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An algorithm for computing multivariate isotonic regression

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1 Introduction

Isotonic regression theory plays a key role in the field of order restricted statistical inference. Most of the theory related to this which appeared in the literature prior to the seventies are reviewed with a historical background in the seminal book of Barlow, Bartholomew, Bremner and Brunk [1]. Since then extensive research has been done in this field filling many gaps in the theory and most of them are contained in the recently published book of Robertson, Wright and Dykstra [7]. A multivariate generalization of isotonic regression including the multivariate extensions of well-known Bartholomew's $\bar{\chi}^2$ and $\bar{E}^2$ is given by Sasabuchi, Inutsuka and Kulatunga [8]. This theory enables us to study statistical inference for ordered vector-valued parameters or sets of ordered parameters. Some of them are discussed by Kulatunga and Sasabuchi [5] and Kulatunga, Inutsuka and Sasabuchi [4]. An algorithm for the computation of bivariate isotonic regression is also demonstrated in Sasabuchi et al.'s paper [8]. This algorithm involves iterative computation of univariate isotonic regression. The main purpose of this paper is to present a multivariate generalization of the algorithm described for the bivariate case.

The definition of multivariate isotonic regression and some important results are stated in section 2. In section 3 we describe the multivariate generalization of the algorithm. Some theorems on convergence of the algorithm are given in section 4.

2 Definitions and basic theorems

First in our notation, we state the definition of univariate isotonic regression (see, Robertson et al. [7, p. 25]).

Let $K = \{1, \dots, k\}$ be a finite set on which a partial order $\ll$ is defined. The partial order on K may or may not be the natural order among positive integers $1 \ll 2 \ll \cdots \ll k$, which is called the simple order.

DEFINITION 2.1. A real vector $(\theta_1, \dots, \theta_k)$ is said to be *isotonic* with respect to the partial order $\ll$, if $\mu, \nu \in K$ and $\mu \ll \nu$ imply $\theta_\mu \le \theta_\nu$.

DEFINITION 2.2. Given real numbers $x_1,\dots,x_k$ and positive numbers $w_1,\dots,w_k$, a vector $(\hat{\theta}_1,\dots,\hat{\theta}_k)$ is said to be the *univariate isotonic regression* of $x_1,\dots,x_k$ with weights $w_1,\dots,w_k$ if it is isotonic and minimizes

$$\sum_{\nu=1}^{k}(x_\nu-\theta_\nu)^2 w_\nu$$

under the restriction that $(\theta_1,\dots,\theta_k)$ is isotonic.

There are various algorithms for computing univariate isotonic regression (see, e.g. Barlow et al. [1, section 2.3], Robertson et al. [7, section 1.4]). Especially for the simple order, the so-called pool adjacent violators algorithm is available (Bartholomew [2]), and a computer program is given by Cran [3] in this case.

Multivariate version of the above definitions and multivariate extensions of some theorems of Barlow et al. [1, section 1.3] are given and proved by Sasabuchi et al. [8], and we state them as follows.

DEFINITION 2.3. A $p\times k$ real matrix $\theta=(\theta_1,\dots,\theta_k)$ is said to be *isotonic* with respect to the partial order $\ll$, if $\mu,\nu\in K$ and $\mu\ll\nu$ imply $\theta_\mu\le\theta_\nu$, where $\theta_\mu\le\theta_\nu$ means all the elements of $\theta_\nu-\theta_\mu$ are nonnegative.

Throughout this paper $\min_\theta^*(\cdot)$ denotes the minimum for all θ isotonic with respect to the partial order $\ll$.

DEFINITION 2.4. Given p-dimensional real vectors $x_1,\dots,x_k$ and $p\times p$ positive definite matrices $\Lambda_1,\dots,\Lambda_k$, a $p\times k$ matrix $(\hat{\theta}_1,\dots,\hat{\theta}_k)$ is said to be the *multivariate isotonic regression*, in fact *p-variate isotonic regression*, of $x_1,\dots,x_k$ with weights $\Lambda_1^{-1},\dots,\Lambda_k^{-1}$ if it is isotonic and satisfies

$$\min_\theta^*\sum_{\nu=1}^{k}(x_\nu-\theta_\nu)'\Lambda_\nu^{-1}(x_\nu-\theta_\nu)=\sum_{\nu=1}^{k}(x_\nu-\hat{\theta}_\nu)'\Lambda_\nu^{-1}(x_\nu-\hat{\theta}_\nu).$$

For brevity, we sometimes say $\hat{\theta}_\nu$ is the multivariate isotonic regression of x_ν with weights Λ_ν^{-1}.

THEOREM 2.1. *Given any partial order and weights, the multivariate isotonic regression exists uniquely for any p-dimensional real vectors $x_1,\dots,x_k$.*

THEOREM 2.2. *A $p\times k$ real matrix $(\hat{\theta}_1,\dots,\hat{\theta}_k)$ is the multivariate isotonic regression of $x_1,\dots,x_k$ with weights $\Lambda_1^{-1},\dots,\Lambda_k^{-1}$ if and only if it is isotonic and satisfies*

$$\sum_{\nu=1}^{k}(x_\nu-\hat{\theta}_\nu)'\Lambda_\nu^{-1}\hat{\theta}_\nu=0 \tag{2.1}$$

and

$$\sum_{\nu=1}^{k}(x_\nu-\hat{\theta}_\nu)'\Lambda_\nu^{-1}\xi_\nu\le 0 \quad \textit{for any } p\times k \textit{ isotonic matrix } (\xi_1,\dots,\xi_k). \tag{2.2}$$

3 An algorithm for computing multivariate isotonic regression

In this section we propose an algorithm for computing multivariate isotonic regression. This algorithm is a generalization of the algorithm demonstrated in Sasabuchi et al. [8] for the bivariate case.

Now the problem is to find a $p \times k$ real matrix

$$\theta = (\theta_1, \ldots, \theta_k) = \begin{pmatrix} \theta_{11} & \cdots & \theta_{1k} \\ \vdots & \vdots \vdots \vdots & \vdots \\ \theta_{p1} & \cdots & \theta_{pk} \end{pmatrix} = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \vdots \\ \theta^p \end{pmatrix}$$

for given p-dimensional real vectors $x_1, \ldots, x_k$ and $p \times p$ positive definite matrices $\Lambda_1, \ldots, \Lambda_k$ which minimizes

$$L(\theta^1, \ldots, \theta^p) = \sum_{\nu=1}^{k} (x_\nu - \theta_\nu)' \Lambda_\nu^{-1} (x_\nu - \theta_\nu)$$

under the condition that θ is isotonic. As seen above, the right hand side is expressed as a function of $\theta_1, \ldots, \theta_k$. But we define L as a function of $\theta^1, \ldots, \theta^p$ because of the convenience in describing our algorithm.

When the weight matrices are diagonal, as in the case of bivariate isotonic regression (cf. (4.1) of Sasabuchi et al. [8]), $L(\theta^1, \ldots, \theta^p)$ can be written as follows

$$L(\theta^1, \ldots, \theta^p) = \sum_{i=1}^{p} \sum_{\nu=1}^{k} (x_{i\nu} - \theta_{i\nu})^2 \lambda_{\nu ii}^{-1},$$

where $x_\nu' = (x_{1\nu}, \ldots, x_{p\nu})$ and $\lambda_{\nu ii}$ is the (i, i)-th element of Λ_ν. Thus the multivariate isotonic regression can be obtained easily by applying the methods of computing univariate isotonic regression to each term of the sum separately.

Now we suppose that at least one weight matrix is not diagonal and in this case L can be rewritten in the following p forms.

$$\begin{aligned} L(\theta^1, \ldots, \theta^p) &= f_1(\theta^2, \ldots, \theta^p) + g_1(\theta^1, \ldots, \theta^p) \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= f_i(\theta^1, \ldots, \theta^{i-1}, \theta^{i+1}, \ldots, \theta^p) + g_i(\theta^1, \ldots, \theta^p) \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= f_p(\theta^1, \ldots, \theta^{p-1}) + g_p(\theta^1, \ldots, \theta^p) \end{aligned}$$

where

$$\begin{aligned} &f_i(\theta^1, \ldots, \theta^{i-1}, \theta^{i+1}, \ldots, \theta^p) = \sum_{\nu=1}^{k} (x_{\nu(i)} - \theta_{\nu(i)})' \Lambda_{\nu(i)}^{-1} (x_{\nu(i)} - \theta_{\nu(i)}), \\ &g_i(\theta^1, \ldots, \theta^p) = \sum_{\nu=1}^{k} w_{\nu:(i)}^{-1} \{(x_{i\nu} - \theta_{i\nu}) - \lambda_{\nu(i)}' \Lambda_{\nu(i)}^{-1} (x_{\nu(i)} - \theta_{\nu(i)})\}^2, \\ &w_{\nu:(i)} = \lambda_{\nu ii} - \lambda_{\nu(i)}' \Lambda_{\nu(i)}^{-1} \lambda_{\nu(i)}, \qquad i = 1, \ldots, p;\ \nu = 1, \ldots, k \end{aligned}$$

$x_{\nu(i)}$ and $\theta_{\nu(i)}$ are the column vectors obtained after deleting the i-th element of x_ν and θ_ν, respectively, $\Lambda_{\nu(i)}$ is the $(p-1)\times(p-1)$ submatrix obtained after deleting the i-th row and column of Λ_ν and $\lambda_{\nu(i)}$ is the vector obtained after deleting the i-th element of the i-th column of Λ_ν.

We describe our algorithm of stepwise approximation to the multivariate isotonic regression, where the n-th approximation of the i-th row vector θ^i computed at (n, i)-step $(n = 1, 2, \ldots; i = 1, \ldots, p)$ is denoted by

$$\theta^{i(n)} = (\theta_{i1}^{(n)}, \ldots, \theta_{ik}^{(n)}), \qquad i = 1, \ldots, p.$$

From the (1, 1)-step to $(1, p-1)$-step of our algorithm, we take the first approximation $\theta^{1(1)}, \ldots, \theta^{p-1(1)}$ of $\theta^1, \ldots, \theta^{p-1}$ arbitrarily. For instance, $\theta^{i(1)} = (x_{i1}, \ldots, x_{ik})$, $i = 1, \ldots, p-1$.

In the $(1, p)$-step, we find the first approximation $\theta^{p(1)}$ of θ^p. In fact we find isotonic $\theta^{p(1)}$ such that

$$\min\nolimits^*_{\theta^p} L(\theta^{1(1)}, \ldots, \theta^{p-1(1)}, \theta^p) = L(\theta^{1(1)}, \ldots, \theta^{p-1(1)}, \theta^{p(1)})$$

or equivalently

$$\min\nolimits^*_{\theta^p} g_p(\theta^{1(1)}, \ldots, \theta^{p-1(1)}, \theta^p) = g_p(\theta^{1(1)}, \ldots, \theta^{p-1(1)}, \theta^{p(1)}).$$

Then $\theta_{p\nu}^{(1)}$ is just the univariate isotonic regression of $\{x_{p\nu} - \lambda'_{\nu(p)} \Lambda_{\nu(p)}^{-1}(x_{\nu(p)} - \theta_{\nu(p)}^{(1)})\}$ with weights $w_{\nu:(p)}^{-1} = (\lambda_{\nu pp} - \lambda'_{\nu(p)} \Lambda_{\nu(p)}^{-1} \lambda_{\nu(p)})^{-1}$.

After computing $\theta^{1(n)}, \ldots, \theta^{p(n)}$ at $(n, 1)$-$, \ldots, (n, p)$-steps, respectively, we proceed to $(n+1, 1)$-step, where $\theta^{1(n+1)}$ is determined by the following relation with the condition that $\theta^{1(n+1)}$ is isotonic:

$$\min\nolimits^*_{\theta^1} L(\theta^1, \theta^{2(n)}, \ldots, \theta^{p(n)}) = L(\theta^{1(n+1)}, \theta^{2(n)}, \ldots, \theta^{p(n)})$$

or equivalently

$$\min\nolimits^*_{\theta^1} g_1(\theta^1, \theta^{2(n)}, \ldots, \theta^{p(n)}) = g_1(\theta^{1(n+1)}, \theta^{2(n)}, \ldots, \theta^{p(n)}).$$

In general, at $(n+1, i)$-step, we find an isotonic $\theta^{i(n+1)}$ such that

$$\begin{aligned}\min\nolimits^*_{\theta^i} L(\theta^{1(n+1)}, \ldots, \theta^{i-1(n+1)}, \theta^i, \theta^{i+1(n)}, \ldots, \theta^{p(n)}) \\ = L(\theta^{1(n+1)}, \ldots, \theta^{i-1(n+1)}, \theta^{i(n+1)}, \theta^{i+1(n)}, \ldots, \theta^{p(n)})\end{aligned}$$

or equivalently

$$\begin{aligned}\min\nolimits^*_{\theta^i} g_i(\theta^{1(n+1)}, \ldots, \theta^{i-1(n+1)}, \theta^i, \theta^{i+1(n)}, \ldots, \theta^{p(n)}) \\ = g_i(\theta^{1(n+1)}, \ldots, \theta^{i-1(n+1)}, \theta^{i(n+1)}, \theta^{i+1(n)}, \ldots, \theta^{p(n)}).\end{aligned}$$

Continuing in this manner, at $(n+1, p)$-step we find an isotonic $\theta^{p(n+1)}$ such that

$$\min\nolimits^*_{\theta^p} L(\theta^{1(n+1)},\dots,\theta^{p-1(n+1)},\theta^p) = L(\theta^{1(n+1)},\dots,\theta^{p-1(n+1)},\theta^{p(n+1)})$$

or equivalently

$$\min\nolimits^*_{\theta^p} g_p(\theta^{1(n+1)},\dots,\theta^{p-1(n+1)},\theta^p) = g_p(\theta^{1(n+1)},\dots,\theta^{p-1(n+1)},\theta^{p(n+1)}).$$

Here we have demonstrated an iterative algorithm in which each cycle of recurrent computations consists of p subcycles; each revises one row vector while the other $p-1$ row vectors are tentatively fixed. In fact this algorithm involves iterative computation of univariate isotonic regressions. Thus at each step $\theta^{1(n)},\dots,\theta^{p(n)}$ are determined uniquely. This follows from the existence and uniqueness of the univariate isotonic regression (see, Theorem 2.1).

4 Convergence of the algorithm

For $\theta^{i(n)}$, $i=1,\dots,p$, in our algorithm, the following important theorem holds.

THEOREM 4.1. *If*

$$\lim\nolimits_{n\to\infty} \theta^{i(n)} = \theta^{i(\infty)}, \qquad i=1,\dots,p$$

exists, then the $p \times k$ *matrix*

$$\theta^{(\infty)} = (\theta^{1(\infty)\prime},\dots,\theta^{p(\infty)\prime})'$$

is the multivariate isotonic regression of $x_1,\dots,x_k$ *with weights* $\Lambda_1^{-1},\dots,\Lambda_k^{-1}$. *That is,* $\theta^{(\infty)}$ *is isotonic and*

$$\min\nolimits^*_\theta L(\theta^1,\dots,\theta^p) = L(\theta^{1(\infty)},\dots,\theta^{p(\infty)}).$$

PROOF. Let

$$\theta^{(\infty)} = \begin{pmatrix} \theta^{1(\infty)} \\ \vdots \\ \theta^{p(\infty)} \end{pmatrix} = \begin{pmatrix} \theta_{11}^{(\infty)} & \cdots & \theta_{1k}^{(\infty)} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \theta_{p1}^{(\infty)} & \cdots & \theta_{pk}^{(\infty)} \end{pmatrix} = (\theta_1^{(\infty)},\dots,\theta_k^{(\infty)}).$$

Now $\theta^{i(n+1)}$ is the univariate isotonic regression of

$$x_{i\nu} - \lambda'_{\nu(i)}\Lambda_{\nu(i)}^{-1}(x_{\nu(i)} - (\theta_{1\nu}^{(n+1)},\dots,\theta_{i-1,\nu}^{(n+1)},\theta_{i+1,\nu}^{(n)},\dots,\theta_{p\nu}^{(n)})')$$

with weights $w_{\nu:(i)}^{-1}$. Substituting this value to (2.1) and (2.2), we have

$$\begin{aligned}\textstyle\sum_{\nu=1}^k \{(x_{i\nu} - \theta_{i\nu}^{(n+1)}) - \lambda'_{\nu(i)}\Lambda_{\nu(i)}^{-1}(x_{\nu(i)} \\ - (\theta_{1\nu}^{(n+1)},\dots,\theta_{i-1,\nu}^{(n+1)},\theta_{i+1,\nu}^{(n)},\dots,\theta_{p\nu}^{(n)})')\}\, w_{\nu:(i)}^{-1}\,\theta_{i\nu}^{(n+1)} = 0, \\ i=1,\dots,p\end{aligned}$$

and

$$\sum_{v=1}^{k} \{(x_{iv} - \theta_{iv}^{(n+1)}) - \lambda'_{v(i)} \Lambda_{v(i)}^{-1} (x_{v(i)} - (\theta_{1v}^{(n+1)}, \ldots, \theta_{i-1,v}^{(n+1)}, \theta_{i+1,v}^{(n)}, \ldots, \theta_{pv}^{(n)})')\} w_{v:(i)}^{-1} \xi_{iv} \leq 0, \quad i = 1, \ldots, p$$

for any $p \times k$ isotonic matrix,

$$\begin{pmatrix} \xi_{11} & \cdots & \xi_{1k} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \xi_{p1} & \cdots & \xi_{pk} \end{pmatrix}.$$

As $n \to \infty$, we have for $i = 1, \ldots, p$

$$\sum_{v=1}^{k} \{(x_{iv} - \theta_{iv}^{(\infty)}) - \lambda'_{v(i)} \Lambda_{v(i)}^{-1} (x_{v(i)} - \theta_{v(i)}^{(\infty)})\} w_{v:(i)}^{-1} \theta_{iv}^{(\infty)} = 0 \tag{4.1}$$

and

$$\sum_{v=1}^{k} \{(x_{iv} - \theta_{iv}^{(\infty)}) - \lambda'_{v(i)} \Lambda_{v(i)}^{-1} (x_{v(i)} - \theta_{v(i)}^{(\infty)})\} w_{v:(i)}^{-1} \xi_{iv} \leq 0, \tag{4.2}$$

where $\theta_{v(i)}^{(\infty)}$ is the vector obtained after deleting the i-th element of $\theta_v^{(\infty)}$.
From (4.1) we have

$$\sum_{i=1}^{p} \sum_{v=1}^{k} \{(x_{iv} - \theta_{iv}^{(\infty)}) - \lambda'_{v(i)} \Lambda_{v(i)}^{-1} (x_{v(i)} - \theta_{v(i)}^{(\infty)})\} w_{v:(i)}^{-1} \theta_{iv}^{(\infty)} = 0. \tag{4.3}$$

Now it can be easily seen that

$$\{(x_{iv} - \theta_{iv}^{(\infty)}) - \lambda'_{v(i)} \Lambda_{v(i)}^{-1} (x_{v(i)} - \theta_{v(i)}^{(\infty)})\} w_{v:(i)}^{-1} = (x_v - \theta_v^{(\infty)})' \Lambda_v^{-1} e_i, \tag{4.4}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$, the i-th column vector of the identity matrix of order p, $i = 1, \ldots, p$. Thus using (4.4), equation (4.3) can be rewritten as

$$\sum_{v=1}^{k} \sum_{i=1}^{p} \{(x_v - \theta_v^{(\infty)})' \Lambda_v^{-1} e_i\} \theta_{iv}^{(\infty)} = 0.$$

Hence

$$\sum_{v=1}^{k} (x_v - \theta_v^{(\infty)})' \Lambda_v^{-1} \theta_v^{(\infty)} = 0.$$

Similarly by using (4.2) we can prove that

$$\sum_{v=1}^{k} (x_v - \theta_v^{(\infty)})' \Lambda_v^{-1} \xi_v \leq 0$$

for any $p \times k$ isotonic matrix $(\xi_1, \ldots, \xi_k)$. Thus from Theorem 2.2 we can conclude that $\theta^{(\infty)}$ is the multivariate isotonic regression of $x_1, \ldots, x_k$ with weights $\Lambda_1^{-1}, \ldots, \Lambda_k^{-1}$. □

For the purpose of considering the convergence of our algorithm first we define

$$a = \max_i \max_\nu \| \Lambda_{\nu(i)}^{-1} \lambda_{\nu(i)} \|,$$
$$b = a(p-1)^{1/2},$$

where $\| \cdot \|$ denotes the Euclidean norm in $(p-1)$-dimensional Euclidean space.

Then we have the following theorems which can be used in giving a condition for convergence of our algorithm and also in evaluating the order of convergence.

THEOREM 4.2. *When* $b \le 1$, *for* $n = 1, 2, \ldots; i = 1, \ldots, p$,

$$\max_\nu |\theta_{i\nu}^{(n)} - \theta_{i\nu}^{(n+1)}| \le b^{n-1} \max_j \max_\nu |\theta_{j\nu}^{(1)} - \theta_{j\nu}^{(2)}|.$$

PROOF. Let, for arbitrary n and $i = 1, \ldots, p$,

$$c_i^{(n)} = \max_\nu |\theta_{i\nu}^{(n)} - \theta_{i\nu}^{(n+1)}|,$$
$$c = \max_j c_j^{(1)}.$$

When $n = 1$, the proof is trivial.

Now $\theta_{1\nu}^{(2)}$ and $\theta_{1\nu}^{(3)}$ are the univariate isotonic regressions of $\{x_{1\nu} - \lambda'_{\nu(1)} \Lambda_{\nu(1)}^{-1} (x_{\nu(1)} - \theta_{\nu(1)}^{(1)})\}$ and $\{x_{1\nu} - \lambda'_{\nu(1)} \Lambda_{\nu(1)}^{-1} (x_{\nu(1)} - \theta_{\nu(1)}^{(2)})\}$ with the same weights $w_{\nu:(1)}^{-1}$, respectively. Therefore by using Lemma 4.1 (ii) of Sasabuchi et al. [8], we have

$$\begin{aligned}
c_1^{(2)} &= \max_\nu |\theta_{1\nu}^{(2)} - \theta_{1\nu}^{(3)}| \\
&\le \max_\nu |\lambda'_{\nu(1)} \Lambda_{\nu(1)}^{-1} (\theta_{\nu(1)}^{(1)} - \theta_{\nu(1)}^{(2)})| \\
&\le \max_\nu (\| \Lambda_{\nu(1)}^{-1} \lambda_{\nu(1)} \| \cdot \| \theta_{\nu(1)}^{(1)} - \theta_{\nu(1)}^{(2)} \|) \\
&\le \max_\nu (\| \Lambda_{\nu(1)}^{-1} \lambda_{\nu(1)} \| \cdot (p-1)^{1/2} \max_{2 \le i \le p} |\theta_{i\nu}^{(1)} - \theta_{i\nu}^{(2)}|) \\
&\le (\max_\nu \| \Lambda_{\nu(1)}^{-1} \lambda_{\nu(1)} \|) \cdot (p-1)^{1/2} \max_{2 \le i \le p} \max_\nu |\theta_{i\nu}^{(1)} - \theta_{i\nu}^{(2)}| \\
&\le a(p-1)^{1/2} \max \{c_2^{(1)}, \ldots, c_p^{(1)}\} \\
&\le bc.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
c_2^{(2)} &= \max_\nu |\theta_{2\nu}^{(2)} - \theta_{2\nu}^{(3)}| \\
&\le b \max \{c_3^{(1)}, \ldots, c_p^{(1)}, c_1^{(2)}\} \\
&\le b \max \{c, bc\} \\
&= bc.
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$c_p^{(2)} \le b \max \{c_1^{(2)}, \ldots, c_{p-1}^{(2)}\}$$

$$\le b^2 c$$

$$\le bc.$$

Hence the result is proved when $n = 2$.

Similarly we can show that

$$c_1^{(3)} \le b \max\{c_2^{(2)}, \ldots, c_p^{(2)}\}$$
$$\le b^2 c.$$

Continuing in this manner, we can show that

$$c_i^{(n)} \le b \max\{c_{i+1}^{(n-1)}, \ldots, c_{i-1}^{(n)}\}.$$

Hence by induction we can show that

$$c_i^{(n)} \le b^{n-1} c.$$

This completes the proof. □

The following theorem can be easily derived from Theorem 4.2.

THEOREM 4.3. *When $b < 1$, the p sequences of vectors $\{\theta^{i(n)}\}_{n=1,2,\ldots}$, $(i = 1,\ldots,p)$ converge and*

$$\max_v |\theta_{iv}^{(n)} - \theta_{iv}^{(\infty)}| \le (1-b)^{-1} b^{n-1} \max_j \max_v |\theta_{jv}^{(1)} - \theta_{jv}^{(2)}|,$$
$$i = 1,\ldots,p;\ n = 1, 2,\ldots$$

COROLLARY 4.1. *When Λ_v's take the following form, for $v = 1,\ldots,k$; $i = 1,\ldots,p$,*

$$\lambda_{vij} = \sigma_v^2 \rho_v,\ i \neq j, \qquad \lambda_{vii} = \sigma_v^2,$$

where λ_{vij} is the (i,j)-th element of Λ_v, the conclusions of Theorem 4.3 hold with

$$b = \max_v \frac{|\rho_v|(p-1)}{|\rho_v(p-1) + (1-\rho_v)|}$$

if $-(2p-3)^{-1} < \rho_v < 1$, $v = 1,\ldots,k$.

PROOF. In this case it can be easily seen that $\Lambda_{v(i)}^{-1} \lambda_{v(i)}$ is a $(p-1)$-dimensional vector with all elements $\rho_v\{\rho_v(p-1) + (1-\rho_v)\}^{-1}$. Thus

$$b = \max_v \frac{|\rho_v|(p-1)}{|\rho_v(p-1) + (1-\rho_v)|}.$$

After some simple calculation we can show that

$$\frac{|\rho_v|(p-1)}{|\rho_v(p-1)+(1-\rho_v)|} < 1 \qquad \text{if } -(2p-3)^{-1} < \rho_v < 1.$$

Hence the assertion of the corollary follows. □

REMARK. When Λ_v's take the above form, by using the fact that they are positive definite we can show that $-(p-1)^{-1} < \rho_v < 1$. Thus, as $p-1 = 2p-3$ when $p=2$, the condition for convergence of our algorithm is satisfied automatically as stated in Corollary 2 of Sasabuchi et al. [8]. But, as $p-1 < 2p-3$ when $p \geq 3$, the condition for convergence of our algorithm does not follow automatically only from the positive definiteness of Λ_v's.

Theorems 4.2 and 4.3, and Corollary 4.1 are multivariate generalizations of Theorems 4.4 and 4.5, and Corollary 2 of Sasabuchi et al. [8], respectively.

Sasabuchi et al. [8] have given a similar condition for convergence of this algorithm in the bivariate case. But it has been stated in Nomakuchi and Shi [6] that the condition imposed by Sasabuchi et al. [8] is not necessary for its convergence. However, as seen above, we have given a simple proof for the convergence of our algorithm under the condition imposed in Theorem 4.3. Also this can be used to evaluate the order of convergence.

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