

The Dirichlet problem for the dissipative Helmholtz equation in a plane domain bounded by closed and open curves

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ABSTRACT. The Dirichlet problem for the dissipative Helmholtz equation in a connected plane region bounded by closed and open curves is studied. The existence of a classical solution is proved by potential theory. The problem is reduced to a Fredholm equation of the second kind, which is uniquely solvable. Our approach holds for both internal and external domains. Moreover, domains bounded by closed curves and exterior of open curves in a plane are particular cases of our problem. In case of strongly dissipative Helmholtz equation, the problem is studied under weakened assumptions.

1. Introduction

The boundary value problems in arbitrary plane domains bounded by closed and open curves were not studied in the theory of partial differential equations before. Problems outside open curves in a plane and problems in domains bounded by closed curves have been studied separately, because different methods for their analysis were used.

The 2-dimensional Dirichlet boundary value problem for the Helmholtz equation in a multiply connected domain bounded by closed curves is considered in text books on mathematical physics, for instance, in [1], [11]. The review on studies of the Dirichlet problem for this equation in the exterior of open curves is given in [4]. The present note is an attempt to join these problems together and to consider domains bounded by closed and open curves. From practical stand-point such domains have great significance, because open curves model cracks, screens or wings in physical problems.

The approach proposed in the present paper can be applied to other elliptic boundary value problems in domains bounded by closed and open curves. In particular, the Neumann problem for the dissipative Helmholtz equation is studied in [6]. The external Dirichlet problem for the propagative Helmholtz equation is considered in [8]. The Neumann problem for the Laplace equation, describing the flow of an ideal fluid over several obstacles,

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including wings, is investigated in [7]. The case of the nonlinear stratified flow is treated in [5].

2. Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [9].

Let γ be a set of curves, which may be closed and open. We say that $\gamma \in C^{2,\lambda}$ (or $\gamma \in C^{1,\lambda}$) if curves γ are of class $C^{2,\lambda}$ (or $C^{1,\lambda}$) with the Hölder exponent $\lambda \in (0, 1]$.

In the plane $x = (x_1, x_2) \in R^2$ we consider the multiply connected domain bounded by simple open curves $\Gamma_1^1, \dots, \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, \dots, \Gamma_{N_2}^2 \in C^{2,\lambda}$, $\lambda \in (0, 1]$, so that the curves do not have points in common. We will consider both the case of an external domain and the case of an internal domain, when the curve Γ_1^2 encloses all others. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \quad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by Γ^2 will be called \mathcal{D} . We assume that each curve Γ_n^k is parametrized by the arc length s :

$$\Gamma_n^k = \{x : x = x(s) = (x_1(s), x_2(s)), s \in [a_n^k, b_n^k]\}, \quad n = 1, \dots, N_k, \quad k = 1, 2,$$

so that

$$a_1^1 < b_1^1 < \dots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \dots < a_{N_2}^2 < b_{N_2}^2$$

and the domain \mathcal{D} is to the right when the parameter s increases on Γ_n^2 . Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2 , which correspond to the same point x for $n = 1, \dots, N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \quad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \quad \bigcup_{k=1}^2 \bigcup_{n=1}^{N_k} [a_n^k, b_n^k]$$

will be denoted by the same symbols, as corresponding sets of curves, that is, by Γ^1 , Γ^2 and Γ respectively.

We put $C^{k,r}(\Gamma_n^2) = \{\mathcal{F}(s) : \mathcal{F}(s) \in C^{k,r}[a_n^2, b_n^2], \mathcal{F}^{(m)}(a_n^2) = \mathcal{F}^{(m)}(b_n^2), m = 0, k\}$, $k = 0, 1$, $r \in [0, 1]$ and

$$C^{k,r}(\Gamma^2) = \bigoplus_{n=1}^{N_2} C^{k,r}(\Gamma_n^2).$$

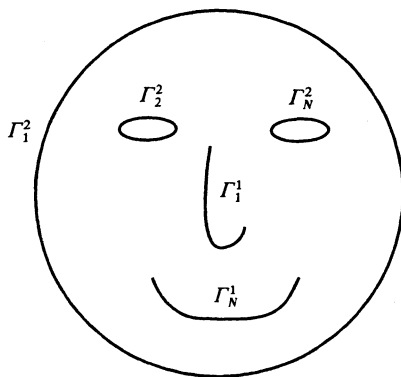


Figure 1. An internal domain

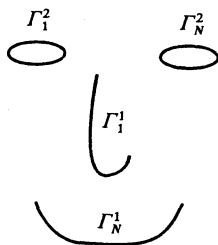


Figure 2. An external domain

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos \alpha(s), \sin \alpha(s))$, where $\cos \alpha(s) = x'_1(s)$, $\sin \alpha(s) = x'_2(s)$. Let $\mathbf{n}_x = (\sin \alpha(s), -\cos \alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of \mathbf{n}_x is chosen such that it will coincide with the direction of τ_x if \mathbf{n}_x is rotated anticlockwise through an angle of $\pi/2$.

We say, that a function $w(x)$ belongs to the smoothness class \mathbf{K} if

- 1) $w \in C^0(\overline{\mathcal{D}}) \oplus C^2(\mathcal{D} \setminus \Gamma^1)$,
- 2) $\nabla w \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$, where X is a point-set, consisting of the end-points of Γ^1 :

$$X = \bigcup_{n=1}^{N_1} (x(a_n^1) \cup x(b_n^1)),$$

- 3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\varepsilon > -1$ the inequality holds

$$(1) \quad |\nabla w| \leq \mathcal{C} |x - x(d)|^\varepsilon,$$

where $x \rightarrow x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \dots, N_1$.

In the definition of the class **K**, we consider Γ^1 as a set of cuts. In particular, by $C^0(\overline{\mathcal{D} \setminus \Gamma^1 \setminus X})$ we denote a class of functions, which are continuously extended on $\Gamma^1 \setminus X$ from the left and right, but their values on $\Gamma^1 \setminus X$ from the left and right can be different, so that these functions may have a jump across $\Gamma^1 \setminus X$.

Let us formulate the Dirichlet problem for the dissipative Helmholtz equation in the domain $\mathcal{D} \setminus \Gamma^1$.

Problem U. To find a function $w(x)$ of the class **K** which satisfies the Helmholtz equation

$$(2a) \quad w_{x_1 x_1}(x) + w_{x_2 x_2}(x) + \beta^2 w(x) = 0, \quad x \in \mathcal{D} \setminus \Gamma^1, \quad \beta = \text{const}, \quad \text{Im } \beta > 0,$$

and the boundary condition

$$(2b) \quad w(x(s))|_{\Gamma} = f(s).$$

If \mathcal{D} is an external domain, then we add the following condition at infinity

$$(2c) \quad w = o(|x|^{-1/2}), \quad |\nabla w(x)| = o(|x|^{-1/2}), \quad |x| = \sqrt{x_1^2 + x_2^2} \rightarrow \infty.$$

All conditions of the problem **U** must be satisfied in the classical sense.

On the basis of the energy equalities we can easily prove the following assertion.

THEOREM 1. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0, 1]$, then the problem **U** has at most one solution.*

The theorem holds for both internal and external domain \mathcal{D} .

3. Integral equations at the boundary

Below we assume that $f(s)$ in (2b) is an arbitrary function in the Banach space $C^{1,\lambda}(\Gamma)$, where the Hölder exponent $\lambda \in (0, 1]$ and $C^{1,\lambda}(\Gamma) = C^{1,\lambda}(\Gamma^1) \oplus C^{1,\lambda}(\Gamma^2)$.

If $\mathcal{B}_1(\Gamma^1)$, $\mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2 , then for functions given on Γ we introduce the Banach space $\mathcal{B}_1(\Gamma^1) \oplus \mathcal{B}_2(\Gamma^2)$ with the norm $\|\cdot\|_{\mathcal{B}_1(\Gamma^1) \oplus \mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}$.

Let us construct a solution of the problem **U**. This solution can be obtained with the help of potential theory for the Helmholtz equation (2a). We seek a solution of the problem in the form of the sum of a single-layer

potential on Γ^1 and a double-layer potential on Γ^2

$$(3) \quad \begin{aligned} w[\mu](x) &= w_1[\mu](x) + w_2[\mu](x), \\ w_1[\mu](x) &= \frac{i}{4} \int_{\Gamma^1} \mu(\sigma) \mathcal{H}_0^{(1)}(\beta|x-y(\sigma)|) d\sigma, \\ w_2[\mu](x) &= \frac{i}{4} \int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x-y(\sigma)|) d\sigma, \end{aligned}$$

where $\mathcal{H}_0^{(1)}(z)$ is the Hankel function of the first kind (see [10])

$$\mathcal{H}_0^{(1)}(z) = \frac{\sqrt{2} \exp(iz - i\pi/4)}{\pi\sqrt{z}} \int_0^\infty \exp(-t) t^{-1/2} \left(1 + \frac{it}{2z}\right)^{-1/2} dt,$$

$y = y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma$, $|x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}$, and $\mu(\sigma)$ is an unknown density.

By $\int_{\Gamma^k} \dots d\sigma$ we mean

$$\sum_{n=1}^{N_k} \int_{a_n^k}^{b_n^k} \dots d\sigma.$$

We will seek $\mu(s)$ in the Banach space $C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$ with the norm $\|\cdot\|_{C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)} = \|\cdot\|_{C_q^\omega(\Gamma^1)} + \|\cdot\|_{C^0(\Gamma^2)}$. We say that $\mu(s) \in C_q^\omega(\Gamma^1)$ if

$$\mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q |s - b_n^1|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with the exponent ω and

$$\|\mu(s)\|_{C_q^\omega(\Gamma^1)} = \left\| \mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q |s - b_n^1|^q \right\|_{C^{0,\omega}(\Gamma^1)}.$$

It can be checked directly [4] that for such $\mu(s)$ the function $w_1[\mu](x)$ belongs to the class **K** and meets all conditions of the problem **U** except the boundary condition (2b). In particular, the inequality (1) holds with $\varepsilon = -q$ if $q \in (0, 1)$. The potential $w_2[\mu](x)$ obeys equation (2a) and belongs to $C^0(\mathcal{D}) \oplus C^2(\mathcal{D})$. In the case of the external domain \mathcal{D} the function (3) meets the condition (2c) at infinity.

To satisfy the boundary condition we put (3) in (2b) and arrive at the integral equation for the density $\mu(s)$:

$$(4) \quad \begin{aligned} & \frac{i}{4} \int_{\Gamma^1} \mu(\sigma) \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) d\sigma + \frac{1}{2} \delta(s) \mu(s) \\ & + \frac{i}{4} \int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) d\sigma = f(s), \quad s \in \Gamma, \end{aligned}$$

where

$$\delta(s) = \begin{cases} 0, & \text{if } s \in \Gamma^1 \\ 1, & \text{if } s \in \Gamma^2 \end{cases}$$

Let us show that any integrable on Γ^1 and continuous on Γ^2 solution of equation (4) belongs to $C^{1,\lambda/2}(\Gamma^2)$. Indeed, there exists a derivative of the integral term from (4) in s on Γ^2 , because $\Gamma^2 \in C^{2,\lambda}$. This derivative is represented in the form of an improper integral and belongs to $C^{0,\lambda/2}(\Gamma^2)$ in s . Since $f(s) \in C^{1,\lambda}(\Gamma^2)$, the solution $\mu(s)$ of (4) belongs to $C^{1,\lambda/2}(\Gamma^2)$. Consequently, the potential $w_2[\mu](x)$ can be integrated by parts and written in the form of an angular potential [4] with the density $\mu'(s) \in C^{0,\lambda/2}(\Gamma^2)$. It follows from the properties of the angular potential [4] that $\nabla w_2[\mu](x) \in C^0(\overline{\mathcal{D}})$ and so, $w_2[\mu](x)$ belongs to the class **K**.

Thus, if $\mu(s)$ is a solution of equation (4) in the space $C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$, then $\mu(s) \in C_q^\omega(\Gamma^1) \oplus C^{1,\lambda/2}(\Gamma^2)$ and the potential (3) satisfies all conditions of the problem U.

The following theorem holds.

THEOREM 2. *Let $\Gamma \in C^{2,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma)$, $\lambda \in (0, 1]$. If equation (4) has a solution $\mu(s)$ in the Banach space $C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$ for some $\omega \in (0, 1]$ and $q \in [0, 1)$, then the function (3) is a solution of the problem U.*

If $s \in \Gamma^2$, then (4) is an equation of the second kind. If $s \in \Gamma^1$, then (4) is an equation of the first kind and its kernel has the logarithmic singularity, because

$$(5) \quad \mathcal{H}_0^{(1)}(z) = \frac{2i}{\pi} \ln \frac{z}{\beta} + h(z),$$

where $h(z)$ is a smooth function [10]. Indeed, as $z \rightarrow 0 + 0$

$$h(z) = \text{const} + O(z^2 \ln z), \quad h'(z) = O(z \ln z), \quad h''(z) = O(\ln z).$$

Our further treatment will aim at the proof of the solvability of (4) in the Banach space $C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$. Moreover, we reduce (4) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

By differentiating (4) on Γ^1 we reduce it to the following singular integral equation on Γ^1

$$\begin{aligned}
 (6a) \quad \frac{\partial}{\partial s} w[\mu](x(s)) &= \frac{1}{2\pi} \int_{\Gamma^1} \mu(\sigma) \frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} d\sigma \\
 &\quad + \frac{i}{4} \int_{\Gamma^1} \mu(\sigma) \frac{\partial}{\partial s} h(\beta|x(s) - y(\sigma)|) d\sigma \\
 &\quad + \frac{i}{4} \int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial s} \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) d\sigma \\
 &= f'(s), \quad s \in \Gamma^1,
 \end{aligned}$$

where the function $h(z)$ is defined by (5), and $\varphi_0(x, y)$ is the angle between the vector \vec{xy} and the direction of the normal \mathbf{n}_x . The angle $\varphi_0(x, y)$ is taken to be positive if it is measured anticlockwise from \mathbf{n}_x and negative if it is measured clockwise from \mathbf{n}_x . Besides, $\varphi_0(x, y)$ is continuous in $x, y \in \Gamma$ if $x \neq y$.

We rewrite equation (4) on Γ^2 in the form

$$(6b) \quad \mu(s) + \int_{\Gamma} \mu(\sigma) A_2(s, \sigma) d\sigma = 2f(s), \quad s \in \Gamma^2,$$

where

$$\begin{aligned}
 A_2(s, \sigma) &= \left\{ \frac{i}{2} (1 - \delta(\sigma)) \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) \right. \\
 &\quad \left. + \frac{i}{2} \delta(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) \right\} \in C^{0,\lambda}(\Gamma^2 \times \Gamma).
 \end{aligned}$$

REMARK. Evidently, $f(a_n^2) = f(b_n^2)$ and $A_2(a_n^2, \sigma) = A_2(b_n^2, \sigma)$ for any $\sigma \in \Gamma$ ($n = 1, \dots, N_2$). Hence, if $\mu(s)$ is a solution of equation (6b) from $C^0\left(\bigcup_{n=1}^{N_2} [a_n^2, b_n^2]\right)$, then, according to the equality (6b), $\mu(s)$ automatically satisfies matching conditions $\mu(a_n^2) = \mu(b_n^2)$ for $n = 1, \dots, N_2$ and therefore belongs to $C^0(\Gamma^2)$. This observation is true for equation (4) also and can be helpful for finding numerical solutions, since we may abandon matching conditions $\mu(a_n^2) = \mu(b_n^2)$ ($n = 1, \dots, N_2$), which are fulfilled automatically.

We note that equation (6a) is equivalent to (4) on Γ^1 if and only if (6a) is accompanied by the following additional conditions

$$(7) \quad w[\mu](x(a_n^1)) = f(a_n^1), \quad n = 1, \dots, N_1.$$

The system (6), (7) is equivalent to the equation (4).

It can be easily proved that

$$\frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \in C^{0,\lambda}(\Gamma^1 \times \Gamma^1)$$

(see [4], [9] for details). Therefore we can rewrite (6a) in the form

$$(8) \quad 2 \frac{\partial}{\partial s} w[\mu](x(s)) = \frac{1}{\pi} \int_{\Gamma^1} \mu(\sigma) \frac{d\sigma}{\sigma - s} + \int_{\Gamma} \mu(\sigma) Y(s, \sigma) d\sigma = 2f'(s), \quad s \in \Gamma^1,$$

where

$$\begin{aligned} Y(s, \sigma) = & \left\{ (1 - \delta(\sigma)) \left[\frac{1}{\pi} \left(\frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \right) \right. \right. \\ & \left. \left. + \frac{i}{2} \frac{\partial}{\partial s} h(\beta|x(s) - y(\sigma)|) \right] \right. \\ & \left. + \frac{i}{2} \delta(\sigma) \frac{\partial}{\partial s} \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) \right\} \in C^{0,p_0}(\Gamma^1 \times \Gamma), \end{aligned}$$

$p_0 = \lambda$ if $0 < \lambda < 1$ and $p_0 = 1 - \varepsilon_0$ for any $\varepsilon_0 \in (0, 1)$ if $\lambda = 1$.

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (8), we arrive at the following integral equation of the second kind [9]:

$$\begin{aligned} (9) \quad \mu(s) + \frac{1}{Q_1(s)} \int_{\Gamma} \mu(\sigma) A_1(s, \sigma) d\sigma + \frac{1}{Q_1(s)} \sum_{n=0}^{N_1-1} G_n s^n \\ = \frac{1}{Q_1(s)} \Phi_1(s), \quad s \in \Gamma^1, \end{aligned}$$

where

$$\begin{aligned} (10) \quad A_1(s, \sigma) &= -\frac{1}{\pi} \int_{\Gamma^1} \frac{Y(\xi, \sigma)}{\xi - s} Q_1(\xi) d\xi, \\ Q_1(s) &= \prod_{n=1}^{N_1} \left| \sqrt{s - a_n^1} \sqrt{b_n^1 - s} \right| \text{sign}(s - a_n^1), \\ \Phi_1(s) &= -\frac{1}{\pi} \int_{\Gamma^1} \frac{2Q_1(\sigma)f'(\sigma)}{\sigma - s} d\sigma, \end{aligned}$$

and G_0, \dots, G_{N_1-1} are arbitrary constants.

It can be shown using the properties of singular integrals [2], [9] that $\Phi_1(s)$, $A_1(s, \sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Consequently, any solution of (9) belongs to $C_{1/2}^\omega(\Gamma^1)$ and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s) = (1 - \delta(s))Q_1(s) + \delta(s), \quad s \in \Gamma.$$

Instead of $\mu(s) \in C_{1/2}^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$ we introduce a new unknown function $\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \oplus C^0(\Gamma^2)$ and rewrite (9), (6b) in the form of one equation

$$(11) \quad \mu_*(s) + \int_{\Gamma} \mu_*(\sigma) Q^{-1}(\sigma) A(s, \sigma) d\sigma + (1 - \delta(s)) \sum_{n=0}^{N_1-1} G_n s^n = \Phi(s), \quad s \in \Gamma,$$

where

$$A(s, \sigma) = (1 - \delta(s))A_1(s, \sigma) + \delta(s)A_2(s, \sigma),$$

$$\Phi(s) = (1 - \delta(s))\Phi_1(s) + 2\delta(s)f(s).$$

To derive equations for G_0, \dots, G_{N_1-1} , we substitute $\mu(s)$ from (9), (6b) into the conditions (7), then in terms of $\mu_*(s)$ we obtain

$$(12) \quad \int_{\Gamma} Q^{-1}(\xi) \mu_*(\xi) l_n(\xi) d\xi + \sum_{m=0}^{N_1-1} B_{nm} G_m = H_n, \quad n = 1, \dots, N_1,$$

where

$$(13) \quad \begin{aligned} l_n(\xi) &= -w[Q^{-1}(\cdot)A(\cdot, \xi)](a_n^1), \\ B_{nm} &= -w[Q^{-1}(\cdot)(1 - \delta(\cdot))(\cdot)^m](a_n^1), \\ H_n &= -w[Q^{-1}(\cdot)\Phi(\cdot)](a_n^1) + f(a_n^1). \end{aligned}$$

By \cdot we denote the variable of integration in the potential (3).

Thus, the system of equations (7), (6) for $\mu(s)$ has been reduced to the system (11), (12) for the function $\mu_*(s)$ and constants G_0, \dots, G_{N_1-1} . It is clear from our consideration that any solution of system (11), (12) gives a solution of system (7), (6).

As noted above, $\Phi_1(s)$ and $A_1(s, \sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [9]), $\Phi_1(s) \in C^{0,p}(\Gamma^1)$, $p = \min\{1/2, \lambda\}$ and $A_1(s, \sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$.

We arrive at the following assertion.

LEMMA 1. Let $\Gamma \in C^{2,\lambda}$, $\lambda \in (0, 1]$ and $\Phi(s) \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2)$, where $p = \min\{\lambda, 1/2\}$. If $\mu_*(s) \in C^0(\Gamma)$ is a solution of equation (11), then $\mu_*(s)$ belongs to $C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2)$.

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2)$ holds if $f(s) \in C^{1,\lambda}(\Gamma)$. Hence below we will seek $\mu_*(s)$ in $C^0(\Gamma)$.

Since $A(s, \sigma) \in C^0(\Gamma \times \Gamma)$, the integral operator in (11):

$$A\mu_* = \int_{\Gamma} \mu_*(\sigma) Q^{-1}(\sigma) A(s, \sigma) d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself.

We rewrite (11) in the operator form

$$(14) \quad (I + A)\mu_* + PG = \Phi,$$

where P is the operator multiplying the row $P = (1 - \delta(s))(s^0, \dots, s^{N_1-1})$ by the column $G = (G_0, \dots, G_{N_1-1})^T$. The operator P is finite-dimensional from E_{N_1} into $C^0(\Gamma)$ and so compact.

Now we rewrite equations (12) in the form

$$(15) \quad I_{N_1}G + L\mu_* + (B - I_{N_1})G = H,$$

where $H = (H_1, \dots, H_{N_1})^T$ is a column of N_1 elements, I_{N_1} is an identity operator in E_{N_1} , B is an $N_1 \times N_1$ matrix consisting of the elements B_{nm} from (13). The operator L maps $C^0(\Gamma)$ into E_{N_1} , so that $L\mu_* = (L_1\mu_*, \dots, L_{N_1}\mu_*)^T$, where

$$L_n\mu_* = \int_{\Gamma} Q^{-1}(\xi)\mu_*(\xi)l_n(\xi) d\xi.$$

The operators $(B - I_{N_1})$, L are finite-dimensional and so compact.

We consider the columns $\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix}$, $\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix}$ in the Banach space $C^0(\Gamma) \times E_{N_1}$ with the norm $\|\tilde{\mu}\|_{C^0(\Gamma) \times E_{N_1}} = \|\mu_*\|_{C^0(\Gamma)} + \|G\|_{E_{N_1}}$.

We write system (14), (15) in the form of one equation

$$(16) \quad (I + R)\tilde{\mu} = \tilde{\Phi}, \quad R = \begin{pmatrix} A & P \\ L & B - I_{N_1} \end{pmatrix},$$

where I is an identity operator in the space $C^0(\Gamma) \times E_{N_1}$. It is clear that R is a compact operator mapping $C^0(\Gamma) \times E_{N_1}$ into itself. Therefore, (16) is a Fredholm equation in this space.

Let us show that the homogeneous equation (16) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation

(16) has a unique solution for any $\tilde{\Phi}$. We will prove this by a contradiction. Let $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$ be a non-trivial solution of the homogeneous equation (16). According to Lemma 1: $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2) \times E_{N_1}$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s) \in C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$ and the column G^0 convert the homogeneous equations (9), (6b), (12) into identities. For instance, (6b) takes the form

$$(17a) \quad \lim_{x \rightarrow x(s) \in \Gamma^2} w[\mu^0](x) = 0, \quad x \in \mathcal{D}.$$

Using the homogeneous identities (9), (6b), we check that the homogeneous identities (12) are equivalent to

$$(17b) \quad w[\mu^0](a_n^1) = 0, \quad n = 1, \dots, N_1.$$

Besides, acting on the homogeneous identity (9) with a singular operator with the kernel $(s - t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (8):

$$(17c) \quad \left. \frac{\partial}{\partial s} w[\mu^0](x(s)) \right|_{\Gamma^1} = 0.$$

It follows from (17) that $\mu^0(s)$ satisfies the homogeneous equation (4). On the basis of Theorem 2, $w[\mu^0](x)$ is a solution of the homogeneous problem U. According to Theorem 1: $w[\mu^0](x) \equiv 0$, $x \in \mathcal{D}$. Using the limit formulas for normal derivatives of a single-layer potential on Γ^1 , we have

$$\lim_{x \rightarrow x(s) \in (\Gamma^1)^+} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) - \lim_{x \rightarrow x(s) \in (\Gamma^1)^-} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) = \mu^0(s) \equiv 0, \quad s \in \Gamma^1.$$

By $(\Gamma^1)^+$ we denote the side of Γ^1 , which is on the left as a parameter s increases, and by $(\Gamma^1)^-$ we denote the other side.

Hence, $w[\mu^0](x) = w_2[\mu^0](x) \equiv 0$, $x \in \mathcal{D}$, and $\mu^0(s)$ satisfies (17a), which can be written as

$$(18) \quad \frac{1}{2} \mu^0(s) + \frac{i}{4} \int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) d\sigma = 0, \quad s \in \Gamma^2.$$

Equation (18) has only the trivial solution $\mu^0(s) \equiv 0$ in $C^0(\Gamma^2)$. This is true for both internal and external domain \mathcal{D} . The detailed proof is presented in the section 5.

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu_*^0(s) = \mu^0(s)Q^{-1}(s) \equiv 0$ and it follows from the homogeneous identity (9) for $\mu^0(s)$ and $G_0^0, \dots, G_{N_1-1}^0$ that $G^0 = (G_0^0, \dots, G_{N_1-1}^0)^T \equiv 0$. Hence, $\tilde{\mu}^0 \equiv 0$ and we arrive at a contradiction to

the assumption that $\tilde{\mu}^0$ is a non-trivial solution of the homogeneous equation (16). Thus, the homogeneous Fredholm equation (16) has only a trivial solution in $C^0(\Gamma) \times E_{N_1}$.

We have proved the following assertion.

THEOREM 3. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0, 1]$, then (16) is a Fredholm equation of the second kind in the space $C^0(\Gamma) \times E_{N_1}$. Moreover, equation (16) has a unique solution $\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$ for any $\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$.*

As a consequence of Theorem 3 and Lemma 1 we obtain the following corollary.

COROLLARY. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0, 1]$, then equation (16) has a unique solution $\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix} \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2) \times E_{N_1}$ for any $\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix} \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2) \times E_{N_1}$, where $p = \min\{\lambda, 1/2\}$.*

We recall that $\tilde{\Phi}$ belongs to the class of smoothness required in the corollary if $f(s) \in C^{1,\lambda}(\Gamma)$. Besides, equation (16) is equivalent to the system (11), (12). As mentioned above, if $\mu_*(s) \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2)$, G_0, \dots, G_{N_1-1} is a solution of system (11), (12), then $\mu(s) = \mu_*(s)Q^{-1}(s) \in C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$ is a solution of system (7), (6) and so $\mu(s)$ satisfies equation (4). We obtain the following assertion.

THEOREM 4. *If $\Gamma \in C^{2,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma)$, $\lambda \in (0, 1]$, then equation (4) has a solution $\mu(s)$ in $C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$. This solution is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_*(s) \in C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2)$ is found by solving the Fredholm equation (16), which is uniquely solvable.*

REMARK. The solution of equation (4) ensured by Theorem 4 is unique in the space $C_{1/2}^{p_0}(\Gamma^1) \oplus C^0(\Gamma^2)$ for any $p_0 \in (0, p]$. The proof can be given by a contradiction to the assumption that the homogeneous equation (4) has a nontrivial solution in this space. The proof is almost the same as the proof of Theorem 3. Consequently, the numerical solution of equation (4) can be obtained by the direct numerical inversion of the integral operator from (4). In doing so, Hölder functions can be approximated by continuous piecewise linear functions, which also obey Hölder inequality. The simplification for numerical solving equation (4) is suggested in the remark to the equation (6b) in the section 3.

Recall that if Theorem 4 holds, then the solution of equation (4) ensured by Theorem 4 belongs to $C_{1/2}^p(\Gamma^1) \oplus C^{1,\lambda/2}(\Gamma^2)$. On the basis of Theorem 2 we arrive at the final result.

THEOREM 5. *If $\Gamma \in C^{2,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma)$, $\lambda \in (0, 1]$, then the solution of the problem U exists and is given by (3), where $\mu(s)$ is a solution of equation (4) in $C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$ ensured by Theorem 4. More precisely, $\mu(s) \in C_{1/2}^p(\Gamma^1) \oplus C^{1,\lambda/2}(\Gamma^2)$.*

It can be checked directly that the solution of the problem U satisfies condition (1) with $\varepsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulas presented in [4].

Theorem 5 ensures the existence of a classical solution of the problem U when $\Gamma \in C^{2,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma)$. The uniqueness of the classical solution follows from Theorem 1. On the basis of our consideration we suggest the following scheme for solving the problem U. First, we find the unique solution of the Fredholm equation (16) in $C^0(\Gamma) \times E_{N_1}$. This solution automatically belongs to $C^{0,p}(\Gamma^1) \oplus C^0(\Gamma^2) \times E_{N_1}$, $p = \min\{\lambda, 1/2\}$. Second, we construct the solution of equation (4) in $C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. This solution automatically belongs to $C_{1/2}^p(\Gamma^1) \oplus C^{1,\lambda/2}(\Gamma^2)$. Finally, putting $\mu(s)$ in (3) we obtain the solution of the problem U. In accordance with the remark to Theorem 4, the unique solution of equation (4) in $C_{1/2}^{p_o}(\Gamma^1) \oplus C^0(\Gamma^2)$, $p_o \in (0, p]$ can be also found directly.

5. Analysis of equation (18)

Equation (18) is well-known in classical mathematical physics. We arrive at (18) when solving the Dirichlet problem for the Helmholtz equation (2a) in the domain \mathcal{D} by a double layer potential. We give analysis of equation (18) in weaker conditions on Γ^2 than in the sections 2–4. Namely, we suppose that $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$ instead of $\Gamma^2 \in C^{2,\lambda}$. If $\Gamma^2 \in C^{1,\lambda}$, then the kernel of the integral term in (18) can be expressed in the form

$$\frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}(\beta|x(s) - y(\sigma)|) = \frac{I_0(s, \sigma)}{s - \sigma} + I_1(s, \sigma),$$

where $I_1(s, \sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma^2)$, $I_0(s, \sigma) \in C^{0,\lambda}(\Gamma^2 \times \Gamma^2)$ and $I_0(s, s) = 0$. From [9, Sect. 5.7] we obtain

$$\frac{I_0(s, \sigma)}{s - \sigma} + I_1(s, \sigma) = \frac{I_2(s, \sigma)}{|s - \sigma|^{1-\lambda/2}} + I_1(s, \sigma).$$

where $I_2(s, \sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma^2)$. On the basis of this representation one can conclude [9, Sect. 51.1] that if $\mu^0(s) \in C^0(\Gamma^2)$, then the integral term in (18) belongs to $C^{0,\lambda/2}(\Gamma^2)$ in s . Therefore any solution of (18) in $C^0(\Gamma^2)$ automatically belongs to $C^{0,\lambda/2}(\Gamma^2)$. Besides, in accordance with [11, Sect. 18.5],

equation (18) is a Fredholm equation of the second kind in $C^0(\Gamma^2)$, because the integral operator in (18) is a compact operator mapping $C^0(\Gamma^2)$ into itself.

Our aim is to prove the following assertion.

LEMMA 2. *If $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$, then there is only the trivial solution of the homogeneous Fredholm equation (18) in $C^0(\Gamma^2)$.*

According to Fredholm alternative, Lemma 2 is proved, if we show that the homogeneous adjoint integral equation has only the trivial solution in $C^0(\Gamma^2)$. The adjoint equation to equation (18) is

$$(19) \quad \frac{1}{2}\zeta^0(s) - \frac{i}{4} \int_{\Gamma^2} \zeta^0(\sigma) \frac{\partial}{\partial \mathbf{n}_x} \mathcal{H}_0^{(2)}(\bar{\beta}|x(s) - y(\sigma)|) d\sigma = 0, \quad s \in \Gamma^2.$$

Here $\bar{\beta} = \text{Re } \beta - i \text{Im } \beta$ and $\mathcal{H}_0^{(2)}(z)$ is the Hankel function of the second kind [10]. We used the fact [10] that the Hankel functions of the first kind and the second kind are complex conjugate, so that $\overline{\mathcal{H}_0^{(1)}(z)} = \mathcal{H}_0^{(2)}(\bar{z})$ and an overline denotes the complex conjugation. In our assumptions the kernel of the integral term in (19) can be represented in the form

$$\frac{\partial}{\partial \mathbf{n}_x} \mathcal{H}_0^{(2)}(\bar{\beta}|x(s) - y(\sigma)|) = \frac{I_3(s, \sigma)}{|s - \sigma|^{1-\lambda/2}} + I_4(s, \sigma),$$

where $I_3(s, \sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma^2)$ and $I_4(s, \sigma) \in C^{0,\lambda/2}(\Gamma^2 \times \Gamma^2)$. It follows from this representation [9, Sect. 51.1] that if $\zeta^0(s) \in C^0(\Gamma^2)$, then the integral term in (19) belongs to $C^{0,\lambda/2}(\Gamma^2)$ in s . Consequently, any solution of (19) in $C^0(\Gamma^2)$ automatically belongs to $C^{0,\lambda/2}(\Gamma^2)$. Now we prove that equation (19) has only the trivial solution in $C^0(\Gamma^2)$. We give a proof by a contradiction. Suppose that equation (19) has a nontrivial solution $\zeta^0(s) \in C^0(\Gamma^2)$, which converts (19) into identity. As mentioned above, $\zeta^0(s)$ automatically belongs to $C^{0,\lambda/2}(\Gamma^2)$. We put $\mathcal{D}_0 = R^2 \setminus (\mathcal{D} \cup \Gamma^2)$ and consider a single layer potential

$$(20) \quad v[\zeta^0](x) = -\frac{i}{4} \int_{\Gamma^2} \zeta^0(\sigma) \mathcal{H}_0^{(2)}(\bar{\beta}|x(s) - y(\sigma)|) d\sigma \\ \in C^2(R^2 \setminus \Gamma^2) \cap C^0(R^2) \cap C^1(\bar{\mathcal{D}}) \cap C^1(\bar{\mathcal{D}}_0)$$

satisfying the following Helmholtz equation

$$(21) \quad v_{x_1 x_1}(x) + v_{x_2 x_2}(x) + \bar{\beta}^2 v(x) = 0, \quad x \in R^2 \setminus \Gamma^2.$$

REMARK. Potential (20) belongs to $C^1(\bar{\mathcal{D}})$ and to $C^1(\bar{\mathcal{D}}_0)$, since its density $\zeta^0(s)$ is a Hölder function, as shown above.

Note that potential (20) meets condition (2c) at infinity owing to properties of Hankel functions [10] and owing to $\text{Im} \bar{\beta} < 0$. Moreover, potential (20) satisfies the homogeneous Neumann boundary condition

$$\lim_{\substack{x^0 \rightarrow x(s) \in \Gamma^2 \\ x^0 \in \mathcal{D}_0}} \frac{\partial}{\partial \mathbf{n}_x} v[\zeta^0](x^0) = 0,$$

because it is equivalent to the identity (19).

Thus, potential (20) is a solution of the homogeneous Neumann problem for the dissipative Helmholtz equation (21) in a domain \mathcal{D}_0 . If \mathcal{D}_0 is an external domain, then potential (20) meets condition (2c) at infinity. This homogeneous Neumann problem has only the trivial solution in $C^2(\mathcal{D}_0) \cap C^1(\bar{\mathcal{D}}_0)$ thanks to the energy equality for equation (21) in the domain \mathcal{D}_0

$$\|\nabla v\|_{L_2(\mathcal{D}_0)}^2 - \bar{\beta}^2 \|v\|_{L_2(\mathcal{D}_0)}^2 = \int_{\Gamma^2} \bar{v} \frac{\partial v}{\partial \mathbf{n}_x} ds, \quad (\text{Im} \bar{\beta}^2 \neq 0 \quad \text{or} \quad \bar{\beta}^2 = -|\bar{\beta}|^2),$$

where we keep in mind condition (2c) if \mathcal{D}_0 is an external domain. Consequently,

$$(22) \quad v[\zeta^0](x) \equiv 0, \quad x \in \bar{\mathcal{D}}_0.$$

Since potential (20) is continuous across Γ^2 , we obtain that it satisfies the homogeneous Dirichlet boundary condition

$$\lim_{\substack{x^0 \rightarrow x(s) \in \Gamma^2 \\ x^0 \in \mathcal{D}}} v[\zeta^0](x^0) = 0.$$

Hence, potential (20) is a solution of a homogeneous Dirichlet problem for the equation (21) in a domain \mathcal{D} . If \mathcal{D} is an external domain, potential (20) meets condition (2c) at infinity. This homogeneous Dirichlet problem has only the trivial solution in $C^2(\mathcal{D}) \cap C^1(\bar{\mathcal{D}})$ thanks to the energy equality for equation (21) in the domain \mathcal{D}

$$\|\nabla v\|_{L_2(\mathcal{D})}^2 - \bar{\beta}^2 \|v\|_{L_2(\mathcal{D})}^2 = - \int_{\Gamma^2} \bar{v} \frac{\partial v}{\partial \mathbf{n}_x} ds, \quad (\text{Im} \bar{\beta}^2 \neq 0 \quad \text{or} \quad \bar{\beta}^2 = -|\bar{\beta}|^2),$$

where we keep in mind condition (2c) if \mathcal{D} is an external domain. Therefore,

$$v[\zeta^0](x) \equiv 0, \quad x \in \bar{\mathcal{D}}.$$

Together with (22) we have

$$v[\zeta^0](x) \equiv 0, \quad x \in R^2.$$

Using the jump relation [1], [4], [11] for the normal derivative of a single layer potential on Γ^2 , we obtain

$$\lim_{\substack{x^0 \rightarrow x(s) \in \Gamma^2 \\ x^0 \in \mathcal{D}_0}} \frac{\partial}{\partial \mathbf{n}_x} v[\zeta^0](x^0) - \lim_{\substack{x^0 \rightarrow x(s) \in \Gamma^2 \\ x^0 \in \mathcal{D}}} \frac{\partial}{\partial \mathbf{n}_x} v[\zeta^0](x^0) = \zeta^0(s) \equiv 0.$$

We arrive at a contradiction to the assumption, that $\zeta^0(s)$ is a nontrivial solution of equation (19). Thus, equation (19) has only the trivial solution in $C^0(\Gamma)$. According to Fredholm alternative, equation (18) also has only the trivial solution in $C^0(\Gamma)$. Lemma 2 is proved.

As a consequence of Lemma 2 we obtain the corollary.

COROLLARY. *If $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$, then the nonhomogeneous Fredholm equations (18), (19) are uniquely solvable in $C^0(\Gamma^2)$ for any right-hand side from $C^0(\Gamma^2)$.*

6. The strongly dissipative Helmholtz equation

Let us consider the general Helmholtz equation in R^m

$$(23) \quad \Delta u(x) + c(x)u(x) = 0, \quad c(x) = c_1(x) + ic_2(x),$$

where $c_1(x)$ and $c_2(x)$ are real functions, and Δ is Laplacian in R^m , $m \geq 1$. Equation (23) is said to be strongly dissipative in a domain $D \subset R^m$, if $c_1(x) < 0$ for any $x \in D$. In this case the principle of maximum modulus holds for solutions of (23).

The maximum modulus principle enables us to prove uniqueness and solvability theorems for the problem U in weaker assumptions to the smoothness of Γ^2 and to the smoothness of $f(s)$ in (2b) than in the sections 2–4.

In the present paper maximum means weak maximum, i.e., a real function $F(x)$ has a maximum at a point x^0 , if $F(x) \leq F(x^0)$ for any x in a neighbourhood of x^0 .

At first we give a simple proof of the maximum modulus principle.

THEOREM 6. *Let Helmholtz equation (23) be strongly dissipative in $D \subset R^m$. If $u(x) \in C^2(D)$ is a solution of (23) in D , then $|u(x)|$ can not reach positive maximum in the internal point of D .*

REMARK 1. We do not introduce any restrictions on the domain D , which may be internal or external. For example, the domain $D \subset R^2$ may be bounded by closed and open curves, so that D may coincide with the domain $\mathcal{D} \setminus \Gamma^1$ considered in the section 2.

REMARK 2. We suppose that in case of an external domain $x \notin D$ if $|x| = \infty$.

REMARK 3. We do not impose smoothness conditions on $c(x)$.

REMARK 4. If $c_2(x) \equiv 0$ and so $c(x) \equiv c_1(x) < 0$, then (23) is a real equation and the maximum principle for its solutions is well-known [3].

Proof of Theorem 6. Let $u(x) = u_1(x) + iu_2(x)$ be a solution of (23) in D , where $u_1(x)$ and $u_2(x)$ are real functions. We rewrite (23) in the form of a system

$$(24a) \quad \Delta u_1(x) + c_1(x)u_1(x) - c_2(x)u_2(x) = 0,$$

$$(24b) \quad \Delta u_2(x) + c_1(x)u_2(x) + c_2(x)u_1(x) = 0,$$

and put $F(x) = \frac{1}{2}|u(x)|^2 = \frac{1}{2}(u_1^2(x) + u_2^2(x))$. Note

$$(25) \quad F_{x_j x_j} = (u_1)_{x_j x_j} u_1 + (u_2)_{x_j x_j} u_2 + ((u_1)_{x_j})^2 + ((u_2)_{x_j})^2, \quad j = 1, \dots, m.$$

Suppose that x^0 is an internal point of D , and $|u(x)|$ has a maximum in x^0 , i.e., for any x in a neighbourhood of x^0 the inequality holds: $|u(x)| \leq |u(x^0)|$. Consequently, in this neighbourhood $F(x) \leq F(x^0)$, so that $F(x)$ also has a maximum in x^0 .

Since $F(x) \in C^2(D)$, the necessary condition for the maximum of $F(x)$ at the point x^0 is

$$F_{x_j x_j}(x^0) \leq 0, \quad j = 1, \dots, m.$$

This inequality follows from the fact that the function

$$F(x_1^0, \dots, x_{j-1}^0, x_j, x_{j+1}^0, \dots, x_m^0)$$

of a variable x_j has a maximum if $x_j = x_j^0$. Consequently,

$$\sum_{j=1}^m F_{x_j x_j}(x^0) = \Delta F(x^0) \leq 0.$$

With the help of (25) we have

$$u_1 \Delta u_1 + u_2 \Delta u_2 + (\nabla u_1)^2 + (\nabla u_2)^2|_{x=x^0} \leq 0.$$

Substituting here Δu_1 and Δu_2 from (24), we get

$$-2c_1(x^0)F(x^0) + (\nabla u_1(x^0))^2 + (\nabla u_2(x^0))^2 \leq 0.$$

In our assumptions $c_1(x^0) < 0$, and therefore $F(x^0) = 0$, so that $|u(x^0)| = 0$. Theorem 6 is proved.

COROLLARY. *In addition to the conditions of Theorem 6 suppose that D is an internal domain and $u(x) \in C^0(\bar{D})$, then*

$$\sup_{x \in \bar{D}} |u(x)| = \sup_{x \in \partial D} |u(x)| = |u(x^0)|, \quad x^0 \in \partial D.$$

PROOF. If $u(x) \equiv 0$ in \bar{D} , the corollary is clear. If $u(x) \not\equiv 0$ in \bar{D} , then $|u(x)|$ reaches a positive maximum at a point $x^0 \in \bar{D}$. According to Theorem 6, x^0 is not internal point of D , therefore $x^0 \in \partial D$. The corollary is proved.

REMARK. The corollary also holds for an external domain D if $u(x)$ uniformly tends to zero as $|x| \rightarrow \infty$.

Now we give an example, which shows that the maximum modulus principle (i.e. Theorem 6) does not hold if $c(x) \equiv \text{const}$ and $c_1 > 0$.

Suppose $c = \beta^2 = \text{const}$, $\beta = \beta_1 + i\beta_2$, $|\beta_2| < |\beta_1|$, and so $c_1 = \beta_1^2 - \beta_2^2 > 0$. In these assumptions the function

$$\begin{aligned} u(x) &= u(x_1) = \cos(\beta(x_1 - x_1^0)) \\ &= \cosh(\beta_2(x_1 - x_1^0)) \cos(\beta_1(x_1 - x_1^0)) \\ &\quad - i \sinh(\beta_2(x_1 - x_1^0)) \sin(\beta_1(x_1 - x_1^0)) \end{aligned}$$

is a solution of (23) for any x_1^0 . We consider an arbitrary domain $D \subset R^m$ containing the point $x^0 = (x_1^0, \dots, x_m^0)$. The function $|u(x)|$ reaches a positive maximum in the point x^0 , i.e., for any x in a neighbourhood of this point: $|u(x)| \leq |u(x^0)| = 1$.

7. The problem U for the strongly dissipative Helmholtz equation

In this section we study the problem U if Helmholtz equation (2a) is strongly dissipative, that is,

$$\text{Im } \beta > |\text{Re } \beta|.$$

As mentioned above, this assumption enables us to prove uniqueness and solvability theorems in weaker conditions on Γ^2 and on $f(s)$ from (2b) then in the sections 2–4.

Further on we follow notations from the sections 2–4.

We suppose that $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$ instead of $\Gamma^2 \in C^{2,\lambda}$ in the sections 2–4. Besides, we abandon the class **K** in the formulation of the problem U. Now we reformulate the problem U for the case of strongly dissipative equation (2a).

Problem U_0 . To find a function $w(x) \in C^2(\mathcal{D} \setminus \Gamma^1) \cap C^0(\bar{\mathcal{D}})$ which satisfies equation (2a), where $\text{Im } \beta > |\text{Re } \beta|$, and satisfies the boundary condition (2b).

In addition, if \mathcal{D} is an external domain, then $w(x)$ must uniformly tend to zero as $|x| \rightarrow \infty$.

The uniqueness theorem holds for the problem U_0 in case of both internal and external domain \mathcal{D} and follows from the corollary to Theorem 6.

THEOREM 7. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0, 1]$, then the problem U_0 has at most one solution.*

Theorem 7 is essentially based on the fact that equation (2a) is strongly dissipative.

To prove the solvability theorem, we assume that $f(s)$ in (2b) belongs to $C^{1,\lambda}(\Gamma^1) \oplus C^0(\Gamma^2)$ instead of $f(s) \in C^{1,\lambda}(\Gamma)$ in the sections 3–4. We seek the solution of the problem U_0 in the form (3), where $\mu(s) \in C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$. The analysis of the problem U_0 can be given in the same way as in the sections 2–4. Instead of Theorem 2 we have

THEOREM 8. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma^1) \oplus C^0(\Gamma^2)$, $\lambda \in (0, 1]$. If equation (4) has a solution $\mu(s)$ in $C_q^\omega(\Gamma^1) \oplus C^0(\Gamma^2)$, where $\omega \in (0, 1]$ and $q \in [0, 1)$, then function (3) is a solution of the problem U_0 .*

Lemma 1, Theorem 3 and the corollary to Theorem 3 hold if $\Gamma^1 \in C^{2,\lambda}$ and $\Gamma^2 \in C^{1,\lambda}$. Theorem 4 holds, if in addition to these conditions on Γ we assume $f(s) \in C^{1,\lambda}(\Gamma^1) \oplus C^0(\Gamma^2)$. We note that the proof of the solvability of equation (16) is essentially based on Theorem 7 and on the analysis of equation (18) presented in the section 5. Finally, we arrive at the solvability theorem for the problem U_0 .

THEOREM 9. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $f(s) \in C^{1,\lambda}(\Gamma^1) \oplus C^0(\Gamma^2)$, $\lambda \in (0, 1]$, then the solution of the problem U_0 is given by (3), where $\mu(s)$ is a solution of equation (4) in $C_{1/2}^p(\Gamma^1) \oplus C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$.*

The existence of a solution of equation (4) mentioned in Theorem 9 is ensured by Theorem 4, modified as described above. Thus, in case of strongly dissipative Helmholtz equation we do not use energy equalities to prove uniqueness theorem for our problem and so we abandon the class \mathbf{K} in the formulation of the problem. Nevertheless, the solution $w(x)$ of the problem U_0 constructed by Theorem 9 satisfies conditions 1) and 3) of the class \mathbf{K} and in addition $\nabla w \in C^0(\mathcal{D} \setminus \overline{\Gamma^1} \setminus \Gamma^2 \setminus X)$, where X is a set of end-points of Γ^1 . Unlike the problem U , gradient of the solution of the problem U_0 may not be continuously extendable to Γ^2 . In case of an external domain, the solution of U_0 presented in Theorem 9 meets condition (2c), though we require weaker condition at infinity in the formulation of U_0 .

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