HIROSHIMA MATH. J.
20 (1990), 443–476

A mathematical study on statistical database designs

Yasuyuki KOBAYASHI

(Received September 4, 1989)

1. Introduction

Let be given a finite set U and non-negative integers $f(x)$ for all $x \in U$. Then, by taking the sum of products of them, we have an integer

$$\mathrm{SP}_f(\mathscr{A}) = \textstyle\sum_{A \in \mathscr{A}} \prod_{x \in A} f(x) \tag{1}$$

for each subfamily $\mathscr{A} \subset 2^U - \{\varnothing\}$, especially, for any covering $\mathscr{A}$ of U; and we can consider the following

PROBLEM. For given U, f as above and a covering $\mathscr{B}$ of U, find effectively $\mathscr{A}$ in such coverings $\mathscr{A}'$ that $\mathscr{B}$ is a refinement of $\mathscr{A}'$ so that the function SP_f in (1) takes the minimum value at $\mathscr{A}$ among such coverings $\mathscr{A}'$ (see Definition 2.3).

We call $\mathscr{A}$ in this problem an MSPD for $\langle U, \mathscr{B}, f \rangle$ simply. Of course, an MSPD exists and any MSPD can be found by calculating $\mathrm{SP}_f(\mathscr{A}')$ for all finitely many such $\mathscr{A}'$; but the number of $\mathscr{A}'$ may increase rapidly as $|U|$ increases. ($|X|$ denotes the number of elements in a finite set X.)

Thus, the purpose of this paper is to establish an effective method of finding an MSPD of special type, which is applicable even when $|U|$ may be large.

Our motivation is in the problem on statistical database designs stated in §5. (For databases, cf. Codd [3–5] and Smith-Smith [23], and for statistical databases, cf. Shoshani [22] and several papers in the reference.)

Let R be a given collection of statistical records, that is, a finite subset of the product $D = \Pi_{i=1}^N D_i$ of domains D_i of i-th field. Then, an aggregation function S can be specified by the category fields $X(S)$, the summary fields $Y(S)$ and the summarizing operators g_j over D_j given for each summary field j in $Y(S)$; and S gives us the summary table $S(R)$ corresponding to $X(S)$, $Y(S)$ and g_j's. Moreover, for any finite set $\mathscr{S}$ of aggregation functions, we have

$$\mathrm{NRec}\,(\mathscr{S}) = \textstyle\sum_{S \in \mathscr{S}} |S(R)|\,, \tag{2}$$

the total number of records of $\{S(R)\colon S \in \mathscr{S}\}$. Thus we have the following

PROBLEM. Let R be a given collection of statistical records. Then, for a finite set of summary tables $\{S_0(R)\colon S_0 \in \mathscr{S}_0\}$ to be derived from the database, find a finite set of base tables $\{S(R): S \in \mathscr{S}\}$ to be organized as the database such that each required table $S_0(R)$ is derivable from some of them and that the total number NRec ($\mathscr{S}$) of records in (2) is minimum among such sets of base tables (see Definition 5.7).

Now, by using the projection pr_i of $D = \Pi_{i=1}^N D_i$ onto the i-th factor D_i, put

$$f(i) = |\mathrm{pr}_i(R)| \quad (i \in \{1, \dots, N\}).$$

Then, for NRec in (2) and SP_f in (1), we see the equality

$$\mathrm{NRec}(\mathscr{S}) = \mathrm{SP}_f(\mathscr{A}(\mathscr{S})) \quad (\mathscr{A}(\mathscr{S}) = \{X(S): S \in \mathscr{S}\})$$

under some conditions on $\mathscr{S}$; and we can prove that a solution $\mathscr{S}$ in the second problem for $\mathscr{S}_0$ with such conditions is obtained by a simple way from an MSPD $\mathscr{A}$ for $\langle U = \langle \mathscr{B} \rangle, \mathscr{B} = \mathscr{A}(\mathscr{S}_0), f|U \rangle$ in the first problem (see Theorem 5.4). ($\langle \mathscr{A} \rangle = \bigcup \{A: A \in \mathscr{A}\}$ for any family $\mathscr{A}$ of sets.)

Besides § 5 and the final note in § 6, we are concerned with the first problem.

By removing some trivial cases noticed in § 2, we assume from § 3 on that

(3) any set in $\mathscr{B}$ does not contain another set in $\mathscr{B}$, and $f(x) \geqq 2$ for $x \in U$.

Then, we can prove that any MSPD for $\langle U, \mathscr{B}, f \rangle$ is of the form

(4) $$\mathscr{A}_\beta = \{\langle \mathscr{C} \rangle: \mathscr{C} \in \beta\} \qquad \text{for some } \beta \in \mathrm{Part}(\mathscr{B}),$$

where Part ($\mathscr{B}$) is the set of all partitions of $\mathscr{B}$ (see Theorem 3.2). Thus any MSPD can be found by calculating $\mathrm{SP}_f(\mathscr{A}_\beta)$ for $\beta \in \mathrm{Part}(\mathscr{B})$. But $|\mathrm{Part}(\mathscr{B})|$ increases still rapidly as $|\mathscr{B}|$ increases, and so does $|\{\mathrm{SP}_f(\mathscr{A}_\beta): \beta \in \mathrm{Part}(\mathscr{B})\}|$ which is seen to be equal to the number $p(n)$ of partitions of the integer $n = |\mathscr{B}|$ in some cases (see Propositions 3.3 and 3.4). (For $p(n)$, cf. [9] and [19].)

Therefore, it is desirable to find an MSPD of special type. For this purpose, we consider an indecomposable family $\mathscr{C}$ characterized by the condition that

(I) any elements a, b in $\langle \mathscr{C} \rangle$ is contained in some $C \in \mathscr{C}$ (see Proposition 4.2).

Then, we can prove in §§ 3–4 the following main result.

THEOREM. *Under the assumption* (3), *there exists an* MSPD *for* $\langle U, \mathscr{B}, f \rangle$ *in the first problem which is of the form* $\mathscr{A}_\beta$ *given in* (4) *for a partition* β *of* $\mathscr{B}$ *satisfying the following conditions* (5)–(7):

(5) *Each* $\mathscr{C} \in \beta$ *is indecomposable* (*Theorem* 3.6).

(6) *For any distinct* $\mathscr{C}$ *and* $\mathscr{C}'$ *in* β, *if* $\mathscr{A} \subset \mathscr{C}$ *and* $\langle \mathscr{A} \rangle \subset \langle \mathscr{C}' \rangle$, *then* $\langle \mathscr{C} - \mathscr{A} \rangle = \langle \mathscr{C} \rangle$ *and* $\mathscr{C} - \mathscr{A}$ *is indecomposable* (*Theorem* 3.8).

(7) $\mathrm{SP}_f(\{\langle \mathscr{C} \rangle\}) < \mathrm{SP}_f(\mathscr{C})$ *holds for each* $\mathscr{C} \in \beta$ *with* $|\mathscr{C}| \geqq 3$ (*Theorem* 4.13).

Moreover, we introduce in §4 and §6 some notions on subfamilies of $\mathscr{B}$ which give us necessary conditions for $\mathscr{C}$ to be indecomposable (see Propositions 4.3 and 6.1); and we give methods of finding all such subfamilies (see Theorem 4.8, Corollaries 4.9–4.12 and Corollary 6.3). Thus we can find all indecomposable subfamilies of $\mathscr{B}$ from $\mathscr{C}$'s satisfying these necessary conditions by investigating the above condition (I). Then we find an MSPD for $\langle U, \mathscr{B}, f\rangle$ by the above theorem. We give an algorithm to find an MSPD in §6 by this way, which is effective when $|U| \leqq 20$ and $|\mathscr{B}| \leqq 200$ and which is applicable for statistical database designs; and an application is announced in [15].

2. Definition of MSPD

In this section, the basic consideration on some concepts related to solutions of the minimum sum of products decomposition (MSPD) problem will be treated. A simple example will be also given.

Definition 2.1. Let be given

1) a finite non-empty set U, and

2) a function f defined on U with values in non-negative integers.

Then, for any family $\mathscr{A}$ of distinct non-empty subsets of U, i.e., a subset $\mathscr{A}$ of $2^U - \{\varnothing\}$, we define the *sum of products* $\mathrm{SP}_f(\mathscr{A})$ to be the non-negative integer

$$\mathrm{SP}_f(\mathscr{A}) = \textstyle\sum_{A \in \mathscr{A}} \prod_{x \in A} f(x) \qquad (\mathrm{SP}_f(\varnothing) = 0)\,.$$

Definition 2.2. For any subsets $\mathscr{A}$ and $\mathscr{B}$ of $2^U - \{\varnothing\}$, we say that $\mathscr{B}$ is a *refinement* of $\mathscr{A}$ and denote it by $\mathscr{A} \geq \mathscr{B}$, if for any $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $A \supset B$. Also, put $\langle\mathscr{A}\rangle = \bigcup\{A: A \in \mathscr{A}\}$, where $\langle\varnothing\rangle = \varnothing$.

It is clear that $\mathscr{A} \geq \mathscr{A}$, and $\mathscr{A} \geq \mathscr{B}$ and $\mathscr{B} \geq \mathscr{C}$ imply $\mathscr{A} \geq \mathscr{C}$; but $\mathscr{A} \geq \mathscr{B}$ and $\mathscr{B} \geq \mathscr{A}$ do not imply $\mathscr{A} = \mathscr{B}$ in general. Also, it is clear that $\mathscr{A} \geq \mathscr{B}$ implies $\langle\mathscr{A}\rangle \supset \langle\mathscr{B}\rangle$.

Definition 2.3. Let be given U and f as in Definition 2.1 and

3) a subset $\mathscr{B}$ of $2^U - \{\varnothing\}$ satisfying $\langle\mathscr{B}\rangle = U$, i.e., a covering $\mathscr{B}$ of U.

Then, we consider the problem to find $\mathscr{A}$ satisfying the following conditions 4) and 5):

4) $\mathscr{A}$ is a subset of $2^U - \{\varnothing\}$ such that $\mathscr{A} \geq \mathscr{B}$.

5) $\mathrm{SP}_f(\mathscr{A}) \leqq \mathrm{SP}_f(\mathscr{A}')$ for any subset $\mathscr{A}'$ of $2^U - \{\varnothing\}$ with $\mathscr{A}' \geq \mathscr{B}$.

We call in the paper such $\mathscr{A}$ a *solution of minimum sum of products decomposition problem for* U, f and $\mathscr{B}$, or simply, an MSPD for $\langle U, \mathscr{B}, f\rangle$.

In the first place, we note trivial cases.

PROPOSITION 2.1. (i) *If* $f(x) = 0$ *for some* $x \in U$, *then any* MSPD $\mathscr{A}$ *for* $\langle U, \mathscr{B}, f \rangle$ *is trivial, that is,* $\mathrm{SP}_f(\mathscr{A}) = 0$.

(ii) *When* f *is the constant map with value* 1, $\mathscr{A}$ *is an* MSPD *for* $\langle U, \mathscr{B}, f \rangle$ *if and only if* $\mathscr{A} = \{U\}$, *that is,* $\mathrm{SP}_f(\mathscr{A}) = 1$.

PROOF. (i) Put $\mathscr{B}_0 = \{B \cup \{x\}: B \in \mathscr{B}\}$. Then $\mathscr{B}_0 \geq \mathscr{B}$, and $\mathrm{SP}_f(\mathscr{B}_0) = 0$ by the assumption. Hence $0 = \mathrm{SP}_f(\mathscr{B}_0) \geqq \mathrm{SP}_f(\mathscr{A}) \geqq 0$ by 5) of Definition 2.3. Thus we see (i).

(ii) In this case, $\mathrm{SP}_f(\mathscr{A})$ is equal to the number $|\mathscr{A}|$ of sets in $\mathscr{A}$ by definition. Hence we see (ii), because $\{U\} \geq \mathscr{B}$.

We give a simple example to understand the problem.

EXAMPLE 2.1. Let $U = \{u_1, u_2, u_3, u_4\}$, $B_1 = \{u_1, u_2, u_4\}$, $B_2 = \{u_1, u_3\}$, $B_3 = \{u_2, u_3\}$, $B_4 = \{u_1, u_4\}$, $B_5 = \{u_2, u_3, u_4\}$, $\mathscr{B} = \{B_1, B_2, B_3, B_4, B_5\}$, $f(u_1) = 3$, $f(u_2) = f(u_3) = 2$, and $f(u_4) = 1$, then we can illustrate $\langle U, \mathscr{B}, f \rangle$ as follows:

	u_1	u_2	u_3	u_4
B_1	1	1		1
B_2	1		1	
B_3		1	1	
B_4	1			1
B_5		1	1	1
f	3	2	2	1

Figure 2.1.

The following two propositions show that we can derive an MSPD for $\langle U, \mathscr{B}, f \rangle$ from that for simpler $\langle U', \mathscr{B}', f' \rangle$.

PROPOSITION 2.2. *Assume that* $B \in \mathscr{B}$ *is contained in some other* $B' \in \mathscr{B}$. *Then* $\mathscr{A}$ *is an* MSPD *for* $\langle U, \mathscr{B}, f \rangle$ *if and only if so is it for* $\langle U, \mathscr{B} - \{B\}, f \rangle$.

PROOF. By assumption, we see easily that $\langle \mathscr{B} - \{B\} \rangle = \langle \mathscr{B} \rangle = U$, and $\mathscr{A} \geq \mathscr{B}$ is equivalent to $\mathscr{A} \geq \mathscr{B} - \{B\}$. Thus the proposition holds by the definition.

PROPOSITION 2.3. *For* U *and* f *in Definition* 2.1, *assume that* $f^{-1}(0) = \varnothing$ *and* $f^{-1}(1) \neq U$, *and consider*

$$U' = U - V, \quad V = f^{-1}(1) \quad \textit{and} \quad f' = f \,|\, U'.$$

Also, put $\mathscr{A} - V = \{A - V: A \in \mathscr{A}\} - \{\varnothing\}$ *for* $\mathscr{A} \subset 2^U - \{\varnothing\}$, *and* $\mathscr{A}' \cup V = \{A' \cup V: A' \in \mathscr{A}'\}$ *for* $\mathscr{A}' \subset 2^{U'} - \{\varnothing\}$.

(i) *If $\mathscr{A}$ is an* MSPD *for* $\langle U, \mathscr{B}, f\rangle$, *then so is* $\mathscr{A} - V$ *for* $\langle U', \mathscr{B} - V, f'\rangle$.

(ii) *Conversely, if* $\mathscr{A}'$ *is an* MSPD *for* $\langle U', \mathscr{B} - V, f'\rangle$, *then so is* $\mathscr{A}' \cup V$ *for* $\langle U, \mathscr{B}, f\rangle$.

PROOF. The proposition is trivial if $V = \varnothing$. Therefore, we assume that $V = f^{-1}(1) \neq \varnothing$, and can take $a_1 \in f^{-1}(1)$ and $a_2 \notin f^{-1}(1)$ by assumption.

(i) Take any MSPD $\mathscr{A}$ for $\langle U, \mathscr{B}, f\rangle$. Then $\mathscr{A} - V \geq \mathscr{B} - V$ since $\mathscr{A} \geq \mathscr{B}$.

Assume that $A \subset V$ for some $A \in \mathscr{A}$. Take $A_1 \in \mathscr{A}$ with $A_1 \ni a_2$, and put $A_2 = A \cup A_1$. Then $\Pi_{x \in A} f(x) = 1$ and $\Pi_{x \in A_1} f(x) = \Pi_{x \in A_2} f(x)$. Therefore, for $\mathscr{C} = (\mathscr{A} - \{A, A_1\}) \cup \{A_2\}$, we have

$$\mathrm{SP}_f(\mathscr{C}) = \mathrm{SP}_f(\mathscr{A}) - 1 \qquad \text{and} \qquad \mathscr{C} \geq \mathscr{A} \geq \mathscr{B}\,.$$

Since $\mathscr{A}$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$, the latter implies $\mathrm{SP}_f(\mathscr{C}) \geqq \mathrm{SP}_f(\mathscr{A})$ which contradicts the former. Thus $A - V \neq \varnothing$ for any $A \in \mathscr{A}$, which implies by the definition that

$$\mathrm{SP}_f(\mathscr{A}) = \mathrm{SP}_{f'}(\mathscr{A} - V)\,.$$

Now take any $\mathscr{A}' \subset 2^{U'} - \{\varnothing\}$ with $\mathscr{A}' \geq \mathscr{B} - V$. Then,

$$\mathrm{SP}_f(\mathscr{A}' \cup V) = \mathrm{SP}_{f'}(\mathscr{A}')$$

by the definition. Take any $B \in \mathscr{B}$. If $B \subset V$, then $B \subset A' \cup V$ for any $A' \in \mathscr{A}'$. If $B - V \neq \varnothing$, then $B - V \subset A'$ for some $A' \in \mathscr{A}'$; hence $B \subset A' \cup V$.

Therefore $\mathscr{A}' \cup V \geq \mathscr{B}$, and $\mathrm{SP}_f(\mathscr{A}' \cup V) \geqq \mathrm{SP}_{f'}(\mathscr{A})$ since $\mathscr{A}$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$. According to the above two equalities, this implies $\mathrm{SP}_{f'}(\mathscr{A}') \geqq \mathrm{SP}_{f'}(\mathscr{A} - V)$. Thus $\mathscr{A} - V$ is a MSPD for $\langle U', \mathscr{B} - V, f'\rangle$.

(ii) Assume that $\mathscr{A}'$ is an MSPD for $\langle U', \mathscr{B} - V, f'\rangle$. Then $\mathscr{A}' \subset 2^{U'} - \{\varnothing\}$ and $\mathscr{A}' \geq \mathscr{B} - V$. In the same manner as in the proof of (i), we can prove that

$$\mathrm{SP}_f(\mathscr{A}' \cup V) = \mathrm{SP}_{f'}(\mathscr{A}') \qquad \text{and} \qquad \mathscr{A}' \cup V \geq \mathscr{B}\,.$$

Take any $\mathscr{A} \subset 2^{U'} - \{\varnothing\}$ with $\mathscr{A} \geq \mathscr{B}$. Then $\mathscr{A} - V \geq \mathscr{B} - V$ and $\mathrm{SP}_f(\mathscr{A} - V) \leqq \mathrm{SP}_f(\mathscr{A})$. Since $\mathscr{A}'$ is an MSPD for $\langle U', \mathscr{B} - V, f'\rangle$, we have

$$\mathrm{SP}_{f'}(\mathscr{A} - V) \geqq \mathrm{SP}_{f'}(\mathscr{A}')\,.$$

According to the equality and the inequality above, this implies

$$\mathrm{SP}_f(\mathscr{A}) \geqq \mathrm{SP}_f(\mathscr{A}' \cup V)\,.$$

Thus $\mathscr{A}' \cup V$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$.

Proposition 2.3 shows that we can derive an MSPD in Example 2.1 from an MSPD for $\langle U', \mathscr{B}', f'\rangle$ illustrated as follows.

	u_1	u_2	u_3
B_1	1	1	
B_2	1		1
B_3		1	1
B_4	1		
f	3	2	2

Figure 2.2.

Moreover, Proposition 2.2 implies that an MSPD for $\langle U', \mathscr{B}' - \{B_4\}, f' \rangle$ in Figure 2.3 is also an MSPD for $\langle U', \mathscr{B}', f' \rangle$.

	u_1	u_2	u_3
B_1	1	1	
B_2	1		1
B_3		1	1
f	3	2	2

Figure 2.3.

3. MSPD of special type

According to Propositions 2.1–2.3, we shall try hereafter to find an MSPD for $\langle U, \mathscr{B}, f \rangle$ under the following assumptions (A1) and (A2):

(A1) Each $B \in \mathscr{B}$ does not contain any $B' \in \mathscr{B} - \{B\}$.
(A2) $f(x) \geqq 2$ for any element x of U.

In general, to find an MSPD for $\langle U, \mathscr{B}, f \rangle$, it is necessary to calculate $\mathrm{SP}_f(\mathscr{A})$ for all $\mathscr{A} \geq \mathscr{B}$. But the number of $\mathscr{A}$ may increase rapidly as $|U|$ increases. Thus, in this section, we study some sufficient conditions for $\mathscr{A}$ to be an MSPD for $\langle U, \mathscr{B}, f \rangle$.

Definition 3.1. For any partition α of $\mathscr{B}$, i.e., a covering α of $\mathscr{B}$ consisting of disjoint non-empty subsets of $\mathscr{B}$, we define a covering $\mathscr{A}_\alpha$ of $U = \langle \mathscr{B} \rangle$ by

$$\mathscr{A}_\alpha = \{\langle \mathscr{A} \rangle : \mathscr{A} \in \alpha\} \qquad \text{with } \mathscr{A}_\alpha \geq \mathscr{B} .$$

Also, we denote by Part ($\mathscr{B}$) the set of all partitions of $\mathscr{B}$.

Lemma 3.1. *Let be given U and f as in Definition* 2.1 *with* (A.2) *satisfied. Then, for any $\mathscr{A} \subset 2^U - \{\emptyset\}$, $A \in \mathscr{A}$ and $B \subset A$ with $A \neq B \neq \emptyset$, we have*

$$\mathrm{SP}_f(\mathscr{A} - \{A\}) < \mathrm{SP}_f((\mathscr{A} - \{A\}) \cup \{B\}) < \mathrm{SP}_f(\mathscr{A}) .$$

PROOF. By assumption, we see that $\mathrm{SP}_f(\{A\}) > \mathrm{SP}_f(\{B\}) > 0$. Hence,

$$\begin{aligned}\mathrm{SP}_f(\mathscr{A}) &= \mathrm{SP}_f(\mathscr{A}') + \mathrm{SP}_f(\{A\}) > \mathrm{SP}_f(\mathscr{A}') + \mathrm{SP}_f(\{B\}) \\ &= \mathrm{SP}_f(\mathscr{A}' \cup \{B\}) > \mathrm{SP}_f(\mathscr{A}') \qquad \text{for} \qquad \mathscr{A}' = \mathscr{A} - \{A\}\,.\end{aligned}$$

THEOREM 3.2. *For* $\langle U, \mathscr{B}, f\rangle$ *with* (A1–2) *above, any* MSPD *is an element of* $\{\mathscr{A}_\alpha\colon \alpha \in \mathrm{Part}(\mathscr{B})\}$.

PROOF. Assume that $\mathscr{A}$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$. Then, $\mathscr{B} \leq \mathscr{A}$ and we can choose a function g of $\mathscr{B}$ to $\mathscr{A}$ such that $g(B) \supset B$ for any $B \in \mathscr{B}$. Therefore, $\mathscr{B}$ is the union of disjoint subsets $g^{-1}(A)$ for $A \in \mathscr{A}$, and we have the partition

$$\alpha = \{g^{-1}(A)\colon A \in \mathscr{A}\} - \{\varnothing\} \in \mathrm{Part}(\mathscr{B})\,.$$

Since $g(B) \supset B$ for $B \in \mathscr{B}$, we have $\langle g^{-1}(A)\rangle \subset A$ for any $A \in \mathscr{A}$.

Suppose that $g^{-1}(A) \neq \varnothing$ and $\langle g^{-1}(A)\rangle \neq A$ for some $A \in \mathscr{A}$. Then, Lemma 3.1 shows that

$$\mathrm{SP}_f(\mathscr{A}') < \mathrm{SP}_f(\mathscr{A}) \qquad \text{for} \qquad \mathscr{A}' = (\mathscr{A} - \{A\}) \cup \{\langle g^{-1}(A)\rangle\}\,.$$

Also, for any $B \in \mathscr{B}$, if $g(B) = A$, then $B \subset \langle g^{-1}(A)\rangle$; and if $g(B) \neq A$, then $B \subset g(B) \in \mathscr{A} - \{A\}$. Hence $\mathscr{B} \leq \mathscr{A}'$, and the above inequality contradicts that $\mathscr{A}$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$. In the same way, we have a contradiction if $g^{-1}(A) = \varnothing$ by taking $\mathscr{A}' = \mathscr{A} - \{A\}$.

Thus $g^{-1}(A) \neq \varnothing$ and $\langle g^{-1}(A)\rangle = A$ for any $A \in \mathscr{A}$. Therefore, $\mathscr{A} = \mathscr{A}_\alpha$ for $\alpha = \{g^{-1}(A)\colon A \in \mathscr{A}\} \in \mathrm{Part}(\mathscr{B})$.

By Theorem 3.2, we can find an MSPD for $\langle U, \mathscr{B}, f\rangle$ by calculating $\mathrm{SP}_f(\mathscr{A})$ only for $\mathscr{A} = \mathscr{A}_\alpha(\alpha \in \mathrm{Part}(\mathscr{B}))$. But, we note that

$$|\mathrm{Part}(\mathscr{B})| \qquad \text{and} \qquad |\{\mathrm{SP}_f(\mathscr{A}_\alpha)\colon \alpha \in \mathrm{Part}(\mathscr{B})\}|$$

($|A|$ denotes the number of elements in a set A) increase rapidly as $|\mathscr{B}|$ increases, by Propositions 3.3 and 3.4 stated below.

PROPOSITION 3.3. *Let* $|\mathscr{B}| = n$ *and* $q(n, i) = |\{\alpha \in \mathrm{Part}(\mathscr{B})\colon |\alpha| = i\}|$. *Then we have*

$$|\mathrm{Part}(\mathscr{B})| = \textstyle\sum_{i=1}^{n} q(n, i)\,, \qquad \textit{and}$$

$$q(n, i) = \begin{cases} 1 & (i = 1) \\ (i^n - \sum_{j=1}^{i-1} j!\,{}_iC_j q(n, j))/i! & (n \geqq i \geqq 2) \end{cases},$$

Here, $i! = \Pi_{j=1}^{i} j$ *and* ${}_iC_j = i!/j!(i-j)!$.

PROOF. The first equality and $q(n, 1) = 1$ are clear.

For $1 \leqq j \leqq i \leqq n$, let $t(n, i, j)$ be the number of all cases in which we divide n elements into ordered i groups such that the number of groups containing at least one element is equal to j. Then, we see that

$$t(n, i, i) = i!q(n, i), \quad \sum_{j=1}^{i} t(n, i, j) = i^n,$$

and $t(n, i, j) = {}_iC_j t(n, j, j)$, by the definition. Thus for $n \geqq i \geqq 2$,

$$\begin{aligned} q(n, i) &= t(n, i, i)/i! = \{i^n - \textstyle\sum_{j=1}^{i-1} {}_iC_j t(n, i, j)\}/i! \\ &= \{i^n - \textstyle\sum_{j=1}^{i-1} {}_iC_j j! q(n, j)\}/i!. \end{aligned}$$

TABLE 3.1. The number of partitions of $\mathscr{B}$

$\lvert\mathscr{B}\rvert$	$\lvert\mathrm{Part}(\mathscr{B})\rvert$
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975

EXAMPLE 3.1. There occurs $\mathscr{A}_\alpha = \mathscr{A}_\beta$ for distinct partitions α and β. For example, if $B_1 = \{b_1, b_2\}$, $B_2 = \{b_3, b_4\}$, $B_3 = \{b_2, b_3\}$, $B_4 = \{b_1, b_4\}$, $\mathscr{B} = \{B_1, B_2, B_3, B_4\}$, $\alpha = \{\{B_1, B_2\}, \{B_3, B_4\}\}$, and $\beta = \{\mathscr{B}\}$, then $\mathscr{A}_\alpha = \mathscr{A}_\beta = \{b_1, b_2, b_3, b_4\}$.

PROPOSITION 3.4. *Let $p(n)$ be the number of partitions of the positive integer $n = |\mathscr{B}|$. Then, the inequality*

$$|\{\mathrm{SP}_f(\mathscr{A}_\alpha) \colon \alpha \in \mathrm{Part}(\mathscr{B})\}| \leqq p(n)$$

holds when $f(x) = m$ for any $x \in U$ and $\mathscr{B} = \{\{x\} \colon x \in U\}$; and the equality

$$|\{\mathrm{SP}_f(\mathscr{A}_\alpha) \colon \alpha \in \mathrm{Part}(\mathscr{B})\}| = p(n)$$

holds when $m > n$ in addition.

PROOF. Let α and β be elements of Part $(\mathscr{B})$. We say that α is equivalent to β if $|\alpha| = |\beta|$ and there exists a one-to-one onto map h of α to β such that $|h(\mathscr{A})| = |\mathscr{A}|$ for any $\mathscr{A} \in \alpha$. When α is equivalent to β, we denote by $\alpha \sim \beta$. Then, $\sim$ is an equivalence relation on Part $(\mathscr{B})$, and the number $|\mathrm{Part}(\mathscr{B})/\sim|$ of all equivalence classes is known to be $p(n)$ for $n = |\mathscr{B}|$.

Now, let $f(x) = m$ for any $x \in U$ and $\mathscr{B} = \{\{x\} \colon x \in U\}$. Then

$$\mathrm{SP}_f(\mathscr{A}_\alpha) = \sum_{\mathscr{A} \in \alpha} m^{|\mathscr{A}|} \qquad \text{for any } \alpha \in \mathrm{Part}(\mathscr{B}),$$

by the definition of SP_f. Hence $\mathrm{SP}_f(\mathscr{A}_\alpha) = \mathrm{SP}_f(\mathscr{A}_\beta)$ if $\alpha \sim \beta$, which implies the desired inequality.

Consider the case $m > n$. Let $(a, a_1, \dots, a_k)$ and $(b, b_1, \dots, b_l)$ be non-increasing sequences of positive integers such that

$$a > b \quad \text{and} \quad a + \sum_{i=1}^{k} a_i = b + \sum_{i=1}^{l} b_i = n' \leqq n .$$

Then,

$$m^b + \sum_{i=1}^{l} m^{b_i} \leqq n' m^b < m^{b+1} \leqq m^a + \sum_{i=1}^{k} m^{a_i} ,$$

since $n' \leqq n < m$. Therefore, we see that $\mathrm{SP}_f(\mathscr{A}_\alpha) \neq \mathrm{SP}_f(\mathscr{A}_\beta)$ if α is not equivalent to β, which implies the desired equality.

To determine the number $p(n)$ of partitions of a positive integer n is an old problem in the field of analytic number theory; and there are many researches on $p(n)$. It is familiar that

$$\log p(n) \sim \Pi(2n/3)^{1/2}$$

by Hardy and Ramanujan [9]; and various properties of $p(n)$ are investigated in detail by Rademacher [19] and so on.

By Propositions 3.3 and 3.4, it is difficult in general to find an MSPD for $\langle U, \mathscr{B}, f \rangle$ by calculating $\mathrm{SP}_f(\mathscr{A}_\alpha)$ for $\alpha \in \mathrm{Part}\,(\mathscr{B})$. Therefore, in the rest of this section, we shall try to find an MSPD of special type.

Definition 3.2. Let U be a finite set. We say that a subset $\mathscr{A}$ of $2^U - \{\varnothing\}$ is *indecomposable* if there holds

$$\langle \mathscr{C} \rangle = \langle \mathscr{A} \rangle \quad \text{or} \quad \langle \mathscr{A} - \mathscr{C} \rangle = \langle \mathscr{A} \rangle \qquad \text{for any } \mathscr{C} \subset \mathscr{A}$$

($\langle \varnothing \rangle = \varnothing$), and that $\mathscr{A}$ is *decomposable* if it is not indecomposable.

Moreover, we say that a partition $\alpha \in \mathrm{Part}\,(\mathscr{B})$ is *indecomposable* if so is each $\mathscr{A} \in \alpha$; and we consider the subset

$$\mathrm{IP}\,(\mathscr{B}) = \{\alpha : \alpha \text{ is an indecomposable partition of } \mathscr{B}\} \subset \mathrm{Part}\,(\mathscr{B}) .$$

The following proposition is clear by the definition and (A1) for $\mathscr{B}$.

Proposition 3.5. *If* $|\mathscr{A}| = 1$, *then* $\mathscr{A}$ *is indecomposable. If* $\mathscr{A} \subset \mathscr{B}$ *and* $|\mathscr{A}| = 2$, *then* $\mathscr{A}$ *is decomposable. Therefore, if* $\alpha \in \mathrm{IP}\,(\mathscr{B})$, *then*

$$|\mathscr{A}| = 1 \quad \textit{or} \quad |\mathscr{A}| \geqq 3 \qquad \textit{for any } \mathscr{A} \in \alpha .$$

We give another examples:

EXAMPLE 3.2. If $U = \{a_1, a_2, a_3\}$, $A_1 = \{a_1, a_2\}$, $A_2 = \{a_2, a_3\}$, $A_3 = \{a_3, a_1\}$, and $\mathscr{A} = \{A_1, A_2, A_3\}$, then $\mathscr{A}$ is indecomposable.

EXAMPLE 3.3. If $U = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{a_1, a_2\}$, $A_2 = \{a_1, a_3\}$, $A_3 = \{a_2, a_4\}$, $A_4 = \{a_3, a_4\}$, and $\mathscr{A} = \{A_1, A_2, A_3, A_4\}$, then $\mathscr{A}$ is decomposable because $\langle\mathscr{C}\rangle \neq \langle\mathscr{A}\rangle \neq \langle\mathscr{A} - \mathscr{C}\rangle$ for $\mathscr{C} = \{A_1, A_2\}$.

THEOREM 3.6. *For $\langle U, \mathscr{B}, f\rangle$ with the assumptions* (A1–2), *there exists at least one* MSPD *of the form $\mathscr{A}_\alpha$ in Definition* 3.1 *for an indecomposable partition* $\alpha \in \mathrm{IP}(\mathscr{B})$.

PROOF. By Theorem 3.2, we have an MSPD $\mathscr{A}_\alpha$ for $\langle U, \mathscr{B}, f\rangle$ where $\alpha \in \mathrm{Part}(\mathscr{B})$.

Assume that there exists a decomposable family $\mathscr{A} \in \alpha$. Then,

$$\langle\mathscr{A}_1\rangle \neq \langle\mathscr{A}\rangle \neq \langle\mathscr{A} - \mathscr{A}_1\rangle \qquad \text{for some } \mathscr{A}_1 \subset \mathscr{A}\,.$$

Hence, $\mathscr{A}_1$ and $\mathscr{A}_2 = \mathscr{A} - \mathscr{A}_1$ are non-empty, and

$$\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2\,, \quad \mathscr{A}_1 \cap \mathscr{A}_2 = \varnothing\,, \quad \langle\mathscr{A}_1\rangle \not\supset \langle\mathscr{A}_2\rangle \quad \text{and} \quad \langle\mathscr{A}_2\rangle \not\supset \langle\mathscr{A}_1\rangle\,.$$

Put

$$C_1 = \langle\mathscr{A}_1\rangle - \langle\mathscr{A}_2\rangle\,, \quad C_2 = \langle\mathscr{A}_1\rangle \cap \langle\mathscr{A}_2\rangle\,, \quad C_3 = \langle\mathscr{A}_2\rangle - \langle\mathscr{A}_1\rangle \quad \text{and}$$

$$M_j = \textstyle\prod_{C_j \ni x} f(x) \qquad \text{for} \qquad j = 1, 2, 3\,.$$

Here $C_j \neq \varnothing$ and $M_j \geqq 2$ for $j = 1, 3$, and $M_2 = 1$ when $C_2 = \varnothing$. Then, we see by the definition of SP_f that

$$\mathrm{SP}_f(\{\langle\mathscr{A}\rangle\}) = M_1 M_2 M_3 \quad \text{and}$$

$$\mathrm{SP}_f(\{\langle\mathscr{A}_1\rangle, \langle\mathscr{A}_2\rangle\}) = M_1 M_2 + M_2 M_3\,,$$

where $M_1 M_2 M_3 - (M_1 M_2 + M_2 M_3) = M_2\{(M_1 - 1)(M_3 - 1) - 1\} \geqq 0$. Hence,

$$\mathrm{SP}_f(\{\langle\mathscr{A}\rangle\}) \geqq \mathrm{SP}_f(\{\langle\mathscr{A}_1\rangle, \langle\mathscr{A}_2\rangle\}) \quad \text{and} \quad \mathrm{SP}_f(\mathscr{A}_\alpha) \geqq \mathrm{SP}_f(\mathscr{A}_{\alpha'})$$

for the partition $\alpha' = (\alpha - \{\mathscr{A}\}) \cup \{\mathscr{A}_1, \mathscr{A}_2\} \in \mathrm{Part}(\mathscr{B})$. Therefore, $\mathrm{SP}_f(\mathscr{A}_\alpha) = \mathrm{SP}_f(\mathscr{A}_{\alpha'})$ and $\mathscr{A}_{\alpha'}$ is also an MSPD for $\langle U, \mathscr{B}, f\rangle$, because so is $\mathscr{A}_\alpha$.

In this proof, we note that

$$|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_2| > |\mathscr{A}_i| \geqq 1 \qquad \text{for} \qquad i = 1, 2\,,$$

and $|\mathscr{A}|$ is finite since so is $|U|$. Therefore, by repeating the above process for any decomposable family finite times, we obtain a partition $\beta \in \mathrm{Part}(\mathscr{B})$ such that $\mathscr{A}_\beta$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$ and β consists of indecomposable families, according to Proposition 3.5.

Thus Theorem 3.6 is proved.

LEMMA 3.7. *For* $\alpha \in \mathrm{Part}(\mathscr{B})$, *assume that there exist distinct elements* $\mathscr{A}_1$, $\mathscr{A}_2$ *of* α *and a subset* $\mathscr{C} \subset \mathscr{A}_2$ *satisfying*

$$\langle \mathscr{A}_1 \rangle \supset \langle \mathscr{C} \rangle \quad \textit{and} \quad \langle \mathscr{A}_2 \rangle \neq \langle \mathscr{A}_2 - \mathscr{C} \rangle .$$

Then $\mathrm{SP}_f(\mathscr{A}_{\alpha'}) < \mathrm{SP}_f(\mathscr{A}_\alpha)$ *holds for*

$$\alpha' = (\alpha - \{\mathscr{A}_1, \mathscr{A}_2\}) \cup \{\mathscr{A}_1 \cup \mathscr{C}, \mathscr{A}_2 - \mathscr{C}\} - \{\varnothing\} \in \mathrm{Part}(\mathscr{B}) .$$

PROOF. $\langle \mathscr{A}_1 \cup \mathscr{C} \rangle = \langle \mathscr{A}_1 \rangle$ by assumption. Hence $\mathrm{SP}_f(\mathscr{A}_{\alpha'}) < \mathrm{SP}_f(\mathscr{A}_\alpha)$ holds by Lemma 3.1.

DEFINITION 3.3. We define $\mathrm{P}_1(\mathscr{B})$ to be the subset of $\mathrm{IP}(\mathscr{B})$ consisting of all $\alpha \in \mathrm{IP}(\mathscr{B})$ satisfying the following condition (P_1):

(P_1) For any distinct $\mathscr{A}_1$, $\mathscr{A}_2 \in \alpha$, if $\mathscr{C} \subset \mathscr{A}_2$ and $\langle \mathscr{C} \rangle \subset \langle \mathscr{A}_1 \rangle$, then $\langle \mathscr{A}_2 - \mathscr{C} \rangle = \langle \mathscr{A}_2 \rangle$ and $\mathscr{A}_2 - \mathscr{C}$ is indecomposable.

Then, we have the following.

THEOREM 3.8. *For* $\langle U, \mathscr{B}, f \rangle$ *with* (A1–2), *there exists at least one* MSPD $\mathscr{A}_\alpha$ *with* $\alpha \in \mathrm{P}_1(\mathscr{B})$.

PROOF. Let $\mathscr{A}_\alpha$ for $\alpha \in \mathrm{IP}(\mathscr{B})$ be an MSPD for $\langle U, \mathscr{B}, f \rangle$ given by Theorem 3.6. Then, we note that the condition

$$\langle \mathscr{A}_2 - \mathscr{C} \rangle = \langle \mathscr{A}_2 \rangle \qquad \text{in } (\mathrm{P}_1)$$

holds automatically. In fact, if $\langle \mathscr{A}_2 - \mathscr{C} \rangle \neq \langle \mathscr{A}_2 \rangle$, then $\mathrm{SP}_f(\mathscr{A}_{\alpha'}) < \mathrm{SP}_f(\mathscr{A}_\alpha)$ for some $\alpha' \in \mathrm{Part}(\mathscr{B})$ by Lemma 3.7, which contradicts that $\mathscr{A}_\alpha$ is an MSPD for $\langle U, \mathscr{B}, f \rangle$.

Now, assume that

$(*)$ there exist $\mathscr{A}_1 \neq \mathscr{A}_2$ in α and $\mathscr{C} \subset \mathscr{A}_2$ such that $\langle \mathscr{C} \rangle \subset \langle \mathscr{A}_1 \rangle$ (hence $\langle \mathscr{A}_2 - \mathscr{C} \rangle = \langle \mathscr{A}_2 \rangle$ by the above note) and $\mathscr{A}_2 - \mathscr{C}$ is decomposable.

Then, by the same proof as that of Theorem 3.6, we obtain an indecomposable partition $\gamma \in \mathrm{IP}(\mathscr{A}_2 - \mathscr{C})$ such that

$$\mathrm{SP}_f(\mathscr{A}_\gamma) \leqq \mathrm{SP}_f(\{\langle \mathscr{A}_2 - \mathscr{C} \rangle\}) .$$

Therefore, we have

$$\beta = (\alpha - \{\mathscr{A}_1, \mathscr{A}_2\}) \cup \{\mathscr{A}_1 \cup \mathscr{C}\} \cup \gamma \qquad \text{in } \mathrm{Part}(\mathscr{B})$$

such that $\mathrm{SP}_f(\mathscr{A}_\beta) \leqq \mathrm{SP}_f(\mathscr{A}_\alpha)$, since $\langle \mathscr{A}_1 \cup \mathscr{C} \rangle = \langle \mathscr{A}_1 \rangle$ by $(*)$. Hence,

$$\mathrm{SP}_f(\mathscr{A}_\beta) = \mathrm{SP}_f(\mathscr{A}_\alpha)$$

and $\mathscr{A}_\beta$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$, because so is $\mathscr{A}_\alpha$. Moreover,

$$|\beta| > |\alpha| \qquad \text{and} \qquad \beta \in \mathrm{IP}(\mathscr{B}).$$

In fact, these are clear by the definition, except for the indecomposability of $\mathscr{A}' = \mathscr{A}_1 \cup \mathscr{C}$. Let $\mathscr{C}' \subset \mathscr{A}'$. Then, $\langle \mathscr{C}'_1 \rangle = \langle \mathscr{A}_1 \rangle$ for $\mathscr{C}'_1 = \mathscr{A}_1 \cap \mathscr{C}'$ or $\mathscr{A}_1 - \mathscr{C}'$ since $\mathscr{A}_1$ is indecomposable; hence

$$\langle \mathscr{A}' \rangle = \langle \mathscr{A}_1 \rangle = \langle \mathscr{C}'_1 \rangle \subset \langle \mathscr{C}_1 \rangle \subset \langle \mathscr{C}' \rangle \qquad \text{and} \qquad \langle \mathscr{C}_1 \rangle = \langle \mathscr{A}' \rangle$$

for $\mathscr{C}_1 = \mathscr{C}'$ or $\mathscr{A}' - \mathscr{C}'$. This shows that $\mathscr{A}_1 \cup \mathscr{C}$ is indecomposable.

Now, repeat the above process for β if (*) holds for β instead of α, and so on. Since $|\beta| \leqq |\mathscr{B}|$ for any $\beta \in \mathrm{IP}(\mathscr{B})$, we obtain $\gamma \in \mathrm{IP}(\mathscr{B})$ by finite processes such that (*) does not hold for γ instead of α. Thus γ satisfies the condition (P_1) and $\mathscr{A}_\gamma$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$ by the above proof.

Thus, Theorem 3.8 is proved.

Now, we try to find an MSPD for $\langle U', \mathscr{B}_1, f'\rangle$ illustrated in Figure 2.3. Theorem 3.8 implies that candidates of MSPD's for $\langle U', \mathscr{B}_1, f'\rangle$ are $\mathscr{B}_1$ and $\{\langle \mathscr{B}_1 \rangle\} = \{U'\}$. Since $\mathrm{SP}_f(\{U'\}) = 12$ and $\mathrm{SP}_f(\mathscr{B}_1) = 16$, $\{U'\}$ is an MSPD for $\langle U', \mathscr{B}_1, f'\rangle$.

4. *C*-maximal indecomposable families

In this section, we consider some methods to find all indecomposable subsets of $\mathscr{B}$, by introducing the notion of C-maximal indecomposable families of $\mathscr{A}$ in $\mathscr{B}$.

In the first place, we notice the following two propositions on decomposability or indecomposability.

PROPOSITION 4.1. *A subset $\mathscr{A}$ of $2^U - \{\varnothing\}$ is decomposable if and only if there exists an element a of $\langle \mathscr{A} \rangle$ such that*

$$\langle \{A \in \mathscr{A} : a \in A\} \rangle \neq \langle \mathscr{A} \rangle .$$

PROOF. Assume that $\mathscr{A}$ is decomposable. Then,

$$\langle \mathscr{C} \rangle \neq \langle \mathscr{A} \rangle \neq \langle \mathscr{A} - \mathscr{C} \rangle \qquad \text{for some } \mathscr{C} \subset \mathscr{A} .$$

Hence, $\langle \mathscr{C} \rangle \not\subset \langle \mathscr{A} - \mathscr{C} \rangle$ and we can take $a \in \langle \mathscr{C} \rangle - \langle \mathscr{A} - \mathscr{C} \rangle$. If $A \in \mathscr{A}$ contains a, then $A \in \mathscr{C}$ since $a \in \langle \mathscr{A} - \mathscr{C} \rangle$. Therefore,

$$\langle \{A \in \mathscr{A} : a \in A\} \rangle \subset \langle \mathscr{C} \rangle \subsetneqq \langle \mathscr{A} \rangle .$$

Conversely, assume that there exists $a \in \langle \mathscr{A} \rangle$ with $\langle \mathscr{C} \rangle \neq \langle \mathscr{A} \rangle$, where $\mathscr{C} = \{A \in \mathscr{A} : a \in A\}$. If $B \in \mathscr{A} - \mathscr{C}$, then $a \notin B$ by the definition of $\mathscr{C}$. Hence $\langle \mathscr{A} - \mathscr{C} \rangle \not\ni a$ and $\langle \mathscr{A} - \mathscr{C} \rangle \neq \langle \mathscr{A} \rangle$. Therefore $\mathscr{A}$ is decomposable.

Proposition 4.2. *A subset $\mathscr{A}$ of $2^U - \{\emptyset\}$ is indecomposable if and only if for any elements a and b of $\langle\mathscr{A}\rangle$, there exists $A \in \mathscr{A}$ with $A \ni a, b$.*

Proof. Assume that $\mathscr{A}$ is indecomposable, and that $a, b \in \langle\mathscr{A}\rangle$. From Proposition 4.1 we have $\langle\{A \in \mathscr{A}: a \in A\}\rangle = \langle\mathscr{A}\rangle$. Then we have $\langle\{A \in \mathscr{A}: a \in A\}\rangle \ni b$. This implies that there exists $A \in \mathscr{A}$ with $A \ni a, b$.

Conversely, assume that for any elements a and b of $\langle\mathscr{A}\rangle$, there exists $A \in \mathscr{A}$ with $A \ni a, b$. Then for any $a \in \langle\mathscr{A}\rangle$, $\langle\{A \in \mathscr{A}: a \in A\}\rangle$ contains any $b \in \langle\mathscr{A}\rangle$, and so $\{A \in \mathscr{A}: a \in A\} = \langle\mathscr{A}\rangle$. Therefore, $\mathscr{A}$ is indecomposable by Proposition 4.1.

We try to find a solution of the following example by using the results of the above discussion.

Example 4.1.

	u_1	u_2	u_3	u_4
B_1	1	1		
B_2	1			1
B_3	1		1	
B_4		1		1
f	2	3	4	3

Figure 4.1.

1) $\{\langle\mathscr{B}\rangle\}$ is decomposable because there exists $u_2 \in \mathscr{B}$ with $\langle\{B \in \mathscr{B}: u_2 \in B\}\rangle \neq \langle\mathscr{B}\rangle$.
2) Let $\mathscr{A}$ be a subset of $\mathscr{B}$ with $|\mathscr{A}| = 3$, and $\mathscr{A} \ni B_3$. Then $\mathscr{A}$ is decomposable since $\langle\{B \in \mathscr{A}: u_3 \in B\}\rangle \neq \langle\mathscr{A}\rangle$. $\{B_1, B_2, B_4\}$ is indecomposable.
3) From 1) and 2), all elements of $\mathrm{P}_1(\mathscr{B})$ are $\alpha = \{\{B_1, B_2, B_4\}, \{B_3\}\}$ and $\beta = \{\{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}\}$.
4) Since $\mathrm{SP}_f(\mathscr{A}_\alpha) = 26$ and $\mathrm{SP}_f(\mathscr{A}_\beta) = 29$, $\mathscr{A}_\alpha$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$.

The following definition is given to obtain a necessary condition stated in Proposition 4.3 for a family to be indecomposable.

Definition 4.1. In addition to $\mathscr{B} \subset 2^U - \{\emptyset\}$ satisfying (A1), let be given

$$C \subset U, \qquad \text{and} \qquad \mathscr{A} \subset \mathscr{B} \qquad \text{with} \qquad \langle\mathscr{A}\rangle \cap C = \emptyset .$$

Then, $\mathscr{C}$ is called a *C-maximal indecomposable family* (or simply, C-MIF) of $\mathscr{A}$ in $\mathscr{B}$, if it satisfies the following conditions 1) and 2):
1) $\mathscr{A} \subset \mathscr{C} \subset \mathscr{B}$, $\langle\mathscr{C}\rangle \cap C = \emptyset$ and $\mathscr{C}$ is indecomposable.

2) For any $\mathscr{C}'$ with $\mathscr{A} \subset \mathscr{C}' \subset \mathscr{B}$ and $\langle \mathscr{C}' \rangle \cap C = \emptyset$, $\mathscr{C}' = \mathscr{C}$ holds if $\mathscr{C}' \supset \mathscr{C}$ and $\mathscr{C}'$ is indecomposable.

In general, a C-MIF of $\mathscr{A}$ in $\mathscr{B}$ is not unique. Example 4.2 shows that $\emptyset$-MIF's of $\{B_1\}$ in $\mathscr{B}$ are $\{B_1, B_2, B_3\}$ and $\{B_1, B_4, B_5\}$.

EXAMPLE 4.2.

	u_1	u_2	u_3	u_4
B_1	1	1		
B_2	1		1	
B_3		1	1	
B_4	1			1
B_5		1		1

Figure 4.2.

The condition 2) of Definition 4.1 means the following

PROPOSITION 4.3. *A subfamily $\mathscr{C}'$ of $\mathscr{B}$ is decomposable if it contains properly some C-MIF of $\mathscr{A}$ in $\mathscr{B}$ and $\langle \mathscr{C}' \rangle \cap C = \emptyset$.*

Therefore, to find an indecomposable family from the set of all subfamilies of $\mathscr{B}$, we can count out any one satisfying the assumptions of Proposition 4.3.

Thus, we study some conditions for $\mathscr{C}$ to be a C-MIF by preparing the following lemmas.

LEMMA 4.4. *For any $\mathscr{C} \subset 2^U - \{\emptyset\}$, $u \in U$ and $B \subset U$, put*

$$\mathscr{C}_u = \{A \in \mathscr{C} : A \ni u\} \qquad \text{and}$$

$$D(\mathscr{C}; B) = \{A \in \mathscr{C} : A \subset \mathscr{C}_b \rangle \qquad \text{for any } b \in B\} .$$

Then, $A \in D(\mathscr{C}; B)$ if and only if A satisfies the following 1) *and* 2):

1) $A \in \mathscr{C}$.

2) *For any $a \in A$ and $b \in B$, there exists $A' \in \mathscr{C}$ containing a and b.*

PROOF. By the definition, $A \in D(\mathscr{C}; B)$ if and only if $A \in \mathscr{C}$ and

$$a \in \langle \mathscr{C}_b \rangle = \{A' \in \mathscr{C} : A' \ni b\} \qquad \text{for any } a \in A \qquad \text{and any } b \in B ;$$

and the latter is equivalent to 2).

LEMMA 4.5. *In the above lemma, if an indecomposable family $\mathscr{C}'$ satisfies $\mathscr{C}' \subset \mathscr{C}$ and $\langle \mathscr{C}' \rangle \supset B$, then $\mathscr{C}' \subset D(\mathscr{C}; B)$.*

PROOF. Let $A \in \mathscr{C}'$ and take $a \in A$ and $b \in B$. Then

$$\{a, b\} \subset A \cup B \subset \langle \mathscr{C} \rangle \qquad \text{and so } \{a, b\} \subset A' \qquad \text{for some } A' \in \mathscr{C}' \subset \mathscr{C},$$

by the assumption and by Proposition 4.2. Hence $A \in D(\mathscr{C}; B)$ by the above lemma; and we have $\mathscr{C}' \subset D(\mathscr{C}; B)$.

DEFINITION 4.2. For $\mathscr{B}$, C and $\mathscr{A}$ in Definition 4.1, we define a sequence

$$D^0(\mathscr{A}, C) \supset D^1(\mathscr{A}, C) \supset \cdots \supset D^i(\mathscr{A}, C) \supset \cdots$$

of subfamilies of $\mathscr{B}$ inductively as follows:

$$D^0(\mathscr{A}, C) = D^0(C) = \{A \in \mathscr{B}: A \cap C = \emptyset\} \quad (\supset \mathscr{A}),$$

$$D^{i+1}(\mathscr{A}, C) = D(D^i(\mathscr{A}, C); \langle \mathscr{A} \rangle) \qquad \text{for } i \geqq 0,$$

where $D(\mathscr{C}; B)$ is the one given in Lemma 4.4. Moreover, we put

$$D^*(\mathscr{A}, C) = D^i(\mathscr{A}, C) \qquad \text{when} \qquad D^i(\mathscr{A}, C) = D^{i+1}(\mathscr{A}, C).$$

LEMMA 4.6. *In this definition, we have the last equality*

$$D^i(\mathscr{A}, C) = D^{i+1}(\mathscr{A}, C) \qquad \textit{for some } i \leqq |D^0(C)|;$$

and then $D^j(\mathscr{A}, C) = D^i(\mathscr{A}, C)$ for any $j > 1$. Thus the family $D^*(\mathscr{A}, C)$ is well-defined by taking the minimum integer of such $i \geqq 0$.

PROOF. There exists such i, since the sequence in Definition 4.2 is non-increasing and $|D^0(C)| \leqq |\mathscr{B}|$ is finite.

Put $D^i = D^i(\mathscr{A}, C)$. If $D^i = D^{i+1}$, then by the definition, we have

$$D^i = D^{i+1} = D(D^i; \langle \mathscr{A} \rangle) = D(D^{i+1}; \langle \mathscr{A} \rangle) = D^{i+2},$$

and so on; hence $D^j = D^i$ for any $j > i$.

LEMMA 4.7. *$\mathscr{C} \subset D^*(\mathscr{A}, C)$ holds for any $\mathscr{C}$ satisfying the condition* 1) *in Definition* 4.1, *and in particular, for any C-MIF $\mathscr{C}$ of $\mathscr{A}$ in $\mathscr{B}$.*

PROOF. Assume that $\mathscr{C}$ satisfies the condition 1) in Definition 4.1. Then

$$D^0(\mathscr{A}, C) = D^0(C) = \{A \in \mathscr{B}: A \cap C = \emptyset\} \subset \mathscr{C},$$

since $\langle \mathscr{C} \rangle \cap C = \emptyset$. If $\mathscr{C} \subset D^i(\mathscr{A}, C)$ for some i, then

$$\mathscr{C} \subset D(D^i(\mathscr{A}, C); \langle \mathscr{A} \rangle) = D^{i+1}(\mathscr{A}, C)$$

by Lemma 4.5, because $\mathscr{C}$ is indecomposable and $\langle \mathscr{C} \rangle \supset \langle \mathscr{A} \rangle$ by assumption. Thus we see $\mathscr{C} \subset D^i(\mathscr{A}, C)$ for any $i \geqq 0$ by induction; hence $\mathscr{C} \subset D^*(\mathscr{A}, C)$.

Now we can prove the following

THEOREM 4.8. *In Definitions* 4.1 *and* 4.2, *there exists a* C-MIF $\mathscr{C}$ *of* $\mathscr{A}$ *in* $\mathscr{B}$ *if and only if the collection*

$$\{\mathscr{C}' : \mathscr{A} \subset \mathscr{C}' \subset D^*(\mathscr{A}, C) \text{ and } \mathscr{C}' \text{ is indecomposable}\}$$

is non-empty; and then $\mathscr{C}$ *is a maximal family in this collection with respect to the order given by the inclusion relation.*

PROOF. $D^*(\mathscr{A}, C) \subset D^0(C) \subset \mathscr{B}$ and $\langle D^0(C) \rangle \cap C = \varnothing$ by Definition 4.2. Hence the above collection is contained in the collection

$$\{\mathscr{C}' : \mathscr{A} \subset \mathscr{C}' \subset \mathscr{B}, \langle \mathscr{C}' \rangle \cap C = \varnothing \text{ and } \mathscr{C}' \text{ is indecomposable}\}$$

$$= \{\mathscr{C} : \mathscr{C} \text{ satisfies the condition 1) in Definition 4.1}\} .$$

Conversely, this is contained in the above one by Lemma 4.7. Therefore, these two collections coincide.

Now, a C-MIF $\mathscr{C}$ of $\mathscr{A}$ in $\mathscr{B}$ is a maximal family in the last collection by Definition 4.2, which exists if and only if the collection is non-empty. Therefore, we see the theorem.

We have several corollaries to Theorem 4.8, the first two of which are the special cases of it.

COROLLARY 4.9. *There exists no* C-MIF *of* $\mathscr{A}$ *in* $\mathscr{B}$ *when* $\mathscr{A}$ *is not contained in* $D^*(\mathscr{A}, C)$, *or when* $D^*(\mathscr{A}, C) = \mathscr{A}$ *and* $\mathscr{A}$ *is decomposable.*

COROLLARY 4.10. *If* $D^*(\mathscr{A}, C) \supset \mathscr{A}$ *and* $D^*(\mathscr{A}, C)$ *is indecomposable, then* $D^*(\mathscr{A}, C)$ *is a unique* C-MIF *of* $\mathscr{A}$ *in* $\mathscr{B}$.

COROLLARY 4.11. *Assume that* $D^*(\mathscr{A}, C) \supset \mathscr{A}$ *and* $D^*(\mathscr{A}, C)$ *is decomposable.*

(1) *If* $\mathscr{C}$ *is a* C-MIF *of* $\mathscr{A} \cup \{B\}$ *in* $\mathscr{B}$ *for some* $B \in \mathscr{B} - \mathscr{A}$ *with* $B \cap C = \varnothing$, *then* $\mathscr{C}$ *is a* C-MIF *of* $\mathscr{A}$ *in* $\mathscr{B}$ *with* $\mathscr{C} \neq \mathscr{A}$.

(2) *If there exists no* $\mathscr{C}$ *satisfying the assumption of* (1), *then a* C-MIF *of* $\mathscr{A}$ *in* $\mathscr{B}$ *is* $\mathscr{A}$ *or does not exist according as* $\mathscr{A}$ *is indecomposable or not.*

PROOF. Definition 4.1 implies immediately (1) and also the converse of (1).

Therefore, the assumption of (2) implies that there exists no C-MIF of $\mathscr{A}$ with $\mathscr{C} \neq \mathscr{A}$; hence the collection in Theorem 4.8 is $\{\mathscr{A}\}$ or empty according as $\mathscr{A}$ is indecomposable or not. Thus, we see (2) by Theorem 4.8.

COROLLARY 4.12. *We can find all C-MIF's of $\mathscr{A}$ in $\mathscr{B}$ by using Corollaries 4.9–4.11.*

PROOF. We note that $D^*(\mathscr{A}, C) \subset D^0(C) = \{B \in \mathscr{B}: B \cap C = \varnothing\} \supset \mathscr{A}$.

If $|D^0(C) - \mathscr{A}| = 0$, then $\mathscr{A} = D^0(C) \supset D^*(\mathscr{A}, C)$. Hence, a C-MIF of $\mathscr{A}$ does not exist or is $\mathscr{A}$ by Corollaries 4.9 and 4.10, and the corollary holds in this case.

We prove the corollary by induction on $|D^0(C) - \mathscr{A}|(\leqq |D^0(C)|)$. Assume that the corollary holds when $|D^0(C) - \mathscr{A}| \leqq k$ for some $k \geqq 0$, and consider the case $|D^0(C) - \mathscr{A}| = k + 1$. Then, Corollaries 4.9 and 4.10 show that a C-MIF of $\mathscr{A}$ does not exist or is $D^*(\mathscr{A}, C)$, except for the case of Corollary 4.11; and the cases (1) and (2) in Corollary 4.11 are seen by the inductive assumption, because $|D^0(C) - (\mathscr{A} \cup \{B\})| = k$ for $B \in D^0(C) - \mathscr{A}$.

Here, we shall find all indecomposable subfamilies of $\mathscr{B}$ in the following example, by using the above discussions.

EXAMPLE 4.3.

	u_1	u_2	u_3	u_4	u_5
B_1	1	1	1		
B_2	1			1	
B_3		1		1	
B_4			1	1	
B_5	1				1
B_6		1			1

Figure 4.3.

In case of $\mathscr{A} = \{B\}$ for $B \in \mathscr{B}$ and $C = \varnothing$, we see $D^j(\{B\}, \varnothing)$ as follows:

$$D^0(\{B_i\}, \varnothing\} = D^0(\varnothing) = \mathscr{B} \qquad (1 \leqq i \leqq 6)\,.$$

u	$D^0(\{B_i\}, \varnothing)_u$	$\langle D^0(\{B_i\}, \varnothing)_u \rangle$
u_1	$\{B_1, B_2, B_4\}$	U
u_2	$\{B_1, B_3, B_6\}$	U
u_3	$\{B_1, B_4\}$	$\{u_1, u_2, u_3, u_4\}$
u_4	$\{B_2, B_3, B_4\}$	$\{u_1, u_2, u_3, u_4\}$
u_5	$\{B_5, B_6\}$	$\{u_1, u_2, u_5\}$

$$D^1(\{B_i\}, \varnothing) = \{B_1, B_2, B_3, B_4\} \qquad (1 \leqq i \leqq 4)\,,$$

$$= \{B_5, B_6\} \qquad (i = 5, 6)$$

$$\begin{aligned}\langle D^1(\{B_i\}, \varnothing)_u\rangle &= \{u_1, u_2, u_3, u_4\} \qquad (u = u_j,\ 1 \leqq i, j \leqq 4)\,,\\ &= \{u_1, u_5\} \qquad (u = u_1, i = 5)\,,\\ &= \{u_2, u_5\} \qquad (u = u_1, i = 6)\,,\\ &= \{u_1, u_2, u_5\} \qquad (u = u_2, i = 5, 6)\,,\\ &= \varnothing \qquad (\text{otherwise})\,.\\ D^2(\{B_i\}, \varnothing) &= \{B_1, B_2, B_3, B_4\} \qquad (1 \leqq i \leqq 4)\,,\\ &= \{B_i\} \qquad (i = 5, 6)\,.\end{aligned}$$

By noticing that $\{B_i\}$ is indecomposable, we have

$$\begin{aligned}D^*(\{B_i\}, \varnothing) &= \{B_1, B_2, B_3, B_4\} \qquad (1 \leqq i \leqq 4)\,,\\ &= \{B_i\} \qquad (i = 5, 6)\,.\end{aligned}$$

Thus $\{B_i\}$ is a unique $\varnothing$-MIF of $\{B_i\}$ for $i = 5$, 6, by Corollary 4.10; hence, if $\mathscr{C} \ni B_5$ or B_6 and $|\mathscr{C}| \geqq 2$, then $\mathscr{C}$ is decomposable by Proposition 4.3.

Now, $\mathscr{B}' = \{B_1, B_2, B_3, B_4\}$ is indecomposable and $\mathscr{B}' - \{B_i\}$ is decomposable for $1 \leqq i \leqq 4$, by Proposition 4.2. These and Proposition 3.5 show that indecomposable subfamilies of $\{B_i\colon 1 \leqq i \leqq 6\}$ are $\{B_1, B_2, B_3, B_4\}$ and $\{B_i\}$ for $1 \leqq i \leqq 6$.

Now, let $|\mathscr{B}| = n$ and $\mathscr{C}$ be a $\varnothing$-MIF of $\{B\}$ in $\mathscr{B}$ where $B \in \mathscr{B}$. Then, among 2^{n-1} subfamilies of $\mathscr{B}$ containing B, there are $2^{n-m} - 1$ families containing properly $\mathscr{C}$ where $m = |\mathscr{C}|$, which are decomposable by Proposition 4.3. Thus we may investigate the indecomposability for the other $2^{n-1} - 2^{n-m} + 1$ ones only, and the number is small if so is m, e.g., is 1 if $m = 1$. When m is large, e.g., when $m = n$ which holds when $\mathscr{B}$ is indecomposable, we try to find a C-MIF of $\{B\}$ in $\mathscr{B}$ for $C \neq \varnothing$ with $B \in D^0(C) = \{A \in \mathscr{B}\colon A \cap C = \varnothing\} \neq \mathscr{B}$, which is a $\varnothing$-MIF of $\{B\}$ in $D^0(C)$, and so on.

After finding all indecomposable subfamilies of $\mathscr{B}$ by the above discussions, we can find an MSPD for $\langle U, \mathscr{B}, f\rangle$ by calculating $\mathrm{SP}_f(\mathscr{A}_\alpha)$ for all α in $\mathrm{P}_1(\mathscr{B})$ according to Theorem 3.8. But $|\mathrm{P}_1(\mathscr{B})|$ may be still large, and $\mathrm{P}_1(\mathscr{B})$ is independent of the function f.

Therefore, we prepare the following definition and theorem to decrease the number of candidates of solutions.

DEFINITION 4.3. We define $\mathrm{P}_2(\mathscr{B}, f)$ to be the subset of $\mathrm{P}_1(\mathscr{B})$ consisting of all $\alpha \in \mathrm{P}_1(\mathscr{B})$ satisfying the following condition (P_2):
(P_2) $|\alpha| = 1$, or $|\alpha| \geqq 3$ and $\mathrm{SP}_f(\{\langle\mathscr{A}\rangle\}) < \mathrm{SP}_f(\mathscr{A})$ for any $\mathscr{A} \in \alpha$.

THEOREM 4.13. *For* $\langle U, \mathscr{B}, f\rangle$ *with* (A1–2), *there exists at least one* MSPD $\mathscr{A}_\alpha$ *with* $\alpha \in \mathrm{P}_2(\mathscr{B}, f)$.

PROOF. We can find an MSPD $\mathscr{A}_\alpha$ for $\langle U, \mathscr{B}, f\rangle$ with $\alpha \in \mathrm{P}_1(\mathscr{B})$, by Theorem 3.8.

Suppose that

$$\mathrm{SP}_f(\{\langle \mathscr{A}\rangle\}) \geqq \mathrm{SP}_f(\mathscr{A}) \qquad \text{for some } \mathscr{A} \in \alpha\,,$$

and put $\beta = (\alpha - \{\mathscr{A}\}) \cup \{\{A\}\colon A \in \mathscr{A}\} \in \mathrm{Part}\,(\mathscr{B})$. If we replace α by β, then the term $\mathrm{SP}_f(\{\langle \mathscr{A}\rangle\})$ in $\mathrm{SP}_f(\mathscr{A}_\alpha)$ is replaced by

$$\textstyle\sum_{A \in \mathscr{A}} \mathrm{SP}_f(\{A\})\ (=\mathrm{SP}_f(\mathscr{A}) \text{ by the definition})$$

in $\mathrm{SP}_f(\mathscr{A}_\beta)$ and the other terms are unchanged. Thus the inequality of the assumption implies that

$$\mathrm{SP}_f(\mathscr{A}_\alpha) \geqq \mathrm{SP}_f(\mathscr{A}_\beta)\,; \qquad \text{hence} \qquad \mathrm{SP}_f(\mathscr{A}_\alpha) = \mathrm{SP}_f(\mathscr{A}_\beta)$$

and $\mathscr{A}_\beta$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$, since so is $\mathscr{A}_\alpha$.

Now, we show that $\beta \in \mathrm{P}_1(\mathscr{B})$. β is in $\mathrm{IP}\,(\mathscr{B})$, because so is α and each $\{A\} \in \beta$ $(A \in \mathscr{A})$ is indecomposable by Proposition 3.5.

To prove the condition (P_1) for β, we take any

$$\mathscr{A}_1 \neq \mathscr{A}_2 \quad \text{in } \beta \text{ and} \quad \mathscr{C} \subset \mathscr{A}_2 \quad \text{with} \quad \langle \mathscr{C}\rangle \subset \langle \mathscr{A}_1\rangle\,,$$

and show that $\langle \mathscr{A}_2 - \mathscr{C}\rangle = \langle \mathscr{A}_2\rangle$ and that $\mathscr{A}_2 - \mathscr{C}$ is indecomposable. Here, the equality holds as noted in the beginning of the proof of Theorem 3.6, because $\mathscr{A}_\beta$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are in $\alpha - \{\mathscr{A}\} = \beta - \{\{A\}\colon A \in \mathscr{A}\}$ then $\mathscr{A}_2 - \mathscr{C}$ is indecomposable, because $\alpha \in \mathrm{P}_1(\mathscr{B})$ satisfies (P_1).

If $\mathscr{A}_2 = \{A\}$ for $A \in \mathscr{A}$, then $\mathscr{C} = \mathscr{A}_2$ or $\varnothing$, and $\mathscr{C} \neq \mathscr{A}_2$ since $\langle \mathscr{A}_2 - \mathscr{C}\rangle = \langle \mathscr{A}_2\rangle$; hence $\mathscr{C} = \varnothing$. If $\mathscr{A}_1 = \{A\}$ for $A \in \mathscr{A}$ and $\mathscr{A}_2 \in \alpha - \{\mathscr{A}\}$, then $\mathscr{C} \subset \mathscr{A}_2$ and $\langle \mathscr{C}\rangle \subset \langle \mathscr{A}_1\rangle$ hold only when $\mathscr{C} = \varnothing$. In fact, if $A' \in \mathscr{C}$, then $A' \subset \langle \mathscr{C}\rangle \subset \langle \mathscr{A}_1\rangle = A$ and $A' \neq A$ since $A' \in \mathscr{A}_2 \in \alpha - \{\mathscr{A}\}$, $A \in \mathscr{A}$ and α is a partition of $\mathscr{B}$; and $A' \subset A \neq A'$ contradicts the assumption (A1). In these cases $\mathscr{A}_2 - \mathscr{C} = \mathscr{A}_2$ is indecomposable, since $\beta \in \mathrm{IP}\,(\mathscr{B})$.

Therefore, we see that $\beta \in \mathrm{P}_1(\mathscr{B})$. By repeating the above process for any $\mathscr{A} \in \alpha$ with $\mathrm{SP}_f(\{\langle \mathscr{A}\rangle\}) \geqq \mathrm{SP}_f(\mathscr{A})$, we obtain $\gamma \in \mathrm{P}_2(\mathscr{B}, f)$ such that $\mathscr{A}_\gamma$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$.

Thus, Theorem 4.13 is proved.

Here, we note that there happens $\mathrm{SP}_f(\{\langle \mathscr{A}\rangle\}) \geqq \mathrm{SP}_f(\mathscr{A})$ for an indecomposable subfamily $\mathscr{A}$ of $\mathscr{B}$, by the following proposition.

PROPOSITION 4.14. *For some integers $n \geqq k \geqq 2$, assume that*

$$|U| = n\,, \quad \mathscr{B} = \{B \subset U \colon |B| = k\},\ f(u) \geqq 2 \qquad \text{for any } u \in U\,,$$

and ${}_nC_k \leqq 2^{n-k}$. *Then $\mathscr{B}$ is indecomposable,* $\langle \mathscr{B} \rangle = U$ *and* $\mathrm{SP}_f(\{U\}) \geqq \mathrm{SP}_f(\mathscr{B})$.

PROOF. $\mathscr{B} = \{B \subset U \colon |B| = k\}(k \geqq 2)$ is indecomposable by Proposition 4.2. Now, by the definition and the assumption, we see that

$$\mathrm{SP}_f(\{U\}) = \prod_{u \in U} f(u) \geqq 2^{n-k} \prod_{u \in B} f(u) \geqq {}_nC_k\, \mathrm{SP}_f(\{B\})$$

for any $B \in \mathscr{B}$, since $|U| = n \geqq k = |B|$ and $f(u) \geqq 2$. Therefore, by taking $B' \in \mathscr{B}$ such that $\mathrm{SP}_f(\{\mathscr{B}'\}) = \mathrm{Max}\,\{\mathrm{SP}_f(\{B\}) \colon B \in \mathscr{B}\}$ and by noticing $|\mathscr{B}| = {}_nC_k$, we see that

$$\mathrm{SP}_f(\{U\}) \geqq |\mathscr{B}|\mathrm{SP}_f(\{B'\}) \geqq \sum_{B \in \mathscr{B}} \mathrm{SP}_f(\{B\}) = \mathrm{SP}_f(\mathscr{B})\,.$$

In this proposition, the inequality ${}_nC_k \leqq 2^{n-k}$ holds when $n \geqq 6$, 10, 14 or 19 if $k = 2$, 3, 4 or 5, respectively.

5. Application to statistical database designs

In this section, we apply an MSPD to design a statistical database with the minimum number of records. In the first place, we give some definitions and their examples related to a statistical database (cf. [3], [4], [20]).

DEFINITION 5.1. Let be given a positive integer N and sets D_i for all integers i with $1 \leqq i \leqq N$. Then, we call a finite subset R of the product $\Pi_{i=1}^N D_i$ a *relation* of $(D_1, \ldots, D_N)$.

Hereafter, we assume that a relation R of $(D_1, \ldots, D_N)$ is given, and use the following notations:

$D_X = \Pi_{i \in X} D_i$ for any non-empty subset X of $\{1, \ldots, N\}$; in particular, $D = D_{\{1,\ldots,N\}} = \Pi_{i=1}^N D_i$ and $D_{\{i\}} = D_i$.

pr_i is the projection of D_X to the i-coordinate D_i for $i \in X$, and so is pr_X of $D_{X'}$ to D_X for $X \subset X'$ such that the composition function $\mathrm{pr}_i \circ \mathrm{pr}_X$ of $D_{X'}$ to D_i coincides with pr_i for any $i \in X$.

EXAMPLE 5.1. Let $D_1 = \{\mathrm{ABC, ACC, NSS, QQQ, WSE, BTT}\}$, $D_2 = \{\mathrm{KYOTO, TOKYO}\}$, and D_3, D_4, D_5, D_6 be the set of non-negative integers. Then the following table is a relation.

TABLE 5.1. Example of a relation

ABC	KYOTO	1	1	200	27
ACC	KYOTO	1	2	310	31
NSS	KYOTO	2	1	140	56
PCC	KYOTO	2	2	100	20
QQQ	TOKYO	2	2	580	21
WSE	TOKYO	2	1	120	74
BTT	TOKYO	1	2	230	35
AXT	TOKYO	1	1	130	18

DEFINITION 5.2. For any set A, we denote by A^* the set of all finite sequences in A. Moreover, we call g a *summarizing operator* over A, if g is a function of A^* to A satisfying

$$g(\mathbf{a}') = g(\mathbf{a}) \qquad \text{for any permutation } \mathbf{a}' \text{ of } \mathbf{a} \in A^* .$$

EXAMPLE 5.2. SUM, MAX and MEDIAN, which take the sum, the maximum and the median, respectively, of any finite sequence of real numbers, are summarizing operators over the set of all real numbers.

DEFINITION 5.3. Let R be a given relation of $(D_1, \ldots, D_N)$ in Definition 5.1. Then, for any non-empty proper subset

$$X \subset \{1, \ldots, N\} \qquad \text{and} \qquad j \in \{1, \ldots, N\} - X ,$$

we can define a function $E_{j,X} = E_{j,X,R}$ of D_X to D_j^* by

$$E_{j,X}(\mathbf{d}) = (\mathrm{pr}_j(\mathbf{r}) \colon \mathbf{r} \in R \text{ with } \mathrm{pr}_j(\mathbf{r}) = \mathbf{d}) \in D_j^*$$

for $\mathbf{d} \in D_X$, where the notations in Definition 5.1 are used.

EXAMPLE 5.3. If $X = \{2\}$ and $i = 5$ in Example 5.1, then

$$E_{i,X,R}(\text{KYOTO}) = (200, 310, 140, 100) ,$$

$$E_{i,X,R}(\text{TOKYO}) = (580, 120, 230, 130) , \qquad \text{and}$$

$$E_{i,X,R}(D_2) = \{(200, 310, 140, 100), (580, 120, 230, 130)\} .$$

DEFINITION 5.4. In addition to Definition 5.3, let

$$G = \{g_j \colon j \in Y(G)\} , \qquad \text{for} \qquad \varnothing \neq Y = Y(G) \subset \{1, \ldots, N\} - X ,$$

be a set such that each $g_j (j \in Y)$ is a summarizing operator over D_j in Definition 5.2. Then, we can define a function $S_{G,X} = S_{G,X,R}$ of D_X to $D_{X \cup Y}$ by

$$S_{G,X}(\mathbf{d}) = (\mathbf{d}, (g_j(E_{j,X}(\mathbf{d}) \colon j \in Y)) \in D_X \times D_Y = D_{X \cup Y}$$

for $\mathbf{d} \in D_X$, by using $E_{j,X} = E_{j,X,R}$ in Definition 5.3; and the composition

$$S_{G,X} \circ \mathrm{pr}_X \quad \text{of} \quad D_{X'} \quad \text{to} \quad D_{X \cup Y} , \quad \text{for} \quad X \subset X' \subset \{1, \ldots, N\} ,$$

by the projection pr_X of $D_{X'}$ to D_X, is also denoted by the same letter $S_{G,X}$.

Such a function $S_{G,X} = S_{G,X} \circ \mathrm{pr}_X$ is called an *aggregation function* on a given relation R of $(D_1, \dots, D_N)$, given by summarizing operators g_j over D_j for $j \in Y$. Moreover, the image

$$S(G, X; R) = S_{G,X}(R) = S_{G,X,R}(R) \subset D_{X \cup Y}$$

is called *the summary data*, or *the summary table*, of R by $S_{G,X}$; and X and Y are called *a set of category fields* and *a set of summary fields*, respectively (cf. [15], [20], [22]).

LEMMA 5.1. *In the above definition, we have*

$$|S(G, X; R)| = |\mathrm{pr}_X(R)| ,$$

where $|A|$ denotes the cardinal number of a finite set A. In particular,

$$|S(G, X; R)| = \textstyle\prod_{i \in X} |\mathrm{pr}_i(R)| \qquad \text{if} \qquad \mathrm{pr}_X(R) \supset \textstyle\prod_{i \in X} \mathrm{pr}_i(R) .$$

PROOF. By the definition, $S_{G,X}(\mathbf{d}) = S_{G,X}(\mathrm{pr}_X(\mathbf{d}))$ and $\mathrm{pr}_X(S_{G,X}(\mathbf{d})) = \mathrm{pr}_X(\mathbf{d})$ for any $\mathbf{d} \in D$. Thus, for any $\mathbf{d}$, $\mathbf{d}' \in D$, we see that

$$S_{G,X}(\mathbf{d}) = S_{G,X}(\mathbf{d}') \text{ if and only if } \mathrm{pr}_X(\mathbf{d}) = \mathrm{pr}_X(\mathbf{d}') .$$

This equivalence shows the first equality.

If $\mathrm{pr}_X(R)$ contains $\Pi_{i \in X} \mathrm{pr}_i(R)$, then these coincide clearly. Hence the first half implies the second one immediately.

EXAMPLE 5.4. Let $X = \{2\}$, $Y = \{5, 6\}$, g_5 and g_6 be SUM, and R be the table in Example 5.1. Then

$$S_{G,X}(R) = \{(\text{KYOTO}, 750, 134), (\text{TOKYO}, 1060, 148)\} .$$

DEFINITION 5.5. A summarizing operator g over a set A in Definition 5.2 is called associative if

$$g(g(\mathbf{a}), g(\mathbf{a}')) = g(\mathbf{a} \cup \mathbf{a}') \qquad \text{for any } \mathbf{a}, \mathbf{a}' \in A^* ,$$

where $\mathbf{a} \cup \mathbf{a}' \in A^*$ is the sequence in A obtained by drawing up $\mathbf{a}$ and $\mathbf{a}'$ in a line.

Then, Sato [20] proved the following

THEOREM 5.2. *Let R be a given relation of $(D_1, \dots, D_N)$, and X, X' and Y be non-empty subsets of $\{1, \dots, N\}$ with $X \subset X'$ and $X' \cap Y = \varnothing$. Moreover, assume that each summarizing operator g_j over D_j in a set $G = \{g_j : j \in Y\}$ is associative. Then, we have*

$$S(G, X; S(G, X'; R)) = S(G, X; R)$$

for the summary table S given in Definition 5.4, where the left hand side $S(G, X; R')$ is the one of $R' = S(G, X'; R)$ considered as a relation of $(D_j\colon i \in X' \cup Y)$.

PROOF. By Definition 5.4, the equality follows immediately from the one

$$S_{G,X,R'}(S_{G,X',R}(\mathbf{r})) = S_{G,X,R}(\mathbf{r}) \qquad \text{for any } \mathbf{r} \in R\,,$$

or $g_j(E_{j,X,R'}(\mathbf{d})) = g_j(E_{j,X,R}(\mathbf{d}))$ for any $j \in Y$ and $\mathbf{d} \in D_X$.

Now, $\mathbf{r}' \in R' = S(G, X'; R) = S_{G,X',R}(R)$ means that

$$\mathrm{pr}_{X'}(\mathbf{r}') = \mathbf{d}' \qquad \text{and} \qquad \mathrm{pr}_j(\mathbf{r}') = g_j(E_{j,X',R}(\mathbf{d}'))\ (j \in Y)$$

for some $\mathbf{d}' \in \mathrm{pr}_{X'}(R)$, and $\mathbf{r}'$ is determined uniquely by $\mathbf{d}'$ (see the proof of Lemma 5.1). Therefore, by Definition 5.3, we see that

$$E_{j,X,R'}(\mathbf{d}) = (g_j(E_{j,X',R}(\mathbf{d}'))\colon \mathbf{d}' \in \mathrm{pr}_{X'}(R) \text{ and } \mathrm{pr}_X(\mathbf{d}') = \mathbf{d})\,,$$

where the sequences $E_{j,X',R}(\mathbf{d}')$ form a partition of the one $E_{j,X,R}(\mathbf{d})$. Thus $g_j(E_{j,X,R'}(\mathbf{d})) = g_j(E_{j,X,R}(\mathbf{d}))$, since g_j is associative; and the theorem is proved.

DEFINITION 5.6. Let R be a given relation of $(D_1, \ldots, D_N)$, and consider the collection FSAF of all finite sets of aggregation functions on R in Definition 5.4.

Then, for $\mathscr{S}$ and $\mathscr{S}'$ in FSAF, we say that $\mathscr{S}$ is *derivable* from $\mathscr{S}'$, denoted by

$$\mathscr{S}' \to \mathscr{S}\,,$$

if these satisfy the following condition (D):

(D) For any $S_{G,X}$, in $\mathscr{S}$, there exists a subset $\mathscr{S}''$ of $\mathscr{S}'$ such that

$$\bigcup \{G'\colon S_{G',X'} \in \mathscr{S}''\} = G$$

and that for each $S_{G',X'} \in \mathscr{S}''$,

$$X' \supset X \qquad \text{and} \qquad S(G', X; S(G', X'; R)) = S(G', X; R)$$

hold under the meaning stated in the above theorem.

EXAMPLE 5.5. Let $X_1 = \{2, 3\}$, $X_2 = \{2, 4\}$, $G_1 = \{g_5\}$, $G_2 = \{g_6\}$, $\mathscr{S}' = \{S_{G_1,X_1}, S_{G_2,X_2}\}$, $X = \{2\}$, $G = \{g_5, g_6\}$, and $\mathscr{S} = \{S_{G,X}\}$ in Example 5.1. Then $S_{G_1,X_1}(R)$, $S_{G_2,X_2}(R)$, and $S_{G,X}(R)$ are given as follows:

$S_{G_1,X_1}(R)$		
KYOTO	1	510
KYOTO	2	240
TOKYO	1	360
TOKYO	2	700

$S_{G_2,X_2}(R)$		
KYOTO	1	83
KYOTO	2	51
TOKYO	1	92
TOKYO	2	56

$S_{G,X}(R)$		
KYOTO	750	134
TOKYO	1060	148

We can easily show that $\mathscr{S}$ is derivable from $\mathscr{S}'$.

DEFINITION 5.7. In the above definition, we consider the number of records

$$\mathrm{NRec}(\mathscr{S}) = \sum |S(G, X; R)| = \sum |S_{G,X}(R)| \qquad \text{for any } \mathscr{S} \in \mathrm{FSAF},$$

where the sum is taken over all $S_{G,X} \in \mathscr{S}$.

Also, we consider the set

$$\mathscr{A}(\mathscr{S}) = \{X \colon S_{G,X} \in \mathscr{S}\} \quad (X \subset \{1, \dots, N\}) \qquad \text{for } \mathscr{S} \in \mathrm{FSAF},$$

and the positive integer

$$k(X) = k(X; \mathscr{S}) = |\{G \colon S_{G,X} \in \mathscr{S}\}| \qquad \text{for } X \in \mathscr{A}(\mathscr{S}).$$

Moreover, we say that a given relation R is *full* over $\mathscr{S}$, if

$$\text{(F)} \qquad \mathrm{pr}_U(R) = \prod_{i \in U} \mathrm{pr}_i(R) \qquad \text{for } U = \langle \mathscr{A}(\mathscr{S}) \rangle \subset \{1, \dots, N\};$$

and consider the positive integer $f(i)$ defined by

$$f(i) = |\mathrm{pr}_i(R)| \qquad \text{for each } i \in \{1, \dots, N\}.$$

PROPOSITION 5.3. *In Definition* 5.7, *we have*

$$\mathrm{NRec}(\mathscr{S}) = \textstyle\sum_{X \in \mathscr{A}(\mathscr{S})} k(X) |\mathrm{pr}_X(R)| \qquad \textit{for any } \mathscr{S} \in \mathrm{FSAF}.$$

If a given relation R is full over $\mathscr{S}$, then

$$\mathrm{NRec}(\mathscr{S}) = \textstyle\sum_{X \in \mathscr{A}(\mathscr{S})} k(X) \prod_{i \in X} f(i);$$

and if $k(X) = 1$ for any $X \in \mathscr{A}(\mathscr{S})$ in addition, then

$$\mathrm{NRec}(\mathscr{S}) = \textstyle\sum_{X \in \mathscr{A}(\mathscr{S})} \prod_{i \in X} f(i) = \mathrm{SP}_f(\mathscr{A}(\mathscr{S})),$$

where SP_f is the function given in Definition 2.1.

PROOF. By the definition, Lemma 5.1 implies the proposition.

DEFINITION 5.8. When $\mathscr{S}_0 \in \mathrm{FSAF}$ is given, we call $\mathscr{S} \in \mathrm{FSAF}$ a *minimum record set* for $\mathscr{S}_0$, if

$$\text{(MRS)} \quad \mathscr{S} \to \mathscr{S}_0 \text{ and } \mathrm{NRec}(\mathscr{S}) = \mathrm{Min}\{\mathrm{NRec}(\mathscr{S}') \colon \mathscr{S}' \in \mathrm{FSAF} \text{ and } \mathscr{S}' \to \mathscr{S}_0\},$$

under the notations in Definitions 5.6 and 5.7; and then we say also that $\mathscr{S}$ is a *solution of statistical database design problem*, or simply, a *solution of* $\mathrm{SDD}(\mathscr{S}_0)$, under a given relation R.

Now, this problem is solved under some assumptions by finding an MSPD stated in Definition 2.3. In fact, we can prove the following theorem, which is a motivation of this paper (cf. [15]).

THEOREM 5.4. *Let R be a given relation of $(D_1, \ldots, D_N)$. For a given finite set $\mathscr{S}_0 \in$ FSAF of aggregation functions on R, put*

$$\mathscr{B} = \mathscr{A}(\mathscr{S}_0) = \{X: S_{G,X} \in \mathscr{S}_0\} \qquad \text{and} \qquad U = \langle \mathscr{B} \rangle \, ;$$

and assume that

(F) *R is full over $\mathscr{S}_0$, i.e., $\mathrm{pr}_U(R) = \prod_{i \in U} \mathrm{pr}_i(R)$, and*

(USAO) *$\mathscr{S}_0$ has a unique set of associative operators, i.e., there exists G_0 such that $G = G_0$ for any $S_{G,X} \in \mathscr{S}_0$, and each $g_j \in G_0$ is associative.*

Then, we have the following (i) *and* (ii):

(i) *If $\mathscr{S}$ is a solution of* SDD($\mathscr{S}_0$), *then*

$$\mathscr{A}(\mathscr{S}) \cap U = \{X \cap U: S_{G,X} \in \mathscr{S}\} - \{\varnothing\}$$

is an MSPD *for $\langle U, \mathscr{B}, f \rangle$ in Definition* 2.3, *where $f(i) = |\mathrm{pr}_i(R)|$ for $i \in U$.*

(ii) *Conversely, if $\mathscr{A}$ is an* MSPD *for $\langle U, \mathscr{B}, f \rangle$, then*

$$\mathscr{S}(\mathscr{A}) = \{S_{G_0,X}: X \in \mathscr{A}\} \quad (G_0 \text{ is the one in (USAO)})$$

is a solution of SDD($\mathscr{S}_0$).

PROOF. Assume that $\mathscr{S} \to \mathscr{S}_0$. Then, $\mathscr{A}(\mathscr{S}) \geq \mathscr{B}$ and $\mathscr{A}(\mathscr{S}) \cap U \geq \mathscr{B}$ by Definition 5.5. Moreover, Proposition 5.3 and the assumption (F) imply that

$$\begin{aligned}\mathrm{NRec}(\mathscr{S}) &= \sum k(X)|\mathrm{pr}_X(R)| \geqq \sum |\mathrm{pr}_X(R)| \\ &= \sum |\mathrm{pr}_{X \cap U}(R)| = \sum |\textstyle\prod_{i \in X \cap U}(R)| \geqq \mathrm{SP}_f(\mathscr{A}(\mathscr{S}) \cap U)\end{aligned}$$

(the sum $\sum$ is taken over all $X \in \mathscr{A}(\mathscr{S})$).

On the other hand, assume that $\mathscr{A} \geq \mathscr{B}$ for $\mathscr{A} \subset 2^U - \{\varnothing\}$. Then,

$$\mathscr{S}(\mathscr{A}) = \{S_{G_0,X}: X \in \mathscr{A}\} \to \mathscr{S}_0$$

by Theorem 5.2 and the assumption (USAO), and

$$\mathscr{A}(\mathscr{S}(\mathscr{A})) = \mathscr{A} \qquad \text{and} \qquad \mathrm{NRec}(\mathscr{S}(\mathscr{A})) = \mathrm{SP}_f(\mathscr{A})$$

by Proposition 5.3 and (F).

(i) Let $\mathscr{S}$ be a solution of SDD($\mathscr{S}_0$). Then $\mathscr{S} \to \mathscr{S}_0$, $\mathscr{A}(\mathscr{S}) \cap U \geq \mathscr{B}$ and

$$\mathrm{SP}_f(\mathscr{A}) = \mathrm{NRec}(\mathscr{S}(\mathscr{A})) \geqq \mathrm{NRec}(\mathscr{S}) \geqq \mathrm{SP}_f(\mathscr{A}(\mathscr{S}) \cap U)$$

for any $\mathscr{A} \subset 2^U - \{\varnothing\}$ with $\mathscr{A} \geq \mathscr{B}$, by the above results and by Definition 5.7. Thus $\mathscr{A}(\mathscr{S}) \cap U$ is an MSPD for $\langle U, \mathscr{B}, f \rangle$ by the definition.

(ii) Let $\mathscr{A}$ be an MSPD for $\langle U, \mathscr{B}, f \rangle$. Then, in the same way, we have

$$\mathrm{NRec}(\mathscr{S}) \geqq \mathrm{SP}_f(\mathscr{A}(\mathscr{S}) \cap U) \geqq \mathrm{SP}_f(\mathscr{A}) = \mathrm{NRec}(\mathscr{S}(\mathscr{A}))$$

for any $\mathscr{S} \in \mathrm{FSAF}$ with $\mathscr{S} \to \mathscr{S}_0$; hence $\mathscr{S}(\mathscr{A})$ is a solution of $\mathrm{SDD}(\mathscr{S}_0)$.

EXAMPLE 5.6. In Example 5.1, put $Y = \{5, 6\}$, $U = \{2, 3, 4\}$, $f_5 = f_6 = \mathrm{SUM}$, $G = \{g_5, g_6\}$, $f(i) = |\mathrm{pr}_i(R)| = 2$ for i $(2 \leqq i \leqq 4)$, $\mathscr{B} = \{\{2, 3\}, \{2, 4\}, \{3, 4\}\}$, and $\mathscr{S}_0 = \{S_{G,B}\colon B \in \mathscr{B}\}$. Then we see easily that $\mathrm{pr}_U(R) = \Pi_{i\in U}\,\mathrm{pr}_i(R)$. Hence in order to find a solution of $\mathrm{SDD}(\mathscr{S}_0)$, we may find an MSPD for $\langle U, \mathscr{B}, f\rangle$ illustrated as follows.

	2	3	4
B_1	1	1	
B_2	1		1
B_3		1	1
f	2	2	2

Since $\{U\}$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$, $\{S_{G,U}\}$ is a solution of $\mathrm{SDD}(\mathscr{S}_0)$.

6. An algorithm to find an MSPD

In addition to a necessary condition of the indecomposability given in Proposition 4.3, we consider another one given by the following

DEFINITION 6.1. Let $\mathscr{A} \subset \mathscr{B}$. Then we say that $\mathscr{C}$ is *maximum* in $\mathscr{A}$ (with respect to $\langle\mathscr{C}\rangle$) *if*

1) $\mathscr{C} \subset \mathscr{A}$, and $\mathscr{C}' \subset \mathscr{C}$ holds for any $\mathscr{C}' \subset \mathscr{A}$ with $\langle\mathscr{C}'\rangle = \langle\mathscr{C}\rangle$;

and we also say that $\mathscr{C}$ is *max-indecomposable* if

2) $\mathscr{C}$ is indecomposable in addition to 1).

PROPOSITION 6.1. *If $\mathscr{C} \subset \mathscr{A}$ and $\mathscr{C}$ is indecomposable, then*

$$\mathscr{C}^* = \{A \in \mathscr{A}\colon A \subset \langle\mathscr{C}\rangle\}$$

is max-indecomposable, and satisfies $\mathscr{C}^* \supset \mathscr{C}$ and $\langle\mathscr{C}^*\rangle = \langle\mathscr{C}\rangle$.

In particular, if $|\mathscr{C}| = 1$, then $\mathscr{C}^* = \mathscr{C}$.

PROOF. By the definition of $\mathscr{C}^*$, we see that $\mathscr{C} \subset \mathscr{C}^* \subset \mathscr{A}$, $\langle\mathscr{C}^*\rangle \subset \langle\mathscr{C}\rangle$ and so $\langle\mathscr{C}^*\rangle = \langle\mathscr{C}\rangle$. These and Proposition 4.2 show that $\mathscr{C}^*$ is indecomposable if so is $\mathscr{C}$. If $\langle\mathscr{C}'\rangle = \langle\mathscr{C}^*\rangle$ for $\mathscr{C}' \subset \mathscr{A}$, then $\mathscr{C}' \subset \mathscr{C}^*$ holds because $A \in \mathscr{C}'$ implies that $A \subset \langle\mathscr{C}'\rangle = \langle\mathscr{C}^*\rangle = \langle\mathscr{C}\rangle$, i.e., $A \in \mathscr{C}^*$. Therefore $\mathscr{C}^*$ is maximum in $\mathscr{A}$. The last assertion follows from the assumption (A1). Thus we see the proposition.

PROPOSITION 6.2. *Let $\mathscr{A} \subset \mathscr{B}$, $A \in \mathscr{A}$ and $C \subset \langle\mathscr{A}\rangle - A$, and consider the following conditions* 1), 2) *and* 3) *for $\mathscr{C} \subset \mathscr{A}$:*

1) *$\mathscr{C}$ is max-indecomposable in $\mathscr{A}$, $\mathscr{C} \ni A$, $\langle\mathscr{C}\rangle \cap C = \emptyset$ and $\mathscr{C} \neq \mathscr{A}$.*

2) *There exist* $u \in \langle \mathscr{A} \rangle - A$ *and a* $(C \cup \{u\})$-MIF $\mathscr{C}'$ *of* $\{A\}$ *in* $\mathscr{A}$ *with* $\mathscr{C}' \supset \mathscr{C}$.

3) $\mathscr{C}$ *is a* C'-MIF *of* $\{A\}$ *in* $\mathscr{A}$ *for some* $C' \subset \langle \mathscr{A} \rangle - A$ *with* $C' \supsetneqq C$.

Then, 1) *implies* 2), *and* 3) *implies* 1).

PROOF. Assume 1). Then $\langle \mathscr{C} \rangle \neq \langle \mathscr{A} \rangle$. In fact, if $\langle \mathscr{C} \rangle = \langle \mathscr{A} \rangle$, then $\mathscr{A} \subset \mathscr{C} \subset \mathscr{A}$ and $\mathscr{C} = \mathscr{A}$, since $\mathscr{C}$ is maximum in $\mathscr{A}$. Therefore, there exists $u \in \langle \mathscr{A} \rangle - \langle \mathscr{C} \rangle \subset \langle \mathscr{A} \rangle - A$. Then $\langle \mathscr{C} \rangle \cap (C \cup \{u\}) = \varnothing$, $\mathscr{C} \supset \{A\}$ and $\mathscr{C}$ is indecomposable; hence there exists $\mathscr{C}'$ in 2) by Definition 4.1. Thus 2) holds.

Assume 3). Then $\langle \mathscr{C} \rangle \subset \langle \mathscr{A} \rangle - C' \subsetneqq \langle \mathscr{A} \rangle - C \subset \langle \mathscr{A} \rangle$; hence $\mathscr{C} \neq \mathscr{A}$ and $\langle \mathscr{C} \rangle \cap C = \varnothing$. $\mathscr{C} \ni A$ and $\mathscr{C}$ is indecomposable by the definition. Therefore, by the above proposition, $\mathscr{C}^*$ is max-indecomposable in $\mathscr{A}$, $\mathscr{C}^* \supset \mathscr{C}$ and $\langle \mathscr{C}^* \rangle = \langle \mathscr{C} \rangle$. Thus $\langle \mathscr{C}^* \rangle \cap C' = \varnothing$ and $\mathscr{C}^* = \mathscr{C}$, since $\mathscr{C}$ is a C'-MIF of $\{A\}$ in $\mathscr{A}$. Thus 1) holds.

COROLLARY 6.3. *We can find all* $\mathscr{C}$ *satisfying* 1) *in the above proposition, by finding* C'-MIF's *of* $\{A\}$ *in* $\mathscr{A}$ *and by using the above proposition.*

PROOF. In this proof, 1), 2) and 3) are the conditions in the above proposition.

If $|\langle \mathscr{A} \rangle - A| = 0$, then 2) does not hold, and there exists no $\mathscr{C}$ with 1), because 1) implies 2).

Assume inductively that the corollary holds when $|\langle \mathscr{A} \rangle - A| \leqq k$ for some $k \geqq 0$; and consider the case $|\langle \mathscr{A} \rangle - A| = k + 1$.

If there exists no $(C \cup \{u\})$-MIF of $\{A\}$ in $\mathscr{A}$ for any $u \in \langle \mathscr{A} \rangle - A$, then there exists no $\mathscr{C}$ satisfying 1).

Assume that there exists a $(C \cup \{u\})$-MIF $\mathscr{C}$ of $\{A\}$ in $\mathscr{A}$ for some $u \in \langle \mathscr{A} \rangle - A$. Then $|\langle \mathscr{C}' \rangle - A| \leqq k$ holds, because $A \subset \langle \mathscr{C} \rangle \subsetneqq \langle \mathscr{C} \rangle \cup \{u\} \subset \langle \mathscr{A} \rangle$. Thus, we can find all max-indecomposable subfamilies $\mathscr{C}'$ in $\mathscr{C}$ satisfying

$$\mathscr{A} \supset \mathscr{C} \supset \mathscr{C}' \supset \{A\}, \quad \langle \mathscr{C}' \rangle \cap C = \varnothing \qquad \text{and} \qquad \mathscr{C}' \neq \mathscr{C},$$

by the inductive assumption. Moreover, $\mathscr{C}$ satisfies 3) and so does 1) by the above proposition. Thus we can obtain all $\mathscr{C}$'s in 1) in this case.

Thus the desired result is proved by induction on $|\langle \mathscr{A} \rangle - A|$.

Now, we give an algorithm to find an MSPD for $\langle U, \mathscr{B}, f \rangle$, which is done by the following steps:

(I) We prepare all $\varnothing$-MIF's of $\{A\}$ in $\mathscr{A}$ $(\subset \mathscr{B})$ for some $A \in \mathscr{A}$ by Theorem 4.8.

(II) We determine all max-indecomposable subfamilies of $\mathscr{A}$ $(\subset \mathscr{B})$ by Corollary 6.3.

(III) Any indecomposable subfamilies of $\mathscr{B}$ are found from the ones with the necessary conditions in Propositions 4.1 and 4.2.

(IV) We determine the set $\mathrm{IP}(\mathscr{B})$ of all indecomposable partitions of $\mathscr{B}$ by (III), and also the subsets $\mathrm{P}_1(\mathscr{B})$ and $\mathrm{P}_2(\mathscr{B}, f)$ by the definition.

(V) We can find an MSPD for $\langle U, \mathscr{B}, f\rangle$ by calculating $\mathrm{SP}_f(\mathscr{A}_\alpha)$ for all α in the set $\mathrm{P}_2(\mathscr{B}, f)$ in (IV) and by investigating their minimum value, according to Theorem 4.13. Here, the larger set $\mathrm{P}_1(\mathscr{B})$ or $\mathrm{IP}(\mathscr{B})$ in (IV) may be useful when $|\mathrm{P}_1(\mathscr{B})|$ or $|\mathrm{IP}(\mathscr{B})|$ is not so large, or may be desirable to find more MSPD's.

Our algorithm is given as follows:

$\langle$Algorithm MAIN$\rangle$

Purpose: To find an MSPD $\mathscr{A}_0$ for $\langle U, \mathscr{B}, f\rangle$.

(S1) Set $\mathscr{A}$ as $\mathscr{B}$.

(S2) If $|\mathscr{A}| \leqq 2$, then

(S3) store $\{\{B\}: B \in \mathscr{A}\}$ into $\mathscr{P}$ and

(S4) go to step S15

(S5) Compute I by Algorithm INDEC with parameter $\mathscr{A}$.

(S6) If $\mathscr{A}$ is decomposable ($I = 1$), then do steps S7 and S8.

(S7) Find $A \in \mathscr{A}$ such that there exist u and u' in $\langle\mathscr{A}\rangle$ satisfying the following conditions i), ii) and iii):

i) $|\{B \in \mathscr{A}: B \ni u\}| \leqq |\{B \in \mathscr{A}: B \ni u'\}|$,

ii) $A \ni u$ and

iii) there exists no element B of $\mathscr{A}$ with $u, u' \in B$.

(S8) Obtain $\mathscr{P}$ by Algorithm ALL with parameters $\mathscr{A}$, A and $\varnothing$.

(S9) If $\mathscr{A}$ is indecomposable ($I = 0$), then do steps (S10–S12).

(S10) If there exists no $\mathscr{A}' \in \mathscr{P}$ with $\mathscr{A} \subset \mathscr{A}'$ and $\langle\mathscr{A}\rangle = \langle\mathscr{A}'\rangle$, then store $\mathscr{A}$ into $\mathscr{P}$.

(S11) Find an element A of $\mathscr{A}$ such that

$$|A| = \mathrm{Max}\{|B|: B \in \mathscr{A}\}\,.$$

(S12) Obtain $\mathscr{P}$ by Algorithm IND with parameters $\mathscr{A}$ and A.

(S13) Remove A from $\mathscr{A}$.

(S14) Go to step S2.

(S15) Obtain $\mathscr{P}_1$ by Algorithm P1 with parameter $\mathscr{P}$.

(S16) Obtain $\mathscr{P}_2$ by Algorithm P2 with parameter $\mathscr{P}_1$.

(S17) Find $\beta \in \mathscr{P}_2$ satisfying $\mathrm{SP}_f(\mathscr{A}_\beta) \leqq \mathrm{SP}_f(\mathscr{A}_\alpha)$ for any $\alpha \in \mathscr{P}_2$, by calculating $\mathrm{SP}_f(\mathscr{A}_\alpha)$ for all elements α in $\mathscr{P}_2$.

(S18) Set $\mathscr{A}_0$ as $\mathscr{A}_\beta$.

(S19) End MAIN.

$\langle$Algorithm IND($\mathscr{A}$, A, $\mathscr{P}$)$\rangle$

Purpose: To store into $\mathscr{P}$ all max-indecomposable subfamilies $\mathscr{C}$ of $\mathscr{A}$ satisfying $\mathscr{A} \supsetneqq \mathscr{C} \supset \{A\}$.

Input parameters: $\mathscr{A}$ and $A(\mathscr{A} \ni A)$.
Output parameters: $\mathscr{P}$.
(S20) Obtain $\mathscr{P}$ by Algorithm ALL with parameters $\mathscr{A}$, $\{A\}$ and $\{u\}$ for all elements u of $\langle \mathscr{A} \rangle - A$.
(S21) End IND.

⟨Algorithm ALL ($\mathscr{A}, A, C, \mathscr{P}$)⟩
Purpose: To store into $\mathscr{P}$ all max-indecomposable subfamilies $\mathscr{C}$ of $\mathscr{A}$ satisfying

$$\mathscr{A} \supsetneq \mathscr{C} \supset \{A\} \text{ and } \langle \mathscr{C} \rangle \cap C = \varnothing .$$

Input parameters: $\mathscr{A}$, A and C ($\mathscr{A} \ni A$, $\langle \mathscr{A} \rangle \supset C$, $A \cap C = \varnothing$).
Output parameters: $\mathscr{P}$.
(S22) Obtain $\mathscr{M}$ by Algorithm MAX with parameters $\mathscr{A}$, $\{A\}$ and C.
(S23) If $\mathscr{M} = \varnothing$, then exit.
(S24) If $\mathscr{M} \neq \varnothing$, then do steps S25 and S26 for all elements $\mathscr{C}$ of $\mathscr{M}$.
(S25) Store $\mathscr{C}$ into $\mathscr{P}$.
(S26) If $\langle \mathscr{C} \rangle \neq A$, then obtain $\mathscr{P}$ by Algorithm ALL with parameters $\mathscr{A}$, A and $(C \cup \{u\})$ for all elements u of $\langle \mathscr{C} \rangle - A$.
(S27) End ALL.

⟨Algorithm MAX($\mathscr{A}, \mathscr{D}, C, \mathscr{M}$)⟩
Purpose: To obtain the set $\mathscr{M}$ of all C-MIF's of $\mathscr{D}$ in $\mathscr{A}$.
Input parameters: $\mathscr{A}$, $\mathscr{D}$ and C ($\mathscr{D} \neq \varnothing$, $\mathscr{A} \supsetneq \mathscr{D}$, $\langle \mathscr{D} \rangle \cap C = \varnothing$).
Output parameters: $\mathscr{M}$.
(S28) Obtain $D^*(\mathscr{D}, C)$.
(S29) If $D^*(\mathscr{D}, C) \not\supset \mathscr{D}$, then set $\mathscr{M}$ as $\varnothing$.
(S30) If $D^*(\mathscr{D}, C) \supset \mathscr{D}$ and $D^*(\mathscr{D}, C)$ is indecomposable, then set $\mathscr{M}$ as $\{D^*(\mathscr{D}, C)\}$.
(S31) If $D^*(\mathscr{D}, C) \supset \mathscr{D}$ and $D^*(\mathscr{D}, C)$ is decomposable, then do steps S32–S35.
(S32) Set $\mathscr{M}$ as $\varnothing$.
(S33) If $D^*(\mathscr{D}, C) = \mathscr{D}$, then exit.
(S34) Set $\mathscr{C}$ as $D^0(C)$
(S35) Repeat step S36 to S39 until $\mathscr{C} = \mathscr{D}$.
(S36) Take B from $\mathscr{C} - \mathscr{D}$.
(S37) Obtain $\mathscr{M}'$ by Algorithm MAX with parameters $\mathscr{A}$, $(\mathscr{C} - \{B\})$ and C.
(S38) Store all elements of $\mathscr{M}'$ into $\mathscr{M}$.
(S39) Remove B from $\mathscr{C}$.
(S40) End MAX.

⟨Algorithm INDEC($\mathscr{A}$, I)⟩
Purpose: To investigate that $\mathscr{A}$ is indecomposable or not.
Input parameters: $\mathscr{A}$.

Output parameters: $I(I = 0$ or 1 according as $\mathscr{A}$ is indecomposable or not).
(S41) Set I as 1.
(S42) If $\langle\{A \in \mathscr{A}: A \ni a\}\rangle \neq \langle\mathscr{A}\rangle$ for some element $a \in \langle\mathscr{A}\rangle$, then exit.
(S43) Set I as 0.
(S44) End INDEC.

$\langle$Algorithm P1$(\mathscr{P}, \mathscr{P}_1)\rangle$
Purpose: To obtain the set $\mathrm{P}_1(\mathscr{B})$ in Theorem 3.8.
Input parameters: $\mathscr{P}$.
Output parameters: $\mathscr{P}_1$.
(S45) For all subset $\mathscr{P}'$ of $\mathscr{P}$,
if there exists indecomposable subset $\mathscr{C}_{\mathscr{A}}$ of $\mathscr{A}$ with $\langle\mathscr{C}_{\mathscr{A}}\rangle = \langle\mathscr{A}\rangle$ for each $\mathscr{A}$ in $\mathscr{P}'$ and $\alpha = \{\mathscr{C}_{\mathscr{A}}: \mathscr{A} \in \mathscr{P}'\}$ satisfies the condition (P_1) in Definition 3.3, then store α into $\mathscr{P}_1$.
(S46) End P1.

$\langle$Algorithm P2$(\mathscr{P}_1, \mathscr{P}_2)\rangle$
Purpose: To obtain the set $\mathrm{P}_2(\mathscr{B}, f)$ in Theorem 4.13.
Input parameters: $\mathscr{P}_1$.
Output parameters: $\mathscr{P}_2$.
(S47) For each element $\alpha \in \mathscr{P}_1$,
if $\mathrm{SP}_f(\mathscr{C}) > \mathrm{SP}_f(\{\langle\mathscr{C}\rangle\})$ for all $\mathscr{C}$'s in α with $|\mathscr{C}| \geqq 3$, then store α into $\mathscr{P}_2$.
(S48) End P2.

THEOREM 6.4. *For a given* $\langle U, \mathscr{B}, f\rangle$ *with* (A1–A2), *the above algorithms can find an* MSPD *by finite processes.*

PROOF. Corollaries 4.12, 6.3 with $C = \varnothing$, 6.3, and Proposition 4.2 imply that the algorithms MAX, IND, ALL and INDEC, respectively, are correct. Assume that $\mathscr{A}$ is a max-indecomposable subfamily of $\mathscr{B}$. If an element A of $\mathscr{A}$ is selected in steps S7 or S11 of the algorithm MAIN, then $\mathscr{A}$ is stored into $\mathscr{P}$ in steps S8 or S12, respectively. If an element A of $\mathscr{A}$ is not selected in steps S7 or S11, then $\mathscr{A}$ is a max-indecomposable subfamily of $\mathscr{B} - \{A\}$.

Since $\mathscr{B}$ is a finite set, the algorithm MAIN can go to step S15 after finite repeats and some element A of $\mathscr{A}$ is selected in step S7 or S11; or $\mathscr{A}$ is stored into $\mathscr{P}$ in step S3 or S10. Thus all max-indecomposable subfamilies in $\mathscr{B}$ are stored in $\mathscr{P}$ before step S15. Theorem 3.8 and Theorem 4.13 imply that there exists $\beta \in \mathscr{P}_2$ with $\mathscr{A}_\beta$ is an MSPD for $\langle U, \mathscr{B}, f\rangle$. This completes the proof.

We demonstrate how to construct an MSPD by the algorithm MAIN and its effectiveness in the following example.

EXAMPLE 6.1.

		U							Part($\mathscr{B}$)		
		u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\mathscr{B}$	$\mathscr{A}_0$	$\{U\}$
$\mathscr{B}$	B_1	1	1	1					12		
	B_2	1	1		1	1			24	24	
	B_3	1		1		1			12	12	
	B_4	1	1				1		12	24	
	B_5	1		1			1		12		192
	B_6		1	1			1		8		
	B_7				1		1		4	4	
	B_8			1		1	1	1	16	16	
	B_9			1	1	1			8	8	
	B_{10}	1		1				1	12	12	
f		3	2	2	2	2	2	2			
SP_f									120	100	192

Let be given U, $\mathscr{B}$ and f in the above table. The table shows the comparison of SP_f of $\mathscr{B}$ itself, $\{U\}$ and an MSPD $\mathscr{A}_0$ for $\langle U, \mathscr{B}, f\rangle$ that is constructed by the algorithm MAIN as follows.

1) Set $\mathscr{A}$ as $\mathscr{B}$ in S1. Then $\mathscr{A}$ is decomposable by S5 and B_8 is found in S7.
2) By Algorithm ALL with parameters $\mathscr{A}$, B_8, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_3, B_5, B_8, B_{10}\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_3, B_5, B_8, B_{10}\}$ is stored into $\mathscr{P}$ in S25.
3) By Algorithm ALL with parameters $\mathscr{A}$, B_8, $\{u_1\}$ and $\mathscr{P}$ in S26, $\{\{B_8\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_8\}$ is stored into $\mathscr{P}$ in S25.
4) $\mathscr{A}$ becomes $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_9, B_{10}\}$ in S13.
5) $\mathscr{A}$ is decomposable by S5 and B_{10} is found in S7.
6) By Algorithm ALL with parameters $\mathscr{A}$, B_{10}, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_{10}\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_{10}\}$ is stored into $\mathscr{P}$ in S25.
7) $\mathscr{A}$ becomes $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_9\}$ in S13.
8) $\mathscr{A}$ is decomposable by S5 and B_2 is found in S7.
9) By Algorithm ALL with parameters $\mathscr{A}$, B_2, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_1, B_2, B_3, B_9\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_1, B_2, B_3, B_9\}$ is stored into $\mathscr{P}$ in S25.

10) By Algorithm ALL with parameters $\mathscr{A}$, B_2, $\{u_3\}$ and $\mathscr{P}$ in S26, $\{\{B_2\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_2\}$ is stored into $\mathscr{P}$ in S25.
11) $\mathscr{A}$ becomes $\{B_1, B_3, B_4, B_5, B_6, B_7, B_9, B_{10}\}$ in S13.
12) $\mathscr{A}$ is decomposable by S5 and B_3 is found in S7.
13) By Algorithm ALL with parameters $\mathscr{A}$, B_3, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_3\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_3\}$ is stored into $\mathscr{P}$ in S25.
14) $\mathscr{A}$ becomes $\{B_1, B_4, B_5, B_6, B_7, B_9\}$ in S13.
15) $\mathscr{A}$ is decomposable by S5 and B_9 is found in S7.
16) By Algorithm ALL with parameters $\mathscr{A}$, B_9, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_9\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_9\}$ is stored into $\mathscr{P}$ in S25.
17) $\mathscr{A}$ becomes $\{B_1, B_4, B_5, B_6, B_7\}$ in S13.
18) $\mathscr{A}$ is decomposable by S5 and B_7 is found in S7.
19) By Algorithm ALL with parameters $\mathscr{A}$, B_7, $\varnothing$ and $\mathscr{P}$ in S8, $\{\{B_7\}\}$ is obtained as $\mathscr{M}$ in S22; and $\{B_7\}$ is stored into $\mathscr{P}$ in S25.
20) $\mathscr{A}$ becomes $\{B_1, B_4, B_5, B_6\}$ in S13.
21) $\mathscr{A}$ is indecomposable by S5 and $\mathscr{A}$ is stored into $\mathscr{P}$ in S10.
22) In Algorithm IND with parameters $\mathscr{A}$, B_1 and $\mathscr{P}$, $\{\{B_1\}\}$ is obtained as $\mathscr{M}$ by S22 of Algorithm ALL with parameters $\mathscr{A}$, B_1, $\{u_6\}$ and $\mathscr{P}$; and $\{B_1\}$ is stored into $\mathscr{P}$ in S25.
23) $\mathscr{A}$ becomes $\{B_4, B_5, B_6\}$ in S13.
24) $\mathscr{A}$ is indecomposable by S5 but $\mathscr{A}$ is not stored into $\mathscr{P}$ in S10, because there exists $\{B_1, B_4, B_5, B_6\}$ satisfying $\langle\{B_1, B_4, B_5, B_6\}\rangle = \langle\{B_4, B_5, B_6\}\rangle$.
25) In Algorithm IND with parameters $\mathscr{A}$, B_4 and $\mathscr{P}$, $\{\{B_4\}\}$ is obtained as $\mathscr{M}$ by S22 of Algorithm ALL with parameters $\mathscr{A}$, B_4, $\{u_3\}$ and $\mathscr{P}$; and $\{B_4\}$ is stored into $\mathscr{P}$ in S25.
26) $\mathscr{A}$ becomes $\{B_5, B_6\}$ in S13.
27) $\{B_5\}$ and $\{B_6\}$ are stored into $\mathscr{P}$ in S3.
28) We have $\mathscr{P} = (\mathscr{A}_i\colon 1 \leqq i \leqq 13\}$, $\mathscr{A}_i = \{B_i\}$ $(1 \leqq i \leqq 10)$, $\mathscr{A}_{11} = \{B_1, B_4, B_5, B_6\}$, $\mathscr{A}_{12} = \{B_1, B_2, B_3, B_9\}$ and $\mathscr{A}_{13} = \{B_3, B_5, B_8, B_{10}\}$ in S15.
29) In Algorithm P1 with parameters $\mathscr{P}$ and $\mathscr{P}_1$, Theorem 3.8 implies that $\{\mathscr{A}_{12}, \mathscr{A}_{11} - \{B_1\}, \mathscr{A}_7, \mathscr{A}_8, \mathscr{A}_{10}\}$ is not an element of $\mathrm{P}_1(\mathscr{B})$, because $\langle\mathscr{A}_{11} - \{B_1\}\rangle \supset B_1$ and $\mathscr{A}_{12} - \{B_1\}$ is decomposable. $\{\mathscr{A}_{13}, \mathscr{A}_{12} - \{B_3\}, \mathscr{A}_4, \mathscr{A}_6, \mathscr{A}_7\}$ is not an element of $\mathrm{P}_1(\mathscr{B})$, because $\langle\mathscr{A}_{12} - \{B_3\}\rangle \supset B_3$ and $\mathscr{A}_{13} - \{B_3\}$ is decomposable. $\{\mathscr{A}_{13}, \mathscr{A}_{11} - \{B_5\}, \mathscr{A}_2, \mathscr{A}_7, \mathscr{A}_9\}$ is not an element of $\mathrm{P}_1(\mathscr{B})$, because $\langle\mathscr{A}_{11} - \{B_5\}\rangle \supset B_5$ and $\mathscr{A}_{13} - \{B_5\}$ is decomposable. Therefore we have $\mathscr{P}_1 = \{\alpha_i\colon 1 \leqq i \leqq 4\}$, where $\alpha_1 = \{\mathscr{A}_i\colon 1 \leqq i \leqq 10\}$, $\alpha_2 = \{\mathscr{A}_i\colon i = 2, 3, 7, 8, 9, 10, 11\}$, $\alpha_3 = \{\mathscr{A}_i\colon i = 4, 5, 6, 7, 8, 10, 12\}$ and $\alpha_4 = \{\mathscr{A}_i\colon i = 1, 2, 4, 6, 7, 9, 13\}$, in S45.
30) By Algorithm P2 with parameters $\mathscr{P}_1$ and $\mathscr{P}_2$ in S16, since

$$\mathrm{SP}_f(\mathscr{A}_{11}) \quad (=44) \quad > \quad \mathrm{SP}_f(\{\langle\mathscr{A}_{11}\rangle\}) \quad (=24)\,,$$

$$\mathrm{SP}_f(\mathscr{A}_{12}) \quad (=56) \quad > \quad \mathrm{SP}_f(\{\langle\mathscr{A}_{12}\rangle\}) \quad (=48) \quad \text{and}$$

$$\mathrm{SP}_f(\mathscr{A}_{13}) \quad (=52) \quad > \quad \mathrm{SP}_f(\{\langle\mathscr{A}_{13}\rangle\}) \quad (=48),$$

we have $\mathscr{P}_2 = \{\alpha_i\colon 1 \leqq i \leqq 4\}$.

31) α_2 is found in S17, because $\mathrm{SP}_f(\mathscr{A}_{\alpha_1}) = 120$, $\mathrm{SP}_f(\mathscr{A}_{\alpha_2}) = 100$, $\mathrm{SP}_f(\mathscr{A}_{\alpha_3}) = 112$ and $\mathrm{SP}_f(\mathscr{A}_{\alpha_4}) = 116$.

32) $\mathscr{A}_0$ as $\mathscr{A}_{\alpha_2}$ in S18.

The above algorithm is not necessary enough to solve a general MSPD problem, but Example 6.1 shows that the algorithm is effective for statistical database designs, because $|U| \leqq 20$ and $|\mathscr{B}| \leqq 200$ in most statistical databases.

Acknowledgements

I would like to express my thanks to Professor Hideto Ikeda, Hiroshima University, for his valuable advices and his encouragement all over this work. I am also grateful to Professor Masahiro Sugawara, Hiroshima University, for his encouragement and his valuable directions during the preparation of this paper. I also wish to thank Miss. Kahoru Futagami, Mr. Masahide Ohtomo and Mr. Fumio Kitagawa, Hiroshima University, for their discussions.

References

[1] D. S. Batory, On searching transposed files, *TODS* (4) **4**(1979), 531–544.

[2] P. Chan and A. Shoshani, SUBJECT: A directory driven system for organizing accessing large statistical databases, *VLDB*'**81**(1981), 553–563.

[3] E. F. Codd, A relational model of data for large shared data banks, *Comm. ACM* (6) **13**(1970), 377–387.

[4] E. F. Codd, Extending the database relational model to capture more meaning, *TODS* (4) **4**(1979), 397–434.

[5] E. F. Codd, Relational database: A practical foundation for productivity, *Comm. ACM* (2) **25**(1982), 109–117.

[6] C. J. Date, *An introduction to database systems*, Volume **II**, Addison Wesley (1983).

[7] D. E. Denning, et al.: TACKER: A threat to statistical database security, *TODS* (1) **4**(1979), 76–96.

[8] S. J. Eggers and A. Shoshani, Efficient access of compressed data, *VLDB*'**80**(1980), 205–211.

[9] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.* (2) **17**(1918), 75–115.

[10] Y. Hotaka and M. Tsubaki, Self-descriptive relational databases, *VLDB*'**77**(1977), 415–426.

[11] H. Ikeda and Y. Kobayashi, Additional facilities of a conventional DBMS to support interactive statistical analysis, *Proc. 1st LBL Workshop on Statistical Database Management* (1981), 25–36.

[12] W. Kent, Choices in practical data design, *VLDB*'**82**(1982), 165–180.

[13] Y. Kobayashi, K. Futagami and H. Ikeda, Implementation of a statistical database system: HSDB, *COMPSTAT*'**82** (1982), 282–287.

[14] Y. Kobayashi, Data dictionary/directory system of a statistical database, *Journal of Information Processing* (4) **7**(1984), 233–239.

[15] Y. Kobayashi and H. Ikeda, Statistical data model and semantics, *Proc. of World Conference on Information Processing/Communication* (1989), 438–443.

[16] E. Malmborg, On the semantics of aggregated data, *Proc. 3rd Int. Workshop on Statistical and Scientific Database Management* (1986), 152–158.

[17] J. L. McCarthy, Metadata management for large statistical databases, *VLDB*'**82**(1982), 234–243.

[18] Z. M. Ozsoyoglu and G. Ozsoyoglu: An extension of relational algebra for summary tables, *Proc. 2nd Int. Workshop on Statistical Database Management* (1983), 202–211.

[19] H. A. Rademacher, Lectures on analytic number theory, *Tata Inst. Fund. Res., Bombay*, 1954–55.

[20] H. Sato, Fundamental concepts of social/regional summary data and inferences in their databases, *Doctorial Thesis, The Faculty of Engineering, Tokyo University* (1983).

[21] H. Sato, T. Nakano, Y. Fukasawa and R. Hotaka, Conceptual schema for a wide-scope statistical database and its application, *Proc. 3rd Int. Workshop on Statistical and Scientific Database Management* (1986), 165–172.

[22] A. Shoshani, Statistical databases: Characteristics, problems and some solutions, *VLDB*' **82**(1982), 208–222.

[23] J. M. Smith and D. C. P. Smith, Database abstractions: Aggregation and generalization, *TODS* (2) **2**(1977), 105–133.

Department of Information Science,
Faculty of Science,
Shimane University