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Legendre character sums

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1. Introduction

Character sum analogue of certain orthogonal polynomials were defined by Evans [1], but he mainly studied the properties of Hermite character sums. On the other hand, Greene [2] studied character sum analogue of hypergeometric series and has shown its usefulness. Classically it is well known certain orthogonal polynomials are deeply connected with the hypergeometric series. So it is natural to study the character sum analogue of orthogonal polynomials systematically and relate them to Greene's work.

In this paper, we consider the analogues of Legendre polynomials and its relation to Greene's hypergeometric character sums.

Throughout this paper we use the same notation as in Evans [1] and Greene [2].

Elementary formulae of Gauss and Jacobi sums which we will make use of are

$$G(N)G(\bar{N}) = qN(-1) \qquad (N \neq \varepsilon),\ G(\varepsilon) = -1 \tag{1.1}$$

$$J(M, N) = \frac{G(M)G(N)}{G(MN)} \qquad (MN \neq \varepsilon) \tag{1.2}$$

$$J(N, \bar{N}) = -N(-1) \qquad (N \neq \varepsilon),\ J(\varepsilon, \varepsilon) = q - 2 \tag{1.3}$$

$$G(\chi^2)G(\phi) = \chi(4)G(\chi)G(\chi\phi) \qquad (\phi^2 = \varepsilon) \tag{1.4}$$

2. Legendre character sums

In this section, we show that Legendre character sums have the similar properties as Legendre polynomials.

Evans [1] defined *Legendre character sums* as follows.

For any character N of $F_q^\times$,

$$P_N(x) = \frac{1}{q}\sum_u N(u)\,\phi(1 - 2xu + u^2).$$

We can show Legendre character sums have the generating function. (See, for example, [5] p. 157)

PROPOSITION 1. *For* $t \in \boldsymbol{F}_q^\times$,

$$\phi(1-2xt+t^2)=\frac{q}{q-1}\sum_N P_N(x)N(t),$$

where N *runs through all characters of* $\boldsymbol{F}_q^\times$.

PROOF. $\frac{q}{q-1}\sum_N P_N(x)N(t)$

$$=\frac{q}{q-1}\sum_N\frac{1}{q}\sum_u N(u)\phi(1-2xu+u^2)N(t)$$

$$=\frac{q}{q-1}\sum_u\{\phi(1-2xu+u^2)\sum_N N(ut)\}$$

$$=\phi\left(1-2x\cdot\frac{1}{t}+\frac{1}{t^2}\right)$$

$$=\phi(1-2xt+t^2).$$

In the next proposition we prove the Fourier coeficients of Legendre character sums can be described by Greene's binomial coeficients.

To prove Proposition 2, we need the following lemma.

LEMMA 1. ([2] Theorem 2.3)

$$A(1-x)=\frac{1}{q-1}\sum_\chi J(A,\bar{\chi})\chi(x).$$

Now we state Proposition 2 which shows that Legendre character sums have the expression analogous to polynomial expression.

PROPOSITION 2. *For* $x \in \boldsymbol{F}_q^\times$,

$$P_N(x)=\frac{1}{q(q-1)}\sum_\chi J(\chi,\chi N)J(\chi\bar{N},\phi)N\bar{\chi}^2(2x).$$

PROOF. By Lemma 1,

$$\phi(1-2xt+t^2)=\frac{1}{q-1}\sum_\chi J(\bar{\chi},\phi)\chi(2xt-t^2)$$

$$=\frac{1}{q-1}\sum_\chi J(\bar{\chi},\phi)\chi(2xt)\chi\left(1-\frac{t}{2x}\right).$$

Also, by Lemma 1,

$$\chi\left(1-\frac{t}{2x}\right)=\frac{1}{q-1}\sum_N J(\bar{N},\chi)N\left(\frac{t}{2x}\right).$$

Consequently,

$$\phi(1-2xt+t^2)=\frac{1}{(q-1)^2}\sum_N\sum_\chi J(\bar{N},\chi)J(\bar{\chi},\phi)\chi(2xt)N\left(\frac{t}{2x}\right)$$
$$=\frac{1}{(q-1)^2}\sum_N\sum_\chi J(\bar{N},\chi)J(\bar{\chi},\phi)\chi\bar{N}(2x)\chi N(t)$$

If we replace χN by M, we obtain

$$\phi(1-2xt+t^2)=\frac{1}{(q-1)^2}\sum_M\sum_\chi J(\chi\bar{M},\chi)J(\bar{\chi},\phi)\chi^2\bar{M}(2x)M(t).$$

Then, by Proposition 1 and the uniqueness of Fourier expansions

$$P_N(x)=\frac{1}{q(q-1)}\sum_\chi J(\chi\bar{N},\chi)J(\bar{\chi},\phi)\chi^2\bar{N}(2x)$$

If we replace χ by $\chi\bar{N}$, we have

$$P_N(x)=\frac{1}{q(q-1)}\sum_\chi J(\bar{\chi},\bar{\chi}N)J(\chi\bar{N},\phi)N\bar{\chi}^2(2x).$$

COROLLARY. *For* $x\in F_q^\times$,

$$P_N(-x)=N(-1)P_N(x).$$

We can calculate the special values of $P_N(x)$ at 0 and 1. (cf. [5] p. 158)

PROPOSITION 3.

$$P_N(1)=\begin{cases}-\dfrac{1}{q} & \text{if } N\neq\varepsilon,\\ 1-\dfrac{1}{q} & \text{if } N=\varepsilon.\end{cases}$$

$$P_N(0)=\begin{cases}0 & \text{if } N \text{ is not a square},\\ \dfrac{M(-1)\{J(\phi,\bar{M})+\phi(-1)J(\phi,\phi\bar{M})\}}{q} & \text{if } N=M^2.\end{cases}$$

PROOF. By the definition of $P_N(x)$,

$$P_N(1)=\frac{1}{q}\sum_u N(u)\phi((1-u)^2)$$
$$=\frac{1}{q}\{\sum_u N(u)-1\}$$

$$= \begin{cases} -\dfrac{1}{q} & \text{if } N \neq \varepsilon, \\ 1 - \dfrac{1}{q} & \text{if } N = \varepsilon. \end{cases}$$

Next we calculate $P_N(0)$. By Lemma 1,

$$\begin{aligned} \phi(1+t^2) &= \frac{1}{q-1}\sum_{\chi} J(\phi, \bar{\chi})\chi(-t^2) \\ &= \frac{1}{q-1}\sum_{\chi} J(\phi, \bar{\chi})\chi(-1)\chi^2(t). \end{aligned}$$

The result now follows from Proposition 1 and the uniqueness of Fourier expansions.

The next Proposition is the analogue of the Rodriguez formula, and it plays an important role to obtain the relation between the Legendre character sums and the hypergeometric character sums.

PROPOSITION 4. *For* $x \in F_q^{\times}$,

$$\begin{aligned} P_N(x) &= \frac{1}{N(2)G(N)} D^N N(x^2 - 1) \\ &\quad + \frac{\delta(N\phi)}{q(q-1)}(\phi(-2) - \lambda(-1))(J(\lambda, \phi) + J(\bar{\lambda}, \phi)) \\ &\quad - \frac{\delta(N)}{q}(q-1)(q-3), \end{aligned}$$

where λ is quartic. If such characters don't exist, we mean that the second term vanishes.

PROOF. For simplicity, we will prove Proposition 4 under the condition $N \neq \varepsilon, \phi$.

If $N = \varepsilon$ or ϕ, by the similar method we can show the results after straightforward calculations.

First we prove the following statement.

$$(2.1) \qquad D^N N(x^2 - 1) = \frac{N(-1)}{(q-1)G(\bar{N})}\sum_{\chi} J(\bar{\chi}, N) J(\chi^2, \bar{N})\chi^2\bar{N}(x)$$

By Lemma 1,

$$N(x^2 - 1) = \frac{N(-1)}{q-1}\sum_{\chi} J(\bar{\chi}, N)\chi^2(x).$$

Thus,

$$D^N N(x^2-1)=\frac{N(-1)}{(q-1)G(\bar N)}\sum_t \bar N(t)\{\sum_\chi J(\bar\chi, N)\chi^2(x-t)\}$$
$$=\frac{N(-1)}{(q-1)G(\bar N)}\sum_\chi J(\bar\chi, N)\left\{\sum_t \bar N\left(\frac{t}{x}\right)\chi^2\left(1-\frac{t}{x}\right)\right\}\chi^2\bar N(x)$$
$$=\frac{N(-1)}{(q-1)G(\bar N)}\sum_\chi J(\bar\chi, N)J(\chi^2, \bar N)\chi^2\bar N(x).$$

The next statement is obvious from Proposition 2.

$$P_N(x)=\frac{1}{q(q-1)}\sum_\chi J(\chi\bar N, \chi)J(\bar\chi, \phi)\bar N(2)\chi(4)\chi^2\bar N(x) \tag{2.2}$$

Finally, Proposition 4 follows from the next statement.

$$\frac{N(-1)}{(q-1)N(2)G(\bar N)G(N)}J(\bar\chi, N)J(\chi^2, \bar N)=\frac{1}{q(q-1)}J(\chi\bar N, \chi)J(\bar\chi, \phi)\chi(4)\bar N(2)$$

This is equivalent to

$$J(\bar\chi, N)J(\chi^2, \bar N)=J(\chi\bar N, \chi)J(\bar\chi, \phi)\chi(4) \tag{2.3}$$

To prove (2.3), we should consider several cases. If $\chi=N$, then we have

the left hand side of (2.3) $=-N(-1)\dfrac{G(N^2)G(\bar N)}{G(N)}$, and

the right hand side of (2.3) $=-\dfrac{G(\bar N)G(\phi)N(4)}{G(\bar N\phi)}$.

So the result follows from the formula (1.4).
If $\chi^2=N$, then we have

the left hand side of (2.3) $=-\dfrac{G(\bar\chi)G(\chi^2)}{G(\chi)}$, and

the right hand side of (2.3) $=-\dfrac{G(\bar\chi)G(\phi)\chi(-4)}{G(\bar\chi\phi)}$.

So the result follows from the formula (1.4).
If $\chi\neq N$, $\chi^2\neq N$, then we have

the left hand side of (2.3) $=\dfrac{qN(-1)G(\bar\chi)G(\chi^2)}{G(\chi^2\bar N)G(\bar\chi N)}$

the right hand side of (2.3) $=\dfrac{G(\chi\bar N)G(\chi)G(\bar\chi)G(\phi)\chi(4)}{G(\chi^2\bar N)G(\bar\chi\phi)}$.

So the result follows from the formulae (1.1) and (1.4).

Consequently we proved (2.3), so the proof of Proposition 4 is obtained.

Next we use Proposition 4 to examine the relation between the Legendre character sums and the hypergeometric character sums.

So we define *the hypergeometric character sums* after Greene [2];

For characters A, B and C of $F_q^\times$ and $x \in F_q^\times$,

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right) = \frac{BC(-1)}{q}\sum_y B(y)B\bar{C}(1-y)A(1-xy) $$

Greene proved that the hypergeometric character sums have the following expression,

$$ {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right) = \frac{BC(-1)}{q(q-1)}\sum_\chi J(\bar{\chi}, \bar{A})J(B\chi, C\bar{B})\chi(x). $$

Now we can state the relation between $P_N(x)$ and ${}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right)$.

THEOREM. *For* $x \neq 0$ *and* 1, we have

$$ \begin{aligned} P_N(x) = {} & N\left(\frac{x-1}{2}\right){}_2F_1\left(\begin{matrix} \bar{N}, & \bar{N} \\ & \varepsilon \end{matrix}\middle| \frac{x+1}{x-1}\right) \\ & + \frac{\delta(N\phi)}{q(q-1)}(\phi(-2) - \lambda(-1))(J(\lambda, \phi) + J(\bar{\lambda}, \phi)) \\ & - \frac{\delta(N)}{q}(q-1)(q-3), \end{aligned} $$

where λ is the same as in Proposition 4.

PROOF. The result easily follows from the next proposition.

PROPOSITION 5. *For* $x \in F_q^\times$,

$$ \begin{aligned} P_N(x) = {} & \frac{N(-1)}{q(q-1)}\sum_\chi J(N, \bar{\chi})J(N, \chi\bar{N})N\bar{\chi}\left(\frac{x-1}{2}\right)\chi\left(\frac{x+1}{2}\right) \\ & + \frac{\delta(N\phi)}{q(q-1)}(\phi(-2) - \lambda(-1))(J(\lambda, \phi) + J(\bar{\lambda}, \phi)) \\ & - \frac{\delta(N)}{q}(q-1)(q-3), \end{aligned} $$

where λ is the same as in Proposition 4.

PROOF. To prove Proposition 5, we need the following two lemmas.

LEMMA 2. *For* $x \in F_q^\times$,

$$D^N M(x) = \frac{M\bar{N}(x)}{G(\bar{N})} J(M, \bar{N})$$

PROOF. $D^N M(x) = \sum_t \bar{N}(t) M(x-t)$

$$= \frac{M\bar{N}(x)}{G(\bar{N})} \sum_t \bar{N}\left(\frac{t}{x}\right) M\left(1 - \frac{t}{x}\right)$$

$$= \frac{M\bar{N}(x)}{G(\bar{N})} J(M, \bar{N}).$$

LEMMA 3. ([1] Theorem 2.2)

$$D^N(E(x)F(x)) = \sum_M \frac{G(\bar{M})G(M\bar{N})}{(q-1)G(\bar{N})} (D^N E(x))(D^{N\bar{M}} F(x)).$$

So by Lemma 2, Lemma 3 and Proposition 4, if $N \neq \varepsilon, \phi$, we have

$$P_N(x) = \frac{1}{N(2)G(N)} D^N\{N(x-1)N(x+1)\}$$

$$= \frac{1}{N(2)G(N)(q-1)} \sum_\chi \frac{G(\bar{\chi})G(\chi\bar{N})}{G(\bar{N})} \{D^\chi N(x-1)\}\{D^{N\bar{\chi}}(x+1)\}$$

$$= \frac{1}{N(2)G(N)(q-1)} \sum_\chi \frac{G(\bar{\chi})G(\chi\bar{N})}{G(\bar{N})} \left\{ \frac{N\bar{\chi}(x-1)}{G(\bar{\chi})} J(N, \bar{\chi}) \right\} \left\{ \frac{\chi(x+1)}{G(\chi\bar{N})} J(N, \chi\bar{N}) \right\}$$

$$= \frac{N(-1)}{q(q-1)} \sum_\chi J(N, \bar{\chi}) J(N, \chi\bar{N}) N\bar{\chi}\left(\frac{x-1}{2}\right) \chi\left(\frac{x+1}{2}\right).$$

Thus we proved Proposition 5 under the condition that $N \neq \phi, \varepsilon$. If $N = \phi$ or ε, we only add the constant terms coming from PROPOSITION 4.

REMARK. Theorem enables us to apply the general properties of hypergeometric character sums obtained by Greene [2] in studying $P_N(x)$.

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