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Cofine boundary behaviour of temperatures

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1. Introduction, notation and terminology

The Dirichlet problem for the heat equation is much less satisfactory than that for Laplace's equation, because the set of irregular boundary points is not necessarily polar, or even negligible. Therefore the behaviour of generalized solutions near irregular boundary points is more important. In this paper, we study such behaviour in the case of irregular points that are also cofine boundary points, that is, boundary points in the fine topology for the adjoint heat equation. We show that cofine limits exist and coincide with the values of the given boundary function at many such points, and characterize the points where this occurs in terms of both barriers and zero limits of the Green function. It is natural to call such points 'cofine regular'. Earlier, Bauer [3] and Doob [6], p. 358, have studied the existence of fine limits (that is, limits in the fine topology for the heat equation itself) of generalized solutions at irregular boundary points, but our results differ from theirs in both the topology and the fact that the limits assumed are the values of the given boundary function.

Non-polar sets of irregular boundary points that are also cofine boundary points, commonly occur within a single characteristic hyperplane. However, such a phenomenon does not occur if cofine irregular points are considered instead of (Euclidean) irregular ones. We prove this using a new reduction (or balayage) operator, which is introduced in Section 3. There Theorem 2 establishes the most important properties, and Theorem 3 uses these to prove a new result on the cofine boundary behaviour of greatest thermic minorants, which easily implies the existence of zero cofine limits of potentials (and hence of Green functions). Section 2 is devoted to proving a result which was stated and given an erroneous proof by Doob in [6]. In fact, we give a slight extension of the result, which is required for the proof of Theorem 2. Section 4 is where the results on cofine limits of generalized solutions are established. The methods are adaptations of classical techniques.

Throughout this paper, D denotes an arbitrary open subset of real Euclidean space $R^{n+1} = \{(x, t): x \in R^n, t \in R\}$. A typical point will be denoted by

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p or (x, t) as convenient. Given $p_0 \in D$, we denote by $\Lambda(p_0, D)$ the set of all $p \in D \setminus \{p_0\}$ which can be joined to p_0 by a polygonal path in D along which t is strictly increasing from p to p_0. The set $\Lambda^*(p_0, D)$ is defined analogously, with 'increasing' replaced by 'decreasing'. A temperature on D is a solution of the heat equation. Supertemperatures are defined in [9], and superparabolic functions in [6]; the equivalence follows from the Riesz decomposition theorems given independently in those works, or from [2], Theorem 2.5. The class of all non-negative supertemperatures on D is denoted by $S^+(D)$. The Green function for D is denoted by G_D, except that the subscript is omitted when $D = R^{n+1}$. If μ is a non-negative, locally finite Borel measure on D, we put

$$G_D\mu(p) = \int_D G_D(p, q)\, d\mu(q)$$

for all $p \in D$, and call $G_D\mu$ a potential if it is finite on a dense subset of D, in which case $G_D\mu \in S^+(D)$. If $v \in S^+(D)$, the greatest thermic (or parabolic) minorant of v on D is that temperature $u \leq v$ which majorizes all others, and is denoted by $GM_D v$. An element $w \in S^+(D)$ is a potential on D if and only if $GM_D w = 0$. For details of these concepts see [6] or [9].

The fine topology is the coarsest topology that makes every supertemperature continuous, and the cofine topology is the corresponding concept for the adjoint heat equation. A polar set is any subset of the infinity set of any supertemperature. A proposition which is true for all points outside a polar set is said to hold quasi-everywhere (q.e.). For any set B, the set of cofine limit points of B is written B^{*f}, and the cofine boundary of B is $\partial^{*f}B$. If v is a function on D which has a cofine limit at a point $q \in D^{*f}$, that limit is denoted by $v^*(q)$. In particular, if v, $w \in S^+(D)$ then $(v/w)^*$ exists q.e. on the subset of D where $v + w > 0$, and is finite q.e. on the subset where $w > 0$ ([6], p. 351). If u is a function on D, its lower semicontinuous smoothing $\hat{u}$ is the lower semicontinuous minorant of u that majorizes all others. Given an arbitrary subset A of D and any $v \in S^+(D)$, the parabolic reduction of v on A is defined by

$$R_v^A = \inf\{w \in S^+(D) \colon w \geq v \text{ on } A\},$$

and $\hat{R}_v^A \in S^+(D)$. The reader is referred to [6] for details of these concepts.

The term 'increasing' is used in the wide sense.

Finally, $u \wedge v$ denotes the pointwise minimum of u and v.

2. An internal limit theorem

Theorem 1 below is a slight extension of a result given by Doob. Unfortunately, Doob's proof ([6], p. 354) is invalid, since it is not necessarily true that (in his notation) 'the restriction of $\dot{v}$ to $\dot{D}_0$ is a potential', because the Riesz measure associated with $\dot{v}$ may have some mass outside $\dot{D}_0$, particularly in $\partial\dot{Z}$. However, a modification of Doob's method reduces the result to the corresponding one for non-negative temperatures on $R^n \times]0, \infty[$, and for this there are several known proofs. Here $u^*(q)$ denotes the cofine limit of $u(p)$ as $p \to q$.

THEOREM 1. *Let v, $w \in S^+(D)$, let p_0 be a limit point of $\{p: w(p) > 0\}$ in D, let $\Lambda = \Lambda(p_0, D)$, and let χ be the characteristic function on D of $D\backslash\bar{\Lambda}$, so that $v\chi$, $w\chi \in S^+(D)$. If ν, ω are the Riesz measures associated with $v\chi$, $w\chi$ respectively, and ν_Λ, ω_Λ are their restrictions to $D \cap \partial\Lambda$, then*

$$\left(\frac{v}{w}\right)^* = \frac{d\nu}{d\omega} = \frac{d\nu_\Lambda}{d\omega_\Lambda} \tag{1}$$

ω-a.e. on the component Γ of $D \cap \partial\Lambda$ that contains p_0.

PROOF. Put $E = \Lambda^*(p_0, D)$. We can assume that $D = \Lambda \cup \Gamma \cup E$. Since p_0 is a limit point of $\{p: w(p) > 0\}$, the quotient v/w is defined q.e. on E. Since E is a deleted cofine neighbourhood of p for every $p \in \Gamma$, we may further suppose that $v = v\chi$ and $w = w\chi$. It is sufficient to prove that (1) holds ω-a.e. on an arbitrary compact subset F of Γ. We can therefore suppose that v and w are potentials of finite measures on D; for if N is an open neighbourhood of F and is relatively compact in D, then $R_v^N \in S^+(D)$, $R_v^N = v$ on N, $R_v^N \le v = 0$ on $\Lambda \cup \Gamma$, and the Riesz measure associated with R_v^N is supported by the compact subset $\bar{N}\backslash\Lambda$ of D and is therefore finite, so we could replace v, w by R_v^N, R_w^N.

We now prove that if $\nu(\Gamma) = 0$ then $(v/w)^* = 0$ ω-a.e. on Γ. Let $b \in]0, \infty[$ and put $B = \{p: v(p) \ge bw(p)\}$. Then (as in [6], p. 354)

$$v \ge \hat{R}_v^B \ge b\hat{R}_w^B = bG_D(\omega\delta_D^B) \ge bG_D\omega' , \tag{2}$$

where for each $p \in D$ the measure $\delta_D^B(p, \cdot)$ is the sweeping over B of the unit mass concentrated at p, and ω' is the restriction of ω onto $B^{*f}\backslash E$, the set of cofine limit points of B outside E. The Green function G_E is the restriction to $E \times E$ of G_D ([9], p. 271, and [6], p. 300). Therefore, since $\nu(D\backslash E) = 0$, the restriction of v to E is a potential. By (2), this potential majorizes the restriction to E of $bG_D\omega'$, which is a temperature and is therefore zero. Hence $\omega(B^{*f}\backslash E) = 0$, and therefore

$$\text{cofine} \limsup_{p \to q} (v/w)(p) \le b$$

for ω-almost every $q \in \Gamma$. Since b is arbitrary, $(v/w)^* = 0$ ω-a.e. on Γ.

It follows from this result that

$$\left(\frac{w - G_D\omega_A}{G_D\omega_A}\right)^* = 0 = \left(\frac{v - G_D\nu_A}{G_D\omega_A}\right)^*$$

ω-a.e. on Γ, so that $(v/w)^* = (G_D\nu_A/G_D\omega_A)^*$ ω-a.e. on Γ. Thus only ν_A and ω_A are relevant, and we can suppose that ν and ω are supported by $\overline{N} \cap \Gamma$.

If ν, ω are defined to be null on $R^{n+1}\backslash D$, there are non-negative temperatures h_v, h_w on D such that

$$G\nu = G_D\nu + h_v\,, \qquad G\omega = G_D\omega + h_w \tag{3}$$

on D ([9], p. 276), so that q.e. on E we have

$$\frac{v}{w} = \left(\frac{G\nu}{G\omega} - \frac{h_v}{G\omega}\right)\Big/\left(1 - \frac{h_w}{G\omega}\right).$$

There are already several proofs that, for any potentials on R^{n+1} of measures on $R^n \times \{0\}$,

$$\left(\frac{G\nu}{G\omega}\right)^* = \frac{d\nu}{d\omega} \qquad \omega - \text{a.e. on } R^n \times \{0\} \tag{4}$$

([5], [8], [4], [6] p. 382). Therefore the result will follow if we show that, for $u \in \{h_v, h_w\}$, $(u/G\omega)^* = 0$ ω-a.e. on Γ. Since all the potentials in (3) are zero on $D\backslash E$, the same is true of u. Since u is continuous on D, it therefore suffices to show that $(G\omega)^* > 0$ ω-a.e. But this follows from (4), because

$$\left(\frac{1}{G\omega}\right)^* = \frac{dm_n}{d\omega} < \infty \qquad \omega - \text{a.e.}$$

REMARKS. (i) It is easy to check that $v\chi = R_v^A$, with $A = D\backslash\Lambda$, in Theorem 1.

(ii) If $w = 1$ on D, then $\omega_\Lambda = m_n$ on Γ. This is clear when $D = R^{n+1}$, and the general case can be deduced from this using [9], Theorem 19.

3. Cooling

We now define the reduction operator, and use Theorem 1 to establish its fundamental properties. Although it can be defined on arbitrary subsets, we restrict its definition to those on which we can prove its idempotence; this is ample for our present purpose.

DEFINITIONS. If $A \subseteq R^{n+1}$ and $t \in R$, we write $A(t)$ for $A \cap (R^n \times \{t\})$. If D is open, A is a subset of D such that $A(t)$ is Borel for all t, and $v \in S^+(D)$, we put

$$\Psi_v^A = \{w \in S^+(D) \colon w^* \geq v^* \text{ q.e. on } A(t) \text{ for all } t\}\,,$$

and define C_v^A, the *cooling of v over A* relative to D, by

$$C_v^A = \inf \Psi_v^A .$$

Where convenient, we write $C^A(v)$ for C_v^A.

Obviously $v \geq C_v^A \geq \hat{C}_v^A$ on D. Also, if $B \subseteq D$ and the symmetric difference of $A(t)$ and $B(t)$ is polar for every t, then $C_v^A = C_v^B$ on D. By the fundamental convergence theorem ([6], p. 314), $\hat{C}_v^A \in S^+(D)$. There are several other conclusions we could draw from that theorem, but they are all improved upon by our next result, which shows that $C_v^A = \hat{C}_v^A$ on D and that cooling is idempotent.

THEOREM 2. *If $A \subseteq D$ and $v \in S^+(D)$, then*
(i) $\hat{C}_v^A = C_v^A$ *on D,*
(ii) $(C_v^A)^* = v^*$ *q.e. on $A(t)$ for all t,*
(iii) $C_v^A = C^A(C_v^A)$ *on D.*
(iv) C_v^A *is a temperature on $D \backslash \bar{A}$.*

PROOF. We begin by proving that $(\hat{C}_v^A)^* = v^*$ q.e. on $A(t)$ for all t. Let $u = \hat{C}_v^A$, fix $t \in R$, and let $E = D \cap (R^n \times]t, \infty[)$. If χ is the characteristic function of E on D, then χ, $v\chi \in S^+(D)$. Let λ, ν be the Riesz measures associated with χ, $v\chi$, respectively. Then λ is the restriction to $D(t)$ of m_n. Let $Z = D \backslash E$, and let ν_Z denote the restriction of ν to Z. Since $v\chi$ is a temperature on Z^0, ν_Z is supported by $D(t)$. There is a sequence $\{p_j\}$ in $D(t)$ such that both E and $D(t)$ remain unchanged if Z^0 is replaced by $\bigcup_{j=1}^{\infty} A(p_j, D)$, and it therefore follows from Theorem 1 that

$$v^* = \left(\frac{v\chi}{\chi}\right)^* = \frac{d\nu_Z}{dm_n} \tag{5}$$

m_n-a.e. on $D(t)$. Since the m_n-null subsets of $D(t)$ are precisely the polar subsets of $D(t)$ ([10]), (5) holds q.e. on $D(t)$.

Let μ_t denote the restriction to $A(t)$ of the absolutely continuous part of ν_Z with respect to m_n, and let $v_t = G_D\mu_t$. Then $v \geq v\chi \geq G_D\nu_Z \geq v_t$ on D, so that for any w in $\Psi_v^{A(t)}$ we have $w^* \geq v_t^*$ q.e. on $A(t)$. The absolute continuity of μ_t and the coincidence of m_n-null sets with polar sets in $A(t)$, imply that polar sets are μ_t-null and that $v_t^* \leq w^*$ μ_t-a.e. on D. Therefore the domination principle ([6], p. 358) implies that $v_t \leq w$ on D. Thus $C_v^A \geq C_v^{A(t)} \geq v_t$ on D, and therefore $u \geq v_t$ on D. Furthermore, by (5) and the corresponding result for v_t,

$$v_t^* = \frac{d\mu_t}{dm_n} = \frac{d\nu_Z}{dm_n} = v^*$$

m_n-a.e. on $A(t)$, hence q.e. on $A(t)$. Thus $u^* \geq v^*$ q.e. on $A(t)$, and since $u \leq v$ on D we deduce that $u^* = v^*$ q.e on $A(t)$. Since t is arbitrary, we have proved that $(\hat{C}_v^A)^* = v^*$ q.e. on $A(t)$ for all t.

It follows immediately that $\hat{C}_v^A \geq C_v^A$, so that equality holds. Thus (i) and (ii) are established.

If $u = C_v^A$ and $w \in \Psi_u^A$, then $w \in \Psi_v^A$ by (ii), so that $w \geq C_v^A = u$. Therefore $C_u^A \geq u$, so that equality holds and (iii) is established.

To prove (iv), let B be an open interval with $\bar{B} \subseteq D$. Following [6], for each $u \in S^+(D)$ we put $\tau_B u = u$ on $D \backslash B$, $\tau_B u$ equal to the (parabolic) Poisson integral $PI(B, u)$ of $u|_{\partial B}$ on B, and $\tau_B u(q) = \lim_{p \to q} PI(B, u)(p)$ for inner points q of the upper boundary of B. If $u \in \Psi_v^A$ and $\bar{B} \subseteq D \backslash \bar{A}$, then $\tau_B u \in \Psi_v^A$ and $\tau_B u \leq u$ on D. Therefore, on B,

$$C_v^A = \inf\{(\tau_B u)|_B : u \in \Psi_v^A\} .$$

By the Harnack convergence theorem ([6], p. 276), C_v^A is a temperature on B, and (iv) follows.

EXAMPLE. Let $-\infty \leq a < 0 < b \leq \infty$, let $D = R^n \times]a, b[$, let μ be a measure supported by $B = R^n \times \{0\}$ such that $v = G_D \mu$ is a potential, let $f = d\mu/dm_n$, let A be a Borel subset of B with characteristic function χ_A, and let

$$d\nu(y) = f(y)\chi_A(y)\, dy .$$

We show that $C_v^A = G_D \nu$ on D. If $u = G_D \nu$ then, by Theorem 1, $v^* = f$ and $u^* = f\chi_A$ m_n-a.e. on B, so that $u^* = v^*$ m_n-a.e. on A. Since m_n-null and polar subsets of B coincide ([10]), $u \in \Psi_v^A$ and so $u \geq C_v^A$. Let $w \in \Psi_v^A$. Since $u^* < \infty$ m_n-a.e. on A and $u^* = v^* \leq w^*$ q.e. on A, we see that $u^* < \infty$ ν-a.e. and $u^* \leq w^*$ q.e. on a Borel support of ν. Therefore $u \leq w$ on D by the domination principle ([6], p. 358), so that $u \leq C_v^A$ and hence $G_D \nu = C_v^A$. It follows that the restriction of μ to B is absolutely continuous with respect to m_n if and only if $C_v^B = v$, and that it is singular if and only if $C_v^B = 0$.

Despite Theorem 2(ii), it is not necessarily true that $C_v^A = v$ on A. For example, if $v = 1$, $A = R^n \times \{0\}$ and $D = R^{n+1}$, then C_v^A is the characteristic function of $R^n \times]0, \infty[$, so that $C_v^A < v$ on A.

We now use Theorem 2 to prove a new result on the cofine boundary behaviour of greatest thermic minorants.

THEOREM 3. *If B is an open subset of D, and $v \in S^+(D)$, then*

$$(GM_B v)^* = v^* < \infty$$

q.e. on $(D \cap \partial^{*f} B)(t)$ for all t.

PROOF. Put $u = GM_B v$ on B, and $u = v$ on $D \backslash B$. Then u is the reduction of v over B relative to D, in the terminology of [9], p. 263 (but not a parabolic reduction in the sense of [6]), so that u is the limit of a decreasing sequence of elements of $S^+(D)$. Therefore u^* is defined and finite q.e. on D ([6], p. 355). By Theorem 2(ii), $(C_v^{D \cap \partial B})^* = v^*$ q.e. on $(D \cap \partial B)(t)$ for all t. Therefore, since the restriction to B of $C_v^{D \cap \partial B}$ is a thermic minorant of v on B (by Theorem 2(iv)), q.e. on $(D \cap \partial^{*f} B)(t)$ for all t we have

$$v^* \geq u^* \geq (C_v^{D \cap \partial B})^* = v^* ,$$

which implies the result of the theorem.

COROLLARY. *If $v = G_D \nu$ is a potential on D, and ν_B is the restriction of ν to B, then $(G_B \nu_B)^* = 0$ q.e. on $(D \cap \partial^{*f} B)(t)$ for all t.*

PROOF. If we put $\nu_B(D \backslash B) = 0$, then $G_D \nu_B \in S^+(D)$ and there is a nonnegative temperature h on B such that $G_D \nu_B = G_B \nu_B + h$ on B ([9], p. 276). Since $G_B \nu_B$ is a potential on B, $h = GM_B G_D \nu_B$. Therefore, by Theorem 3, $h^* = (G_D \nu_B)^* < \infty$ q.e. on $(D \cap \partial^{*f} B)(t)$ for all t, and the result follows.

EXAMPLE. Taking $D = R^{n+1}$ and ν to be the unit mass at a point $r \in B$, we see that $G_B(\cdot, r)^* = 0$ q.e. on $(\partial^{*f} B)(t)$ for all t.

4. The Dirichlet problem

Throughout this section, g denotes a given function on ∂D (which includes the point at infinity if D is unbounded) and H_g, $\overline{H}_g$, $\underline{H}_g$ denote, respectively, the solution (if it exists), upper solution, and lower solution of the Dirichlet problem for g in the PWB sense ([6]).

The example at the end of Section 3, combined with standard techniques, yields new information about the behaviour of H_g, $\overline{H}_g$ and $\underline{H}_g$ at certain irregular boundary points.

DEFINITIONS. A positive supertemperature u on D will be called a *weak cofine barrier* for D at a point $q \in \partial^{*f} D$ if $u^*(q)$ exists and is zero. It will be called a *cofine barrier* for D at q if, in addition, $\inf_{D \backslash B} u > 0$ whenever B is a (Euclidean) neighbourhood of q.

If E is an open subset of D, and $q \in \partial^{*f} E \cap \partial^{*f} D$, then the restriction to E of a cofine barrier for D at q is a cofine barrier for E at q. Conversely, suppose that there is an open neighbourhood V of q such that $V \cap D = V \cap E$, and that there is a cofine barrier u for E at q. If $\alpha = \inf_{E \backslash V} u$, then $\alpha > 0$ and the function v, defined by putting $v = \alpha \wedge u$ on $D \cap V$, $v = \alpha$ on $D \backslash V$, is a cofine barrier for D at q. Thus the existence of a cofine barrier is a local property.

THEOREM 4. *If there is a weak cofine barrier for D at q, then there is a cofine barrier for D at q that is also a temperature on D.*

PROOF. Follow [6], p. 333, choosing v to be continuous at 0 and taking the limit at the end in the cofine topology.

THEOREM 5. *If there is a cofine barrier for D at q, and if g is bounded above, then*

$$\text{cofine} \limsup_{p \to q} \overline{H}_g(p) \leq \limsup_{p \to q} g(p).$$

In particular, if g is bounded on ∂D and continuous at q, then

$$\overline{H}_g^*(q) = \underline{H}_g^*(q) = g(q).$$

PROOF. It is easy to adapt the proof for the classical case in [6], p. 126, to the present situation.

DEFINITION. A point $q \in \partial^{*f} D$ is called *cofine regular* if, whenever g is real-valued and continuous on ∂D,

$$H_g^*(q) = g(q).$$

THEOREM 6. *A point $q \in \partial^{*f} D$ is cofine regular if and only if there is a cofine barrier for D at q.*

PROOF. The proof follows that for the classical case in [6], p. 127, except that the suggestion therein for a barrier is no use in thermic case (despite the remark apparently to the contrary in [6], p. 334). Let f denote the distance from q, let $g = f \wedge 1$, take H_g as a weak cofine barrier for D at q, and apply Theorem 4 above.

It follows from Theorem 6 that cofine regularity is a local property.

EXAMPLE. If q is a cofine regular point of $\partial^{*f} D$ and $r \in D$, then $G_D(\cdot, r)^*(q) = 0$. This follows because $G_D(\cdot, r) = G(\cdot, r) - H_g$ with g the restriction of $G(\cdot, r)$ to ∂D ([6], p. 331).

This example, together with that in Section 3, suggests that cofine regular boundary points might be characterized in terms of Green functions. This is, indeed, the case, as we show in Theorem 7 below. It seems that the corresponding result for (ordinary) regularity has not been given before, so we draw attention to its details after the proof of the theorem. To some extent, there is a parallel between the roles played by the components of D in the case of Laplace's equation and by the sets $\Lambda^*(p, D)$ here. The main differences occur because the relation $r \in \Lambda^*(p, D)$ is not symmetric in r and p, and because $\partial \Lambda^*(p, D) \nsubseteq \partial D$. In Theorem 7, we also use the results of

Section 3 to prove that, for each t, quasi-every point of $(\partial^{*f}D)(t)$ is cofine regular.

We use the following terminology.

DEFINITION. If $\{p_k\}$ is a sequence of points in D such that the family $\{\Lambda^*(p_k, D): k \in N\}$ covers D, we call $\{p_k\}$ an *ancestral sequence* for D.

The Lindelöf property ensures the existence of ancestral sequences.

THEOREM 7. *Let $q \in \partial^{*f}D$, let $\{p_k\}$ be any ancestral sequence for D, and let $I_q = \{k: q \in \partial^{*f}\Lambda^*(p_k, d)\}$.*

(i) *If $I_q = \phi$, then q is cofine regular for D.*

(ii) *If $I_q \neq \phi$, then q is cofine regular for D if and only if q is cofine regular for every $\Lambda^*(p_k, D)$ such that $k \in I_q$.*

(iii) *If $I_q \neq \phi$, then q is cofine regular for D if and only if $G_D(\cdot, p_k)^*(q) = 0$ for all $k \in I_q$.*

(iv) *For each t, quasi-every point of $(\partial^{*f}D)(t)$ is cofine regular for D.*

PROOF.

(i) Put

$$u = \sum_{k=1}^{\infty} (G_D(\cdot, p_k) \wedge 2^{-k}). \tag{6}$$

Since $G_D(\cdot, p_k) > 0$ on $\Lambda^*(p_k, D)$ for each k, and $\{p_k\}$ is ancestral for D, $u > 0$ on D. The series converges uniformly, so that $u \in S^+(D)$. For each $j \in N$, there is a cofine neighbourhood V_j of q such that the j-th partial sum of the series is zero on $V_j \cap D$, because $G_D(\cdot, p_k) = 0$ outside $\Lambda^*(p_k, D)$ ([9], p. 271) and $I_q = \phi$. Therefore $u \leq 2^{-j}$ on $V_j \cap D$, and it follows that $u^*(q) = 0$. Hence u is a weak cofine barrier for D at q, and q is cofine regular for D.

(ii) Suppose that q is cofine regular for D, and let v be a cofine barrier for D at q. Given any $k \in I_q$, the restriction of v to $\Lambda^*(p_k, D)$ is a cofine barrier for that subset at q, so that q is cofine regular for $\Lambda^*(p_k, D)$. Conversely, suppose that q is cofine regular for every $\Lambda^*(p_k, D)$ such that $k \in I_q$. Given any $k \in I_q$, put $E = \Lambda^*(p_k, D)$ and choose a sequence $\{r_i^k\}$ in E with limit p_k. The Green function G_E is the restriction of G_D to $E \times E$ ([9], p. 271). Therefore, on E,

$$G_D(\cdot, r_i^k) = G_E(\cdot, r_i^k) = G(\cdot, r_i^k) - H_g$$

([6], p. 331), where g is the restriction to ∂E of $G(\cdot, r_i^k)$ and H_g is taken relative to E. The cofine regularity of q for E implies that $G_D(\cdot, r_i^k)^*(q) = 0$ for every i. Since $\{r_i^k\}$ converges to p_k it is ancestral for E, and because this holds for each $k \in I_q$ the countable set

$$\{r_i^k: i \in N,\ k \in I_q\} \cup \{p_k: k \in N \backslash I_q\}$$

can be arranged as an ancestral sequence $\{\pi_j\}$ for D. This sequence has the property that, if $q \in \partial^{*f} \Lambda^*(\pi_j, D)$ for some j then $G_D(\cdot, \pi_j)^*(q) = 0$. Therefore, if

$$w = \sum_{j=1}^{\infty} (G_D(\cdot, \pi_j) \wedge 2^{-j}),$$

then w is a weak cofine barrier for D at q. It follows that q is cofine regular for D.

(iii) Suppose that $G_D(\cdot, p_k)^*(q) = 0$ for all $k \in I_q$, and let u be defined by (6). Then u is a weak cofine barrier for D at q, so that q is cofine regular for D. The converse follows from the example in this section.

(iv) Given t and k, we know from the example in Section 3 that $G_D(\cdot, p_k)^* = 0$ q.e. on $(\partial^{*f} D)(t)$. Therefore, given t, we have $G_D(\cdot, p_k)^* = 0$ for all k, q.e. on $(\partial^{*f} D)(t)$. The result now follows from (iii).

Results analogous to Theorem 7 (i)–(iii) hold for (ordinary) regularity. Let $q \in \partial D$, replace I_q by $\{k: q \in \partial \Lambda^*(p_k, D)\}$, delete the word 'cofine' and replace cofine limits by Euclidean limits, throughout the statements and proofs.

Theorem 7 (iv) shows that cofine regular points occur in sufficient quantity to be of interest. In [9], a subset $ab_2(\partial D)$ of ∂D was defined by the conditions that $\Lambda(q, B(q, \varepsilon)) \subseteq D$ and $\Lambda^*(q, B(q, \varepsilon)) \cap D \neq \phi$ for some ball $B(q, \varepsilon)$ in R^{n+1}. By [9], Lemma 31, $ab_2(\partial D)$ is contained in a countable union of hyperplanes of the form $R^n \times \{t\}$. It therefore follows from Theorem 7 (iv) that quasi-every point of $ab_2(\partial D) \cap \partial^{*f} D$ is cofine regular.

The implication of Theorem 7(iv) is that more complete information about boundary regularity can be obtained if we consider non-Euclidean topologies. This was the idea behind the treatment of the Dirichlet problem in [9], but the half-ball topology used there is not as appropriate as the cofine topology, as can be seen by comparing Theorem 7(iv) with the results in [1]. Perhaps the neatest Dirichlet problem will turn out to be one where the cofine topology replaces the Euclidean one in both the boundary limits and the boundary itself.

After the research for this paper was completed, I discovered that closely related questions had been asked in [7].

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