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A method of iterations for quasi-linear evolution equations in nonreflexive Banach spaces

Kazuo KOBAYASI AND Nobuhiro SANEKATA

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Introduction

This paper is concerned with the Cauchy problem for an abstract quasi-linear evolution equation of the form

$$(CP)\qquad \begin{aligned} &du(t)/dt + A(t, u(t))u(t) = f(t, u(t)), \qquad 0 \le t \le T, \\ &u(0) = u_0, \end{aligned}$$

in a pair of Banach spaces $X \supset Y$. Here $A(t, w)$ is a linear operator in X depending on t and w varies on an open subset W of Y. In [6] T. Kato established an existence theorem for (CP) in the pair of reflexive Banach spaces $X \supset Y$. To construct C^1-solutions of (CP) in the space X it is assumed in [6] that the operators $A(t, w)$ are the negative generators of (C_0)-semigroups $\{\exp[-sA(t, w)]\}_{s\ge 0}$ on X such that $\|\exp[-sA(t, w)]\|_X \le e^{\beta s}$ for $s \ge 0$, $0 \le t \le T$, $w \in W$ and some constant β. In the subsequent papers [3] and [7] (see also [8] and the references therein) he extended the results in [6] to the case in which $\|\exp[-sA(t, w)]\|_X \le Me^{\beta s}$ for $s \ge 0$, $0 \le t \le T$, $w \in W$ and some constants $\beta \ge 0$ and $M \ge 1$.

To emphasize the two cases mentioned above we use the class of negative generators of (C_0)-semigroups. Given $M \ge 1$ and $\beta \ge 0$, $G(X, M, \beta)$ denotes the set of all negative generators A of (C_0)-semigroups $\{e^{-tA}\}$ on X satisfying $\|e^{-tA}\|_X \le Me^{\beta t}$ for $t \ge 0$. In [6] the family $\{A(t, w)\}$ is contained in $G(X, 1, \beta)$ and in [3] and [7] it is contained in $G(X, M, \beta)$. However in these papers except [7] the Banach spaces X and Y are assumed to be reflexive. In [7] the reflexivity condition is not assumed for X and Y, but only weak solutions of (CP) are constructed.

In [11] we eliminated the reflexivity assumption for X and Y and showed an existence theorem of C^1-solutions of (CP) in general Banach spaces under appropriate assumptions which were also employed in [6]. In particular, Theorems 4.5 and 5.2 in [11] shows that the conclusions of [8; Theorem A] remains valid without assuming the reflexivity of X and Y.

This paper is a continuation of the previous paper [11], and the purpose here is to extend the results given in [11] to the case in which the family $\{A(t, w)\}$ is contained in $G(X, M, \beta)$ for some $M \geq 1$ and $\beta \geq 0$. We will obtain C^1-solutions of (CP) in general Banach spaces under weaker assumptions than those of [11]. The proof of our Main Theorem is based on the theory of linear evolution equations advanced in [2], [4], [5] and [9] and a method of successive approximations is applied to construct C^1-solutions to (CP). Our approach is similar to but different from that of [6]. In [6] the convergence of the successive approximations is shown only in the X-norm, and then it is shown via the reflexivity of Y that the limit of the approximations remains in Y and is the unique solution of (CP). In this paper we prove the convergence in the stronger norm, the Y-norm, as well as in the X-norm, and by this result we show directly (without assuming the reflexivity) that the limit is the solution of (CP). This is the reason why we can obtain C^1-solutions of (CP) in general Banach spaces.

In [11] we employed the method of the difference approximations. This approach gives a more direct proof without going through linear theory, and also gives an extension of [1] to the case of general Banach spaces. However the proof given in [11] seems to be somewhat complicated, since the assumption of [11; $(A2)$] on the t-dependence of $A(t, w)$ is weaker than the corresponding one in [1]. Our argument in this paper is parallel to [11] though it is rather simple, and so this paper would point out the essentials of the previous work [11]. It is not difficult to extend our results here in the direction as mentioned in [8], but the proof will be more complicated.

Finally, we mention that a simple proof for the convergence of the difference approximations in the Y-norm is given in the forthcoming paper [12].

§1. Preliminary results on linear evolution equations

In this paper, we consider two real Banach spaces X and Y. We denote the norms of X and Y by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The symbol $B(Y, X)$ denotes the set of all bounded linear operators from Y to X. The operator norm of $A \in B(Y, X)$ is denoted by $\|A\|_{Y,X}$. For brevity in notation we write respectively $B(X)$ for $B(X, X)$, $B(Y)$ for $B(Y, Y)$, $\|A\|_X$ for $\|A\|_{X,X}$ and $\|A\|_Y$ for $\|A\|_{Y,Y}$ if there is no ambiguity. The domain of an operator A is denoted by $D(A)$. Throughout this paper we impose the following condition on the pair (X, Y) of Banach spaces:

(X) *Y is densely and continuously embedded in X. There is an isomorphism S of Y onto X.*

Under assumption (X), there is $c_0 > 0$ such that

$$\|y\|_X \leq c_0 \|y\|_Y \quad \text{for all } y \in Y. \tag{1.1}$$

In this section we summarize basic results on the Cauchy problem for the time-dependent linear evolution equation

$$(L) \qquad du(t)/dt + A(t)u(t) = f(t), \qquad 0 \leq t \leq T, \qquad u(0) = u_0$$

in X obtained in [2], [4], [5] and [9], where $T > 0$, $u_0 \in X$ and $f \in L^1(0, T; X)$. Our objective here is to seek the solution u of (L) satisfying

$$(1.2) \qquad u \in C([0, T]; Y) \cap C^1([0, T]; X).$$

Let $T > 0$ and let $\{A(t); 0 \leq t \leq T\}$ be a family of negative generators of (C_0)-semigroups $\{\exp[-sA(t)]\}_{s \geq 0}$ on X. The family $\{A(t); 0 \leq t \leq T\}$ is said to be *stable* if there exist constants M and β such that

$$(1.3) \qquad \| \prod_{j=1}^{k} \exp[-s_j A(t_j)] \|_X \leq \exp[\beta(s_1 + \cdots + s_k)]$$

for every finite family $s_j \geq 0$, $1 \leq j \leq k$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and $k \geq 1$. In (1.3) the operator product on the left is assumed to be *time-ordered*; namely, $\exp[-s_j A(t_j)]$ is on the left of $\exp[-s_i A(t_i)]$ if $t_j > t_i$. The pair (M, β) is called the *stability index* for $\{A(t); 0 \leq t \leq T\}$. The set of all stable families $\{A(t); 0 \leq t \leq T\}$ $(T > 0)$ in X with the stability index (M, β) is denoted by $S(X, M, \beta)$. At the beginning of this paper we have introduced a class $G(X, M, \beta)$ of negative generators of (C_0)-semigroups. It should be noted that each $A(t)$ belongs to the class $G(X, M, \beta)$, if $\{A(t); 0 \leq t \leq T\} \in S(X, M, \beta)$.

For the operators $\{A(t); 0 \leq t \leq T\}$ in (L) we assume (i), (ii) and (iii) below:

(i) *There exist constants M and β such that*

$$\{A(t); 0 \leq t \leq T\} \in S(X, M, \beta).$$

(ii) *There is a strongly measurable operator valued function $B(\cdot)$ on $[0, T]$ to $B(X)$ such that*

$$SA(t)S^{-1} = A(t) + B(t) \qquad \textit{for } t \in [0, T]$$

and that

$$\sup_{0 \leq t \leq T} \| B(t) \|_X \leq \lambda \qquad \textit{for some } \lambda > 0.$$

(iii) *$Y \subset D(A(t))$ for each $t \in [0, T]$ and $A(t)$ is strongly continuous in $B(Y, X)$ on $[0, T]$.*

Detailed explanations concerning conditions (i), (ii) and (iii) are seen in [4].

A major part of the study of (L) consists in constructing an *evolution operator* $\{U(t, s)\} \subset B(X)$ associated with $\{A(t)\}$ which may be formulated below.

THEOREM 1.1. *Under assumptions (X), (i), (ii) and (iii), there exists a unique family $\{U(t, s)\} \subset B(X)$ defined on the triangle*

$$\Delta = \{(t, s); 0 \le s \le t \le T\},$$

with the following properties:

(*a*) $U(t, s)$ *is strongly continuous in* $B(X)$ *on* Δ *and*

$$\| U(t, s) \|_X \le M e^{\beta(t-s)} \qquad \text{for } (t, s) \in \Delta.$$

(*b*) $U(t, s)U(s, r) = U(t, r)$ *and* $U(s, s) = 1$ *for* $(t, s) \in \Delta$ *and* $(s, r) \in \Delta$.

(*c*) $U(t, s)(Y) \subset Y$, $U(t, s)$ *is strongly continuous in* $B(Y)$ *on* Δ *and*

$$\| U(t, s) \|_Y \le \bar{M} e^{\bar{\beta}(t-s)} \qquad \text{for } (t, s) \in \Delta,$$

where $\bar{M} = M \| S \|_{Y,X} \| S^{-1} \|_{X,Y}$ *and* $\bar{\beta} = \lambda M + \beta$.

(*d*) $\partial U(t, s)/\partial t = -A(t)U(t, s)$ *and* $\partial U(t, s)/\partial s = U(t, s)A(s)$, *both of which exist in the strong sense in* $B(Y, X)$ *and are strongly continuous in* $B(Y, X)$ *on* Δ.

The family $\{U(t, s)\}$ obtained by Theorem 1.1 is called *the evolution operator generated by* $\{A(t)\}$. Theorem 1.1 was previously proved in [9] under a stronger assumption that $B(\cdot)$ is strongly continuous. However, the family $\{U(t, s)\}$ satisfying (*a*) and (*b*) of Theorem 1.1 can be obtained in the same way as in [9]. We here verify only the third assertion (*c*) referring to Dorroh [2]. The last assertion (*d*) is obtained from (*c*) via a standard argument.

We first introduce the solution $\{W(t, s)\} \subset B(X)$ of the Volterra-type integral equation

$$W(t, s) = U(t, s) - \int_s^t W(t, \sigma)B(\sigma)U(\sigma, s)d\sigma. \tag{1.4}$$

The integral in (1.4) is taken with respect to the strong topology of $B(X)$. The solution $W(t, s)$ of (1.4) is unique and is given by

$$W(t, s) = \textstyle\sum_{n=0}^{\infty}(-1)^n \cdot K_n(t, s). \tag{1.5}$$

Here $\{K_n(t, s)\} \subset B(X)$ is defined by

$K_0(t, s) = U(t, s)$ and

$$K_n(t, s) = \int_s^t U(t, \sigma)B(\sigma)K_{n-1}(\sigma, s)d\sigma \left(= \int_s^t K_{n-1}(t, \tau)B(\tau)U(\tau, s)d\tau \right)$$

for $n \ge 1$. Each K_n: $\Delta \to B(X)$ is strongly continuous and satisfies

$$\| K_n(t, s) \|_X \le M \frac{[\lambda M(t-s)]^n}{n!} e^{\beta(t-s)}$$

for $(t, s) \in \Delta$. This implies that the series in (1.5) converges uniformly on Δ with respect to $B(X)$-norm and that

$$\|W(t, s)\|_X \leq M\exp[(\lambda M + \beta)(t - s)] \tag{1.6}$$

for $(t, s) \in \Delta$. By (1.4) and (1.5) we have

$$W(t, s) = U(t, s) - \int_s^t U(t, \sigma)B(\sigma)W(\sigma, s)d\sigma. \tag{1.7}$$

Let us consider another integral equation

$$Z(t, s) = S^{-1}U(t, s) - \int_s^t Z(t, \sigma)B(\sigma)U(\sigma, s)d\sigma. \tag{1.8}$$

(1.8) has a unique solution $Z = S^{-1}W$. On the other hand it is shown (for instance see [2]) that $Z = US^{-1}$ also satisfies (1.8). Therefore, we have

$$U(t, s)S^{-1} = S^{-1}W(t, s) \quad \text{and} \quad \|U(t, s)\|_Y \leq \bar{M}e^{\bar{\beta}(t-s)} \tag{1.9}$$

by (1.6). Now the relations (1.9) imply the assertion (*c*) of Theorem 1.1. These are proved in [2] and [9]. See also [4], [5] and [11; Section 3].

With the evolution operator $\{U(t, s)\}$, the solution u of (L) is formally given by

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s)ds, \qquad 0 \leq t \leq T. \tag{1.10}$$

The function u on $[0, T]$ defined by (1.10) is called a *mild solution* of (L). For the mild solution u to be differentiable, we need further assumptions on u_0 and f.

THEOREM 1.2. *Suppose that conditions* (X), (*i*), (*ii*) *and* (*iii*) *are satisfied, and that* $u_0 \in Y$ *and* $f \in L^1(0, T; Y) \cap C([0, T]; X)$. *Then the mild solution* u *of* (L) *is a unique solution of* (L) *satisfying* (1.2).

Theorem 1.2 follows immediately from Theorem 1.1 and (1.10). Before closing this section, we state here the following lemma which will be used in Section 3.

LEMMA 1.3. *Let* $u_0 \in Y$ *and* $f \in L^1(0, T; Y) \cap C([0, T]; X)$. *Then the solution* u *of* (L) *satisfies*

$$Su(t) = U(t, 0)Su_0 + \int_0^t U(t, s)\{Sf(s) - B(s)Su(s)\}ds.$$

PROOF. By (1.9) and (1.10), we have

$$\begin{aligned} u(t) &= U(t, 0)u_0 + \int_0^t U(t, s)f(s)ds \\ &= S^{-1}\left\{W(t, 0)Su_0 + \int_0^t W(t, s)Sf(s)ds\right\}. \end{aligned} \tag{1.11}$$

By (1.7) and (1.11), we have

$$\begin{aligned} & W(t, 0)Su_0 \\ & = U(t, 0)Su_0 - \int_0^t U(t, \sigma)B(\sigma)W(\sigma, 0)Su_0\, d\sigma \\ & = U(t, 0)Su_0 - \int_0^t U(t, \sigma)B(\sigma)\{Su(\sigma) - \int_0^\sigma W(\sigma, s)Sf(s)ds\}\, d\sigma \end{aligned}$$

and

$$\begin{aligned} & \int_0^t W(t, s)Sf(s)ds \\ & = \int_0^t U(t, s)Sf(s)ds - \int_0^t \left\{ \int_s^t U(t, \sigma)B(\sigma)W(\sigma, s)Sf(s)d\sigma \right\} ds \\ & = \int_0^t U(t, s)Sf(s)ds - \int_0^t d\sigma \int_0^\sigma U(t, \sigma)B(\sigma)W(\sigma, s)Sf(s)ds. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} Su(t) & = W(t, 0)Su_0 + \int_0^t W(t, s)Sf(s)ds \\ & = U(t, 0)Su_0 + \int_0^t U(t, s)\{Sf(s) - B(s)Su(s)\}\, ds. \end{aligned}$$

Q.E.D.

§2. Basic hypotheses and main result

In this section we set up basic hypotheses on the operators appearing in the Cauchy problem (CP) along with some comments. We consider two real Banach spaces X and Y satisfying condition (X) described in Section 1. For the operators $A(t, w)$ in (CP) we assume the three conditions $(A1)$, $(A2)$ and $(A3)$ below.

$(A1)$ *There exist an open subset W of Y and $T_0 > 0$ satisfying the following properties: $A(t, w)$ is a linear operator in X defined for each $t \in [0, T_0]$ and $w \in W$. For each $\rho \geq 0$ there are $M \geq 1$ and $\beta \geq 0$ such that*

$$\{A(t, v(t)); 0 \leq t \leq T_0\} \in S(X, M, \beta)$$

for all $v(\cdot) \in D_\rho$.

Here D_ρ is defined by

$$D_\rho = \{v \in C([0, T_0]; W);\ \|v(t) - v(s)\|_X \leq \rho |t - s| \quad \text{for } 0 \leq s < t \leq T_0\}.$$

By $(A1)$, each operator $A(t, w)$ is the negative generator of a (C_0)-semigroup $\{\exp[-sA(t, w)]\}_{s\geq 0}$ on X. In what follows, we always consider W as a metric space with respect to the metric d defined by

$$d(w, z) = \|w - z\|_Y \qquad \text{for } w, z \in W.$$

We will find the solutions u of (CP) satisfying

$$u \in C([0, T]; W) \cap C^1([0, T]; X) \tag{2.1}$$

for some $T \in (0, T_0]$ which may depend on the initial value $u_0 \in W$ of (CP).

In [3] and [7] it is assumed that for each $t \in [0, T_0]$ and $w \in W$ there is a norm $\|\cdot\|_{(t,w)}$ of X which is equivalent to $\|\cdot\|_X$ with the following properties:

$(N1)$ $\qquad \|x\|_X \leq \lambda_X \|x\|_{(t,w)}$ *and* $\|x\|_{(t,w)} \leq \lambda_X \|x\|_X$

for $t \in [0, T_0]$, $w \in W$ *and* $x \in X$.

$(N2)$ $\qquad \|x\|_{(t,w)} \leq \|x\|_{(s,z)} \cdot \exp[\mu(\|w - z\|_X + |t - s|)]$

for $t, s \in [0, T_0]$, $w, z \in W$ *and* $x \in X$.

Here $\lambda_X \geq 1$ and $\mu \geq 0$ are constants independent of t, s, w, z and x. With these equivalent norms the following condition is assumed in [3] and [7]:

$(A1')$ *There is* $\beta \geq 0$ *such that*

$$A(t, w) \in G(X_{(t,w)}, 1, \beta) \qquad \textit{for } t \in [0, T_0] \textit{ and } w \in W,$$

where $X_{(t,w)}$ denotes the Banach space X with the norm $\|\cdot\|_{(t,w)}$. We will show that these conditions imply $(A1)$.

PROPOSITION 2.1. *Let W be an open subset of Y and let $T_0 > 0$. Suppose that* $(N1)$, $(N2)$ *and* $(A1')$ *are satisfied. Then for every $\rho \geq 0$ and $v \in D_\rho$, we have*

$$\{A(t, v(t)); 0 \leq t \leq T_0\} \in S(X, \lambda_X^2 \exp[\mu(\rho + 1)T_0], \beta).$$

PROOF. Let $\rho \geq 0$, $v \in D_\rho$ and $x \in X$. Then for every finite family

$$s_j \geq 0,\ 1 \leq j \leq k \text{ and } 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T_0,\ k \geq 1,$$

we have

$$\begin{aligned} &\|\{\textstyle\prod_{j=1}^k \exp[-s_j A(t_j, v(t_j))]\} x\|_X \\ &\leq \lambda_X \|\{\textstyle\prod_{j=1}^k \exp[-s_j A(t_j, v(t_j))]\} x\|_k, \end{aligned}$$

by $(N1)$, where we write $\|\cdot\|_j$ for $\|\cdot\|_{(t_j, v(t_j))}$, $1 \leq j \leq k$. By $(A1')$ and $(N2)$, we have

$$\begin{aligned} &\|\{\textstyle\prod_{j=1}^k \exp[-s_j A(t_j, v(t_j))]\} x\|_k \\ &\leq e^{\beta s_k} \|\{\textstyle\prod_{j=1}^{k-1} \exp[-s_j A(t_j, v(t_j))]\} x\|_k \end{aligned}$$

$$\leq \{\exp[\beta s_k + \mu(\|v(t_k) - v(t_{k-1})\|_X + |t_k - t_{k-1}|)]\}$$
$$\times \|\{\textstyle\prod_{j=1}^{k-1} \exp[-s_j A(t_j, v(t_j))]\}x\|_{k-1}$$
$$\leq \{\exp[\beta s_k + \mu(\rho + 1)(t_k - t_{k-1})]\}$$
$$\times \|\{\textstyle\prod_{j=1}^{k-1} \exp[-s_j A(t_j, v(t_j))]\}x\|_{k-1},$$

where $\prod_{j=1}^{k-1} \exp[\cdots] = 1$ if $k = 1$. Therefore, we have

$$\|\{\textstyle\prod_{j=1}^{k} \exp[-s_j A(t_j, v(t_j))]\}x\|_X$$
$$\leq \lambda_X\{\exp[\mu(\rho + 1)(t_k - t_1)]\} \cdot \{\exp[(s_1 + \cdots + s_k)\beta]\} \cdot \|x\|_1$$
$$\leq \lambda_X^2\{\exp[\mu(\rho + 1)T_0]\} \cdot \{\exp[(s_1 + \cdots + s_k)\beta]\} \cdot \|x\|_X. \qquad \text{Q.E.D.}$$

The second assumption $(A2)$ below is stronger than the assumption employed in [7]. In fact, the assumption imposed in [7] ensures the existence of weak solutions. However, the condition $(A2)$ is essential for obtaining C^1-solutions.

$(A2)$ *For each $w \in W$, there is a strongly measurable operator valued function $B(\cdot, w)$ on $[0, T_0]$ into $B(X)$ such that*

$$SA(t, w)S^{-1} = A(t, w) + B(t, w) \qquad \text{for } t \in [0, T_0] \text{ and } w \in W.$$

There are positive numbers λ_B and μ_B such that

$$\|B(t, w)\|_X \leq \lambda_B \quad \text{and} \quad \|B(t, w) - B(t, z)\|_X \leq \mu_B \|w - z\|_Y \tag{2.2}$$

for $t \in [0, T_0]$ and $w, z \in W$.

By $(A1)$ and $(A2)$, the semigroup $\{\exp[-sA(t, w)]\}$ leaves Y invariant and the restriction of $\{\exp[-sA(t, w)]\}$ to Y (which will also be denoted by the same symbol) forms a (C_0)-semigroup on Y. The infinitesimal generator of this restriction on Y is the *part of* $A(t, w)$ *in* Y in the sense of [4; p.242], which will also be denoted by the same symbol $A(t, w)$. Furthermore, for $\rho \geq 0$ and $v \in D_\rho$ we have

$$\{A(t, v(t));\ 0 \leq t \leq T_0\} \in S(Y, M\|S\|_{Y,X} \cdot \|S^{-1}\|_{X,Y}, \beta + M\lambda_B),$$

where (M, β) is the stability index for $\{A(t, v(t))\}$ in X determined by $(A1)$. See [4].

Assume for the moment that $(N1)$, $(N2)$, $(A1')$ and $(A2)$ are satisfied. Put

$$|y|_{(t,w)} \equiv \|Sy\|_{(t,w)} \qquad \text{for } y \in Y,\ w \in W \text{ and } t \in [0, T_0],$$

and let $Y_{(t,w)}$ be the Banach space Y equipped with the norm $|\cdot|_{(t,w)}$. Then it is easy to see that

$$(N1') \qquad \|y\|_Y \leq \lambda_Y |y|_{(t,w)} \qquad \text{and} \qquad |y|_{(t,w)} \leq \lambda_Y \|y\|_Y$$

for $t \in [0, T_0]$, $w \in W$ *and* $y \in Y$,

$$(N2') \qquad |y|_{(t,w)} \leq |y|_{(s,z)} \cdot \exp[\mu(\|w - z\|_X + |t - s|)]$$

for $t, s \in [0, T_0]$, $w, z \in W$ *and* $y \in Y$, *and*

$$(A2') \qquad A(t, w) \in G(Y_{(t,w)}, 1, \beta_Y) \qquad \text{for } t \in [0, T_0] \text{ and } w \in W,$$

where $\lambda_Y = \lambda_X \cdot \max\{\|S\|_{Y,X}, \|S^{-1}\|_{X,Y}\}$ and $\beta_Y = \beta + \lambda_B \lambda_X$. Instead of ($A1$) and ($A2$), it is assumed in [7] that the equivalent norms $\|\cdot\|_{(t,w)}$ and $|\cdot|_{(t,w)}$ exist for each $t \in [0, T_0]$ and $w \in W$, and that conditions ($N1$), ($N1'$), ($N2$), ($N2'$), ($A1'$) and ($A2'$) hold for some λ_X, λ_Y, μ, β and β_Y. Under these assumptions together with ($A3$), (2.5) and (2.6) below, the existence of *weak solutions* of (CP) is proved in [7]. Our condition ($A2$) is stronger than the corresponding ones in [7] in the sense described above, however, we obtain sharper results concerning C^1-solutions than those of [7] as mentioned in the Introduction.

The third assumption on $A(t, w)$ is concerned with the (t, w)-dependence of $A(t, w)$.

($A3$) *For each* $t \in [0, T_0]$ *and* $w \in W$, $D(A(t, w)) \supset Y$ (*and hence* $A(t, w) \in B(Y, X)$ *by the closed graph theorem*). *For each* $w \in W$, $A(\cdot, w)$ *is strongly continuous in* $B(Y, X)$ *on* $[0, T_0]$. *There is a positive number* μ_A *such that*

$$(2.3) \qquad \|A(t, w) - A(t, z)\|_{Y,X} \leq \mu_A \|w - z\|_X \qquad \text{for } t \in [0, T_0] \text{ and } w, z \in W.$$

From ($A3$), it follows that for any bounded subset B of W there is $c(B) > 0$ satisfying

$$(2.4) \qquad \|A(t, w)\|_{Y,X} \leq c(B) \qquad \text{for } t \in [0, T_0] \text{ and } w \in B.$$

We next make an assumption on the operators $f(t, w)$.

(f) *For each* $t \in [0, T_0]$ *and* $w \in W$, $f(t, w)$ *is defined and belongs to* Y. *For each* $w \in W$, $f(\cdot, w)$ *is continuous in* X *on* $[0, T_0]$ *and is strongly measurable in* Y. *There are positive numbers* λ_f, μ_f *and* $\bar{\mu}_f$ *such that*

$$(2.5) \qquad \|f(t, w)\|_Y \leq \lambda_f,$$

$$(2.6) \qquad \|f(t, w) - f(t, z)\|_X \leq \mu_f \|w - z\|_X \qquad \text{and}$$

$$\|f(t, w) - f(t, z)\|_Y \leq \bar{\mu}_f \|w - z\|_Y$$

for $t \in [0, T_0]$ *and* $w, z \in W$.

Now our main result in this paper is stated as follows:

MAIN THEOREM. *Suppose that conditions* (X), ($A1$) *through* ($A3$) *and* (f) *are satisfied. Then for each initial value* $u_0 \in W$, *there is a* $T \in (0, T_0]$ *such that* (CP) *has a unique solution* u *satisfying* (2.1).

To prove the Main Theorem, we prepare some notations and lemmas. For any initial value $u_0 \in W$ of (CP), we choose $r_0 > 0$ and $\phi_0 \in W$ so that

$$u_0 \in B(\phi_0, r_0) \subset W,$$

where $B(\phi_0, r_0) = \{w \in Y;\ \|w - \phi_0\|_Y \leq r_0\}$, and then we put

$$\rho_0 = c_0 \lambda_f + c(B(\phi_0, r_0))(\|\phi_0\|_Y + r_0).$$

Here, c_0, λ_f and $c(B(\phi_0, r_0))$ are the constants as mentioned in (1.1), (2.5) and (2.4), respectively. Let (M, β) be the stability index given by $(A1)$ for $\rho = \rho_0$. We choose $r > 0$ and $\phi \in W$ so that

$$\begin{aligned} &\|u_0 - \phi\|_Y < r/(Me^{\beta T_0}\|S\|_{Y,X} \cdot \|S^{-1}\|_{X,Y}) \quad (\leq r) \quad \text{and} \\ &B(\phi, r)(\equiv \{w \in Y;\ \|w - \phi\|_Y \leq r\}) \subset B(\phi_0, r_0). \end{aligned} \tag{2.7}$$

In the rest of this paper we fix

$$u_0 \in W,\ \rho_0 > 0,\ (M, \beta) \text{ and } B(\phi, r) \tag{2.8}$$

as defined above. Let E be the set of all v satisfying

$$v \in C([0, T]; Y),\ v(t) \in B(\phi, r) \text{ for all } t \in [0, T] \text{ and} \tag{2.9}$$

$$\|v(t) - v(s)\|_X \leq \rho_0 |t - s| \qquad \text{for } 0 \leq s \leq t \leq T. \tag{2.10}$$

Here $T \in (0, T_0]$ will be determined after Lemma 3.5. With this T we define the triangle Δ appearing in Theorem 1.1. It should be noted that E is considered as a suset of D_{ρ_0}. For brevity in notation we write

$$A^v(t) = A(t, v(t)),\ B^v(t) = B(t, v(t)) \text{ and } f^v(t) = f(t, v(t))$$

for each $v \in E$ and $t \in [0, T]$.

Lemma 2.2. *The families* $\{A^v(t); 0 \leq t \leq T, v \in E\}$ *and* $\{f^v; v \in E\}$ *satisfy the following*:

(*a*) $\{A^v(t); 0 \leq t \leq T\} \in S(X, M, \beta)$ *for all* $v \in E$.

(*b*) $SA^v(t)S^{-1} = A^v(t) + B^v(t)$, *and* $\|B^v(t)\|_X \leq \lambda_B$ *for all* $t \in [0, T]$ *and* $v \in E$. $B^v(\cdot)$ *is strongly measurable on* $[0, T]$ *into* $B(X)$ *for each* $v \in E$.

(*c*) $Y \subset D(A^v(t))$ *for each* $t \in [0, T]$ *and* $v \in E$, *and* $A^v(\cdot)$ *is strongly continuous on* $[0, T]$ *into* $B(Y, X)$.

(*d*) $f^v \in L^\infty(0, T; Y) \cap C([0, T]; X)$ *and* $\|f^v(t)\|_Y \leq \lambda_f$ *for all* $t \in [0, T]$ *and* $v \in E$.

Here (M, β) is the stability index in (2.8). Lemma 2.2 follows immediately from $(A1)$, $(A2)$, $(A3)$ and (f). By Lemma 2.2, we can apply Theorem 1.1 to the family $\{A^v(t)\}$.

COROLLARY 2.3. *Let (X), $(A1)$, $(A2)$, $(A3)$ and (f) hold. Then for each $v \in E$ there exists a unique evolution operator*

$$\{U^v(t, s); (t, s) \in \Delta\} \subset B(X) \cap B(Y)$$

generated in the sense of Theorem 1.1 *by* $\{A^v(t); 0 \leq t \leq T\}$. *In particular, we have*

$$\| U^v(t, s) \|_X \leq M e^{\beta(t-s)} \qquad \text{and} \qquad \| U^v(t, s) \|_Y \leq \bar{M} e^{\bar{\beta}(t-s)}$$

for $(t, s) \in \Delta$ *and* $v \in E$, *where* $\bar{M} = M \| S \|_{Y,X} \cdot \| S^{-1} \|_{X,Y}$ *and* $\bar{\beta} = \beta + M\lambda_B$.

It should be noted that (M, β) and $(\bar{M}, \bar{\beta})$ are independent of $(t, s) \in \Delta$ and $v \in E$. By Corollary 2.3, we have:

LEMMA 2.4. *For each* $v, w \in E$, $y \in Y$ *and* $(t, s) \in \Delta$, *we have*

$$\begin{aligned} &\| U^v(t, s)y - U^w(t, s)y \|_X \\ (2.11) \qquad &\leq \mu_A M \bar{M} e^{\bar{\beta}(t-s)} \| y \|_Y \cdot \int_s^t \| v(\sigma) - w(\sigma) \|_X \, d\sigma. \end{aligned}$$

PROOF. By Corollary 2.3, we have

$$U^v(t, s)y - U^w(t, s)y = \int_s^t U^v(t, \sigma)[A^w(\sigma) - A^v(\sigma)] U^w(\sigma, s)y \, d\sigma,$$

which is obtained by differentiating $U^v(t, \sigma) U^w(\sigma, s)y$ in σ and then integrating the resultant derivative over $\sigma \in [s, t]$. On the other hand, by (2.3) and Corollary 2.3, we have

$$\begin{aligned} &\| U^v(t, \sigma)[A^w(\sigma) - A^v(\sigma)] U^w(\sigma, s)y \|_X \\ &\leq \mu_A M \bar{M} e^{\bar{\beta}(t-s)} \| y \|_Y \cdot \| v(\sigma) - w(\sigma) \|_X. \end{aligned}$$

Therefore, we have (2.11). Q.E.D.

In the rest of this section, we give an outline of the proof of the Main Theorem. Firstly, we construct approximate solutions $\{u^n\}$ of (CP). These are defined inductively by $u^0 = u_0$ (the initial value of (CP)) and

$$u^n(t) = U_{n-1}(t, 0)u_0 + \int_0^t U_{n-1}(t, s) f(s, u^{n-1}(s)) \, ds, \qquad 0 \leq t \leq T, \qquad n \geq 1,$$

where $\{U_n(t, s)\}$ is the evolution operator generated by $\{A(t, u^n(t))\}$ for each $n \geq 0$. We can choose $T \in (0, T_0]$ so that $\{u^n\} \subset E$. We then show that the limit

$$u(t) \equiv \lim_{n \to \infty} u^n(t)$$

exists in $C([0, T]; X)$. It should be noted that if X and Y are both reflexive, then

$$(2.12) \qquad u(t) \in B(\phi, r) \qquad \text{for each } t \in [0, T],$$

and u satisfies (CP) since $B(\phi, r)$ is closed in X. See [6]. However, in general, $B(\phi, r)$ is not closed in X and so one cannot conclude (2.12) at once. Therefore we need more detailed argument.

Secondly, we show the strong convergence in X of $\{U_n(t, s)\}$:

$$(2.13) \qquad \bar{U}(t, s)x \equiv \lim_{n\to\infty} U_n(t, s)x \qquad \text{for } (t, s) \in \Delta \text{ and } x \in X.$$

After the Main Theorem is proved, we will see that $\{\bar{U}(t, s)\}$ defined by (2.13) is the evolution operator generated by $\{A(t, u(t))\}$.

Thirdly, we consider the integral equation

$$(I)\ S\bar{u}(t) = \bar{U}(t, 0)Su_0 + \int_0^t \bar{U}(t, s)\{Sf(s, \bar{u}(s)) - B(s, \bar{u}(s))S\bar{u}(s)\}ds, \quad 0 \le t \le T.$$

In view of Lemma 1.3, we see that the solution of (CP) must be the solution $\bar{u}$ of (I) if it exists. In fact, we find the solution $\bar{u}$ of (I) satisfying

$$\bar{u} \in C([0, T]; B(\phi, r)),$$

and then we obtain

$$\lim_{n\to\infty} \sup_{0\le t\le T} \| u^n(t) - \bar{u}(t) \|_Y = 0.$$

It follows immediately that $\bar{u} = u$ is the solution of (CP) satisfying (2.1).

§ 3. Proof of Main Theorem

We begin by defining an operator Ψ from E into $C([0, T]; Y)$. We put

$$[\Psi v](t) \equiv U^v(t, 0)u_0 + \int_0^t U^v(t, s)f^v(s)\, ds$$

for $0 \le t \le T$ and $v \in E$. By Corollary 2.3, Ψ is a mapping from E into $C([0, T]; Y) \cap C^1([0, T]; X)$, and $u = \Psi v$ satisfies the linear evolution equation

$$(L^v) \qquad du(t)/dt + A^v(t)u(t) = f^v(t), \qquad 0 \le t \le T, \qquad u(0) = u_0.$$

LEMMA 3.1. *There is $T \in (0, T_0]$ such that $\Psi v \in E$ for $v \in E$.*

PROOF. Let $v \in E$ and $u = \Psi v$. Then, Corollary 2.3 and Lemma 1.3 together imply $u \in C([0, T]; Y) \cap C^1([0, T]; X)$ and the relation

$$Su(t) = U^v(t, 0)Su_0 + \int_0^t U^v(t, s)\{Sf^v(s) - B^v(s)Su(s)\}ds.$$

Therefore, we have

$$\begin{aligned}
\|u(t)-\phi\|_Y &\le \|S^{-1}\|_{X,Y}\cdot\|Su(t)-S\phi\|_X\\
&\le \|S^{-1}\|_{X,Y}\cdot\|U^v(t,0)Su_0-S\phi\|_X\\
&\quad+\|S^{-1}\|_{X,Y}\cdot\int_0^t\|U^v(t,s)\{Sf^v(s)-B^v(s)Su(s)\}\|_X\,ds,
\end{aligned}$$

and

$$\begin{aligned}
&\|U^v(t,0)Su_0-S\phi\|_X\\
&\le\|U^v(t,0)S(u_0-\phi)\|_X+\|U^v(t,0)(S\phi-y)\|_X+\|U^v(t,0)y-y\|_X\\
&\qquad\qquad+\|y-S\phi\|_X\\
&\le Me^{\beta T}\|S\|_{Y,X}\cdot\|u_0-\phi\|_Y+(Me^{\beta T}+1)\|S\phi-y\|_X+\|U^v(t,0)y-y\|_X
\end{aligned}$$

for $y\in Y$. Furthermore,

$$U^v(t,0)y-y=-\int_0^t A^v(\tau)U^v(\tau,0)y\,d\tau,$$

and $\|A^v(t)\|_{Y,X}\le c(B(\phi,r))$ by (2.9) and (2.4), and so we have

$$\begin{aligned}
&\|S^{-1}\|_{X,Y}\cdot\|U^v(t,0)Su_0-S\phi\|_X\\
&\le Me^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}\cdot\|u_0-\phi\|_Y\\
&\quad+\|S^{-1}\|_{X,Y}\{(Me^{\beta T}+1)\|S\phi-y\|_X+c(B(\phi,r))T\bar{M}e^{\bar{\beta}T}\|y\|_Y\}.
\end{aligned}$$

We write r_1 for the right hand side of the above inequality. Since $Me^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}\cdot\|u_0-\phi\|_Y<r$ by (2.7) and Y is dense in X, we can choose $y\in Y$ and $T\in(0,T_0]$ so that $r_1<r$. Next, by (2.2) and (2.5) we have

$$\begin{aligned}
&\int_0^t\|U^v(t,s)\{Sf^v(s)-B^v(s)Su(s)\}\|_X\,ds\\
&\le Me^{\beta T}\int_0^t\{\lambda_f\|S\|_{Y,X}+\lambda_B\|S\|_{Y,X}(\|u(s)-\phi\|_Y+\|\phi\|_Y)\}\,ds\\
&\le Me^{\beta T}\|S\|_{Y,X}(\lambda_f+\lambda_B\|\phi\|_Y)T+\lambda_BMe^{\beta T}\|S\|_{Y,X}\int_0^t\|u(s)-\phi\|_Y\,ds.
\end{aligned}$$

Thus, letting $r_2=r_1+Me^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}(\lambda_f+\lambda_B\|\phi\|_Y)T$, we have

$$\|u(t)-\phi\|_Y\le r_2+\lambda_BMe^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}\int_0^t\|u(s)-\phi\|_Y\,ds,$$

which implies that

$$\|u(t)-\phi\|_Y\le r_2\exp[(\lambda_BMe^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y})T].$$

Since $r_1 < r$, we can choose a smaller number $T \in (0, T_0]$ so that

$$\|u(t) - \phi\|_Y < r$$

for $t \in [0, T]$ and $v \in E$. This implies (2.9). Finally, since $u = \Psi v$ is the solution of (L^v) and $u(t) \in B(\phi_0, r_0)$ for $t \in [0, T]$, (2.4) implies

$$\|du/dt\|_X \leq c_0 \lambda_f + c(B(\phi_0, r_0)) \|u(t)\|_Y \leq \rho_0.$$

From this we obtain the Lipschitz condition (2.10). Q.E.D.

In what follows, let $T \in (0, T_0]$ be an arbitrary but fixed positive number satisfying $\Psi(E) \subset E$. We make E into a metric space by the distance function

$$d_X(v, w) = \sup_{0 \leq t \leq T} \|v(t) - w(t)\|_X \qquad \text{for } v, w \in W.$$

It should be noted that E is not always complete.

LEMMA 3.2. *Let $c_1 = \mu_A M \bar{M} e^{\bar{\beta}}(\|u_0\|_Y + \lambda_f T) + \mu_f M e^{\beta T}$. Then we have*

$$d_X(\Psi^n v, \Psi^n w) \leq \frac{(c_1 T)^n}{n!} \cdot d_X(v, w) \qquad \textit{for } v, w \in W \textit{ and } n = 1, 2, \cdots. \tag{3.1}$$

PROOF. Let $v, w \in E$. By Lemma 2.4 and (2.6), we have

$$\begin{aligned}
&\|[\Psi v](t) - [\Psi w](t)\|_X \\
&\leq \|U^v(t, 0)u_0 - U^w(t, 0)u_0\|_X \\
&\quad + \int_0^t \|[U^v(t, s) - U^w(t, s)]f^v(s)\|_X \, ds \\
&\quad + \int_0^t \|U^w(t, s)[f^v(s) - f^w(s)]\|_X \, ds \\
&\leq \mu_A M \bar{M} e^{\bar{\beta} T} \|u_0\|_Y \cdot \int_0^t \|v(s) - w(s)\|_X \, ds \\
&\quad + \lambda_f \mu_A M \bar{M} e^{\bar{\beta} T} \int_0^t ds \int_s^t \|v(\sigma) - w(\sigma)\|_X \, d\sigma \\
&\quad + \mu_f M e^{\beta T} \cdot \int_0^t \|v(s) - w(s)\|_X \, ds \\
&\leq c_1 \cdot \int_0^t \|v(s) - w(s)\|_X \, ds.
\end{aligned}$$

It follows that

$$\|[\Psi^n v](t) - [\Psi^n w](t)\|_X \leq c_1 \cdot \int_0^t \{[c_1 (t - s)]^{n-1}/(n-1)!\} \|v(s) - w(s)\|_X \, ds,$$

for $n = 1, 2, \cdots$. From this (3.1) follows. Q.E.D.

We define a sequence $\{u^n\}$ in E as follows:

(3.2) $u^0(t) = u_0$ on $[0, T]$ and $u^n = \Psi u^{n-1}$ for $n = 1, 2, \cdots$.

Then we have

COROLLARY 3.3. *The sequence $\{u^n(t)\}$ converges in X uniformly on $[0, T]$.*

Corollary 3.3 follows directly from Lemma 3.2. It will be proved that the limit

$$u(t) = \lim_{n\to\infty} u^n(t) \tag{3.3}$$

gives a unique solution of (CP) satisfying (2.1). In what follows $\{U_n(t, s)\}$ denotes the evolution operator generated by $\{A(t, u_n(t))\}$.

LEMMA 3.4. *There is a family $\{\bar{U}(t, s); (t, s) \in \Delta\}$ of operators in $B(X)$ such that (a) and (b) of Theorem 1.1 hold with U replaced by $\bar{U}$, and such that*

$$\lim_{n\to\infty} \sup_{(t,s)\in\Delta} \| U_n(t, s)x - \bar{U}(t, s)x \|_X = 0 \qquad \textit{for } x \in X. \tag{3.4}$$

PROOF. For $x \in X$, $y \in Y$ and $1 \le m \le n$, we have

$$\begin{aligned} &\| U_m(t, s)x - U_n(t, s)x \|_X \\ &\le \| U_m(t, s)(x - y) \|_X + \| U_m(t, s)y - U_n(t, s)y \|_X + \| U_n(t, s)(y - x) \|_X \\ &\le 2Me^{\beta T} \| x - y \|_X + \mu_A M\bar{M}T \| y \|_Y \, e^{\bar{\beta} T} \cdot d_X(u^m, u^n), \end{aligned}$$

by Lemma 2.4 and Corollary 2.3. Since Y is dense in X and $d_X(u^m, u^n) \to 0$ as $m, n \to \infty$ by Corollary 3.3, we have

$$\sup_{(t,s)\in\Delta} \| U_m(t, s)x - U_n(t, s)x \|_X \longrightarrow 0,$$

as $m, n \to \infty$. From this it follows that there is a family $\{\bar{U}(t, s)\}$ of operators in $B(X)$ satisfying (3.4), and that $\bar{U}(t, s)$ has the properties (*a*) and (*b*) since each $U_n(t, s)$ has the corresponding properties. Q.E.D.

We denote by E_Y the set of all functions v in $C([0, T]; Y)$ satisfying

$$v(t) \in B(\phi, r) \qquad \text{for } t \in [0, T].$$

We make E_Y into a metric space through the distance function

$$d_Y(v, w) = \sup_{0 \le t \le T} \| v(t) - w(t) \|_Y$$

for $v, w \in E_Y$. E_Y is a complete metric space. For each $v \in E_Y$, we put

$$[\Phi v](t) = S^{-1}\bar{U}(t, 0)Su_0 + \int_0^t S^{-1}\bar{U}(t, s)\{Sf^v(s) - B^v(s)Sv(s)\}\, ds.$$

Φ is a mapping from E_Y into $C([0, T]; Y)$. In the following, we will find a smaller $T\in(0, T_0]$ so that Φ has a fixed point.

LEMMA 3.5. *There is $T\in(0, T_0]$ such that $\Phi v\in E_Y$ for $v\in E_Y$.*

PROOF. Let $v\in E_Y$. In the same way as in the proof of Lemma 3.1, we have

$$\begin{aligned}
&\|\Phi v-\phi\|_Y \\
&\leq \|S^{-1}\|_{X,Y}\cdot\{\|\bar{U}(t,0)S(u_0-\phi)\|_X+\|\bar{U}(t,0)S\phi-S\phi\|_X\} \\
&\quad+\|S^{-1}\|_{X,Y}\cdot\int_0^t\|\bar{U}(t,s)\{Sf^v(s)-B^v(s)Sv(s)\}\|_X\,ds \\
&\leq Me^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}\cdot\|u_0-\phi\|_Y \\
&\quad+\|S^{-1}\|_{X,Y}\cdot\sup\nolimits_{0\leq t\leq T}\ \|\bar{U}(t,0)S\phi-S\phi\|_X \\
&\quad+TMe^{\beta T}\|S\|_{Y,X}\cdot\|S^{-1}\|_{X,Y}\cdot\{\lambda_f+\lambda_B(r+\|\phi\|_Y)\}.
\end{aligned}$$

Therefore, by (2.7), we can choose $T\in(0, T_0]$ so that

$$\|\Phi v-\phi\|_Y<r,$$

since $\sup_{0\leq t\leq T}\ \|\bar{U}(t,0)S\phi-S\phi\|_X\to 0$ as $T\downarrow 0$. Q.E.D.

In the rest of this section, we fix $T\in(0, T_0]$ so that $\Psi(E)\subset E$ and $\Phi(E)\subset E$.

LEMMA 3.6. *There is a unique fixed point $\bar{u}\in E_Y$ of Φ.*

PROOF. We apply the contracting mapping principle. Let $v, w\in E_Y$. Then we have

$$\begin{aligned}
&\|[\Phi v](t)-[\Phi w](t)\|_Y \\
&\leq\|S^{-1}\|_{X,Y}\cdot\int_0^t\|\bar{U}(t,s)S\{f^v(s)-f^w(s)\}\|_X\,ds \\
&\quad+\|S^{-1}\|_{X,Y}\cdot\int_0^t\|\bar{U}(t,s)\{B^v(s)-B^w(s)\}Sv(s)\|_X\,ds \\
&\quad+\|S^{-1}\|_{X,Y}\cdot\int_0^t\|\bar{U}(t,s)B^w(s)S\{v(s)-w(s)\}\|_X\,ds \\
&\leq c_2\cdot\int_0^t\|v(s)-w(s)\|_Y\,ds,
\end{aligned}$$

where $c_2=\|S^{-1}\|_{X,Y}\cdot\|S\|_{Y,X}\cdot Me^{\beta T}\{\bar{\mu}_f+(r+\|\phi\|_Y)\mu_B+\lambda_B\}$. It follows that

$$d_Y(\Phi^n v,\Phi^n w)\leq\frac{(c_2T)^n}{n!}\cdot d_Y(v,w),$$

for $n = 1, 2, \cdots$. Q.E.D.

LEMMA 3.7. *Let* $\{u^n\} \subset E$ *be the sequence defined by* (3.2) *and* $\bar{u} \in E_Y$ *the unique fixed point of* Φ. *Then we have*

$$\lim_{n\to\infty} \sup_{0\le t\le T} \| u^n(t) - \bar{u}(t) \|_Y = 0.$$

PROOF. Since $\bar{u} = \Phi\bar{u}$, we have

$$S\bar{u}(t) = \bar{U}(t, 0)Su_0 + \int_0^t \bar{U}(t, s)\{Sf(s, \bar{u}(s)) - B(s, \bar{u}(s))S\bar{u}(s)\}\ ds.$$

Since $u^n = \Psi u^{n-1}$, Lemma 1.3 gives

$$Su^n(t) = U_{n-1}(t, 0)Su_0 + \int_0^t U_{n-1}(t, s)\{Sf_{n-1}(s) - B_{n-1}(s)Su^n(s)\}\ ds,$$

where we write

$$f_n(s) = f(s, u^n(s)) \qquad \text{and} \qquad B_n(s) = B(s, u^n(s))$$

for $n \ge 0$, and $\{U_n(t, s)\}$ is the evolution operator generated by $\{A(t, u^n(t))\}$ as before. Then we have

$$\begin{aligned}
&\| Su^n(t) - S\bar{u}(t) \|_X \\
&\le \| U_{n-1}(t, 0)Su_0 - \bar{U}(t, 0)Su_0 \|_X \\
&\quad + \int_0^t \| U_{n-1}(t, s)S\{f(s, u^{n-1}(s)) - f(s, \bar{u}(s))\} \|_X\ ds \\
&\quad + \int_0^t \| \{U_{n-1}(t, s) - \bar{U}(t, s)\}Sf(s, \bar{u}(s)) \|_X\ ds \\
&\quad + \int_0^t \| U_{n-1}(t, s)B_{n-1}(s)S\{u^n(s) - \bar{u}(s)\} \|_X\ ds \\
&\quad + \int_0^t \| U_{n-1}(t, s)\{B(s, u^{n-1}(s)) - B(s, \bar{u}(s))\}S\bar{u}(s) \|_X\ ds \\
&\quad + \int_0^t \| \{U_{n-1}(t, s) - \bar{U}(t, s)\}B(s, \bar{u}(s))S\bar{u}(s) \|_X\ ds \\
&\le \varepsilon_n + \int_0^t \{c_3 \| S[u^n(s) - \bar{u}(s)] \|_X + c_4 \| S[u^{n-1}(s) - \bar{u}(s)] \|_X\}\ ds,
\end{aligned}$$

where $c_3 = \lambda_B M e^{\beta T}$, $c_4 = M e^{\beta T} \| S \|_{Y,X} \cdot \| S^{-1} \|_{X,Y} \cdot \{\bar{\mu}_f + \mu_B(r + \| \phi \|_Y)\}$ and

$$\begin{aligned}
\varepsilon_n &= \sup_{0\le t\le T} \| U_{n-1}(t, 0)Su_0 - \bar{U}(t, 0)Su_0 \|_X \\
&\quad + \sup_{0\le t\le T} \int_0^t \| \{U_{n-1}(t, s) - \bar{U}(t, s)\}Sf(s, \bar{u}(s)) \|_X\ ds
\end{aligned}$$

$$+ \sup_{0\le t\le T} \int_0^t \|\{U_{n-1}(t, s) - \bar{U}(t, s)\} B(s, \bar{u}(s)) S\bar{u}(s)\|_X \, ds.$$

Put $p_n(t) = \|Su^n(t) - S\bar{u}(t)\|_X$ for $n = 0, 1, \cdots$. Then by the Gronwall's inequality, we have

$$p_n(t) \le \varepsilon_n \cdot \exp(c_3 t) + c_4 \cdot \int_0^t p_{n-1}(s) \cdot \exp[c_3(t-s)] \, ds.$$

Therefore, for any $0 \le m < n$, we have

$$\begin{aligned} p_n(t) \le \exp(c_3 t) \textstyle\sum_{k=0}^m \varepsilon_{n-k}(c_4 t)^k/k! \\ + c_4^{m+1} \int_0^t \{(t-s)^m/m!\} \cdot p_{n-m-1}(s) \cdot \exp[c_3(t-s)] \, ds. \end{aligned}$$

Put $\delta_k = \sup_{i\ge k} \varepsilon_i$ for $k \ge 0$. Then $p_k(s) \le 2r\|S\|_{Y,X}$ for $k \ge 0$, and so the above inequality reduces to

$$p_n(t) \le \delta_{n-m} \cdot \exp[(c_3 + c_4)T] + 2r\|S\|_{Y,X} \cdot \exp(c_3 T) \cdot (c_4 T)^{m+1}/(m+1)!$$

for $0 \le m < n$. We have $\lim_{k\to\infty} \delta_k = 0$ by Lemma 3.4 and the dominated convergence theorem, and hence

$$\limsup_{n\to\infty} [\sup_{0\le t\le T} p_n(t)] \le 2r\|S\|_{Y,X} \cdot \exp(c_3 T) \cdot (c_4 T)^{m+1}/(m+1)!$$

for any $m \ge 0$. Thus we have

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|Su^n(t) - S\bar{u}(t)\|_X = 0. \qquad \text{Q.E.D.}$$

We are now in a position to complete the proof of our Main Theorem.

PROOF OF MAIN THEOREM. Let u be the function defined by (3.3) and $\bar{u}$ the unique fixed point of Φ. By Lemma 3.7, we have $u = \bar{u} \in E_Y$ and

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|u^n(t) - u(t)\|_Y = 0.$$

Furthermore, we have

$$\|u(t) - u(s)\|_X \le \rho_0 |t - s| \qquad \text{for } (t, s) \in \Delta,$$

since each u^n belongs to E. Therefore, we have $u \in E$ and we see that both $A^u(t)$ and $f^u(t)$ are well defined. We will prove that $\{\bar{U}(t, s)\}$ is the evolution operator generated by $\{A^u(t)\}$. For each $y \in Y$, $0 \le \sigma \le s \le t \le T$ and $n = 1, 2, \cdots$, we have

$$U_n(t, s)y - U_n(t, \sigma)y = \int_\sigma^s U_n(t, \tau) A(\tau, u_n(\tau)) y \, d\tau.$$

Passing to the limit as $n \to \infty$, we have

$$\bar{U}(t, s)y - \bar{U}(t, \sigma)y = \int_\sigma^s \bar{U}(t, \tau)A^u(\tau)y \, d\tau.$$

This implies that

$$(\partial/\partial s)\bar{U}(t, s)y = \bar{U}(t, s)A^u(s)y \qquad \text{for } (t, s) \in \Delta \text{ and } y \in Y.$$

Let $\{U^u(t, s)\}$ be the evolution operator generated by $\{A^u(t)\}$. Then for $y \in Y$ and $0 \le \sigma \le s \le t \le T$, we have

$$(\partial/\partial s)\bar{U}(t, s)U^u(s, \sigma)y = \bar{U}(t, s)\{A^u(s) - A^u(s)\}U^u(s, \sigma)y = 0.$$

This implies that $\bar{U}(t, s)y = U^u(t, s)y$. Since Y is dense in X, it follows that

$$\bar{U}(t, s)x = U^u(t, s)x$$

for $x \in X$ and $(t, s) \in \Delta$. Now, using the relations

$$u^u(t) = U_{n-1}(t, 0)u_0 + \int_0^t U_{n-1}(t, s)f(s, u^{n-1}(s)) \, ds \qquad \text{for } 0 \le t \le T,$$

we have

$$u(t) = U^u(t, 0)u_0 + \int_0^t U^u(t, s)f^u(s) \, ds \qquad \text{for } 0 \le t \le T. \tag{3.5}$$

By Theorem 1.2, this implies that u is a solution of (CP) satisfying (2.1). To prove the uniqueness, let v be any solution of (CP) satisfying (2.1). Then, in view of the identity

$$\begin{aligned}&(\partial/\partial s)U^u(t, s)v(s)\\ &\quad = U^u(t, s)\{A^u(s) - A^v(s)\}v(s) + U^u(t, s)f^v(s),\end{aligned}$$

we have

$$\begin{aligned} v(t) = U^u(t, 0)u_0 &+ \int_0^t U^u(t, s)f^u(s) \, ds \\ &+ \int_0^t U^u(t, s)\{A^u(s) - A^v(s)\}v(s) \, ds \\ &+ \int_0^t U^u(t, s)\{f^v(s) - f^u(s)\} \, ds. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have

$$\|u(t) - v(t)\|_X \le Me^{\beta T}\{\mu_A \cdot \max_s \|v(s)\|_Y + \mu_f\} \cdot \int_0^t \|u(s) - v(s)\|_X \, ds.$$

Thus we conclude that $u = v$. Q.E.D.

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Department of Mathematics,
Sagami Institute of Technology
and
Department of Mathematics,
Faculty of Education,
Okayama University