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An integral representation and fine limits at infinity for functions whose Laplacians iterated *m* times are measures

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Abstract. Our aim in this paper is to discuss the behavior at infinity of functions u for which $\Delta^m u \geqq 0$ on $\mathbf{R}^n$ in the weak sense. For this purpose we give a representation of u by means of modified Riesz kernels of order $2m$.

1. Statement of results

A function u is called polyharmonic of order m in an open set $G \subset \mathbf{R}^n$ if $\Delta^m u = 0$ on G, where Δ denotes the Laplace operator, or Laplacian.

We study the existence of fine limits at infinity for functions u on $\mathbf{R}^n$ such that $\Delta^m u$ is a nonnegative measure. To do so, we first consider a condition for polyharmonic functions to be polynomials, and establish an integral representation for u, as a generalization of Riesz decomposition theorem for superharmonic functions.

For a multi-index $j = (j_1, \dots, j_n)$ and a point $x = (x_1, \dots, x_n)$, we follow the usual notation:

$$|j| = j_1 + \cdots + j_n,$$
$$j! = j_1! \times \cdots \times j_n!,$$
$$x^j = x_1^{j_1} \times \cdots \times x_n^{j_n}$$

and

$$D^j = \left(\frac{\partial}{\partial x}\right)^j = \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{j_n}.$$

Consider the Riesz kernel of order $2m$

$$R_{2m}(x) = \begin{cases} |x|^{2m-n} & \text{if } 2m < n \text{ or if } 2m - n \text{ is a positive odd integer,} \\ |x|^{2m-n} \log(1/|x|) & \text{if } 2m - n \text{ is a nonnegative even integer} \end{cases}$$

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and its remainder term of Taylor's expansion

$$R_{2m,\ell}(x, y) = R_{2m}(x - y) - \sum_{|j| \leqq \ell} \frac{x^j}{j!} [D^j R_{2m}](-y),$$

where ℓ is a nonnegative integer. Letting $B(x, r)$ denote the open ball centered at x with radius r, we consider the function

$$K_{m,\ell}(x) = \begin{cases} R_{2m}(x - y) & \text{when } y \in B(0, 1), \\ R_{2m,\ell}(x, y) & \text{when } y \in \mathbf{R}^n - B(0, 1) \end{cases}$$

(cf. Hayman-Kennedy [3]).

Here note that R_{2m} is polyharmonic of order m outside the origin and

$$\Delta^m R_{2m} = c^{-1} \delta_0 \tag{1}$$

with the Dirac measure δ_x at x and a constant $c \neq 0$. As will be seen later, $K_{m,\ell}(\cdot, y)$ is also polyharmonic of order m outside y for any fixed $y \in \mathbf{R}^n$.

For a nonnegative measure μ on $\mathbf{R}^n$, we define

$$K_{m,\ell}\mu(x) = \int_{\mathbf{R}^n} K_{m,\ell}(x, y) d\mu(y).$$

We first give a condition for this potential to have a meaning.

THEOREM 1. *Let μ be a nonnegative measure on $\mathbf{R}^n$ and ℓ be a nonnegative integer such that $\ell \geqq 2m - n$. If*

$$\int_{\mathbf{R}^n} (1 + |y|)^{2m-n-\ell-1} d\mu(y) < \infty \tag{2}$$

holds, then

$$\int_{\mathbf{R}^n} |K_{m,\ell}(x, y)| d\mu(y) \in L^1_{loc}(\mathbf{R}^n). \tag{3}$$

Moreover, in case $2m \leqq n$, (2) is equivalent to (3).

Next we give an integral representation for functions u such that $\Delta^m u$ is a positive measure on $\mathbf{R}^n$, as a generalization of Riesz decomposition theorem.

THEOREM 2. *Let u be a function on $\mathbf{R}^n$ such that $\mu = \Delta^m u \geqq 0$ in the weak sense. If there exists a nonnegative integer ℓ such that $\ell \geqq 2m - n$ and*

$$\limsup_{r \to \infty} r^{-\ell-n} \int_{B(0,r)} |u(x)| dx < \infty, \tag{4}$$

then u is of the form

$$u(x) = c \int_{\mathbf{R}^n} K_{m,\ell}(x, y)d\mu(y) + P(x),$$

where c is the constant in (1) *and P is a polynomial of degree at most* ℓ.

In case $2m < n$, we consider the usual Riesz capacity of order $2m$, which is defined by

$$C_{2m}(E) = \inf \mu(\mathbf{R}^n)$$

for a set $E \subset \mathbf{R}^n$, where the infimum is taken over all nonnegative measures μ on $\mathbf{R}^n$ such that

$$R_{2m}\mu(x) = \int R_{2m}(x - y)d\mu(y) \geqq 1 \qquad \text{whenever } x \in E.$$

In case $2m = n$, we define the logarithmic capacity

$$C_n(E) = \inf \mu(B(0, 1))$$

for a set $E \subset B(0, 1)$, where the infimum is taken over all nonnegative measures μ on $B(0, 1)$ such that

$$\int \log \frac{2}{|x - y|} d\mu(y) \geqq 1 \qquad \text{whenever } x \in E.$$

Finally we are concerned with the fine limits at infinity for the generalized potentials $K_{m,\ell}\mu$.

THEOREM 3. *Let* ℓ *be a nonnegative integer,* $2m \leqq n$ *and* $0 < a \leqq 1$. *If* μ *is a nonnegative measure on* $\mathbf{R}^n$ *satisfying*

$$\int_{\mathbf{R}^n} (1 + |y|)^{2m-n-\ell-a} d\mu(y) < \infty, \tag{5}$$

then there exists a set $E \subset \mathbf{R}^n$ *such that*

$$\lim_{|x| \to \infty,\, x \in \mathbf{R}^n - E} |x|^{-\ell-a} K_{m,\ell}\mu(x) = 0$$

and E is 2m-thin at infinity, that is,

$$\sum_{i=1}^{\infty} C_{2m}(E_i') < \infty, \tag{6}$$

where $E_i' = \{x : 2^{-2} \leqq |x| < 2^{-1}, 2^{i+2}x \in E\}$.

The case $m = 1$ was proved in [4, Theorem 1].

REMARK 1. In case $2m < n$, (6) may be replaced by

$$\sum_{i=1}^{\infty} 2^{-i(n-2m)} C_{2m}(E_i) < \infty,$$

where $E_i = \{x \in E : 2^i \leqq |x| < 2^{i+1}\}$.

2. Polyharmonic functions

Let us begin with a condition under which polyharmonic functions are polynomials. In fact we show the following result.

THEOREM 4. *Let u be a polyharmonic function of order m on $\mathbf{R}^n$. If there exists $a \geqq 0$ for which*

$$\liminf_{r \to \infty} r^{-a-n} \int_{B(0,r)} u^+(x) dx = 0, \tag{7}$$

then u is a polynomial, where u^+ denotes the positive part of u, that is, $u^+(x) =$ $\max\{u(x), 0\}$.

For the harmonic case, see the book of Hayman-Kennedy [3]. If u satisfies two sided inequalities:

$$|u(x)| \leqq M(1 + |x|)^a,$$

then the conclusion of Theorem 4 is clearly true by considering the Fourier transform of $\Delta^m u$. We also note that Theorem 4 was essentially proved by Armitage [1]; in fact, his theorem states that a polyharmonic function u is a polynomial if

$$\lim_{r \to \infty} r^{-a-n+1} \int_{S(0,r)} u^+(x) dS(x) = 0 \tag{8}$$

for some $a \geqq 0$, where $S(0, r)$ denotes the spherical surface $\partial B(0, r)$.

In this paper, we use the symbol M to denote an absolute positive constant whose value is unimportant and may change from line to line.

REMARK 2. If (8) holds, then

$$\lim_{r \to \infty} r^{-a-n} \int_{B(0,r)} u^+(x) dx = 0.$$

We know a mean-value inequality for polyharmonic functions:

LEMMA 1 (cf. [6, Lemma 2]). *If u is polyharmonic of order m in $B(x, r)$, then*

$$|\nabla^k u(x)| \leqq M r^{-k-n} \int_{B(x,r)} |u(y)|\, dy,$$

where $M = M(k, m)$ *is a positive constant independent of* x *and* r, *and* ∇^k *denotes the gradient iterated* k *times.*

LEMMA 2. *If* u *is polyharmonic of order* m *in* $\mathbf{R}^n$, *then*

$$\lim_{r\to\infty} r^{-n-k} \int_{B(0,r)} u(y)\,dy = 0 \tag{9}$$

whenever $k > 2m-2$.

This is an easy consequence of finite Almansi expansion (cf. [2, Proposition 1.3]), which states that u is written as

$$u(x) = \sum_{i=1}^{m} |x|^{2i-2} u_i(x)$$

with harmonic functions u_i. By the mean value property, we have

$$\begin{aligned}\int_{B(0,r)} u(x)\,dx &= \int_0^r \left(\int_{\partial B(0,1)} u(t\Theta)\,d\Theta\right) t^{n-1}\,dt \\ &= \sum_{i=1}^{m} \int_0^r \left(\int_{\partial B(0,1)} u_i(t\Theta)\,d\Theta\right) t^{2i-2+n-1}\,dt \\ &= \sum_{i=1}^{m} [M u_i(0)] \int_0^r t^{2i-2+n-1}\,dt \\ &= \sum_{i=1}^{m} M_i u_i(0) r^{2i-2+n},\end{aligned}$$

which proves (9).

PROOF OF THEOREM 4. Since $|u| = 2u^+ - u$, Lemma 1 gives

$$\begin{aligned}|\nabla^k u(x)| &\leqq M r^{-n-k} \int_{B(x,r)} |u(y)|\,dy \\ &\leqq M r^{-n-k} \int_{B(0,2r)} |u(y)|\,dy \\ &= 2M r^{-n-k} \int_{B(0,2r)} u^+(y)\,dy - M r^{-n-k} \int_{B(0,2r)} u(y)\,dy \\ &= I_1 - I_2\end{aligned}$$

for $x \in B(0, r)$. By our assumption,

$$\liminf_{r \to \infty} I_1 = 0$$

for $k \geqq a$. On the other hand, in view of Lemma 2,

$$\lim_{r \to \infty} I_2 = 0$$

when $k > 2m - 2$. Thus, if $k > a + 2m - 2$, then

$$|\nabla^k u(x)| = 0 \qquad \text{for any } x \in \mathbf{R}^n,$$

which implies that u is a polynomial.

REMARK 3. In view of Lemma 2, (7) may be replaced by

$$\liminf_{r \to \infty} r^{-a-n} \int_{B(0,r)} |u(x)| \, dx = 0.$$

REMARK 4. Professor Suita kindly informed the author that Theorem 4 can be proved by the use of the expansion into spherical harmonics, instead of our Lemma 1.

3. Proof of Theorem 1

First we prepare some lemmas, as generalizations of the corresponding lemmas in [4] concerning the case $m = 1$.

LEMMA 3. *For* $r > 0$, $R_{2m,\ell}(rx, ry) = r^{2m-n} R_{2m,\ell}(x, y)$.

LEMMA 4. *If T is a rotation about the origin, then*

$$K_{m,\ell}(Tx, Ty) = K_{m,\ell}(x, y).$$

PROOF. For $t > 0$, let $f(t) = R_{2m}(tx - y)$. Then note that

$$R_{2m,\ell}(x, y) = f(1) - f(0) - \cdots - (\ell!)^{-1} f^{(\ell)}(0).$$

If T is a rotation about the origin, then $|tTx - Ty| = |tx - y|$, so that

$$R_{2m}(tTx - Ty) = R_{2m}(tx - y).$$

Hence it follows that $R_{2m,\ell}(Tx, Ty) = R_{2m,\ell}(x, y)$. Now the required assertion is proved.

LEMMA 5. *If $2m \leqq n$, then there exists $\delta > 0$ such that*

$$A = \liminf_{r \to \infty} \left(\inf_{y \in B(x,\delta)} r^{n-2m+\ell+1} |K_{m,\ell}(x, ry)| \right) > 0$$

for any $x \in \mathbf{R}^n$ with $|x| = 1$, where A does not depend on x.

PROOF. Let $x \in \mathbf{R}^n$ with $|x| = 1$ be fixed. For $t > 0$ and $y \in \mathbf{R}^n$, let $f(t) = f(t; y) = R_{2m}(tx - y)$. Then we have by Lemma 3

$$\begin{aligned} t^{2m-n-\ell-1} R_{2m,\ell}(x, t^{-1}y) &= t^{-\ell-1} R_{2m,\ell}(tx, y) \\ &= t^{-\ell-1}[f(t) - f(0) - \cdots - (\ell!)^{-1} t^{\ell} f^{(\ell)}(0)] \\ &\to [(\ell+1)!]^{-1} f^{(\ell+1)}(0) \end{aligned}$$

uniformly for $y \in B(x, 1/2)$ as $t \to 0$. Note that $f(t; x) = R_{2m}(tx - x) = |1 - t|^{2m-n}$ when $2m < n$ and $f(t; x) = -\log|1 - t|$ when $2m = n$. Hence we see that

$$\lim_{t \to 0} t^{2m-n-\ell-1} R_{2m,\ell}(x, t^{-1}x)$$

is a non-zero constant. Therefore there exists $\delta > 0$ such that

$$A = \liminf_{r \to \infty} \left(\inf_{y \in B(x,\delta)} r^{n-2m+\ell+1} |R_{2m,\ell}(x, ry)| \right) > 0.$$

In view of Lemma 4, we see that A does not depend on x, and the required assertion now follows.

The following lemma can be derived by the use of mean value theorem (cf. [4] and [5]).

LEMMA 6. *Let $\ell \geqq 2m - n$. If $|y| \geqq 1$ and $|y| \geqq 2|x|$, then*

$$|K_{m,\ell}(x, y)| \leqq M |x|^{\ell+1} |y|^{2m-n-\ell-1}.$$

PROOF OF THEOREM 1. First suppose (2) holds. For $R > 1$, write

$$\begin{aligned} |K_{m,\ell}| \mu(x) &= \int_{\mathbf{R}^n - B(0,2R)} |K_{m,\ell}(x, y)| d\mu(y) + \int_{B(0,2R)} |K_{m,\ell}(x, y)| d\mu(y) \\ &= u_R(x) + v_R(x). \end{aligned}$$

In view of Lemma 6, $u_R(x)$ is bounded on $B(0, R)$. On the other hand, since

$$v_R(x) \leqq \int_{B(0,2R)} |R_{2m}(x - y)| d\mu(y) + \sum_{|j| \leqq \ell} \left| \frac{x^j}{j!} \right| \int_{B(0,2R) - B(0,1)} |[D^j R_{2m}](-y)| d\mu(y),$$

we see that v_R is locally integrable on $\mathbf{R}^n$. Thus $|K_{m,\ell}| \mu$ is integrable on $B(0, R)$. Since R is arbitrary, (3) follows.

Next let $2m \leqq n$ and suppose (3) holds. Then there exists $x_0 \neq 0$ such that

$$\int_{\mathbf{R}^n} |K_{m,\ell}(x_0, y)| d\mu(y) < \infty.$$

In view of Lemma 5, we can find $\delta > 0$ and $R > 1$ such that

$$|K_{m,\ell}(x, y)| \geqq M|y|^{2m-n-\ell-1} \quad \text{whenever } |x| = 1,\ |y| > R \text{ and } y/|y| \in B(x, \delta),$$

so that Lemma 3 gives

$$|K_{m,\ell}(x_0, y)| = |x_0|^{2m-n}|K_{m,\ell}(x_0/|x_0|, y/|x_0|)| \geqq M|x_0|^{\ell+1}|y|^{2m-n-\ell-1}$$

whenever $|y| > R|x_0|$ and $|y/|y| - x_0/|x_0|| < \delta$. Hence we have

$$\infty > \int_{\Gamma(x_0,\delta)-B(0,R|x_0|)} |K_{m,\ell}(x_0, y)|\,d\mu(y)$$
$$\geqq M|x_0|^{\ell+1}\int_{\Gamma(x_0,\delta)-B(0,R|x_0|)} |y|^{2m-n-\ell-1}\,d\mu(y),$$

so that

$$\int_{\Gamma(x_0,\delta)-B(0,R|x_0|)} |y|^{2m-n-\ell-1}\,d\mu(y) < \infty,$$

where $\Gamma(x_0, \delta) = \{y : |y/|y| - x_0/|x_0|| < \delta\}$. Since $K_{m,\ell}\mu$ is finite almost everywhere on $B(0, 2) - B(0, 1)$, we can find a finite family $\{x_j\} \subset B(0, 2) - B(0, 1)$ such that

$$\partial B(0, 1) \subset \bigcup_j B(x_j/|x_j|, \delta)$$

and

$$\int_{\Gamma(x_j,\delta)-B(0,2R)} |y|^{2m-n-\ell-1}\,d\mu(y) < \infty.$$

Thus (2) is seen to hold.

4. Proof of Theorem 2

We need the following properties of $K_{m,\ell}$, which are found in [5, Lemmas 1 and 3].

LEMMA 7. *For each $y \in \mathbf{R}^n$, the function $x \to K_{m,\ell}(x, y)$ is polyharmonic of order m in $\mathbf{R}^n - \{y\}$; in fact,*

$$\Delta^m K_{m,\ell}(\cdot, y) = \Delta^m R_{2m}(\cdot - y) = c^{-1}\delta_y$$

with the constant c in (1).

For this, it suffices to note that

$$f^{(k)}(0) = \sum_{|j|=k} \frac{x^j}{j!}[D^j R_{2m}](-y) \qquad \text{with } f(t) = R_{2m}(tx - y)$$

is a polyharmonic polynomial for any nonnegative integer k and any fixed $y \neq 0$ (cf. [3, Lemma 4.4.1]).

LEMMA 8. *Let $\ell \geqq 2m - n$. If $1 \leqq |y| < 2|x|$ and $|x - y| \geqq 2^{-1}|x|$, then*

$$|K_{m,\ell}(x, y)| \leqq M|x|^{\ell}|y|^{2m-n-\ell} \log(4|x|/|y|).$$

LEMMA 9. *If $|y| \geqq 1$ and $|x - y| < 2^{-1}|x|$, then*

$$|K_{m,\ell}(x, y)| \leqq M|x - y|^{2m-n} \qquad \textit{in case } 2m < n,$$

$$|K_{m,\ell}(x, y)| \leqq M[|x|^{2m-n} + |x - y|^{2m-n} \log(|x|/|x - y|)] \qquad \textit{in case } 2m \geqq n.$$

Suppose $\mu = \Delta^m u$ is a nonnegative measure on $\mathbf{R}^n$ and (4) holds. Let φ be a nonnegative function in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi = 1$ on $B(0, 1)$ and $\varphi = 0$ outside $B(0, 2)$. For $r > 0$, set $\varphi_r(x) = \varphi(r^{-1}x)$. If r is large enough, then we have

$$\begin{aligned} \mu(B(0, r)) &\leqq \int \varphi_r(x) d\mu(x) \\ &= \int u(x) \Delta^m \varphi_r(x) dx \\ &\leqq Mr^{-2m} \int_{B(0,2r)} |u(x)| dx \\ &\leqq Mr^{-2m+\ell+n}, \end{aligned}$$

so that

$$\int_{\mathbf{R}^n} (1 + |y|)^{2m-n-\ell-1} d\mu(y) = \int_0^\infty \mu(B(0, r)) d(-(1 + r)^{2m-n-\ell-1}) < \infty.$$

Thus (2) is satisfied, and hence we can consider the potential $K_{m,\ell}\mu$. For $R > 0$, write

$$\begin{aligned} K_{m,\ell}\mu(x) &= \int_{B(0,2R)} K_{m,\ell}(x, y) d\mu(y) + \int_{\mathbf{R}^n - B(0,2R)} K_{m,\ell}(x, y) d\mu(y) \\ &= k_1(x) + k_2(x). \end{aligned}$$

Then, in view of Lemmas 6 and 7, k_2 is absolutely convergent in $B(0, R)$ and

$$\Delta^m k_2 = 0 \qquad \text{in } B(0, R).$$

By Lemma 7, we have

$$\Delta^m k_1 = c^{-1}\mu \qquad \text{in } B(0, R).$$

Hence it follows that $\Delta^m K_{m,\ell}\mu = c^{-1}\mu$. Now, letting

$$P(x) = u(x) - cK_{m,\ell}\mu(x),$$

we see that

$$\Delta^m P = \Delta^m u - \mu = 0,$$

which implies that P is polyharmonic of order m in $\mathbf{R}^n$.

Let $r > 2$. We have by Lemma 6

$$\int_{B(0,r)\cap B(0,|y|/2)} |K_{m,\ell}(x, y)|\,dx \leqq M|y|^{2m-n-\ell-1} \min\{r^{n+\ell+1}, |y|^{n+\ell+1}\}$$

when $|y| \geqq 1$. If $1 \leqq |y| < 2r$, then it follows from Lemmas 8 and 9 that

$$\int_{B(0,r)-B(0,|y|/2)-B(y,|y|)} |K_{m,\ell}(x, y)|\,dx \leqq Mr^{n+\ell}|y|^{2m-n-\ell} \log\frac{4r}{|y|}$$

and

$$\begin{aligned}\int_{B(y,|y|)} |K_{m,\ell}(x, y)|\,dx &\leqq M\int_{B(y,|y|)} \left(|x|^{2m-n} + |x-y|^{2m-n}\log\frac{2|y|}{|x-y|}\right)dx \\ &\leqq M|y|^{2m}\end{aligned}$$

since $|K_{m,\ell}(x, y)| \leqq M|x|^{2m-n}$ when $|y| \geqq 1$ and $2^{-1}|x| \leqq |x-y| < |y|$ on account of Lemmas 6 and 8. Hence we establish

$$\int_{B(0,r)} |K_{m,\ell}(x, y)|\,dx \leqq Mr^{n+\ell+1}|y|^{2m-n-\ell-1}$$

when $|y| \geqq 2r$, and

$$\int_{B(0,r)} |K_{m,\ell}(x, y)|\,dx \leqq Mr^{n+\ell}|y|^{2m-n-\ell}\log\frac{4r}{|y|}$$

when $1 \leqq |y| < 2r$. If $|y| < 1$, then

$$\int_{B(0,r)} |K_{m,\ell}(x, y)|\,dx \leqq \int_{B(0,r)} |x-y|^{2m-n}(1 + |\log|x-y||)\,dx \leqq Mr^{2m}\log r.$$

Consequently we derive

$$\int_{B(0,r)} |K_{m,\ell}\mu(x)|\,dx \leqq \int\left(\int_{B(0,r)} |K_{m,\ell}(x, y)|\,dx\right)d\mu(y)$$

$$\leqq Mr^{n+\ell+1}\int_{\mathbf{R}^n-B(0,2r)} |y|^{2m-n-\ell-1}\,d\mu(y)$$

$$+ Mr^{n+\ell}\int_{B(0,2r)-B(0,1)} |y|^{2m-n-\ell}\log\frac{4r}{|y|}\,d\mu(y)$$

$$+ M[r^{2m}\log r]\mu(B(0,1)),$$

so that (2) implies that

$$\lim_{r\to\infty} r^{-\ell-1-n}\int_{B(0,r)} |K_{m,\ell}\mu(x)|\,dx = 0$$

because $2m - n - \ell - 1 < 0$. Using assumption (4), we establish

$$\lim_{r\to\infty} r^{-\ell-1-n}\int_{B(0,r)} |P(x)|\,dx = 0. \tag{10}$$

Now Theorem 4 implies that P is a polynomial. We also see from (10) that the degree of P is at most ℓ.

REMARK 5. In view of Lemma 1, (10) implies that

$$\lim_{|x|\to\infty} |x|^{-\ell-1}|P(x)|\,dx = 0.$$

5. Proof of Theorem 3

For $|x| > 1$, write

$$K_{m,\ell}\mu(x) = \int_{\mathbf{R}^n-B(0,2|x|)} K_{m,\ell}(x, y)\,d\mu(y)$$

$$+ \int_{B(0,2|x|)-B(x,|x|/2)} K_{m,\ell}(x, y)\,d\mu(y)$$

$$+ \int_{B(x,|x|/2)} K_{m,\ell}(x, y)\,d\mu(y)$$

$$= U_1(x) + U_2(x) + U_3(x).$$

By Lemma 6, we have

$$|U_1(x)| \leqq M|x|^{\ell+1}\int_{\mathbf{R}^n-B(0,2|x|)} |y|^{2m-n-\ell-1}\,d\mu(y),$$

which together with (5) gives

$$\lim_{|x|\to\infty} |x|^{-\ell-a} U_1(x) = 0.$$

By Lemma 8, we find

$$|U_2(x)| \leqq M|x|^{2m-n}[1 + \log|x|]\mu(B(0, 1)) + M|x|^{\ell} \int_{B(0,2|x|)-B(0,1)} |y|^{2m-n-\ell} \log(|x|/|y|) d\mu(y).$$

Since $a > 0$, we derive for $|x| > R > 1$,

$$\begin{aligned} |U_2(x)| \leqq & M[|x|^{2m-n} \log|x|]\mu(B(0, 1)) \\ & + M|x|^{\ell} \int_{B(0,R)-B(0,1)} |y|^{2m-n-\ell} \log(|x|/|y|) d\mu(y) \\ & + M|x|^{\ell+a} \int_{B(0,2|x|)-B(0,R)} |y|^{2m-n-\ell-a} d\mu(y). \end{aligned}$$

Consequently it follows that

$$\limsup_{|x|\to\infty} |x|^{-\ell-a}|U_2(x)| \leqq M \int_{\mathbf{R}^n-B(0,R)} |y|^{2m-n-\ell-a} d\mu(y),$$

which proves

$$\lim_{|x|\to\infty} |x|^{-\ell-a} U_2(x) = 0.$$

Finally we are concerned with the fine limit of U_3 at infinity. For this purpose, note from Lemma 9 that

$$|U_3(x)| \leqq M \int_{B(x,|x|/2)} |x-y|^{2m-n} d\mu(y) \qquad \text{in case } 2m < n,$$

$$|U_3(x)| \leqq M \int_{B(x,|x|/2)} \log(|x|/|x-y|) d\mu(y) \qquad \text{in case } 2m = n$$

for $|x| \geqq 1$. By (5) we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and

$$\sum_{i=1}^{\infty} a_i 2^{-i(n-2m+\ell+a)} \mu(B_i) < \infty, \tag{11}$$

where $B_i = \{x : 2^{i-1} < |x| < 2^{i+2}\}$. Consider

$$E_i = \{x : 2^i \leqq |x| < 2^{i+1}, |U_3(x)| > a_i^{-1} 2^{i(\ell+a)}\}.$$

In what follows, we treat the case $2m = n$ only, because the case $2m < n$ can be treated similarly. If $x \in E_i$, then $B(x, |x|/2) \subset B_i$, so that

$$a_i^{-1} 2^{i(\ell+a)} < |U_3(x)| \leqq M \int_{B_i} \log(2^{i+3}/|x-y|)d\mu(y)$$

$$= M \int_{B_i} \log(2/|2^{-i-2}x - 2^{-i-2}y|)d\mu(y).$$

Hence, setting $E_i' = 2^{-i-2}E_i$, we have

$$C_{2m}(E_i') \leqq M a_i 2^{-i(\ell+a)} \mu(B_i),$$

which together with (11) gives

$$\sum_{i=1}^{\infty} C_{2m}(E_i') < \infty.$$

If we set

$$E = \bigcup_{i=1}^{\infty} E_i,$$

then E is seen to have all the required properties.

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