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Travelling wave solutions to a perturbed Korteweg-de Vries equation

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1. Introduction

In fluid dynamics, many authors have tried to explain wave motions on a liquid layer over an inclined plane. To study this problem they have proposed several models which seem to be very interesting also from the mathematical point of view [1], [13]. One of the models is the following partial differential equation derived by Topper and Kawahara [18]:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \tag{1.1}$$

Here, the wave motion is assumed to depend only on the gradient direction x of the plane. The variable u means the height of the wave at the point x and time t. The physical parameters α, β and γ are all positive. Let us assume that the inclined plane is infinitely long toward the direction of x, that is, $x \in (-\infty, \infty)$. Then (1.1) can be considered as a 1-parameter equation by taking an appropriate scale transformation of u, x and t. For example, (1.1) can be transformed to the following ε-family of equations:

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \qquad t \geq 0, \ -\infty < x < \infty. \tag{1.2}$$

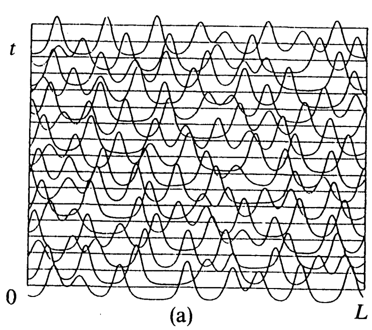
Here ε is a positive parameter. The equation (1.2) is regarded as the Korteweg-de Vries equation when the backward diffusion (u_{xx}) and dissipation (u_{xxxx}) terms are absent [19]. On the other hand, when ε is large it is expected to be close to the Kuramoto-Sivashinsky equation:

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \qquad t \geq 0, \ -\infty < x < \infty,$$

which describes chemical turbulence ([10], [11]) or instability of flame front ([17]) and exhibits very complicated patterns. Therefore, one find that ε is a very important parameter, which determines the character of the solutions of the equation. Numerical solutions to (1.2) are shown in Kawahara and Toh [7] with different values of ε. In Figure 1 one of the same numerical solutions to (1.2) with small ε is shown.

In this paper we are mainly concerned with the equation (1.2) when ε is small. At first one might expect that they have similar solutions to those of the KdV equation. To answer this question is significant not only mathematically but also in the application point of view. Because some physical circumstances, for example the Reynolds number is large or surface tension is small, correspond to the case when ε is small. It is also important in the sense of understanding the role of dispersion, dissipation and instability in nonlinear wave systems.

In Figure 1 a typical numerical simulation of the initial and boundary value problem for (1.2) with small ε is shown, where the boundary condition is periodic. Also there is the one for the KdV equation with multi soliton initial data. For the soliton solution, see Zabusky and Kruskal [19] and Lax [12]. It is a well-known fact that in the KdV equation there are permanent waves of different amplitudes and each wave retains their form even after joint interaction. On the other hand for the equation (1.2), it can be said that at the first stage many pulses appear and each pulse moves to the same direction interacting each other. In the next stage the amplitudes of them become equal to each other and consequently they move at the same speed. Kawahara and Toh's numerical results [8] show that this wave train is very close to the KdV cnoidal wave solution, which are a well-known periodic wave solution to the equation, for suitably determined amplitudes and wave velocities. To analyze this second stage in detail, let us put specific initial data and compare the result for the KdV equation with that for (1.2). In Figure 2 we put a 2-soliton-like initial data. In the perturbed equation (1.2) the two waves interact each other and vary their shapes and amplitudes to become equal gradually. In Figure 3 two cnoidal waves of different amplitudes are given for (1.2). In these cases the amplitudes of them are also modulatd and converge to some value.



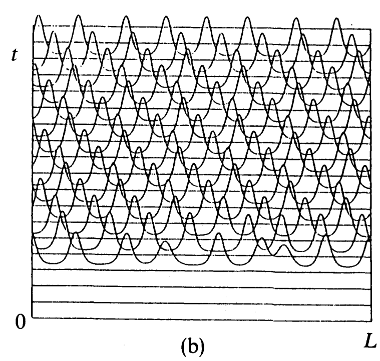


Figure 1. Numerical simulations of the initial and boundary value problem in the interval $(0, L)$.
(a) KdV equation. $\alpha = 0$, $\gamma = 0$ and $\beta = 20$ in (1.1). Multi soliton-like initial data is given. $L = 200$.
(b) $\alpha = 1$, $\gamma = 1$ and $\beta = 20$ in (1.1). It corresponds to the case $\varepsilon = 1/20$ in (1.2). Some small amplitude initial data is given. $L = 200$.
Boundary conditions are all periodic.

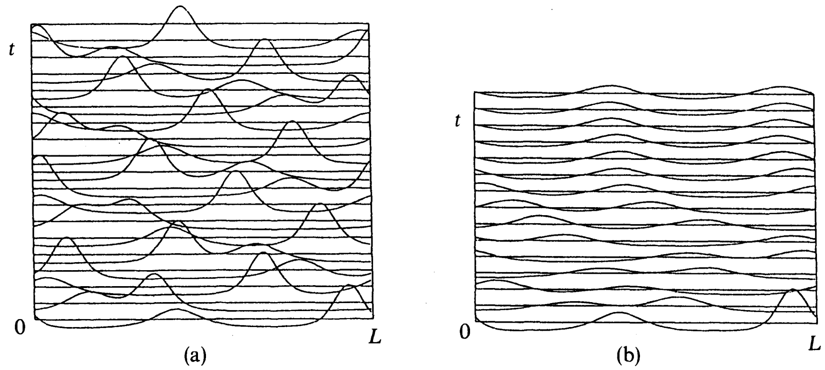


Figure 2. Numerical simulations of the initial and boundary value problem in the interval $(0, L)$. 2-soliton-like initial data are given in all cases. Also boundary conditions are all periodic.

(a) KdV equation. $\alpha = 0$, $\gamma = 0$ and $\beta = 15$ in (1.1). $L = 37$.

(b) $\alpha = 1$, $\gamma = 1$ and $\beta = 15$ in (1.1). $L = 37$.

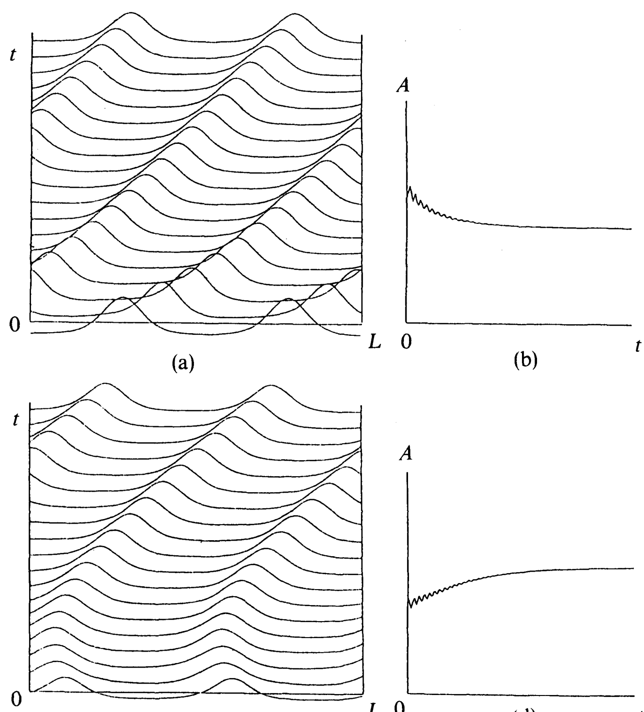


Figure 3. Numerical simulations of the initial and boundary value problem in the interval $(0, L)$.

(a), (c) (1.1) with $\alpha = 1$, $\gamma = 1$ and $\beta = 20$. Two cnoidal waves which have different amplitudes are given for the initial data. $L = 50$. Boundary conditions are all periodic.

(b), (d) Evolutions of corresponding amplitudes are shown.

Motivated by the above, we address the following questions:

(P1) Can KdV pulse and cnoidal wave solutions persist when ε is small in the equation (1.2) as is shown in the numerical examples?

And if so,

(P2) is there any amplitude or speed selection principle in (1.2)?

Kawahara and Toh performed formal perturbation analysis for these problems to equation (1.1) in [7] and determine the amplitudes and the shapes of travelling wave solutions to (1.2). Here we consider this problem from a mathematically rigorous point of view, by using bifurcation methods in Hamiltonian system, so that the structures of solutions can be understood clearly together with the roles of instability and dissipation terms.

For this purpose let us first investigate travelling wave solutions to the equation (1.2). By using the travelling coordinate $z = x - ct$, where c is speed, we get the following equation for travelling wave solution to (1.2):

$$- cu' + uu' + u''' + \varepsilon(u'' + u'''') = 0. \tag{1.3}$$

Here $'$ denotes the derivative by z. After integration and suitable scale transformation, the problem is reduced to finding the homoclinic and periodic solutions of

$$- U + \frac{1}{2}U^2 + \ddot{U} + \varepsilon\left(\frac{1}{\sqrt{c}}\dot{U} + \sqrt{c}\dddot{U}\right) = 0. \tag{1.4}$$

Here $U = u/c$ and denotes the derivative by $\tau = \sqrt{c}z$.

At a glance, this problem looks like a singular perturbation problem because ε is small. However, it can be reduced to a regular perturbation problem. It is convenient to rewrite (1.4) as a system of first order differential equations as follows.

$$\begin{cases} \dot{U} = V, \\ \dot{V} = W, \\ \varepsilon\sqrt{c}\dot{W} = U - \frac{1}{2}U^2 - W - \frac{\varepsilon}{\sqrt{c}}V. \end{cases} \tag{1.5}$$

Then, by putting $\varepsilon = 0$ formally, one can expect that there exists a 2-dimensional invariant manifold near the surface $\{U - U^2/2 - W = 0\}$ when ε is small. Fenichel [6] has studied such kind of singular perturbation problem and his theorem assures the existence of the invariant manifold in this case. Therefore the problem can be reduced to a regular perturbation on this

2-dimensional manifold. Moreover, the unperturbed system of this perturbation equation is the following Hamiltonian system which is equivalent to the travelling wave equation for KdV, in other words, it can be get from the lowest order approximation of the invariant manifold.

$$\begin{cases} \dot{U} = V, \\ \dot{V} = U - \dfrac{1}{2}U^2. \end{cases} \tag{1.6}$$

And by using higher approximation of the invariant manifold we can conclude that the regular perturbation equation takes the form:

$$\begin{cases} \dot{U} = V, \\ \dot{V} = U - \dfrac{1}{2}U^2 + \varepsilon\left(\sqrt{c}(U-1)V - \dfrac{1}{\sqrt{c}}V\right) + O(\varepsilon^2). \end{cases} \tag{1.7}$$

This reduction is the first step.

Next, to study the regular perturbation problem (1.7), we have only to calculate the following integral (See Carr [2]).

$$\Phi(c, \varepsilon) = \int \dot{H}(U, V)d\tau$$

Here, $H = V^2/2 + U^2/2 - U^3/6$ is a Hamiltonian function for the unperturbed system (1.6), and the integral is performed along the orbit of (1.7). $\Phi(c, \varepsilon) = 0$ implies that there exists a homoclinic or periodic ordit for (1.7). Simple calculation shows that

$$\Phi(c, \varepsilon) = \varepsilon\sqrt{c}\left(\int \dot{U}^2 d\tau - c\int \ddot{U}^2 d\tau\right) + O(\varepsilon^2).$$

And by taking the limit ε tends to zero in $\Phi/\varepsilon = 0$, we can conclude that the limit speed c_0 must satisfy the following condition.

$$\int \dot{U}^2 d\tau - c_0\int \ddot{U}^2 d\tau = 0. \tag{1.8}$$

Here, in this case, the integration is performed along the orbit of unperturbed system (1.6). And if one can show that c_0 is positive then the inplicit function theorem would give us the solution, *i.e.*, there exists a smooth function $c(\varepsilon)$ such that $\Phi(c(\varepsilon), \varepsilon) = 0$.

We can obtain the same condition as (1.8) in another way. By multiplying $\dot{U}$ to equation (1.4) and integrating it, we get the equation (1.8) as the necessary condition for the existence of travelling wave solutions to (1.2).

Therefore, for the third step, it is necessary to calculate these integrals of $\dot{U}^2$ and $\ddot{U}^2$ in the condition (1.8). The speed of travelling wave solution is described as the ratio of these Abelian integrals. We can analyze these functions by using the theory of analytic functions and algebraic geometry, which is clearly reported by Carr, Chow and Hale [3] and Cushman and Sanders [4]. As a result, we will get monotonicity properties for the speed of travelling waves, by which the selection principle can be understood. Recently Derks and Gils [5] have studied the similar problem to ours, however, their interest lies only in the necessary condition such as (1.8) and other points still remain unclear. They have not discussed the relation between the amplitude and the wavelength which we will study in the last section.

These are summarized into three steps: i) *reduction into an regular perturbation problem*, ii) *the regular perturbation analysis for a Hamiltonian system* and iii) *calculation of Abelian integrals*. We will study the third step first in Section 3 and later, the former two steps will be considered in Section 4.

In this paper we restrict ourselves to the existence and the structure of travelling wave solutions to the perturbed system (1.2). However as is stated in Kawahara and Toh [8] and [9], the solution structure has very rich character when ε is globally varied. Therefore this work is only a first step for studying this kind of nonlinear wave equations. These are attractive problems for further study.

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2. Some notations and the main result

Our purpose is to find travelling wave solutions to the equation (1.2). Therefore, as was mentioned in Section 1, it can be translated to the following by the travelling coordinate of the speed c, $z = x - ct$:

$$- cu' + uu' + u''' + \varepsilon(u'' + u'''') = 0. \tag{2.1}$$

Here $'$ denotes the derivative by z. On the other hand, these equations have an integral:

$$- cu + \frac{u^2}{2} + u'' + \varepsilon(u' + u''') = C, \tag{2.2}$$

for some integral constant C. In (2.2), without loss of generality, we can put C equal to zero, because (2.1) is invariant under Galilei transformation:

$$\hat{u} = u - k, \quad \hat{c} = c - k.$$

Moreover if we perform the scalings $u = cU$ and $z = \tau/\sqrt{c}$ to the equation (2.2), the final equation is given as

$$- U + \frac{1}{2}U^2 + \ddot{U} + \varepsilon\left(\frac{1}{\sqrt{c}}\dot{U} + \sqrt{c}\dddot{U}\right) = 0. \tag{2.3}$$

Here, denotes the derivative by τ. If we find a bounded solution (U, ε, c) to (2.3) then the corresponding (u, ε, c) is our travelling wave solution to the equation (2.1) and, therefore, to the original equation (1.2).

Now the unperturbed system of (2.3) is the following, whose solutions are travelling wave solutions to the KdV equation.

$$- U + \frac{1}{2}U^2 + \ddot{U} = 0. \tag{2.4}$$

It has an equialent form:

$$\begin{cases} \dot{U} = V, \\ \dot{V} = U - \dfrac{1}{2}U^2, \end{cases} \tag{2.5}$$

which is a Hamiltonian system with the Hamiltonia n function:

$$H = -\frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{6}U^3. \tag{2.6}$$

Consider a level curve of the form $H = \kappa$ in the region $\{U > 0\}$. It corresponds to a periodic orbit of (2.5) if κ satisfies $0 < \kappa < 2/3$, and when $\kappa = 0$ it includes a homoclinic orbit. By *homoclinic* we mean the solution whose ω-limit and α-limit set are the same point. These are a well-known cnoidal wave solution and a 1-solition solution to the KdV equation, respectively. Therefore we can parameterize all travelling wave solutions to (2.4) by κ. By using this parametrization, we can describe the existence result of travelling wave solutions. The proof will be dene in the following two sections.

THEOREM 2.1. *Let I_U be any bounded interval in* $\mathbf{R}$. *Then there are some*

ε^* such that the family of solutions to (2.3):

$$\left\{U_{\kappa,\varepsilon,c(\varepsilon,\kappa)}(\tau) : 0 \le \kappa < \frac{2}{3},\ 0 < \varepsilon < \varepsilon^*,\ \dot{U}_{\kappa,\varepsilon,c(\varepsilon,\kappa)}(0) = 0,\ \ddot{U}_{\kappa,\varepsilon,c(\varepsilon,\kappa)}(0) < 0\right\}$$

exists, where each $U_{\kappa,\varepsilon,c}$ satisfies (2.3) *for ε and c. And $U_{\kappa,\varepsilon,c}(\tau)$ is a homoclinic or periodic solution when $\kappa = 0$ or $0 < \kappa < 2/3$, respectively. On the contrary, if we find a homoclinic or periodic solution $U(\tau)$ of* (2.3) *which satisfies $U(\tau) \in I_U$ for all τ when ε is in $(0, \varepsilon^*)$, then there are some θ and κ such that*

$$U(\tau) = U_{\kappa,\varepsilon,c(\varepsilon,\kappa)}(\tau - \theta).$$

Moreover $c(\varepsilon, \kappa)$ is a smooth function of ε and κ and when ε tends to zero $c(\varepsilon, \kappa)$ converges to $c_0(\kappa)$, where $c_0(\kappa)$ is a smooth decreasing function on $\kappa \in [0, 2/3]$, $c_0(0) = 7/5$ and $c_0(2/3) = 1$. And also $U_{\kappa,\varepsilon,c}(\tau)$ converges to $U_\kappa(\tau)$ uniformly in τ, wher $U_\kappa(\tau)$ is a homoclinic or periodic solution of (2.4) *on the level curve $\{H = \kappa\}$ with $U_\kappa(0) = 0$ and $U_\kappa(0) < 0$.*

REMARK 2.2. The simple pictures of these limits when e tends to zero, i.e. the bifurcating points, are shown in Figure 4.

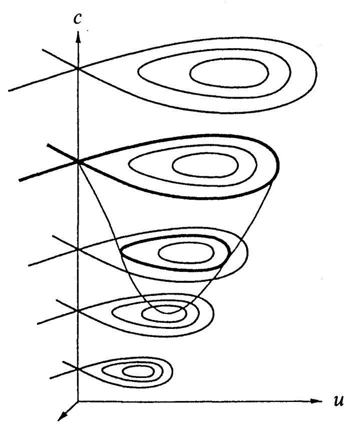

Figure 4.

REMARK 2.3. Although there exist travelling wave solutions to the KdV equation for any positive sped, in the perturbed systems some appropriate speed $c(\varepsilon, \kappa)$ and consequently amplitude are selected by the balance between the instability and dissipation effects.

REMARK 2.4. In this theorem only the relation between the speed and κ (the level of the Hamiltonian) is treated, however, the relation between the speed and wavelength is also a matter of significance. It will be discussed in Section 5.

3. Analysis by the Abelian integral theory

It is already mentioned in Section 1 that there is a necessary condition (1.8) for the existence of travelling wave solutions to (2.3). In this section we assume U is a solution to (2.4) in (1.8). Because U is expected to be close to the solution to (2.4). Rigorous treatment of the perturbation procedure will be mentioned in the next section. And there, (1.8) is given quite naturally in the perturbation procedure. Here we concentrate ourselves on calculating the limit speed, when ε tends to zero, of the travelling wave solutions from (1.8).

First, let Q and R be

$$Q = \frac{1}{2}\int \ddot{U}^2 d\tau \quad \text{and} \quad R = \frac{1}{2}\int \dot{U}^2 d\tau.$$

And let the two non-negative roots of $U^2 - U^3/3 = 2\kappa = K$ be $\alpha(K)$ and $\beta(K)$, where $\alpha(K) < \beta(K)$. Here, as in Section 2, $0 \le \kappa < 2/3$ and $0 \le K < 4/3$. As mentioned above, the orbit $\{(U(\tau), V(\tau))\}$ is on the level curve $\{H = \kappa = K/2\}$, where $V = dU/d\tau$, therefore we have

$$Q = \int_\alpha^\beta \frac{\left(U - \frac{1}{2}U^2\right)^2}{E(U)} dU, \quad R = \int_\alpha^\beta E(U)\, dU,$$

by using equations (2.5). Here, $E(U) = \sqrt{U^2 - U^3/3 - K}$.
Now Q and R are the functions of only K. The purpose of this section is to prove the following proposition which will assert the monotonicity of the speed with K.

PROPOSITION 3.1. *Let* $X(K) = \dfrac{Q}{R}$. *We have* $X'(K) > 0$ *for* $0 < K < 4/3$. *Moreover*

$$\lim_{k \downarrow 0} X(K) = \frac{5}{7} \quad \text{and} \quad \lim_{K \uparrow 4/3} X(K) = 1.$$

To prove the proposition, it is convenient to represent Q and R by the following integrals:

$$J_n(K) = \int_\alpha^\beta U^n E(U) dU, \; n = 0, 1, 2, \cdots. \tag{3.1}$$

Then it holds

$$\int_{\alpha}^{\beta} \frac{U^n}{E(U)} dU = 2J_n'(K). \tag{3.2}$$

Therefore Q and R are represented as follows:

$$Q = -2J_2'(K) + 2J_3'(K) - \frac{1}{2}J_4'(K) \text{ and}$$

$$R = J_0.$$

At first, let us study the basic properties of J_0 and J_1 by the following four lemmas.

LEMMA 3.2. $\lim_{K\downarrow 0} J_0 = \frac{12}{5}$, $\lim_{K\downarrow 0} J_1 = \frac{144}{35}$.

PROOF. Direct calculations of J_0 and J_1 with $K = 0$.

LEMMA 3.3. $\lim_{K\uparrow 4/3} \frac{J_1(K)}{J_0(K)} = 2$.

PROOF. $$\lim_{K\uparrow 4/3} \frac{J_1}{J_0} = \lim_{K\uparrow 4/3} \frac{\int_{\alpha}^{\beta} UE(U)dU}{\int_{\alpha}^{\beta} E(U)dU} = \lim_{U\to 2} U = 2.$$

LEMMA 3.4. $$\begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \Lambda(K)\begin{pmatrix} J_0' \\ J_1' \end{pmatrix}, \tag{3.3}$$

where $$\Lambda(K) = \begin{pmatrix} \frac{6}{5}K & -\frac{4}{5} \\ \frac{12}{35}K & \frac{3}{7}\left(2K - \frac{16}{5}\right) \end{pmatrix}.$$

PROOF. From the relation

$$E^2 = U^2 - \frac{U^3}{3} - K \quad \text{and hence} \quad 2E\frac{dE}{dU} = 2U - U^2,$$

J_0 can be calculated as follows.

$$\begin{aligned} J_0 &= \int E dU = \int E^2 \frac{dU}{E} = \int \left(U^2 - \frac{1}{3}U^3 - K\right)\frac{dU}{E} \\ &= \int \left\{\left(1 - \frac{1}{3}U\right)\left(2U - 2E\frac{dE}{dU}\right) - K\right\}\frac{dU}{E} \end{aligned}$$

$$= \int (2U - K)\frac{dU}{E} - \frac{2}{3}\int U^2 \frac{dU}{E} - 2\int \left(1 - \frac{1}{3}U\right)\frac{dE}{dU}dU$$
$$= \int (2U - K)\frac{dU}{E} - \frac{2}{3}\int \left(2U - 2E\frac{dE}{dU}\right)\frac{dE}{E} - 2\int \left(1 - \frac{1}{3}U\right)\frac{dE}{dU}dU$$
$$= \int \left(\frac{2}{3}U - K\right)\frac{dU}{E}\int (1 - U)\frac{dE}{dU}dU$$
$$= \int \left(\frac{2}{3}U - K\right)\frac{dU}{E} - \frac{2}{3}\int E dU$$
$$= 2KJ_0' - \frac{4}{3}J_1' - \frac{2}{3}J_0.$$

This implies the first equation in the lemma. On the other hand, J_1 is calculated almost simply as J_0.

$$J_1 = \int UE(U)dU = \int UE^2\frac{dU}{E} = \int U\left(U^2 - \frac{U^3}{3} - K\right)\frac{dU}{E}$$
$$= \cdots$$
$$= \frac{4}{5}KJ_0' + \left(2K - \frac{16}{5}\right)J_1' - \frac{4}{3}J_1.$$

This proves the lemma.

LEMMA 3.5. $$\begin{pmatrix} J_0'' \\ J_1'' \end{pmatrix} = \frac{1}{\varDelta}\begin{pmatrix} -\frac{1}{2}K & \frac{1}{3} \\ -K & \frac{1}{2}K \end{pmatrix}\begin{pmatrix} J_0' \\ J_1' \end{pmatrix},$$

where $\varDelta = K(3K - 4)$.

PROOF. By the previous lemma, $J'' = A^{-1}(I - A')J'$ holds, where I denotes the identity matrix. We have

$$I - A' = \begin{pmatrix} -\frac{1}{5} & 0 \\ -\frac{12}{35} & \frac{1}{7} \end{pmatrix} \quad \text{and} \quad A^{-1}(K) = \frac{35}{12\varDelta}\begin{pmatrix} \frac{3}{7}\left(2K - \frac{16}{5}\right) & \frac{4}{5} \\ -\frac{12}{35}K & \frac{6}{5}K \end{pmatrix},$$

hence,

$$\Lambda^{-1}(I - \Lambda')J' = \frac{1}{\Delta}\begin{pmatrix} -\frac{1}{2}K & \frac{1}{3} \\ -K & \frac{1}{2}K \end{pmatrix}.$$

This implies the lemma.

We note that all J_n's can be represented only by J_0 and J_1. For example the following lemma holds:

LEMMA 3.6. $J_2 = 2J_1$, $J_3 = -\frac{6}{11}KJ_0 + \frac{48}{11}J_1$ *and*

$$J_4 = -\frac{180}{11 \cdot 13}KJ_0 + \frac{12}{13}\left(\frac{120}{11} - K\right)J_1.$$

PROOF. J_2, J_3 and J_4 are calculated similarly to the proof of LEMMA 3.4.

$$\begin{aligned} J_2 &= \int U^2 E dU \\ &= \cdots \\ &= 6J_1 - 2J_2. \\ J_3 &= \int U^3 E dU = \cdots \\ &= -2KJ_0 + 16J_1 - \frac{8}{3}J_3. \\ J_4 &= \int U^4 E dU = \int U^2\left(2U - 2E\frac{dE}{dU}\right)EdU \\ &= \cdots \\ &= -4KJ_1 + 10J_3 - \frac{10}{3}J_4. \end{aligned}$$

Therefore we get

$$\frac{13}{3}J_4 = -\frac{60}{11}K + \left(\frac{480}{11} - 4K\right)J_1.$$

This completes the proof.

To analyze Q and R, let us represent them by J_0 and J_1. Applying lemmas 3.4, 3.5 and 3.6,

$$
\begin{aligned}
Q &= -2J_2' + 2J_3' - \frac{1}{2}J_4' \\
&= -\frac{6}{7}KJ_0' + \frac{2}{7}(3K-2)J_1' \\
&= -J_0 + J_1. && (3.4)
\end{aligned}
$$

$$R = J_0,\ R' = J_0'. \tag{3.5}$$

We can now show the monotonicity of $X = Q/R$. By the above relation (3.4) and (3.5) we have

$$X = \frac{Q}{R} = \frac{J_1}{J_0} - 1, \tag{3.6}$$

therefore we consider J_1/J_0 instead of X. Let $\tilde{X} = J_1/J_0$ and $Z = J_1'/J_0'$. Then by LEMMA 3.5

$$
\begin{aligned}
Z' &= \frac{1}{(J_0')^2}(J_1''J_0' - J_1'J_0'') \\
&= -\frac{1}{3\Delta}\{3K - 3KZ + Z^2\}. && (3.7)
\end{aligned}
$$

LEMMA 3.7. $Z' = \dfrac{1}{3\Delta}\{3K - 3KZ + Z^2\} > 0$, *for* $0 < K < 4/3$.

PROOF. $3K - 3KZ + Z^2$

$$= \left(Z - \frac{3K}{2}\right)^2 + \frac{3K(4-3K)}{4} > 0.$$

And $\Delta < 0$ from LEMMA 3.5, we get the lemma.

Equation (3.7) corresponds to the Ricatti equation in Carr, Chow and Hale [3] and Cushman and Sanders [4].

LEMMA 3.8. If $\tilde{X}'(K_0) = 0$ *for some* $0 < K_0 < 4/3$, *then* $\tilde{X}''(K_0) < 0$.

PROOF. By the definitions of $\tilde{X}$ and Z, we have

$$
\begin{aligned}
&J_0'\tilde{X} + J_0\tilde{X}' = J_1', \\
&J_0''\tilde{X} + 2J_0'\tilde{X}' + J_0\tilde{X}'' = J_1'', \\
&J_0'Z = J_1', \\
&J_0''Z + J_0'Z' = J_1''.
\end{aligned}
$$

Since $\tilde{X}'(K_0) = 0$, we have $\tilde{X}''(K_0) = \dfrac{J_0'(K_0)Z'(K_0)}{J_0(K_0)}$. By (3.1) and (3.2), $J_0(K) > 0$ and $J_0'(K) < 0$. Therefore by LEMMA 3.7, we can conclude $\tilde{X}''(K_0) < 0$.

LEMMA 3.9. *If* $\tilde{X}'(K_0) = 0$ *for some* $0 < K_0 < 4/3$, *then* $12/7 < \tilde{X}(K_0) < 2$.

PROOF. By the equations (3.3), *i.e.*

$$J_0 = \frac{6}{5}KJ_0' - \frac{4}{5}J_1' \quad \text{and} \quad J_1 = \frac{12}{35}KJ_0' + \frac{3}{7}\left(2K - \frac{16}{5}\right)J_1',$$

we get

$$12J_0 - 7J_1 = 6K(2J_0' - J_1').$$

$$\frac{J_1}{J_0} - \frac{12}{7} = \frac{6K}{7J_0}J_0'\left(\frac{J_1'}{J_0'} - 2\right).$$

If $\tilde{X}'(K_0) = 0$, then $\tilde{X}(K_0) = \dfrac{J_1'}{J_0'}$ and

$$\tilde{X} - \frac{12}{7} = \frac{6K_0}{7}\frac{J_0'}{J_0}(\tilde{X} - 2).$$

Hence

$$\left(\frac{12}{7} - \tilde{X}\right)(\tilde{X} - 2) > 0,$$

and we have the lemma.

Moreover, from lemmas 3.2 and 3.3, we have

$$\lim_{K\downarrow 0}\frac{J_1}{J_0} = \frac{12}{7}, \qquad \lim_{K\uparrow 4/3}\frac{J_1}{J_0} = 2.$$

Therefore combining these facts with lemmas 3.8 and 3.9 we get the following:

LEMMA 3.10. *For* $0 < K < 4/3$, $\tilde{X}'(K) > 0$.

By noting the relation (3.6), this proves the monotonicity of X in the PROPOSITION 3.1.

4. Perturbation analysis

Our purpose here is to find a homoclinic and periodic solutions to the equation (2.3):

$$- U + \frac{1}{2}U^2 + \ddot{U} + \varepsilon\left(\frac{1}{\sqrt{c}}\dot{U} + \sqrt{c}\,\dddot{U}\right) = 0.$$

It is a singular perturbation problem when ε is small as was mentioned in Section 1.

In contrast, if we did not have the perturbation term of u_{xxxx} in (1.2) then the equation corresponding to (2.3) would be the following regular perturbation problem:

$$- U + \frac{1}{2}U^2 + \ddot{U} + \frac{\varepsilon}{\sqrt{c}}\dot{U} = 0, \tag{4.1}$$

which is equivalent to

$$\begin{cases} \dot{U} = V, \\ \dot{V} = U - \frac{1}{2}U^2 - \frac{\varepsilon}{\sqrt{c}}V. \end{cases} \tag{4.2}$$

Along the orbit of (4.2) the derivative of the Hamiltonian is

$$\dot{H} = \frac{\varepsilon}{\sqrt{c}}V^2 \geq 0.$$

Therefore neither homoclinic nor periodic solution can exists in this case.

Let us go back to (2.3). The equivalent first order ODE system is

$$\begin{cases} \dot{U} = V, \\ \dot{V} = W, \\ \varepsilon\sqrt{c}\,\dot{W} = U - \frac{1}{2}U^2 - W - \frac{\varepsilon}{\sqrt{c}}V. \end{cases} \tag{4.3}$$

Assume c is in some bounded interval $[c_1, c_2]$ with $c_1 > 0$. Because of the small parameter ε the dynamics of (4.3) is very fast at the points apart from the surface $\{U - U^2/2 - W = 0\}$ and these points will be absorbed toward the surface. Therefore the essential part of the dynamics lies on this surface, more accurately, on the invariant manifold close to this surface. Our first aim is to show the existence of sucha invariant manifold and extract the reduced dynamics on it from (4.3).

By putting $Z = W - U + U^2/2$, (4.3) is equivalent to

$$
\begin{cases}
\dot{U} = V, \\
\dot{V} = U - \dfrac{1}{2}U^2 + Z, \\
\dot{Z} = -\dfrac{1}{\varepsilon\sqrt{c}}Z + UV - \left(1 + \dfrac{1}{c}\right)V, \\
\dot{\varepsilon} = 0.
\end{cases}
\tag{4.4}
$$

Here, for convenience we consider the parameter ε as a variable so that we can study the flow in $(U, V, Z, \varepsilon) \in \mathbf{R}^3 \times \mathbf{R}$. And we mean the original dynamics of $(U, V, Z) \in \mathbf{R}^3$ by the words 3-*D dynamics*, 3-*D flow or* 3-*D equations* of (4.4) to distinguish it from the full dynamics of (4.4). The set $\{(U, V, Z, \varepsilon): \varepsilon = \text{constant.}\}$ is invariant under the full dynamics (4.4). Moreover by the slow time scaling $\sigma = \tau/\varepsilon$, (4.4) takes the form

$$
\begin{cases}
U' = \varepsilon V, \\
V' = \varepsilon\left(U - \dfrac{1}{2}U^2 + Z\right), \\
Z' = -\dfrac{1}{\sqrt{c}}Z + \varepsilon\left\{UV - \left(1 + \dfrac{1}{c}\right)V\right\}, \\
\varepsilon' = 0.
\end{cases}
\tag{4.5}
$$

Consider the plane $\Xi = \{(U, V, Z) \in \mathbf{R}^3 : Z = 0\}$. Ξ is consisting entirely of equilibrium points of the 3-D flow of (4.5). Linearized equation of this 3-D equations around each point in Ξ has exactly one negative eigenvalue and two zero eigenvalues whose eigenspace corresponds to the tangent space of Ξ. In this situation, we can apply the theorem on normally hyperbolic vector field by Fenichel (refer to Theorem 9.1 in Fenichel [6].). Consequently, we have 3-dimensional center manifold M in the full dynamics of (4.5) near S, that is a locally invariant manifold, $M \supset S \times \{0\}$ for fixed compact set S in Ξ and M is tangent to Ξ at all the points in $S \times \{0\}$. Moreover it can be represented as the graph of

$$
Z = \Gamma(U, V, \varepsilon), \quad (U, V) \in S, \quad \varepsilon \in I.
\tag{4.6}
$$

Here, I is some neighborhood of zero in $\mathbf{R}$ and $\Gamma(U, V, 0) = 0$, $D\Gamma(U, V, 0) = 0$ for all $(U, V) \in S$. Because of a local invariant property of a center manifold, by differentiating (4.6) along the orbit of (4.5) we get

$$
\frac{1}{\sqrt{c}}(-\Gamma(U, V, \varepsilon)) + \varepsilon\left\{UV - \left(1 + \frac{1}{c}\right)V\right\} = D\Gamma(U, V, \varepsilon)\begin{pmatrix} U' \\ V' \\ \varepsilon' \end{pmatrix} = O(\varepsilon^2).
$$

And it gives us the first order approximation of the center manifold:

$$\Gamma(U, V, \varepsilon) = \varepsilon\sqrt{c}\left\{UV - \left(1 + \frac{1}{c}V\right)\right\} + O(\varepsilon^2). \tag{4.7}$$

LEMMA 4.1. *Let us fix any compact subset S in $\mathbf{R}^2$ and integer k. Then there exists some neighborhood I of zero such that we have a C^k manifold M*

$$M = \{(U, V, Z, \varepsilon): Z = \Gamma(U, V, \varepsilon), (U, V, \varepsilon) \in S \times I\},$$

which has the following four properties.
i) *$\Gamma \in C^k$, $\Gamma(U, V, 0) = 0$ and $D\Gamma(U, V, 0) = 0$ for all $(U, V) \in S$.*
ii) *M is locally invariant, i.e., if $P_0 = (U_0, V_0, Z_0, \varepsilon_0) \in M$ and let $P(\sigma)$ be the solution of* (4.5) *with $P(0) = P_0$ then $P(\sigma) \in M$ when $|\sigma|$ is small enough.*
iii) *If $(U(\sigma), V(\sigma), Z(\sigma), \varepsilon)$ is a solution to* (4.5) *for all $\sigma \in \mathbf{R}$ and $(U(\sigma), V(\sigma)) \in M$ for all σ, then $(U(\sigma), V(\sigma), Z(\sigma), \varepsilon) \in M$ for all σ.*
iv) *Γ satisfies* (4.7).

Now we can reduce the 3-D dynamics onto a 2-dimensional manifold $M^{\varepsilon_1} = M \cap \{(U, V, Z, \varepsilon): \varepsilon = \varepsilon_1\}$. From the previous lemma we have only to consider the dynamics on M^ε for small ε to find a homoclinic or periodic solution. By substituting (4.7) into (4.4) we eventually get the reduced dynamics on M^ε:

$$\begin{cases} \dot{U} = V, \\ \dot{V} = U - \dfrac{1}{2}U^2 + \varepsilon\sqrt{c}\left\{UV - \left(1 + \dfrac{1}{c}\right)V\right\} + O(\varepsilon^2). \end{cases} \tag{4.8}$$

It is a regular perturbation problem.

If we once get a regular perturbation problem like (4.8) in (U, V)-plane, we can easily check if a periodic orbit persists or not as follows. First, remember the dynamics of the unperturbed system (2.5), which can be understood by the level curve of H. Fix an initial data $(\alpha, 0)$ with $0 < \alpha < 2$. Now let $(U(\tau), V(\tau))$ be the solutio of (4.8) with $(U, V)(0) = (\alpha, 0)$. Then there exists $\tau_1 > 0$ and $\tau_2 < 0$ such that

$$V(\tau) > 0 \text{ for } 0 < \tau < \tau_1,\ V(\tau_1) = 0,\ V(\tau) < 0 \text{ for } 0 > \tau > \tau_2 \text{ and } V(\tau_2) = 0. \tag{4.9}$$

Let us define a function Φ as follows. (See chapter 4 of Carr [2] in detail.)

$$\Phi(\alpha, c, \varepsilon) = \int_{\tau_1}^{\tau_2} H(U, V)d\tau. \tag{4.10}$$

Here,

$$\dot{H} = \varepsilon\sqrt{c}\left\{UV^2 - \left(1 + \frac{1}{c}\right)V^2\right\} + O(\varepsilon^2).$$

$\Phi(\alpha, c, \varepsilon)$ denotes difference of the level between the two points on the U-axis:

$$\Phi(\alpha, c, \varepsilon) = H(U(\tau_1), V(\tau_1)) - H(U(\tau_2), V(\tau_2)).$$

Therefore, $\Phi(\alpha, c, \varepsilon) = 0$ if and only if it is a periodic solution. And our aim is to solve $\Phi = 0$. Let $\tilde{\Phi}(\alpha, c, \varepsilon) = \Phi(\alpha, c, \varepsilon)/\varepsilon$, then $\tilde{\Phi}(\alpha, c, \varepsilon)$ has a limit when ε tends to zero:

$$\tilde{\Phi}_0(\alpha, c) = \lim_{\varepsilon \downarrow 0} \tilde{\Phi}(\alpha, c, \varepsilon) = \sqrt{c}\int\left\{(U_0 - 1)V_0^2 - \frac{1}{c}V_0^2\right)\right\}d\tau.$$

Here, by noting (4.9), (U_0, V_0) is a solution of (2.5) and this integral is performed on a level curve $\{H = H(\alpha, 0)\}$, where $H(\alpha, 0) = \kappa \in (0, 2/3)$. Therefore,

$$\begin{aligned}\tilde{\Phi}_0(\alpha, c) &= \sqrt{c}\int\left\{(U_0 - 1)\dot{U}_0^2 - \frac{1}{c}\dot{U}_0^2\right\}d\tau \\ &= -\sqrt{c}\int \dot{U}_0 U\ddot{U}_0 d\tau - \frac{1}{\sqrt{c}}\int \dot{U}_0^2 d\tau \\ &= \frac{1}{\sqrt{c}}\left(c\int \ddot{U}_0^2 - \int \dot{U}_0^2\right) = 0.\end{aligned}$$

Thus we have to determine the limit speed c_0 by

$$\int \dot{U}_0^2 - c_0\int \ddot{U}_0^2 = 0. \tag{4.11}$$

We can define the similar function for a homoclinic solution as

$$\Psi(c, \varepsilon) = \int_{-\infty}^{0} \dot{H}(U, V)d\tau - \int_{0}^{\infty} \dot{H}(U, V)d\tau.$$

Here the former part is integrated along a solution $(U(\tau), V(\tau))$ on the one dimensional unstable manifold of the origin with $V(\tau) > 0$ for $-\infty < \tau < 0$ and $V(0) = 0$. The latter is similar. $\tilde{\Psi}(c, \varepsilon)$ and $\tilde{\Psi}_0(c)$ are also defined similarly:

$$\tilde{\Psi}(c, \varepsilon) = \frac{1}{\varepsilon}\Psi(c, \varepsilon) \quad \text{and} \quad \tilde{\Psi}_0(c) = \lim_{\varepsilon \downarrow 0} \tilde{\Psi}(c, \varepsilon).$$

Consequently, we get

$$\tilde{\Psi}_0(c) = \frac{1}{\sqrt{c}}\left(c\int \ddot{U}_0^2 - \int \dot{U}_0^2\right) = 0,$$

where U_0 is a solution of (2.4) and the integration is on the curve $\{H = 0\}$, more precisely, on the homoclinic solution of (2.5) in this case. Therefore the condition for the limit speed is the same as (4.11).

Let c_0 be

$$c_0 = X(K)^{-1}. \tag{4.12}$$

Then from PROPOSITION 3.1 in the previous section, we have

LEMMA 4.2. *For $0 \le K < 4/3$, the pairs (U_0, c_0) satisfies the limit speed condition* (4.11) *for periodic and homoclnic solutions with $\alpha = \alpha(K)$, where $\alpha(K)$ is defined in Section* 3. *Moreover, $dc_0/dK < 0$*

$$\lim_{K \downarrow 0} c_0 = \frac{7}{5} \quad \textit{and} \quad \lim_{K \uparrow 4/3} c_0 = 1.$$

Let us calculate

$$\frac{\partial \tilde{\Phi}}{\partial c}(\alpha(K), c_0, 0) = \frac{1}{2\sqrt{c_0}}\int \ddot{U}_0^2 + \frac{1}{2c_0\sqrt{c_0}}\int \dot{U}_0^2 > 0$$

and similarly $\partial\tilde{\Psi}/\partial c > 0$ so that we can solve the equations $\tilde{\Phi} = 0$ and $\tilde{\Psi} = 0$ by the implicit function theorem. More precisely, there exists a unique smooth function $c_K(\varepsilon) = c(\varepsilon, K)$ for each K such that

$\tilde{\Phi}(\alpha(K), c(\varepsilon, K), \varepsilon) = 0$ for $0 < K < 4/3$ and $0 < \varepsilon < \varepsilon^*$ and

$\tilde{\Psi}(c(\varepsilon, K), \varepsilon) = 0$ for $0 < \varepsilon < \varepsilon^*$.

Therefore we get THEOREM 2.1.

5. Wavelength and speed

Through Sections 1 to 4, we concentrate only on the relation between the speed of the travelling wave solutions and their level κ or K. However, from a practical view point, the relation between the speed and the wavelength is more important. Because κ is not a practical parameter for the equation (1.2) but it is used in convenience of parametrizing the travelling waves. If we fix the interval $[0, L]$ and assume that the boundary condition is periodic as in the numerical simulation Figure 1, then there are at most countable number of possibilities in wave-length, i.e. L/n for $n = 1, 2, \cdots$.

Noting a scale transformation $z = \tau/\sqrt{c}$ to get the equation (2.3), the wavelength ξ_0 for the original equation (1.2) is $\xi_0 = T/\sqrt{c_0}$. Here, T is a period of the travelling wave solution U in THEOREM 2.1. Let us consider the limit of $\varepsilon \downarrow 0$, then T is now a period of $U_\kappa(\tau)$, *i.e.*,

$$\int_\alpha^\beta \frac{dU}{V} = \int_0^{T/2} d\tau = \frac{T}{2},$$

$$T = 2\int \frac{dU}{E(U)} = -4J_0'(K).$$

LEMMA 5.1. $\lim_{K \downarrow 0} T = +\infty$, $\lim_{K \uparrow 4/3} T = 2\pi$.

PROOF. The first one is easy to calculate. For the second one, it follows from lemmas 3.4 and 3.3 that

$$\begin{aligned}
\lim_{K \uparrow 4/3} T &= -\lim_{K \uparrow 4/3} \frac{1}{3\Delta}\{3(10K-16)J_0 + 28J_1\} \\
&= -\lim_{K \uparrow 4/3} \frac{J_0}{3K(3K-4)}\left\{3(10K-16) + 28\frac{J_1}{J_0}\right\} \\
&= 12 \lim_{K \uparrow 4/3} \frac{J_0}{4-3K} \\
&= 2\pi.
\end{aligned}$$

LEMMA 5.2. *For* $0 < K < 4/3$, $T'(K) < 0$.

PROOF. From lemmas 3.5 and 3.4, we have

$$T'(K) = -4J_0''(K) = \frac{5}{3D}J_0 < 0.$$

LEMMA 5.3. *For* $0 < K < 4/3$, $\xi_0'(K) < 0$. *Moreover*

$$\lim_{K \downarrow 0} \xi_0(K) = +\infty \quad \text{and} \quad \lim_{K \uparrow 4/3} \xi_0(K) = 2\pi.$$

PROOF. Let $\xi = \xi_0^2/16$, then using (4.12), (3.4) and (3.5), we obtain

$$\xi = \frac{J_0'^2(-J_0 + J_1)}{J_0}.$$

We can prove the monotonicity of ξ by the technique almost similar to the argument in Section 3. Let $m = J_0'^2(-J_0 + J_1)$ and $n = J_0$. And let $\rho = \dfrac{m'}{n'}$.

What we have to show is that ρ is positive. By simple calculation,

$$\rho = 2J_0''(-J_0 + J_1) + J_0'(-J_0' + J_1').$$

Here, $J_0'' = -5J_0/12\varDelta$ and also $-J_0 + J_1 = Q$ are positive. While $J_0' = -\frac{1}{2}\int \frac{dU}{E}$ and also $-J_0' + J_1' = \frac{1}{2}\int \frac{1-U}{E} dU$ are negative. Therefore we have $\rho(K) > 0$ and the remainder part of the proof is similar.

Combining lemmas 4.2 and 5.3, we can conclude the following.

THEOREM 5.4. *In the limit ε tends to zero, the speed c_0 and the wavelength ξ_0 of travelling wave solutions to* (1.2) *have the relation*:

$$c_0 = \tilde{c}_0(\xi_0), \quad 2\pi < \xi_0 < +\infty, \quad \tilde{c}_0' > 0.$$

REMARK 5.5. Especially, there is a unique travelling wave solution which has a fixed wavelength, and moreover there is no travelling wave solution which has wavelength less that 2π to equation (1.2). It is also interesting to note that if we fix the interval $[0, L]$ then there are only a finite number of travelling wave solutions under periodic boundary condition, because L/n must be larger than 2π.

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