HIROSHIMA MATH. J.
26 (1996), 623–633

Decomposability of the mod p Whitehead element

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

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(Received August 4, 1995)

ABSTRACT. We give necessary and sufficient conditions for the mod p Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$ to be represented as a composition of some elements of positive stems in the homotopy groups of spheres. We also give necessary and sufficient conditions for w_n to be represented as Toda bracket of some elements of positive stems in the homotopy groups of spheres. Our problem is the odd primary version of the one studied by Iriye and Morisugi who treated the Whitehead product $[\iota_{2n-1}, \iota_{2n-1}]$ for the identity map ι_{2n-1} of S^{2n-1}.

1. Introduction

Let p be a fixed prime. In this paper we always assume that spaces are all localized at p. We study the decomposability of the mod p Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$. Let $C(n)$ be the homotopy fiber of the double suspension $\Sigma^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$, and $\varepsilon : C(n) \to S^{2n-1}$ the inclusion of the fiber. Then, it is known that $C(n)$ is $(2np-4)$-connected and $\pi_{2np-3}(C(n)) \cong \boldsymbol{Z}/p$. We denote a generator of $\pi_{2np-3}(C(n))$ by z. Then, according to the terminology due to [2], the mod p Whitehead element w_n is defined as $w_n = \varepsilon_*(z) \in \pi_{2np-3}(S^{2n-1})$. We will be concerned with w_n for an odd prime p.

Our main results are stated as follows; let $\alpha_i \in \pi^S_{2i(p-1)-1} \cong \boldsymbol{Z}/p$ for $i = 1, 2$ be a generator and $\langle -, -, - \rangle$ denote the Toda bracket [7].

THEOREM A. *Let p be an odd prime and $n \geq 2$. Then, w_n is decomposed as $w_n = \sum_i a_i b_i$ for some elements $\{a_i, b_i\}$ of positive stems in the homotopy groups of spheres if and only if one of the following holds*:

(1) *p is odd and $n = 2$, for which $w_n = \alpha_1 \alpha_1$;*

(2) *$p = 3$ and $n = 3$, for which $w_n = \alpha_1 \alpha_2$.*

THEOREM B. *Let p be an odd prime and $n \geq 2$. Then, w_n is represented as $w_n \in \sum_i \langle a_i, b_i, c_i \rangle$ for some elements $\{a_i, b_i, c_i\}$ of positive stems in the*

*Partially supported by Grand-in-Aid for Scientific Research C-07640123 from the Ministry of Education

1991 *Mathematics Subject Classification numbers.* Primary 55Q15; Secondary 55Q40

Key words and phrases. Whitehead element, decomposability, Toda bracket.

homotopy groups of spheres if and only if one of the following holds:

(1) $p \geq 5$ *and* $n = 3$, *for which* $w_n \in \langle \alpha_1, \alpha_1, \alpha_1 \rangle$;

(2) $p = 3$ *and* $n = 3$, *for which* $w_n \in 3\langle \alpha_1, \alpha_1, \alpha_1 \rangle$;

(3) $p = 3$ *and* $n = 4$, *for which* $w_n \in \langle \alpha_1, \alpha_2, \alpha_1 \rangle$.

The decompositions of w_n in Theorem A and the representation of w_n by Toda brackets in Theorem B are shown by Toda [7], that is, the if part of each theorem is already known. The present paper is devoted to prove the only if part of each theorem, that is, we show that such decompositions or representations of w_n occur only when p and n satisfy the conditions in Theorem A or Theorem B respectively.

Our problem is the odd primary version of the one studied by Iriye and Morisugi [3] who treated the Whitehead product $[\iota_{2n-1}, \iota_{2n-1}]$ of a generator $\iota_{2n-1} \in \pi_{2n-1}(S^{2n-1}) \cong Z_{(p)}$, which is equal to w_n when $p = 2$. They gave the following result:

THEOREM ([3, TH. B, D]). *Let* $n \neq 1, 2$ *or* 4. *Then the following holds*:

(a) *The Whitehead product* $[\iota_{2n-1}, \iota_{2n-1}]$ *is written as* $[\iota_{2n-1}, \iota_{2n-1}] = \sum_i a_i b_i$ *for some elements* $\{a_i, b_i\}$ *of positive stems in the homotopy groups of spheres if and only if* $n = 3, 5, 6$ *or* 8;

(b) *The Whitehead product* $[\iota_{2n-1}, \iota_{2n-1}]$ *can be represented as* $[\iota_{2n-1}, \iota_{2n-1}] \in \sum_i \langle a_i, b_i, c_i \rangle$ *for some elements* $\{a_i, b_i, c_i\}$ *of positive stems in the homotopy groups of spheres if and only if* $n = 3, 5, 6, 7, 8, 9, 10$ *or* 12.

In the course of the proof of our main theorems, we get the next proposition which, we believe, has its own interest.

PROPOSITION C. *Let W be a* $(2n-1)$*-connected CW-complex with* $\dim W \leq 2np - 3$. *Suppose that there exists a map* $\varphi : S^{2np-3} \to W$ *such that* $\varphi_* = 0 : H_{2np-3}(S^{2np-3}; Z/p) \to H_{2np-3}(W; Z/p)$. *Then, the followings are equivalent*:

(1) *There exists a map* $u : W \to S^{2n-1}$ *such that* $w_n = u\varphi$.

(2) *There exists* $\mu : \Sigma^2 C_\varphi \to S^{2n+1}$ *such that* $\mathscr{P}^n \neq 0$ *on* $H^{2n+1}(C_\mu; Z/p)$, *where* $\mathscr{P}^n$ *is the reduced p-th power operation.*

(3) *There exists* $\delta : \Sigma C_\varphi \to \Omega S^{2n+1}$ *such that* $u^p \neq 0$ *for a generator* $u \in H^{2n}(C_\delta; Z/p) \cong Z/p$.

(4) *There exists* $\bar{\delta} : C_\varphi \to \Omega^2 S^{2n+1}$ *such that* $\bar{\delta}_* \neq 0$ *on* $H_{2np-2}(C_\varphi; Z/p)$. *Here,* C_h *denotes the cofiber of a map h.*

In the proof of our main theorems, we only use the fact that (1) implies (2) in the above proposition.

The paper is organized as follows: In § 2, we outline the method of the proofs of Theorems A and B. In § 3 we prove Proposition C, and in § 4 we complete the proof of the main theorems.

The authors would like to express their hearty thanks to Prof. M. Imaoka for his encouragement.

2. Outline of the proofs of Theorems A and B

In this section, we state the methods to prove Theorems A and B by assuming Proposition C. Here and throughout the paper, a homotopy class $\alpha \in [X, Y]$ is identified with the map $\alpha : X \to Y$ itself as conventionally.

We assume that $2np - 4 > d > e > 2n - 1$. Then, for $a \in \pi_e(S^{2n-1})$, $b \in \pi_d(S^e)$ and $c \in \pi_{2np-4}(S^d)$ with $ab = bc = 0$, the Toda bracket $\langle a, b, c\rangle \subset \pi_{2np-3}(S^{2n-1})$ is defined as a class formed by compositions uv, where $v : S^{2np-3} \to C_b$ and $u : C_b \to S^{2n-1}$ are maps induced from relations $ab = 0$ and $bc = 0$.

Concerning Theorem B, if $w_n \in \sum_{i=1}^{l} \langle a_i, b_i, c_i\rangle$ holds, then, putting $W = \bigvee_{i=1}^{l} C_{b_i}, v = \bigvee_{i=1}^{l} v_i : S^{2np-3} \to W$ and $u = \bigvee_{i=1}^{l} u_i : W \to S^{2n-1}$ for $u_i v_i \in \langle a_i, b_i, c_i\rangle$, we have $w_n = uv$. Conversely, $w_n = uv$ for some $v : S^{2np-3} \to W$ and $u : W \to S^{2n-1}$ implies $w_n \in \sum_{i=1}^{l} \langle a_i, b_i, c_i\rangle$.

Similarly, concerning Theorem A, the necessary and sufficient condition for w_n to be $w_n = \sum_{i=1}^{l} a_i b_i$ is also $w_n = uv$ by taking $W = \bigvee_{i=1}^{l} S^{d_i}$, $v = \bigvee_{i=1}^{l} b_i : S^{2np-3} \to W$ and $u = \bigvee_{i=1}^{l} a_i : W \to S^{2n-1}$ instead, where $a_i : S^{d_i} \to S^{2n-1}$ and $b_i : S^{2np-3} \to S^{d_i}$ for $1 \le i \le l$.

From now on, we assume that w_n is represented as $w_n = uv \in \sum_{i=1}^{l} \langle a_i, b_i, c_i\rangle$. Then, the equivalence of (1) and (2) of Proposition C yields the following:

Corollary 2.1. *Under the above situation, for any integer m with $m > n$, there exists a map $\mu : \Sigma^{2(m-n)+1} C_v \to S^{2m}$ for which $\mathscr{P}^n \neq 0$ on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$.*

From the above fact, once Proposition C is proved, Theorem B will be established by showing the following: $\mathscr{P}^n \neq 0$ on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$ $(m > n)$ occurs only when p and n satisfy one of (1)–(3) of Theorem B. In order to examine such property about $\mathscr{P}^n$, we use the result of Atiyah [1] about the complex K-theory.

For a finite complex Y, the p-localized K-group $K(Y)$ has the filtration defined by $K_q(Y) = \mathrm{Ker}[j^* : K(Y) \to K(Y^{q-1})]$, where Y^q is the q-skeleton of Y. Then, we have the associated graded ring $G^t K(Y) = K_t(Y)/K_{t+1}(Y)$. If Y has no p-torsion, then we have an isomorphism $G^{2t}K(Y) \otimes \boldsymbol{Z}/p \cong H^{2t}(Y; \boldsymbol{Z}/p)$. In this case, we denote by $\bar{a} \in H^{2t}(Y; \boldsymbol{Z}/p)$ the element corresponding to $a \in K_{2t}(Y)$ through the isomorphism. Then, the following is known:

Theorem ([1]). *Assume that Y has no p-torsion, and let $x \in K_{2m}(Y)$. Then,*

(1) *there exists* $x_i \in K_{2m+2i(p-1)}(Y)$ $(0 \le i \le m)$ *such that* $\psi^p(x) = \sum_{i=0}^{m} p^{m-i} x_i$, *where* ψ^p *is the Adams operation on* $K(Y)$;

(2) *for the elements* $\bar{x}_i \in H^{2m+2i(p-1)}(Y; \boldsymbol{Z}/p)$ *corresponding to* x_i *of* (1), $\bar{x}_i = \mathscr{P}^i(\bar{x})$.

Suppose that we have a map $\mu : \Sigma^{2(m-n)+1} C_v \to S^{2m}$ with $\mathscr{P}^n \neq 0$ on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$. Note that C_μ has the following cell structure:

$$C_\mu = S^{2m} \cup \bigvee_j (e^{2m+q_j} \cup e^{2m+r_j}) \cup e^{2m+2n(p-1)},$$

where $4 \le q_j + 2 \le r_j \le 2n(p-1) - 2$. Thus, we have $K_{2m}(C_\mu)/K_{2m+2}(C_\mu) \cong H^{2m}(C_\mu; \boldsymbol{Z}_{(p)}) \cong \boldsymbol{Z}_{(p)}$ and $K_{2m+2n(p-1)}(C_\mu) \cong \boldsymbol{Z}_{(p)}$, and we choose x and w which represent the respective generators. Then, $\bar{x} \in H^{2m}(C_\mu; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$ is a generator, and thus $\mathscr{P}^n(\bar{x}) \neq 0$ by Corollary 2.1. Using these properties and Atiyah's theorem, we can represent the Adams operations on $K(C_\mu)$ as in the following lemma, we note (2) corresponds to the fact that $\mathscr{P}^n(\bar{x}) \neq 0$.

LEMMA 2.2. (1) *The Adams operation has the following forms on* $K(C_\mu)$.

$$\psi^k(x) = k^m x + \sum_{i=1}^{l} a_i(k) y_i + \sum_{i=1}^{l} b_i(k) z_i + c(k) w$$

$$\psi^k(y_i) = k^{m+t_i(p-1)} y_i + d_i(k) z_i + e_i(k) w \qquad (1 \le i \le l)$$

$$\psi^k(z_i) = k^{m+s_i(p-1)} z_i + f_i(k) w \qquad (1 \le i \le l)$$

$$\psi^k(w) = k^{m+n(p-1)} w.$$

Here, $y_i \in K_{2m+2t_i(p-1)}(C_\mu)$ *and* $z_i \in K_{2m+2s_i(p-1)}(C_\mu)$ *are some elements with* $t_i < s_i$, *and* $\{a_i(k), b_i(k), c(k), d_i(k), e_i(k), f_i(k)\}$ *are some p-localized integers.*

(2) *When* $k = p$ *in* (1), $c(p) = p^{m-n}\beta$ *for some* $\beta \not\equiv 0 \bmod p$.

In the next section, we prove Proposition C, and in §4 we show that all the relations in Lemma 2.2 hold only when p and n satisfy one of (1)–(3) in Theorem B. In this way, we establish the proof of Theorem B. As for Theorem A, the corresponding lemma and corollary to Lemma 2.1 and Corollary 2.2 respectively hold, and the proof is almost the same.

3. Proof of Proposition C

In this section we prove Proposition C, and so we assume that W is a $(2n-1)$-connected CW-complex with $\dim W \le 2np-3$ and the map $\varphi : S^{2np-3} \to W$ satisfies $\varphi_* = 0 : H_{2np-3}(S^{2np-3}; \boldsymbol{Z}/p) \to H_{2np-3}(W; \boldsymbol{Z}/p)$.

(I) The proof of (1) $\Longleftrightarrow$ (4): First we remark that $\Sigma^2 : [W, S^{2n-1}] \to [W, \Omega^2 S^{2n+1}]$ is surjective. This follows from the Toda fibrations $F \to \Omega S^{2n+1} \to \Omega S^{2np+1}$ and $S^{2n-1} \to \Omega F \to \Omega S^{2np-1}$ ([6]) since $\dim W \le 2np - 3$, where $F = J_{p-1}(S^{2n})$ is the $(p-1)$-th space in the James construction.

Now, if $u : W \to S^{2n-1}$ is given and satisfies $u\varphi = \varepsilon z = w_n$, then there exists $\bar{\delta} : C_\varphi \to \Omega^2 S^{2n+1}$ which makes the following diagram homotopy commutative:

$$\begin{array}{ccccc} C(n) & \xrightarrow{\varepsilon} & S^{2n-1} & \xrightarrow{\Sigma^2} & \Omega^2 S^{2n+1} \\ \uparrow{\scriptstyle z} & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle \bar{\delta}} \\ S^{2np-3} & \xrightarrow{\varphi} & W & \xrightarrow{i} & C_\varphi \end{array}$$

Then the following diagram is commutative:

$$\begin{array}{ccccc} H_{2np-3}(C(n)) & \underset{\cong}{\longleftarrow} & H_{2np-2}(S^{2n-1}, C(n)) & \underset{\cong}{\xrightarrow{p*}} & H_{2np-2}(\Omega^2 S^{2n+1}) \\ \uparrow{\scriptstyle z_*} & & \uparrow & & \uparrow{\scriptstyle \bar{\delta}_*} \\ H_{2np-3}(S^{2np-3}) & \xleftarrow{\cong} & H_{2np-2}(W, S^{2np-3}) & \xrightarrow{\cong} & H_{2np-2}(C_\varphi), \end{array}$$

where all the coefficients of the homologies are in $\boldsymbol{Z}/p$. Hence, $\bar{\delta}_* \neq 0$ on $H_{2np-2}(C_\varphi)$, since $z \in \pi_{2np-3}(C(n)) \cong \boldsymbol{Z}/p$ is a generator.

Conversely, if $\bar{\delta} : C_\varphi \to \Omega^2 S^{2n+1}$ is given, then there exists $\bar{u} : W \to S^{2n-1}$ such that $\Sigma^2 \bar{u} \simeq \bar{\delta} i$. Since $\Sigma^2 \bar{u}\varphi \simeq \bar{\delta} i \varphi \simeq *$, there exists $\bar{z} = az : S^{2np-3} \to C(n)$ such that $\varepsilon \bar{z} \simeq \bar{u}\varphi$, where $a \in \boldsymbol{Z}/p$ is a unit. Then, $u = a^{-1}\bar{u} : W \to S^{2n-1}$ is the required map with $u\varphi \simeq \varepsilon z$.

(II) The proof of (2) $\Longleftrightarrow$ (3): Let $\mu : \Sigma^2 C_\varphi \to S^{2n+1}$ for which $\mathscr{P}^n \neq 0$ on $H^{2n+1}(C_\mu; \boldsymbol{Z}/p)$ be the map given in (2). We define $\delta : \Sigma C_\varphi \to \Omega S^{2n+1}$ by the adjoint of μ. Then, we have the following homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^2 C_\varphi & \xrightarrow{\mu} & S^{2n+1} & \longrightarrow & C_\mu \\ \| & & \uparrow{\scriptstyle e} & & \uparrow{\scriptstyle \tilde{e}} \\ \Sigma^2 C_\varphi & \xrightarrow{\Sigma\delta} & \Sigma\Omega S^{2n+1} & \longrightarrow & \Sigma C_\delta \end{array}$$

where e is the evaluation map and $\tilde{e}$ is the map induced on the cofibers. Using the five lemma we have an isomorphism $\tilde{e}^* : H^{2n+1}(C_\mu; \boldsymbol{Z}/p) \xrightarrow{\cong} H^{2n+1}(\Sigma C_\delta; \boldsymbol{Z}/p)$, and the following diagram:

$$
\begin{array}{ccccc}
H^{2n+1}(C_\mu; \boldsymbol{Z}/p) & \xrightarrow{\mathscr{P}^n} & H^{2np+1}(C_\mu; \boldsymbol{Z}/p) & \xleftarrow{\cong} & H^{2np}(\Sigma^2 C_\varphi; \boldsymbol{Z}/p) \\
\cong \downarrow & & \downarrow & & \| \\
H^{2n+1}(\Sigma C_\delta; \boldsymbol{Z}/p) & \xrightarrow{\mathscr{P}^n} & H^{2np+1}(\Sigma C_\delta; \boldsymbol{Z}/p) & \xleftarrow{\partial_*} & H^{2np}(\Sigma^2 C_\delta; \boldsymbol{Z}/p) \\
\| & & \| & & \\
H^{2n}(C_\delta; \boldsymbol{Z}/p) & \xrightarrow{\mathscr{P}^n} & H^{2np}(C_\delta; \boldsymbol{Z}/p). & &
\end{array}
$$

Note that ∂_* is injective. The three squares in the diagram are commutative, and $\mathscr{P}^n(u) = u^p$ for a generator $u \in H^{2n}(C_\delta; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$. Therefore, δ is the required map as in (3). By taking the converse way in the above, it is obvious that, if δ is given as in (3), then the required map μ of (2) is given by the adjoint.

(III) The proof of (3) $\Longleftrightarrow$ (4): Let $h : K(\boldsymbol{Z}/p, 2n) \to K(\boldsymbol{Z}/p, 2np)$ be the map defined by $h^*(\iota_{2np}) = \mathscr{P}^n(\iota_{2n}) = \iota_{2n}^p$, and $r : E \to K(\boldsymbol{Z}/p, 2n)$ the homotopy fiber of h, where $\iota_j \in H^j(K(G,j); G)$ denotes the fundamental class for any abelian group G. Since $\Omega h^*(\iota_{2np}) = \mathscr{P}^n(\iota_{2n-1}) = 0$, it follows $\Omega h \simeq *$, and hence $\Omega E \simeq K(\boldsymbol{Z}/p, 2n-1) \times K(\boldsymbol{Z}/p, 2np-2)$. Thus, we put $x = (\Omega r)^*(\iota_{2n-1}) \in H^{2n-1}(\Omega E; \boldsymbol{Z}/p)$ and $y \in H^{2np-2}(\Omega E; \boldsymbol{Z}/p)$ with $(\Omega i)^*(y) = \iota_{2np-2}$. Then, we have the following lemma, the proof of which is postponed until the last of this section.

LEMMA 3.1. *There exists a map $f : \Omega S^{2n+1} \to E$ such that $(\Omega f)^*(x)$ and $(\Omega f)^*(y)$ are generators of $H^{2n-1}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$ and $H^{2np-2}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$ respectively.*

In the lemma, we use the following well known fact:

REMARK 3.2. *For $k < 2n(p+1) - 3$, we have*

$$
H^k(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p) \cong \begin{cases} \boldsymbol{Z}/p & \text{for } k = 0,\ 2n-1,\ 2np-2,\ 2np-1 \\ 0 & \text{otherwise.} \end{cases}
$$

Now, we continue the proof of the equivalence of (3) and (4). Let $f : \Omega S^{2n+1} \to E$ be the map in Lemma 3.1. First, we assume that $\delta : \Sigma C_\varphi \to \Omega S^{2n+1}$ is given as in (3), and prove (4). Let $\Sigma C_\varphi \xrightarrow{\delta} \Omega S^{2n+1} \xrightarrow{\kappa} C_\delta \xrightarrow{\lambda} \Sigma^2 C_\varphi$ be the cofiber sequence given from δ. Since W is $(2n-1)$-connected, $\kappa^* : H^{2n}(C_\delta; \boldsymbol{Z}/p) \to H^{2n}(\Omega S^{2n+1}; \boldsymbol{Z}/p)$ is an isomorphism. Hence, there exist $f_2 : C_\delta \to K(\boldsymbol{Z}/p, 2n)$ with $f_2\kappa \simeq rf$, and maps $f_1 : \Sigma C_\varphi \to K(\boldsymbol{Z}/p, 2np-1)$ and $f_3 : \Sigma^2 C_\varphi \to K(\boldsymbol{Z}/p, 2np)$ which are adjoint to each other and make the following left diagram homotopy commutative up to sign:

$$\begin{array}{ccc}
\Sigma C_\varphi & \xrightarrow{f_1} & K(\boldsymbol{Z}/p, 2np-1) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle i} \\
\Omega S^{2n+1} & \xrightarrow{f} & E \\
\downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle r} \\
C_\delta & \xrightarrow{f_2} & K(\boldsymbol{Z}/p, 2n) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle h} \\
\Sigma^2 C_\varphi & \xrightarrow{f_3} & K(\boldsymbol{Z}/p, 2np),
\end{array}
\qquad
\begin{array}{ccc}
C_\varphi & \xrightarrow{\bar{f}} & K(\boldsymbol{Z}/p, 2np-2) \\
\downarrow{\scriptstyle \bar{\delta}} & & \downarrow{\scriptstyle \Omega_i} \\
\Omega^2 S^{2n+1} & \xrightarrow{\Omega f} & \Omega E,
\end{array}$$

where the right sequence in the left diagram is the fiber sequence mentioned before Lemma 3.1. By assumption, $u = f_2^*(\iota_{2n})$ is a generator of $H^{2n}(C_\delta; \boldsymbol{Z}/p)$ and $u^p \neq 0$. Then, $f_3^*(\iota_{2np}) \neq 0$ since $\lambda^* f_3^*(\iota_{2np}) = f_2^* h^*(\iota_{2np}) = f_2^*(\iota_{2n}^p) = f_2^*(\iota_{2n})^p = u^p \neq 0$. Let $\bar{\delta}$ and $\bar{f}$ be the adjoint of δ and f_1 respectively. Then, the above right diagram is homotopy commutative, and it follows $\bar{\delta}^*(\Omega f)^*(y) = \bar{f}^*(\Omega i)^*(y) = \bar{f}^*(\iota_{2np-2}) \neq 0$. Hence, $\bar{\delta}_* \neq 0$ on $H_{2np-2}(C_\varphi; \boldsymbol{Z}/p)$, and $\bar{\delta}$ is the required map of (4).

Conversely, we assume that $\bar{\delta} : C_\varphi \to \Omega^2 S^{2n+1}$ is given as in (4), and show that the adjoint $\delta : \Sigma C_\varphi \to \Omega S^{2n+1}$ of $\bar{\delta}$ is the required map. Consider the above right diagram. Then, it holds $\bar{\delta}^*(\Omega f)^*(y) \neq 0$, since $(\Omega f)^*(y) \in H^{2np-2}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$ is a generator by Lemma 3.1. Let $\bar{f}$ be the map defined by $\bar{f}^*(\iota_{2np-2}) = \bar{\delta}^*(\Omega f)^*(y)$. Then, $(\Omega i)\bar{f} \simeq (\Omega f)\bar{\delta}$. In fact, $\Omega E \simeq K(\boldsymbol{Z}/p, 2n-1) \times K(\boldsymbol{Z}/p, 2np-2)$ and $\bar{f}^*(\Omega i)^*(x) = \bar{\delta}^*(\Omega f)^*(x) = 0$ since C_φ is $(2n-1)$-connected. Since $\bar{f}^*(\Omega i)^*(y) = \bar{\delta}^*(\Omega f)^*(y)$ by definition, we have $(\Omega i)\bar{f} \simeq (\Omega f)\bar{\delta}$. Let f_1 be the adjoint of $\bar{f}$, and f_3 the adjoint of f_1. Then, there is a map f_2 which makes the above left diagram homotopy commutative up to sign. Let $u = f_2^*(\iota_{2n})$. Then, $u^p = (hf_2)^*(\iota_{2np}) = \lambda^* f_3^*(\iota_{2np})$. But $f_3^*(\iota_{2np}) \neq 0$ since $\bar{f}^*(\iota_{2np-2}) \neq 0$. Since $H^{2np-1}(\Omega S^{2n+1}; \boldsymbol{Z}/p) = 0$, λ^* is a monomorphism, and hence $u^p \neq 0$, which completes the proof of Proposition C.

Proof of Lemma 3.1. Let $\bar{r} : \bar{E} \to K(\boldsymbol{Z}_{(p)}, 2n)$ be the homotopy fiber of the composition $K(\boldsymbol{Z}_{(p)}, 2n) \xrightarrow{\rho} K(\boldsymbol{Z}/p, 2n) \xrightarrow{h} K(\boldsymbol{Z}/p, 2np)$, where ρ is the map induced from the mod p reduction. Then, we have the following homotopy commutative diagram:

$$\begin{array}{ccccccc}
K(\boldsymbol{Z}/p, 2np-1) & \xrightarrow{i} & E & \xrightarrow{r} & K(\boldsymbol{Z}/p, 2n) & \xrightarrow{h} & K(\boldsymbol{Z}/p, 2np) \\
\| & & \uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \rho} & & \| \\
K(\boldsymbol{Z}/p, 2np-1) & \xrightarrow{\bar{i}} & \bar{E} & \xrightarrow{\bar{r}} & K(\boldsymbol{Z}_{(p)}, 2n) & \xrightarrow{h\rho} & K(\boldsymbol{Z}/p, 2np) \\
\| & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \xi} & & \| \\
K(\boldsymbol{Z}/p, 2np-1) & \longrightarrow & K(\boldsymbol{Z}_{(p)}, 2np) & \xrightarrow[\times p]{} & K(\boldsymbol{Z}_{(p)}, 2np) & \xrightarrow[\rho]{} & K(\boldsymbol{Z}/p, 2np),
\end{array}$$

where all horizontal sequences are fiber sequences. In fact, ξ is defined by $\xi^*(\iota_{2np}) = \iota_{2n}^p$, and η and ζ are induced maps from the homotopy commutativity of the right squares of the above diagram.

Let $\bar{x} = (\Omega\eta)^*(x)$ and $\bar{y} = (\Omega\eta)^*(y)$. Then, it is sufficient to find the map $\bar{f} : \Omega S^{2n+1} \to \bar{E}$ such that $(\Omega\bar{f})^*(\bar{x})$ and $(\Omega\bar{f})^*(\bar{y})$ are respective generators. Let $h_1 : \Omega S^{2n+1} \to K(\boldsymbol{Z}_{(p)}, 2n)$ and $h_2 : \Omega S^{2n+1} \to K(\boldsymbol{Z}_{(p)}, 2np)$ be maps such that $h_1^*(\iota_{2n})$ and $h_2^*(\iota_{2np})$ are generators of $H^{2n}(\Omega S^{2n+1}; \boldsymbol{Z}_{(p)}) \cong \boldsymbol{Z}_{(p)}$ and $H^{2np}(\Omega S^{2n+1}; \boldsymbol{Z}_{(p)}) \cong \boldsymbol{Z}_{(p)}$ respectively. We can choose h_1 and h_2 to satisfy $h_1^*\xi^*(\iota_{2np}) = h_1^*(\iota_{2n})^p = p!h_2^*(\iota_{2np})$ since $H^*(\Omega S^{2n+1}; \boldsymbol{Z}_{(p)})$ is the divided polynomial algebra. Since the lower square in the middle of the above diagram is a weak pull back diagram, there exists $\bar{f} : \Omega S^{2n+1} \to \bar{E}$ such that $\bar{r}\bar{f} \simeq h_1$ and $\zeta\bar{f} \simeq (p-1)!h_2$. Then, we have $(\Omega\bar{f})^*(\bar{x}) = (\Omega\bar{f})^*(\Omega\bar{r})^*\rho^*(\iota_{2n-1}) = (\Omega h_1)^*\rho^*(\iota_{2n-1})$ and $(\Omega\bar{f})^*(\Omega\zeta)^*\rho^*(\iota_{2np-1}) = (p-1)!(\Omega h_2)^*\rho^*(\iota_{2np-1})$. Thus, $(\Omega\bar{f})^*(\bar{x})$ is a generator of $H^{2n-1}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p)$ as required, and also $(\Omega\bar{f})^*(\Omega\zeta)^*\rho^*(\iota_{2np-1})$ is a generator of $H^{2np-1}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p)$.

Now $(\Omega\bar{i})^*(\Omega\zeta)^*\rho^*(\iota_{2np-1}) = \beta\iota_{2np-2} = (\Omega\bar{i})^*\beta(\bar{y})$, where β is the Bockstein operation. Since $\Omega\bar{E} \simeq K(\boldsymbol{Z}_{(p)}, 2n-1) \times K(\boldsymbol{Z}/p, 2np-2)$, we have $H^{2np-1}(\Omega\bar{E}; \boldsymbol{Z}/p) \cong H^{2np-1}(K(\boldsymbol{Z}_{(p)}, 2n-1); \boldsymbol{Z}/p) \oplus H^{2np-1}(K(\boldsymbol{Z}/p, 2np-2); \boldsymbol{Z}/p)$ for dimensional reason. Thus there exists $\bar{\bar{y}} \in \mathrm{Im}(\Omega\bar{r})^*$ so that $(\Omega\zeta)^*\rho^*(\iota_{2np-1}) = \beta\bar{y} + \bar{\bar{y}}$. $(\Omega\zeta)^*\rho^*(\iota_{2np-1})$ and $\beta\bar{y}$ are primitive by definition, so is $\bar{\bar{y}}$. Thus $\bar{\bar{y}} = (\Omega\bar{r})^*\theta\rho^*(\iota_{2n-1}) = \theta(\bar{x})$ for some Steenrod operation θ, and we have $(\Omega\bar{f})^*(\bar{\bar{y}}) = \theta(\Omega\bar{f})^*(\bar{x}) = 0$ by Remark 3.2. Hence, we have $\beta(\Omega\bar{f})^*(\bar{y}) = (\Omega\bar{f})^*(\Omega\zeta)^*\rho^*(\iota_{2np-1})$ and thus $\beta(\Omega\bar{f})^*(\bar{y})$ is a generator of $H^{2np-1}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p)$. Therefore, $(\Omega\bar{f})^*(\bar{y})$ is itself a generator of $H^{2np-2}(\Omega^2 S^{2n+1}; \boldsymbol{Z}/p)$, and we have completed the proof of Proposition 3.1. q.e.d.

4. Proof of Main theorems

In this section, we compute the Adams operation to complete the proofs of Theorems A and B. Since the computations are similar for both the theorems, we write them for Theorem B mainly and only outline them for Theorem A.

First, we prepare the next lemma, which is necessary in the proof. We omit the proof of it, since it is easy. We use the notation that $v(n)$ is the exponent of p in the primary decomposition of the integer n.

LEMMA 4.1. *If p is an odd prime, then* $v((p+1)^m - 1) = v(m) + 1$.

The action of the Adams operation on $K(C_\mu)$ is given in Lemma 2.2. We prove Theorem B by dividing the following two cases (I) and (II).

(I) The case of $n = p^N$ for some $N > 0$: First we compare the coefficients of y_i on the both sides of $\psi^{p+1}\psi^p x = \psi^p\psi^{p+1}x$. Then, we get $p^m(p^{t_i(p-1)} - 1)a_i(p+1) = (p+1)^m((p+1)^{t_i(p-1)} - 1)a_i(p)$. Hence, $a_i(p) \equiv 0 \bmod p^{m-v(t_i)-1}$. Since $v(t_i) < N$ we have

$$a_i(p) \equiv 0 \bmod p^{m-N} \qquad (1 \le i \le l). \tag{1}$$

Next we compare the coefficients of z_i on the both side of $\psi^{p+1}\psi^p y_i = \psi^p\psi^{p+1}y_i$. By computing similarly, we get $d_i(p) \equiv 0 \bmod p^{m+t_i(p-1)-v(s_i-t_i)-1}$. Since $v(s_i - t_i) < N$ we have

$$d_i(p) \equiv 0 \bmod p^{m+(p-1)-N} \qquad (1 \le i \le l). \tag{2}$$

We compare the coefficients of z_i on the both side of $\psi^{p+1}\psi^p x = \psi^p\psi^{p+1}x$. Then, by considering them mod p^{m-N} and applying (1) and (2), we get $(p+1)^m((p+1)^{s_i(p-1)} - 1)b_i(p) \equiv 0 \bmod p^{m-N}$. But $v((p+1)^{s_i(p-1)} - 1) = v(s_i) + 1$ by Lemma 4.1, and we get $b_i(p) \equiv 0 \bmod p^{m-N-v(s_i)-1}$. Since $v(s_i) < N$, we have

$$b_i(p) \equiv 0 \bmod p^{m-2N} \qquad (1 \le i \le l). \tag{3}$$

We compare the coefficients of w on the both side of $\psi^{p+1}\psi^p z_i = \psi^p\psi^{p+1}z_i$. Then, $(p+1)^{m+s_i(p-1)}((p+1)^{(n-s_i)(p-1)} - 1)f_i(p) = p^{m+s_i(p-1)}(p^{(n-s_i)(p-1)} - 1)f_i(p+1)$. But, $v((p+1)^{(n-s_i)(p-1)} - 1) = v(n - s_i) + 1$ by Lemma 4.1, and we get $f_i(p) \equiv 0 \bmod p^{m+s_i(p-1)-v(n-s_i)-1}$. Since $v(n - s_i) < N$ and $s_i \ge 2$, we have

$$f_i(p) \equiv 0 \bmod p^{m+2(p-1)-N} \qquad (1 \le i \le l). \tag{4}$$

We compare the coefficients of w on the both side of $\psi^{p+1}\psi^p y_i = \psi^p\psi^{p+1}y_i$. Then, by considering them mod $p^{m-N+(p-1)}$ and applying (2) and (4), we get $(p+1)^{m+t_i(p-1)}((p+1)^{(n-t_i)(p-1)} - 1)e_i(p) \equiv 0 \bmod p^{m-N+(p-1)}$. Hence we get $e_i(p) \equiv 0 \bmod p^{m-N+(p-1)-v(n-t_i)-1}$. Since $v(n - t_i) < N$ we have

$$e_i(p) \equiv 0 \bmod p^{m-2N+(p-1)} \qquad (1 \le i \le l). \tag{5}$$

Finally we compare the coefficients of w on the both side of $\psi^{p+1}\psi^p x = \psi^p\psi^{p+1}x$. Then, by considering them mod p^{m-2N} and applying (1), (3), (4) and (5), we get $(p+1)^m((p+1)^{n(p-1)}-1)c(p) \equiv 0 \bmod p^{m-2N}$. Since $\nu(c(p)) = m-n$ by Lemma 2.2, $(p+1)^{n(p-1)}-1 \equiv 0 \bmod p^{n-2N}$. Since $n = p^N$, using Lemma 4.1, we have $\nu((p+1)^{n(p-1)}-1) = N+1$. As a conclusion of this case (I), we obtain $N+1 \geq n-2N$, and it holds only if $N=1$, $n=3$ and $p=3$, which establishes Theorem B (2).

(II) The case of $n \neq p^N$ for any $N \geq 0$: In this case, $\mathscr{P}^n$ is a decomposable element in the Steenrod algebra. We use the following theorem on the non existence of the mod p Hopf invariant 1 due to [4] or [5].

THEOREM 4.2. *Let p be an odd prime. Then, the necessary and sufficient condition for a complex $K = S^t \cup_f e^{t+2m(p-1)}$ to satisfy $\mathscr{P}^m H^t(K; \boldsymbol{Z}/p) \neq 0$ is that $m=1$ and $f = \alpha_1$.*

Now, by the cell structure of C_μ shown in the above of Lemma 2.2, $\mathscr{P}^n$ should not be 4-fold decomposable, and it is sufficient to consider the following cases (i) and (ii).

(i) The case that $\mathscr{P}^n$ is 3-fold decomposable: By Theorem 4.2, the chance for $\mathscr{P}^n$ to operate non trivially on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$ in this case is that $p \geq 5$ and $\mathscr{P}^n$ has a factor of the form of $\mathscr{P}^1\mathscr{P}^1\mathscr{P}^1 = 3!\mathscr{P}^3$ in its decomposition. Therefore, we have $n=3$, and it yields Theorem B (1).

(ii) The case that $\mathscr{P}^n$ is not 3-fold decomposable: By Theorem 4.2, in this case $\mathscr{P}^n$ operates non trivially on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$ only if $\mathscr{P}^n$ has a factor of the form of $\mathscr{P}^1\mathscr{P}^{p^M}$ or $\mathscr{P}^{p^M}\mathscr{P}^1$ in its decomposition for some $M \geq 1$. We compute similarly as in the case (I). Then the conclusions (1), (2) and (4) are just the same as (I) replacing simply N by M. The conclusions (3) and (5) are the same other than the points that $\nu(s_i) \leq M$ and $\nu(n-t_i) \leq M$. For the last step, doing the same way, we have $(p+1)^m((p+1)^{n(p-1)}-1)c(p) \equiv 0 \bmod p^{m-2M-1}$. Then, we get $\nu((p+1)^{n(p-1)}-1) = 1$. Thus, we get $2M+1 \geq p^M$, and it holds only when $M=1$, $n=4$ and $p=3$, which establishes Theorem B (3). This completes the proof of Theorem B.

PROOF OF THEOREM A. For the cases of $n = p^N$ for some $N \geq 1$, we can compute similarly as in (I) above using Lemma 2.2 and Lemma 4.1, and we establish Theorem A (2), that is, $n=3$ and $p=3$.

For the cases of $n \neq p^N$ for any $N \geq 1$, by Theorem 4.2 and the cell structure of C_μ, $\mathscr{P}^n$ operates non trivially on $H^{2m}(C_\mu; \boldsymbol{Z}/p)$, in this case, only if $\mathscr{P}^n$ has a factor of the form of $\mathscr{P}^1\mathscr{P}^1 = 2!\mathscr{P}^2$ in its decomposition. Thus, this yields Theorem A (1), that is, $n=2$ and $p \geq 3$. q.e.d.

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