THEOREM. *Assume the following conditions for the initial function* u_0:
(A.1) u_0 *is a probability density,*
(A.2) u_0 *is a function of bounded variation,*
(A.3) $\int_{\mathbf{R}} |x| u_0(x) dx$ *is finite and*
(A.4) $(u_0)^\alpha$ *has a derivative of bounded variation.*
Then there exist a unigue weak solution u *of* (1.1) *and a diffusion process* ($X = \{X(t)\}$, P^λ) *for each* $\lambda \in [0, 1]$ *such that* $u(t, \cdot)$ *is the probability density of the distribution of* $X(t)$ *under* P^λ *for all* $t \geqq 0$ *and the generator of* (X, P^λ) *is*

$$\left\{\mathscr{G}_t^\lambda = \frac{\lambda}{2}u(t, x)^{\alpha-1}\frac{\partial^2}{\partial x^2} - \frac{1-\lambda}{2}\cdot\frac{(u^\alpha)_x(t,x)}{u(t,x)}\cdot\frac{\partial}{\partial x}\middle| t \geq 0\right\}.$$

In the previous paper [6], we introduced a story of the construction of the diffusion process when $\lambda = 1$. Our purpose in this paper is not only to propose this theorem as an existence of diffusion processes but also to explain the construction of the processes as a limit theorem in the probability theory.

§2. Construction of diffusion processes

In this section we explain the outline of our construction of diffusion processes associated with (1.1). The details are postponed to the subsequent sections. The essential idea of the construction is due to Khintchine [9]. He proved a central limit theorem which appeared in a difference approximation of the heat equation. We shall extend his method to the porous medium equation. That is, we define the diffusion processes as limits of some Markov chains.

In §3 we consider the difference approximation of the porous medium equation. Let h be a positive number and $\tau = h^{\alpha+1}$. We consider the mesh $\{(t, x) = (n\tau, jh) | n = 0, 1, \cdots, j \in \mathbf{Z}\}$ in $[0, \infty) \times \mathbf{R}$. Denote $u_j^n = u(n\tau, jh)$ for any function u on the mesh. We consider the following difference scheme

$$\text{(2.1}a) \qquad \frac{u_j^{n+1} - u_j^n}{\tau} = \frac{(u_{j+1}^n)^\alpha - 2(u_j^n)^\alpha + (u_{j-1}^n)^\alpha}{2h^2}, \; n = 0, 1, \cdots, j \in \mathbf{Z},$$

$$\text{(2.1}b) \qquad u_j^0 = u_0(jh)/c_h, \; j \in \mathbf{Z}$$

where $u_0(x)$ is an initial function of the Cauchy problem (1.1) with the condition $(A.1) \sim (A.4)$ and $c_h = \sum_{j \in Z} u_0(jh)h$ is a normalized constant which converges to 1 as $h \to 0$.

Let u_h be the function on $[0, \infty) \times \mathbf{R}$ defined by

$$\text{(2.2)} \qquad u_h((n+\theta_1)\tau, (j+\theta_2)h) = \begin{cases} u_j^n + \theta_1(u_{j+1}^{n+1} - u_{j+1}^n) + \theta_2(u_{j+1}^n - u_j^n), & \text{if } 0 \leq \theta_1 \leq \theta_2 \leq 1, \\ u_j^n + \theta_1(u_j^{n+1} - u_j^n) + \theta_2(u_{j+1}^{n+1} - u_j^{n+1}), & \text{if } 0 \leq \theta_2 \leq \theta_1 \leq 1. \end{cases}$$

PROPOSITION 1. *Assume* $(A.1)$ *and* $(A.2)$, *then there exists a unigue weak solution* u *of* (1.1) *such that*

$$\text{(2.3}a) \qquad \lim_{h\to 0} \| u_h - u \|_N = 0$$

and

$$\text{(2.3}b) \qquad \lim_{\varepsilon,\delta\to 0} \| u(\cdot + \varepsilon, \cdot + \delta) - u \|_N = 0$$

for each $N>0$ *where* $\|u\|_N=\sup_{0\le t\le N}\int_{-N}^{N}|u(t,x)|dx$. *Moreover if* u_0 *satisfies the condition* (*A*.4) *too, then* u_h *converges to* u *uniformly in any bounded domain of* $[0,\infty)\times\mathbf{R}$ *and* u^α *has a bounded derivative* $(u^\alpha)_x$ *satisfying*

$$\lim_{h\to 0}\|(u_h^\alpha)_x-(u^\alpha)_x\|_N=0 \tag{2.4a}$$

and

$$\lim_{\varepsilon,\delta\to\infty}\|(u^\alpha)_x(\cdot+\varepsilon,\ \cdot+\delta)-(u^\alpha)_x\|_N=0 \tag{2.4b}$$

for each $N>0$.

In §4 we consider the following Markov chains. Put $\mathbf{Z}^{\mathbf{N}}=\{\omega=(\omega_0,\omega_1,\dots)|\omega_n\in\mathbf{Z}\}$ and $S_n(\omega)=\omega_n$ for all $\omega\in\mathbf{Z}^{\mathbf{N}}$. For each $\lambda\in[0,1]$ and $h>0$, let P_h^λ be the Markov measure on $\mathbf{Z}^{\mathbf{N}}$ characterized by

$$\begin{aligned}(2.5a)\qquad & P_h^\lambda(S_{n+1}=j\pm 1|S_n=j)\\ &=\frac{\lambda}{2}(P_h^\lambda(S_n=j))^{\alpha-1}+\frac{1-\lambda}{2}\cdot\frac{\{(P_h^\lambda(S_n=j))^\alpha-(P_h^\lambda(S_n=j\pm 1))^\alpha\}_+}{P_h^\lambda(S_n=j)},\end{aligned}$$

$$P_h^\lambda(S_{n+1}=j|S_n=j)=1-P_h^\lambda(S_{n+1}=j+1|S_n=j)-P_h^\lambda(S_{n+1}=j-1|S_n=j)$$

and

$$P_h^\lambda(S_0=j)=u_0(jh)h/c_h, \tag{2.5b}$$

where $\{x\}_+=\max\{x,0\}$ and $c_h=\sum_{j\in\mathbf{Z}}u_0(jh)h$. Put

$$u_j^n=P_h^\lambda(S_n=j)h^{-1},$$

then the sequence $\{u_j^n\}$ satisfies the difference equation (2.1) for all $\lambda\in[0,1]$. Therefore $u_j^n=P_h^\lambda(S_n=j)h^{-1}$ is independent of λ.

We will show the convergence of the law of the Markov chains. Let $\mathscr{C}$ be the metric space of all continuous functions w: $[0,\infty)\to\mathbf{R}$ with the metric $d(w,w')=\sum_{n=1}^{\infty}2^{-n}\{\sup_{0\le t\le 2^n}|w(t)-w'(t)|\wedge 1\}$ and $\mathscr{F}$ be the σ-field generated by all cylinder sets in $\mathscr{C}$. Let X_h be the $\mathscr{C}$-valued random variable on $(\mathbf{Z}^{\mathbf{N}},P_h^\lambda)$ such that, for each $\omega\in\mathbf{Z}^{\mathbf{N}}$, $X_h(\omega)$ is the polygonal function whose value at a point $t>0$ is

$$X_h(t,\omega)=hS_{[t/\tau]}(\omega)+h((t/\tau)-[t/\tau])\{S_{[t/\tau]+1}(\omega)-S_{[t/\tau]}(\omega)\},$$

where $[x]$ is the integer part of x. Let $P_{X_h}^\lambda$ be the probability measure on $(\mathscr{C},\mathscr{F})$ such that $P_{X_h}^\lambda(A)=P_h^\lambda(X_h\in A)$ for all $A\in\mathscr{F}$.

PROPOSITION 2. *Assume* $(A.1)\sim(A.4)$, *then the family of probability measures* $\{P_{X_h}^\lambda|h>0\}$ *on* $(\mathscr{C},\mathscr{F})$ *is tight for each* $\lambda\in[0,1]$.

By the tightness of $\{P^{\lambda}_{X_h} | h > 0\}$, there exist a sequence $\{h_n\}$ of $\{h\}$ and a probability measure P^{λ} on $(\mathscr{C}, \mathscr{F})$ such that $P^{\lambda}_{X_{hn}}$ converges to P^{λ} weakly. Let $X(t)$ be the function on $\mathscr{C}$ defined by $X(t, w) = w(t)$ for all $w \in \mathscr{C}$.

In §5 we will prove that the density function of the distribution of $X(t)$ under P^{λ} is a weak solution of (1.1) given in Proposition 1 and consider a martingale problem about the process $X = \{X(t)\}$ on $(\mathscr{C}, \mathscr{F}, P^{\lambda})$.

PROPOSITION 3. *Assume* $(A.1) \sim (A.3)$, *then we have*

$$P^{\lambda}(X(t) \in dx) = u(t, x)dx \tag{2.6}$$

and

$$E^{\lambda}[|X(t)|] \leq \int_{\mathbf{R}} |x| u_0(x)dx + (\| u_0 \|_{\infty}{}^{\alpha-1} t)^{1/2} \tag{2.7}$$

for all $t \geq 0$ *and* $\lambda \in [0, 1]$ *where* $u = u(t, x)$ *is a weak solution of* (1.1) *which is constructed in Proposition* 1. *Further if* u_0 *satisfies the condition* $(A.4)$ *too, then we have*

$$E^{\lambda}\left[\left|\frac{(u^{\alpha})_x(t, X(t))}{u(t, X(t))}\right|^{\alpha}\right] < \alpha(\| (u_0^{\alpha})_x \|_{\infty})^{\alpha-1} \mathrm{V}(u_0) \tag{2.8}$$

for all $t \geq 0$ *and* $\lambda \in [0, 1]$ *where* $\mathrm{V}(f)$ *denotes the total variation of a function* f *on* $\mathbf{R}$.

PROPOSITION 4. *Assume* $(A.1) \sim (A.4)$. *Then, for each* $\lambda \in [0, 1]$ *and* $f \in C_0^{\infty}(\mathbf{R} \to \mathbf{R})$, *the process*

$$\left\{f(X(t)) - \frac{\lambda}{2}\int_0^t u(s, X(s))^{\alpha-1} f''(X(s))ds + \frac{1-\lambda}{2}\int_0^t \frac{(u^{\alpha})_x(s, X(s))}{u(s, X(s))} f'(X(s))ds\right\} \tag{2.9}$$

is an $\{\mathscr{F}^{\lambda}_t\}$ — *martingale on* $(\mathscr{C}, P^{\lambda})$ *where* $\mathscr{F}^{\lambda}_t$ *is the* σ-*field generated by* $\{X(s) | s \leq t\}$ *and all* P^{λ}-*null sets*.

COROLLARY. *Assume* $(A.1) \sim (A.4)$. *For each* $\lambda \in [0, 1]$, *the process* $X = \{X(t)\}$ *on* $(\mathscr{C}, \mathscr{F}, P^{\lambda})$ *satisfies the following stochastic differential equation*

$$X(t) = X(0) + \int_0^t \{\lambda u(s, X(s))^{\alpha-1}\}^{1/2} dB(s) - \frac{1-\lambda}{2}\int_0^t \frac{(u^{\alpha})_x(s, X(s))}{u(s, X(s))} ds. \tag{2.10}$$

Finally in §6 we will show the Markov property of the process X on $(\mathscr{C}, \mathscr{F}, P^{\lambda})$.

PROPOSITION 5. *Assume* $(A.1) \sim (A.4)$. *For each* $\lambda \in [0, 1]$, *the process* $X = \{X(t)\}$ *on* $(\mathscr{C}, \mathscr{F}, P^{\lambda})$ *is a diffusion process with the generator*

(2.11)
$$\left\{\mathscr{G}_t^\lambda = \frac{\lambda}{2}u(t,\ x)^{\alpha-1}\frac{\partial^2}{\partial x^2} - \frac{1-\lambda}{2}\cdot\frac{(u^\alpha)_x(t,\ x)}{u(t,\ x)}\cdot\frac{\partial}{\partial x}\,|\, t \geq 0\right\}.$$

Thus we obtain our main theorem.

REMARK. (i) In case of $\lambda = 1$, we can show our theorem without the assumption $(A.4)$. (ii) In case of $\lambda = 0$, the process $X = \{X(t)\}$ is a deterministic flow satisfying

$$\int_{-\infty}^{X(t)} u(t,\ x)dx = \int_{-\infty}^{X(0)} u_0(x)dx$$

with probability 1 for all $t \geq 0$.

§3. Difference approximation

The difference approximation of the porous medium equation was studied by Baklanovskaya, Nakaki and so on. We shall use their results. For each $h > 0$, let u_h be the function on $[0,\ \infty) \times \mathbf{R}$ defined by (2.2). In this section, we prove that u_h converges to a weak soluton of (1.1) (i.e. Proposition 1).

Firstly we see the stability of u_h. By the conditions $(A.1)$ and $(A.2)$ in §1, u_0 is bounded and $c_h = \sum_{j \in \mathbf{Z}} u_0(jh)h \to 1$ as $h \to 0$. Hence there exists a constant $h_0 > 0$ such that

$$1/2 < c_h < 2$$

for each $h \in (0,\ h_0)$. Put $M = \alpha(2\|u_0\|_\infty)^{\alpha-1}$ and $h_1 = \min\{h_0,\ M^{-1/\alpha-1}\}$.

LEMMA 3.1. *Assume the conditions* $(A.1)$ *and* $(A.2)$. *For each* $h < h_1$, *let* $\{u_j^n\}$ *be a sequence satisfying* (2.1). *Then we have*

(3.1)
$$0 \leq u_j^n \leq \|u_0\|_\infty/c_h, \quad n = 0,\ 1, \cdots,\ j \in \mathbf{Z}$$

and

(3.2)
$$\textstyle\sum_{j \in \mathbf{Z}} |u_{j+1}^n - u_j^n| \leq \mathrm{V}(u_0)/c_h,\ n = 0,\ 1, \cdots,$$

where $\mathrm{V}(f)$ *is the total variation of a function* f: $\mathbf{R} \to \mathbf{R}$.

PROOF. The estimate (3.1) were introduced by Baklanovskaya [2]. The estimate (3.2) is easily shown by the relation

$$u_{j+1}^{n+1} - u_j^{n+1} = [1 - ra_j^n](u_{j+1}^n - u_j^n) + \frac{r}{2}a_{j+1}^n(u_{j+2}^n - u_{j+1}^n) + \frac{r}{2}a_{j-1}^n(u_j^n - u_{j-1}^n)$$

where $r = h^{\alpha-1}$ and

$$a_j^n = \{(u_{j+1}^n)^\alpha - (u_j^n)^\alpha\}/(u_{j+1}^n - u_j^n) \in (0,\ M).$$

LEMMA 3.2 (T. Nakaki [11]). *Assume the conditions* (*A*.1), (*A*.2) *and* (*A*.4). *For each* $h < h_1$, *we have*

(3.3) $$\sup_{j\in\mathbf{Z}}|(u_{j+1}^n)^\alpha - (u_j^n)^\alpha|h \leq \|(u_0^\alpha)_x\|_\infty (c_h)^{-\alpha},$$

(3.4) $$\sum_{j\in\mathbf{Z}}|(u_{j+1}^n)^\alpha - (u_j^n)^\alpha| \leq \mathrm{V}(u_0{}^\alpha)(c_h)^{-\alpha} \text{ and}$$

(3.5) $$\tau^{-1}\sum_{j\in\mathbf{Z}}|(u_j^{n+1})^\alpha - (u_j^n)^\alpha|h \leq M\mathrm{V}(u_0{}^\alpha)(c_h)^{-\alpha}, \quad \text{for all } n = 0, 1, \cdots.$$

Next we prepare the compactness of a function space.

LEMMA 3.3. *Let* U *be a set of functions* u: $[0, \infty) \times \mathbf{R} \to \mathbf{R}$. Suppose

$$\sup_{u\in U}(\|u\|_{L^\infty([0,\infty)\times\mathbf{R})} + \sup_{t\geq 0}\mathrm{V}(u(t,\cdot))) < \infty$$

and $\left\{(f*u)(t, x) = \int_{\mathbf{R}} f(x-y)u(t, y)\,dy : u \in U\right\}$ is equicontinuous for each $f \in C_0^\infty(\mathbf{R})$. *If* U *ia an infinite set, then there exists a sequence* $\{u_n\} \subset U$ *and a function* u_∞: $[0, \infty) \times \mathbf{R} \to \mathbf{R}$ *such that*

$$\lim_{n\to\infty}\|u_n - u_\infty\|_N = 0$$

and

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}\|u_\infty(\cdot + \varepsilon, \cdot + \delta) - u_\infty\|_N = 0$$

for each $N > 0$ *where* $\|u\|_N$ *is as in Proposition* 1. *Especially if*

$$U_{a,k} = \{u \in U \,|\, \sup_{t\geq 0}|u(t,x) - u(t,y)| \leq K|x-y|^a\}$$

is an infinite set for some a, $K > 0$, *then there exists a sequence* $\{u_n\} \subset U_{a,K}$ *such that* u_n *converges uniformly in any bounded domain of* $[0, \infty) \times \mathbf{R}$.

PROOF. Let $\rho \in C_0^\infty$ be a probability density function on $\mathbf{R}$ satisfying $\mathrm{supp}(\rho) \subset (-1, 1)$. Put $\rho_n(x) = n\rho(nx)$, then $\{u*\rho_n | u \in U\}$ is uniformly bounded and equicontinuous for each $n \in \mathbf{N}$. We can choose a sequence $\{u_n\} \subset U$ such that $\{u_n*\rho_n | n \geq N\}$ is a Cauchy seqeunce with respect to the norm $\|\ \|_N$ for each $N \in \mathbf{N}$. Putting

$$u_\infty = \lim_{n\to\infty} u_n*\rho_n$$

we get Lemma 3.3, because

$$\|u*\rho_n - u\|_N \leq \frac{1}{n}\sup_{0\leq t\leq N}\mathrm{V}(u(t,\cdot))$$

for any $u \in U$ and

$$\sup_{\substack{0\leq t\leq\infty\\ x\in\mathbf{R}}}|(u*\rho_n)(t,x) - u(t,x)| \leq Kn^{-\alpha}$$

for any $u \in U_{a,K}$.

LEMMA 3.4. *Assume the conditions* $(A.1)$ *and* $(A.2)$. *Then there exist a weak solution* u *of* (1.1) *and sequence* $\{h_n\}$ *such that* $\|u_{h_n} - u\|_N \to 0$ *as* $n \to \infty$ *and* $\|u(\cdot + \varepsilon, \cdot + \delta) - u\|_N \to 0$ *as* $\varepsilon, \delta \to 0$ *for each* $N > 0$.

PROOF. By the definition of u_h and the estimate (3.1), we see

$$|(f*u_h)(t,x) - (f*u_h)(s,y)| \leq \|f'\|_\infty |x-y| + \frac{1}{2}\|\triangle_h f\|_\infty C_0^{\alpha-1}|t-s|$$

for all $t, s \geq 0$, $x \in \mathbf{R}$ and $f \in C_0^\infty(\mathbf{R} \to \mathbf{R})$ where $C_0 = 2\|u_0\|_\infty$ and

$$(\triangle_h f)(x) = \{f(x+h) - 2f(x) + f(x-h)\}h^{-2}.$$

Put $U = \{u_h | 0 < h < h_1\}$, then U satisfies the conditions of Lemma 3.3. Hence there exist a function u and a sequence $\{h_n\}$ of $\{h\}$ such that $\|u_{h_n} - u\|_N \to 0$ as $n \to \infty$. Note $\sum_{j=\infty}^{\infty} u_j^n h = 1$ and (3.1), then we get

$$\|u\|_{L^1([0,T]\times\mathbf{R})} \leq T \quad \text{and} \quad \sup_{t\geq 0, x\in\mathbf{R}} |u(t, x)| \leq C_0$$

which implies $u \in L^1([0, T] \times \mathbf{R}) \cap L^\infty([0, T] \times \mathbf{R})$ for all $T > 0$. Note that, for each $\varphi \in C_0^\infty(\mathbf{R})$, $\left\{\int_{\mathbf{R}} \varphi(t,x)u_h(t,x)dx \,|\, 0 < h < h_1\right\}$ is equicontinuous on $[0,\infty)$ and $\varphi(t,x)u_h(t,x)$ is a function of bounded variation with respect to $x \in \mathbf{R}$ for all $t \geq 0$. Put $\varphi_j^n = \varphi(n\tau, jh)$, $u_j^n = u_h(n\tau, jh)$. From the difference equation (2.1), we have

$$\begin{aligned}
\int_0^\infty dt \int_{\mathbf{R}} u(t,x)\varphi_t(t,x)dx &= \lim_{h\to 0} \tau h \textstyle\sum_{n=0}^{\infty} \sum_{j\in\mathbf{Z}} u_j^n \dfrac{\varphi_j^{n+1} - \varphi_j^n}{\tau} \\
&= \lim_{h\to 0}\left\{ -\tau h \textstyle\sum_{n=0}^{\infty} \sum_{j\in\mathbf{Z}} (u_j^n)^\alpha \dfrac{\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n}{2h^2} - h \sum_{j\in\mathbf{Z}} u_j^0 \varphi_j^0 \right\} \\
&= -\int_0^\infty dt \int_{\mathbf{R}} u^\alpha \cdot \frac{1}{2}\varphi_{xx}\, dx - \int_{\mathbf{R}} u_0(x)\varphi(0,x)dx.
\end{aligned}$$

It follows that u is a weak solution of (1.1).

Using Lemma 3.2, Nakaki [11] proved the following

LEMMA 3.5. *Assume the conditions* $(A.1)$, $(A.2)$ and $(A.4)$. *Then there exist a weak solution* u *of* (1.1) *and a suquence* $\{h_n\}$ *such that* u_{h_n} *converges to* u *uniformly in any bounded domain and* $\|(u_{h_n}^\alpha)_x - (u^\alpha)_x\|_N \to 0$ *as* $n \to \infty$ *and* $\|(u^\alpha)_x(\cdot + \varepsilon, \cdot + \delta) - (u^\alpha)_x\|_N \to 0$ *as* $\varepsilon, \delta \to 0$ *for each* $N > 0$.

By Lemmas 3.4, and 3.5 and [14], Proposition 1 is proved.

§4. Markov chains and tightness

Let $\lambda \in [0, 1]$ and $h > 0$. In this section we consider the Markov chain $\{S_n\}$ on $(\mathbf{Z}^{\mathbf{N}}, P_h^\lambda)$ defined in §2 and we prove Proposition 2. Firstly we consider the martingale property of the Markov chain. Let $u_j^n = P_h^\lambda(S_n = j)h^{-1}$ and $\mathscr{B}_n$ be the σ-field on $\mathbf{Z}^{\mathbf{N}}$ generated by $\{S_i: i \leq n\}$. Put

$$d_n^{\pm}(j) = \frac{1-\lambda}{2}\{(u_j^n)^\alpha - (u_{j\mp1}^n)^\alpha\}_+/(u_j^n h).$$

Remember the definition of the Markov measure P_h^λ and the relation $u_j^n = u_h(n\tau, jh)$, then we can show the following

LEMMA 4.1. *For each function $f: \mathbf{Z} \to \mathbf{R}$, the process*

$$\left\{f(S_n) - \sum_{i=0}^{n-1}\left[\frac{\lambda}{2}(u_h(i\tau, S_ih))^{\alpha-1}(f(S_i+1) - 2f(S_i) + f(S_i-1))h^{-2}\right.\right.$$
$$\left.\left. + d_i^+(S_i)(f(S_i+1) - f(S_i))h^{-1} + d_i^-(S_i)(f(S_i-1) - f(S_i))h^{-1}\right]\tau\right\}$$

is a $\{\mathscr{B}_n\}$-martingale on $(\mathbf{Z}^{\mathbf{N}}, P_h^\lambda)$.

Put

$$D_n^{\pm} = d_n^+(S_n) \pm d_n^-(S_n)$$

and

$$M_n = S_n - S_0 - \sum_{i=0}^{n-1} D_i^- \tau h^{-1}.$$

By Lemma 4.1, the process $\{M_n\}$ is a $\{\mathscr{B}_n\}$-martingale on $(\mathbf{Z}^{\mathbf{N}}, P_h^\lambda)$ satisfying

(4.1) $E_h^\lambda[|M_n - M_k|^2] = E_h^\lambda[\sum_{i=k}^{n-1}\{\lambda(u_h(i\tau, S_ih))^{\alpha-1} + D_i^+ h - (D_i^-)^2\tau\}\tau h^{-2}]$.

Moreover we have

LEMMA 4.2. *Assume (A.1) and (A.2). Then there exists a constant $K_1 > 0$ depending only on α, λ and $\|u_0\|_\infty$ such that*

$$E_h^\lambda[|M_n - M_k|^4] \leq K_1\{(n-k)^2\tau^2 + (n-k)^{3/2}\tau^{3/2}h + (n-k)\tau h^2\}h^{-4} \tag{4.2}$$

for all $n > k \geq 0$. If we assume (A.4) too, then

$$E_h^\lambda[|d_n^{\pm}(S_n)|^\alpha] \leq \alpha(\|(u_0^\alpha)_x\|_\infty)^{\alpha-1}\mathrm{V}(u_0)c_h^{-\alpha(\alpha-1)-1} \tag{4.3}$$

for all $n \geq 0$.

PROOF. Let $v_n = \lambda u_h(n\tau, S_nh)^{\alpha-1}\tau h^{-2} + D_n^+\tau h^{-1}$ and $d_n = D_n^-\tau h^{-2}$. For each $m \in \mathbf{N}$, we see

$$E_h^\lambda[|M_n - M_k|^{2m}]$$
$$= E_h^\lambda[\textstyle\sum_{i=k}^{n-1}\{(M_i - d_i - M_k)^{2m} - (M_i - M_k)^{2m}$$
$$+ v_i\textstyle\sum_{\ell=1}^{m} {}_{2m}C_{2\ell}(M_i - d_i - M_k)^{2m-2\ell}$$
$$+ d_i\textstyle\sum_{\ell=1}^{m} {}_{2m}C_{2\ell-1}(M_i - d_i - M_k)^{2m-2\ell+1}\}].$$

Put $m = 2$, then we have (4.2). By Lemma 3.2, we get

$$E_h^\lambda[|d_n^\pm(S_n)|^\alpha] \le \textstyle\sum_{j\in\mathbf{Z}} \dfrac{\{(u_j^n)^\alpha - (u_{j\pm1}^n)^\alpha\}_+^\alpha}{(u_j^n)^\alpha h^\alpha}\cdot u_j^n h$$
$$\le (\|(u_0^\alpha)_x\|_\infty c_h^{-\alpha})^{\alpha-1}\textstyle\sum_{j\in\mathbf{Z}} \dfrac{\{(u_j^n)^\alpha - (u_{j\pm1}^n)^\alpha\}_+}{(u_j^n)^{\alpha-1}|u_j^n - u_{j\pm1}^n|}\cdot|u_j^n - u_{j\pm1}^n|$$
$$\le \alpha(\|(u_0^\alpha)_x\|_\infty)^{\alpha-1}V(u_0)c_h^{-\alpha(\alpha-1)-1}$$

which implies (4.3).

Now we consider the convergence of the Markov chains. Let $P_{X_h}^\lambda$ be the probability measure on $(\mathscr{C}, \mathscr{F})$ defined in §2. We show the tightness of $\{P_{X_h}^\lambda | h > 0\}$.

PROOF OF PROPOSITION 2. It is enough to show

$$\lim_{N\to\infty}\limsup_{h\to0} P_{X_h}^\lambda(|X(0)| > N) = 0 \tag{4.4}$$

and

$$\lim_{\delta\downarrow0}\limsup_{h\to0} P_{X_h}^\lambda(\max_{\substack{0\le t,s\le T\\|t-s|<\delta}}|X(t) - X(s)| > \varepsilon) = 0 \tag{4.5}$$

for each $\varepsilon > 0$ and $T > 0$. By the definition of $P_{X_h}^\lambda$, we have

$$P_{X_h}^\lambda(|X(0)| > N) \le \frac{h}{N}E_h^\lambda[|S_0|].$$

By the assumption $(A.3)$, $\lim_{h\to0}E_h^\lambda[|S_0|] = \int_{\mathbf{R}}|x|u_0(x)dx$ is finite, which implies (4.4). Next we show (4.5). Fix $\varepsilon > 0$, $T > 0$ and $\delta > 0$. For each $h\in(0, (\varepsilon/2)\wedge h_1)$, put

$$\varepsilon' = (\varepsilon - 2h)/2h,\ N = [2T/\delta]$$
$$k_i = [i\delta/2\tau] \text{ and } m_i = [(i+2)\delta/2\tau] + 1.$$

Then we have

$$P_{X_h}^\lambda(\max_{\substack{0\le t,s\le T\\|t-s|<\delta}}|X(t) - X(s)| > \varepsilon) \le \textstyle\sum_{i=0}^{N-1}P_h^\lambda(\max_{k_i\le n\le m_i}|S_n - S_{k_i}| > \varepsilon')$$
$$\le \textstyle\sum_{i=0}^{N-1}P_h^\lambda(\max_{k_i\le n\le m_i}|M_n - M_{k_i}| > \varepsilon'/2$$
$$+ \textstyle\sum_{i=0}^{N-1}P_h^\lambda(\max_{k_i\le n\le m_i}|\sum_{\ell=k_i}^{n-1}D_\ell^-\tau h^{-1}| > \varepsilon'/2)$$

$$\leq \sum_{i=0}^{N-1}\{(2/\varepsilon')^4 E_h^\lambda[|M_{m_i}-M_{k_i}|^4]+(2/\varepsilon')^\alpha E_h^\lambda[(\sum_{\ell=k_i}^{m_i-1}|D_\ell^- \tau h^{-1}|)^\alpha]\}$$
$$\leq \sum_{i=0}^{N-1}\{(2/\varepsilon')^4\ K_1((m_i-k_i)^2\tau^2+(m_i-k_i)^{3/2}\tau^{3/2}h$$
$$+(m_i-k_i)\tau h^2)h^{-4}+(2/\varepsilon')^\alpha\ K_2(m_i-k_i)^\alpha\tau^\alpha h^{-\alpha}\}$$
$$\leq N\{(2/h\varepsilon')^4\ K_1((\delta+2\tau)^2+(\delta+2\tau)^{3/2}h+(\delta+2\tau)h^2)+(2/h\varepsilon')^\alpha\ K_2(\delta+2\tau)^\alpha\},$$

where $K_2=\alpha(\|(u_0^\alpha)_x\|_\infty)^{\alpha-1}V(u_0)2^{\alpha(\alpha-1)+1}$. It follows that

$$\limsup_{h\to 0}P_{X_h}^\lambda(\max_{\substack{0\leq t,s\leq T\\ |t-s|<\delta}}|X(t)-X(s)|>\varepsilon)\leq 2T\{(4/\varepsilon)^4K_1\delta+(4/\varepsilon)^\alpha K_2\delta^{\alpha-1}\},$$

which implies (4.5).

§5. Martingale problem

By the tightness of $\{P_{X_h}^\lambda\}$, there exist a sequence $\{h_n\}$ of $\{h\}$ and a probability measure P^λ on $(\mathscr{C},\mathscr{F})$ such that $P_{X_{h_n}}^\lambda$ converges to P^λ weakly. In this section we shall prove Propositions 3 and 4.

Lemma 5.1. *Assume* $(A.1)\sim(A.3)$. *Then we have*

$$P^\lambda(X(t)\in dx)=u(t,x)dx \tag{5.1}$$

and

$$E^\lambda[|X(t)|]\leq\int_{\mathbf{R}}|x|u_0(x)dx+((\|u_0\|_\infty)^{\alpha-1}t)^{1/2} \tag{5.2}$$

for all $t>0$ *and* $\lambda\in[0,1]$ *where* $u(t,x)$ *is a weak solution of* (1.1).

Proof. Firstly we show (5.1). For each $f\in C_0^1(\mathbf{R})$, we have

$$E^\lambda[f(X(t))]=\lim_{h\to 0}E_{X_h}^\lambda[f(X(t))]=\lim_{h\to 0}E_h^\lambda[f(X_h([t/\tau]\tau))]$$
$$=\lim_{h\to 0}\sum_{j\in\mathbf{Z}}f(jh)u_h([t/\tau]\tau,jh)h$$
$$=\lim_{h\to 0}\int_{\mathbf{R}}f(x)u_h([t/\tau])\tau,\ x)dx=\int_{\mathbf{R}}f(x)u(t,x)dx,$$

which implies (5.1). Next we show (5.2). Take an increasing sequence of functions $\{f_n\}\subset C_0^1(\mathbf{R})$ satisfying $f_n(x)\uparrow|x|$ as $n\to\infty$. By (5.1), we have

$$E^\lambda[f_n(X(t))]=E^1[f_n(X(t))]\leq\limsup_{h\to 0}E_h^1[|S_kh|]$$

where $k=[t/\tau]$. By (4.1) at $\lambda=1$. we have

$$E_h^1[|S_kh|]\leq E_h^1[|S_0h|]+(E_h^1[|M_kh|^2])^{1/2}$$
$$\leq\sum_{j\in\mathbf{Z}}|jh|u_0(jh)h/c_h+((\|u_0\|_\infty)^{\alpha-1}k\tau)^{1/2}c_h^{-(\alpha-1)/2}$$

It follows that

$$E^{\lambda}[|X(t)|] = \lim_{n\to\infty} E^{\lambda}[f_n(X(t))] \leq \int_{\mathbf{R}} |x| u_0(x)dx + ((\|u_0\|_{\infty})^{\alpha-1}t)^{1/2},$$

as was to be proved.

LEMMA 5.2. *Assume* $(A.1) \sim (A.4)$. *Then we have*

$$E^{\lambda}\left[\left|\frac{(u^{\alpha})_x(t, X(t))}{u(t, X(t))}\right|^{\alpha}\right] < \alpha(\|(u_0^{\alpha})_x\|_{\infty})^{\alpha-1}\mathrm{V}(u_0) \tag{5.3}$$

for all $t \geq 0$ *and* $\lambda \in [0, 1]$.

PROOF. By the uniform convergence of u_h, we have

$$E^{\lambda}\left[\left|\frac{(u^{\alpha})_x(t, X(t))}{u(t, X(t))}\right|^{\alpha} | u(t, X(t)) > \varepsilon\right]$$
$$\leq \limsup_{h\to 0} E_h^{\lambda}\left[\left|\frac{(u_h^{\alpha})_x(n\tau, S_n h)}{u_h(n\tau, S_n h)}\right|^{\alpha} | u_h(n\tau, S_n h) > \varepsilon/2\right] = (*)$$

for each $\varepsilon > 0$ where $n = [t/\tau]$. By Lemma 4.2, we have

$$(*) \leq \alpha(\|(u_0^{\alpha})_x\|_{\infty})^{\alpha-1}\mathrm{V}(u_0)$$

for each $\varepsilon > 0$. Letting $\varepsilon \to 0$ we have (5.3).

Lemmas 5.1 and 5.2 together imply Propostion 3.

PROOF OF PROPOSITION 4. Fix $f \in C_0^{\infty}(\mathbf{R} \to \mathbf{R})$ and $g \in C_b(\mathbf{R} \to \mathbf{R})$. We show that

$$\begin{aligned} E^{\lambda}&[f(X(t))g(X(s))] \\ &= E^{\lambda}\left[\left\{f(X(s)) + \int_s^t \frac{\lambda}{2}(\theta, X(\theta))^{\alpha-1} f''(X(\theta))d\theta \right.\right. \\ &\quad \left.\left. - \int_s^t \frac{1-\lambda}{2} \cdot \frac{(u^{\alpha})_x(\theta, X(\theta))}{u(\theta, X(\theta))} f'(X(\theta))d\theta\right\} g(X(s))\right] \end{aligned} \tag{5.4}$$

for each $t > s \geq 0$. By Lemma 4.1, we get

$$E^{\lambda}[f(X(t))g(X(s))]$$
$$= \lim_{h\to 0} E_h^{\lambda}\left[\left\{f(S_k h)) + \sum_{i=k}^{n-1} \frac{\lambda}{2} u_h(i\tau, S_i h)^{\alpha-1} f''(S_i h)\tau - \sum_{i=k}^{n-1} D_i^- f'(S_i h)\tau\right\} g(S_k h)\right],$$

where $n = [t/\tau]$ and $k = [s/\tau]$. By the uniform convergence of u_h, we have

$$\lim_{h\to 0} E_h^\lambda\left[\left\{f(S_kh) + \sum_{i=k}^{n-1} \frac{\lambda}{2} u_h(i\tau, S_ih)^{\alpha-1} f''(S_ih)\tau\right\} g(S_kh)\right]$$
$$= E^\lambda\left[\left\{f(X(s)) + \int_s^t \frac{\lambda}{2} u(\theta, X(\theta))^{\alpha-1} f''(X(\theta))d\theta\right\} g(X(s))\right].$$

Next we put

$$E_h^\lambda[\{\textstyle\sum_{i=k}^{n-1} D_i^- f'(S_ih)\tau\} g(S_kh)] = \textstyle\sum_{i=k}^{n-1}(J_i + \varepsilon_i)\tau,$$

where

$$J_i = E_h^\lambda[D_i^- f'(S_ih)\chi_{\{u(i\tau,S_ih)>\varepsilon\}} g(S_kh)]$$

and

$$\varepsilon_i = E_h^\lambda[D_i^- f'(S_ih)\chi_{\{u(i\tau,S_ih)\le\varepsilon\}} g(S_kh)]$$

for each $\varepsilon > 0$. Then we have

$$|\varepsilon_i| \le (1-\lambda)\textstyle\sum_{j\in\mathbf{Z}} |(u_j^i)^\alpha - (u_{j+1}^i)^\alpha| |f'(jh)| \chi_{\{u(i\tau,jh)\le\varepsilon\}} \|g\|_\infty$$
$$\le (1-\lambda)\|f'\|_\infty \|g\|_\infty \textstyle\sum_{j\in\mathbf{Z}} \alpha(u_j^i \vee u_{j+1}^i)^{\alpha-1} |u_j^i - u_{j+1}^i| \chi_{\{u(i\tau,jh)\le\varepsilon\}}.$$

By the uniform convergence of u_h, we get

$$\limsup_{h\to 0} |\textstyle\sum_{i=k}^{n-1} \varepsilon_i\tau| \le \|f'\|_\infty \|g\|_\infty |t-s| \alpha(2\varepsilon)^{\alpha-1} \mathrm{V}(u_0).$$

On the other hand, we have

$$\left| J_i - E_h^\lambda\left[-\frac{(u^\alpha)_x(i\tau, S_ih)}{u(i\tau, S_ih)} f'(S_ih)\chi_{\{u(i\tau,S_ih)>\varepsilon\}} g(S_kh)\right]\right| \le I_i^+ + I_i^-,$$

where

$$I_i^\pm = E_h^\lambda\left[\left|\frac{\{(u_{S_i}^i)^\alpha - u_{S_i\pm 1}^i)^\alpha\}_+}{u_{S_i}^i h} - \frac{\{\mp (u^\alpha)_x(i\tau, S_ih)\}_+}{u(i\tau, S_ih)}\right| |f'(S_ih)| \chi_{\{u(i\tau,S_ih)>\varepsilon\}} |g(S_kh)|\right].$$

By the $\|\ \|_N$-convergence of $(u_h^\alpha)_x$ and the uniform convergence of u_h, we get

$$\lim_{h\to 0} \textstyle\sum_{i=k}^{n-1}(I_i^+ + I_i^-)\tau = 0.$$

By the uniform continuity of u, we have

$$\lim_{h\to 0} \textstyle\sum_{i=k}^{n-1} J_i\tau$$
$$= \lim_{h\to 0} E_h^\lambda\left[-\frac{1-\lambda}{2}\int_s^t \frac{(u^\alpha)_x(\tau_\theta, X(\tau_\theta))}{u(\theta, X(\theta))} f'(X(\theta))\chi_{\{u(\theta,X(\theta))>\varepsilon\}} d\theta g(X(s))\right]$$

where $\tau_\theta = [\theta/\tau]\tau$. By the continuity of $(u^\alpha)_x$ with respect to the norm $\|\ \|_N$ (see Lemma 3.5), the above limit is

$$E^{\lambda}\left[-\frac{1-\lambda}{2}\int_s^t \frac{(u^{\alpha})_x(\theta, X(\theta))}{u(\theta, X(\theta))} f\,(X(\theta))\chi_{\{u(\theta,X(\theta))>\varepsilon\}} d\theta g(X(s))\right].$$

It follows that

$$\begin{aligned}
&E^{\lambda}[f(X(t))g(X(s))] \\
&= E^{\lambda}\left[\left\{f(X(s)) + \int_s^t \frac{\lambda}{2} u(\theta, X(\theta))^{\alpha-1} f''(X(\theta))d\theta\right.\right. \\
&\quad \left.\left. - \frac{1-\lambda}{2}\int_s^t \frac{(u^{\alpha})_x(\theta, X(\theta))}{u(\theta, X(\theta))} f'(X(\theta))\chi_{\{u(\theta,X(\theta))>\varepsilon\}} d\theta\right\} g(X(s))\right] + \lim_{h\to 0}\textstyle\sum_{i=k}^{n-1}\varepsilon_i\tau
\end{aligned}$$

for each $\varepsilon > 0$. Letting $\varepsilon \to 0$ we have (5.4), because

$$\begin{aligned}
&\left|E^{\lambda}\left[\int_s^t \frac{(u^{\alpha})_x(\theta,\ X(\theta))}{u(\theta,\ X(\theta))} f'(X(\theta))\chi_{\{u(\theta,X(\theta))>\varepsilon\}} d\theta g(X(s))\right]\right. \\
&\qquad\qquad \left. - E^{\lambda}\left[\int_s^t \frac{(u^{\alpha})_x(\theta,X(\theta))}{u(\theta,\ X(\theta))} f'(X(\theta))d\theta g(X(s))\right]\right| \\
&\le \|g\|_{\infty}\|f'\|_{\infty}\int_s^t d\theta\left(E^{\lambda}\left[\left|\frac{(u^{\alpha})_x(\theta,\ X(\theta))}{u(\theta,\ X(\theta))}\right|^{\alpha}\right]\right)^{1/\alpha}\left(E^{\lambda}[\chi_{\{u(\theta,X(\theta))\le\varepsilon\}}]\right)^{(\alpha-1)/\alpha}
\end{aligned}$$

$\to 0$ (as $\varepsilon \to 0$). Using the same method as above, we have the same formula as (5.4) for $g(X(s))g_1(X(s_1))\cdots g_m(X(s_m))$ in place of $g(X(s))$ for any $s_1,\cdots, s_m \in [0, s)$ and $g_1,\ldots, g_m \in C_b(\mathbf{R}\to\mathbf{R})$. Therefore we have

$$\begin{aligned}
&E^{\lambda}[f(X(t))|\mathscr{F}_s^{\lambda}] \\
&= E^{\lambda}\left[\left\{f(X(s)) + \int_s^t \frac{\lambda}{2} u(\theta,\ X(\theta))^{\alpha-1} f''(X(\theta))d\theta\right.\right. \\
&\quad \left.\left. - \frac{1-\lambda}{2}\int_s^t \frac{(u^{\alpha})_x(\theta,\ X(\theta))}{u(\theta,\ X(\theta))} f'(X(\theta))d\theta\right\}\middle|\mathscr{F}_s^{\lambda}\right],
\end{aligned}$$

for all $t > s \ge 0$. Thus the process (2.9) is an $\{\mathscr{F}_t^{\lambda}\}$-martingale on $(\mathscr{C},\ \mathscr{F},\ P^{\lambda})$.

PROOF OF COROLLARY OF PROPOSITION 4. We prove the stochastic differential equation (2.10). Put

$$D(t) = -\frac{1-\lambda}{2}\int_0^t \frac{(u^{\alpha})_x(s,\ X(s))}{u(s,\ X(s))} ds$$

and

$$M(t) = X(t) - X(0) - D(t),$$

then $M(t)$ is an $\{\mathscr{F}_t^{\lambda}\}$-martingale on $(\mathscr{C},\ \mathscr{F},\ P^{\lambda})$ satisfying

$$E^{\lambda}[|M(t)-M(s)|^2]=E^{\lambda}\left[\int_s^t \lambda u(\theta,\ X(\theta))^{\alpha-1}d\theta\right]$$

and

$$E^{\lambda}[|M(t)|^4]\leq 3(\|u_0\|_{\infty})^{2(\alpha-1)}t^2.$$

If $\lambda\neq 0$, then

$$B(t)\equiv\int_0^t \{\lambda u(s,\ X(s))^{\alpha-1}\}^{-1/2}dM(s)$$

is an $\{\mathscr{F}_t^{\lambda}\}$-Brownian motion on $(\mathscr{C},\ \mathscr{F},\ P^{\lambda})$ and

$$M(t)=\int_0^t \{\lambda u(s,\ X(s))^{\alpha-1}\}^{1/2}dB(s),$$

which implies (2.10) for each $\lambda\in(0,\ 1]$. If $\lambda=0$, then, by the equation (4.1), we have

$$\begin{aligned}
&E^0\left[\left|X(t)-X(0)+\frac{1}{2}\int_0^t\frac{(u^{\alpha})_x(s,\ X(s))}{u(s,\ X(s))}ds\right|^2\right]\\
&=\lim_{h\to 0}E_h^0\left[\left|S_{[t/\tau]}h-S_0h+\textstyle\sum_{i=1}^{[t/\tau]-1}\frac{1}{2}(d_i^+(S_i)-d_i^-(S_i))\tau\right|^2\right]\\
&=\lim_{h\to 0}E_h^0[\textstyle\sum_{i=0}^{[t/\tau]-1}(D_i^+h-(D_i^-)^2\tau)\tau]\\
&\leq\lim_{h\to 0}2t\{\alpha(\|(u_0^{\alpha})_x\|_{\infty})^{\alpha-1}\mathrm{V}(u_0)\}^{1/\alpha}h=0.
\end{aligned}$$

It follows that the process X on $(\mathscr{C},\ \mathscr{F},\ P^0)$ satisfies (2.10) with probability 1.

§6. Markov property

In this section we show that the process X on $(\mathscr{C},\ \mathscr{F},\ P^{\lambda})$ is a Markov process. Let p_h^{λ} and S_n be as in §2. Firstly we prepare the following.

LEMMA 6.1. *Let f be a function of bounded variation on* **R** *into* **R**. *Fix an integer $n>0$. Put*

$$p_m^k=E_h^{\lambda}[f(S_nh)|S_k=m]$$

for $k\leq n$ and $m\in\mathbf{Z}$. Then we have

$$\textstyle\sum_{m\in\mathbf{Z}}|p_m^k-p_{m-1}^k|\leq\mathrm{V}(f)\tag{6.1}$$

for all $0\leq k\leq n$.

PROOF. By the definition of p_m^k, it satisfies the following backward difference equation

$$\frac{p_m^{k-1}-p_m^k}{\tau}=\frac{\lambda}{2}(u_m^{k-1})^\alpha\frac{p_{m+1}^k-2p_m^k+p_{m-1}^k}{h^2}$$
$$+\frac{1-\lambda}{2}\left\{a_m^{k-1}\frac{p_{m+1}^k-p_m^k}{h}+b_m^{k-1}\frac{p_{m-1}^k-p_m^k}{h}\right\},\ k\le n,\ m\in\mathbf{Z},$$

and

$$p_m^n=f(mh),\ m\in\mathbf{Z},$$

where

$$u_m^k=P_h^\lambda(S_k=m)h^{-1}$$

and

$$a_m^k=\{(u_m^k)^\alpha-(u_{m+1}^k)^\alpha\}_+/(u_m^k h),\quad b_m^k=\{(u_m^k)^\alpha-(u_{m-1}^k)^\alpha\}_+/(u_m^k h).$$

Put $c_m^k=p_{m+1}^k-p_m^k$, then we have

$$c_m^{k-1}=\left[1-\frac{\lambda}{2}\{(u_{m+1}^{k-1})^{\alpha-1}+(u_m^{k-1})^{\alpha-1}\}r-\frac{1-\lambda}{2}\{a_m^{k-1}+b_{m+1}^{k-1}\}rh\right]c_m^k$$
$$+\left\{\frac{\lambda}{2}(u_{m+1}^{k-1})^{\alpha-1}+\frac{1-\lambda}{2}a_{m+1}^{k-1}h\right\}rc_{m+1}^k+\left\{\frac{\lambda}{2}(u_m^{k-1})^{\alpha-1}+\frac{1-\lambda}{2}b_m^{k-1}h\right\}rc_{m-1}^k$$

where $r=\tau h^{-2}$. Using the estimate $0\le(u_m^k)^{\alpha-1}\tau h^{-2}<1$, we get

$$\textstyle\sum_{m\in\mathbf{Z}}|c_m^{k-1}|\le\sum_{m\in\mathbf{Z}}|c_m^k|\le\sum_{m\in\mathbf{Z}}|c_m^n|=\sum_{m\in\mathbf{Z}}|f((m+1)h)-f(mh)|\le \mathrm{V}(f)$$

which is (6.1).

Proof of Propsition 5. For simplicity of the notation, we write $h>0$ in stead of $h_n>0$ in §5. Fix $f\in C_0^\infty(\mathbf{R})$ and $t>s\ge 0$. For each $h>0$, let

$$p_m=E_h^\lambda[f(S_n h)|S_k=m]$$

where $n=[t/\tau]$ and $k=[s/\tau]$. By Lemma 6.1, we see

$$\textstyle\sum_{m\in\mathbf{Z}}|p_{m+1}-p_m|\le\mathrm{V}(f).$$

Let p_h be the continuous function on $\mathbf{R}$ defined by

$$p_h(x)=p_{[x/h]}+(x/h-[x/h])(p_{[x/h]+1}-p_{[x/h]}).$$

Then we have

$$\mathrm{V}(p_h)\le\mathrm{V}(f).$$

By Lemma 3.3, there exist a function $p(x)$ and a sequence $\{h_n\}$ of $\{h\}$ such that

$$\lim_{n\to\infty}\int_{-N}^{N}|p_{h_n}(x)-p(x)|dx=0,$$

and

$$\lim_{\varepsilon \to 0} \int_{-N}^{N} |p(x + \varepsilon) - p(x)| dx = 0$$

for each $N > 0$. Let g be a bounded continuous function on $\mathbf{R}$ with compact support and $g_1, \cdots, g_m$ be bounded continuous functions on $\mathbf{R}$. For any $s > s_1 > \cdots > s_m \geq 0$, we have

$$E^{\lambda}[f(X(t))g(X(s))g_1(X(s_1)) \cdots g_m(X(s_m))]$$
$$= \lim_{h \to 0} E_h^{\lambda}[f(S_{\ell} h)g(S_k h)g_1(S_{k_1} h) \cdots g_m(S_{k_m} h)]$$

where $\ell = [t/\tau]$, $k = [s/\tau]$, $k_1 = [s_1/\tau], \cdots$, $k_m = [s_m/\tau]$. By the Markov property of $\{S_n\}$ and the convergence of p_{h_n}, the above limit is equal to

$$\lim_{n \to \infty} E_{h_n}^{\lambda}[p_{h_n}(S_k h_n)g(S_k h_n)g_1(S_{k_1} h_n) \cdots g_m(S_{k_m} h_n)]$$
$$= E^{\lambda}[p(X(s))g(X(s))g_1(X(s_1)) \cdots g_m(X(s_m))].$$

It follows that

$$E^{\lambda}[f(X(t))|X(s),\ X(s_1), \cdots,\ X(s_m)] = p(X(s)) = E^{\lambda}[f(X(t))|X(s)]$$

for any $s_1, s_2, \cdots, s_m \in [0, s)(m \in \mathbf{N})$ and $f \in C_0^{\infty}(\mathbf{R} \to \mathbf{R})$. Therefore the Markov property of the process X on $(\mathscr{C}, \mathscr{F}, P^{\lambda})$ has been proved. Next we calculate the generator of it. Fix $f \in C_0^{\infty}(\mathbf{R})$ and $t > 0$. By Proposition 4, we get

$$E^{\lambda}[f(X(t + \varepsilon))|X(t) = x]$$
$$= E^{\lambda}\Big[f(X(t)) + \int_t^{t+\varepsilon} \frac{\lambda}{2} f''(X(s))\, u(s,\ X(s))^{\alpha - 1} ds$$
$$- \frac{1 - \lambda}{2} \int_t^{t+\varepsilon} f'(X(s)) \frac{(u^{\alpha})_x(s, X(s))}{u(s, X(s))} ds \Big| X(t) = x\Big]$$

for each $\varepsilon > 0$. Therefore the generator of X is

$$(\mathscr{G}_t^{\lambda} f)(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{E^{\lambda}[f(X(t + \varepsilon))|X(t) = x] - f(x)\}$$
$$= \frac{\lambda}{2} f''(x) u(t, x)^{\alpha - 1} - \frac{1 - \lambda}{2} f'(x) \frac{(u^{\alpha})_x(t, x)}{u(t, x)},$$

as was to be proved.

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