Hiroshima Math. J.
28 (1998), 129–137

Invariance principle for a Brownian motion with large drift in a white noise environment

Kiyoshi Kawazu and Hiroshi Tanaka

(Received November 14, 1996)

Abstract. This paper discusses an invariance principle for a Brownian motion with drift coefficient $\kappa/4$ in a white noise environment under the assumption that κ is large. Our method clarifies the relation between the environment-wise invariance principle discussed in [7] and the present result (the invariance principle in random environment).

Introduction

Let W be the space of continuous functions on $\mathbf{R}$ vanishing at 0 that is equipped with the Wiener measure P. For an element $w \in W$ let us denote by w_κ the element of W defined by $w_\kappa(x) = w(x) - (\kappa x/2)$ where κ is a given positive constant. For $w \in W$, P_w denotes the probability measure on $\Omega = C[0,\infty)$ such that $\mathbf{X}_x = \{\omega(t), t \geq 0, P_w\}$ is a diffusion process with generator

$$\mathscr{L}_w = \frac{1}{2} e^{w_\kappa}(x) \frac{d}{dx}\left(e^{-w_\kappa}(x) \frac{d}{dx}\right)$$

starting at 0, where $\omega(t)$ is the value of a function $\omega(\in \Omega)$ at time t. We regard $\omega(t)$ as a process defined on the probability space $\{W \times \Omega, \mathscr{P}\}$ where $\mathscr{P}(dw\,d\omega) = P(dw)P_w(d\omega)$. Then symbolically

$$d\omega(t) = dB(t) + \frac{\kappa}{4}\,dt - \frac{1}{2} w'(\omega(t))\,dt,$$

where $B(t)$ is a standard Brownian motion independent of the white noise $\{w'(x)\}$. We call the process $\mathbf{X} = \{\omega(t), t \geq 0, \mathscr{P}\}$ a Brownian motion with drift in a white noise environment; in [2] [6] [7] it is called a diffusion process in a Brownian environment with drift. The present authors obtained some limit theorems for $\mathbf{X}$ in [2] (see [8] for further results; see also [6] for a brief survey on related problems), which are analogous to those of [3] and [5]; however, some problems remain open. The present paper is a continuation of [7] and

1991 *Mathematics Subject Classification.* 60J60
Key words and phrases. Invariance principle, Brownian motion, Random environment.

discusses the central limit theorem, or more precisely speaking, invariance principle *in random environment* in the case $\kappa > 2$.

We set

$$M_x(= M(x)) = 2\int_0^x dy \int_{-\infty}^y e^{w_\kappa(y)-w_\kappa(z)}\, dz, x \in \mathbf{R},$$

$$\mu(t) = \text{the inverse function of } \{M_x, x \in \mathbf{R}\}, t \in \mathbf{R},$$

$$T_x(= T(x)) = \inf\{t > 0 : \omega(t) > x\}, x \geq 0.$$

We use also the following notation:

$$\bar{\omega}(t) = \max\{\omega(s) : 0 \leq s \leq t\}, \underline{\omega}(t) = \inf\{\omega(s) : s \geq t\}, (\omega \in \Omega),$$

$$\gamma = (\kappa - 1)/2, m = 4/(\kappa - 1) = 2/\gamma,$$

$$A = 64(\kappa - 1)^{-2}(\kappa - 2)^{-1} = 16\gamma^{-2}(2\gamma - 1)^{-1},$$

$$B = 64(\kappa - 1)^{-3}(\kappa - 2)^{-1} = 8\gamma^{-3}(2\gamma - 1)^{-1},$$

$$C = A + B = 64\kappa(\kappa - 1)^{-3}(\kappa - 2)^{-1} = 8(2\gamma + 1)\gamma^{-3}(2\gamma - 1)^{-1}.$$

The following theorem was proved in [7].

THEOREM A (*Environment-wise invariance principle, see* [7]). *When* $\kappa > 2$, *we have the following*:

(i) *For almost all* $w \in W$ *with respect to* P, *the process*

$$\left\{\frac{T_{\lambda x} - M_{\lambda x}}{\sqrt{A\lambda}}, x \geq 0, P_w\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$ (*in the sense of convergence of probability measures on the Skorohod space*).

(ii) *For almost all* w, *the process*

$$\left\{\frac{\omega(\lambda t) - \mu(\lambda t)}{\sqrt{m^{-3}A\lambda}}, t \geq 0, P_w\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$ (*in the sense of convergence of probability measures on* $C[0, \infty)$). *The same is true when* $\omega(\lambda t)$ *is replaced by either of* $\bar{\omega}(\lambda t)$ *and* $\underline{\omega}(\lambda t)$.

Our main theorems are the following ($\kappa > 2$ is assumed throughout).

THEOREM 1. (i) *The process*

$$\left\{\frac{M_{\lambda x} - \lambda m x}{\sqrt{B\lambda}}, x \in \mathbf{R}, P\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$.

(ii) *The process*

$$\left\{\frac{\mu(\lambda t) - \lambda m^{-1} t}{\sqrt{m^{-3} B\lambda}}, t \in \mathbf{R}, P\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$.

THEOREM 2 (*Invariance principle in random environment*). (i) *The process*

$$\left\{\frac{T_{\lambda x} - \lambda m x}{\sqrt{C\lambda}}, x \geq 0, \mathcal{P}\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$.

(ii) *The process*

$$\left\{\frac{\omega(\lambda t) - \lambda m^{-1} t}{m^{-3} C\lambda}, t \geq 0, \mathcal{P}\right\}$$

converges in law to a Brownian motion as $\lambda \to \infty$. *The same is true when* $\omega(\lambda t)$ *is replaced by either of* $\bar{\omega}(\lambda t)$ *and* $\underline{\omega}(\lambda t)$.

As in [7] we introduce a one parameter family of measure preserving transformations θ_t, $t \in \mathbf{R}$, on (W, P) defined by $(\theta_t w)(x) = w(x+t) - w(t)$, $x \in \mathbf{R}$. Clearly $\theta_t \theta_s = \theta_{t+s}$ and $\{\theta_t\}$ is ergodic. Set

$$f_0(w) = \int_{-\infty}^{0} e^{-w_\kappa(t)}\, dt. \tag{0.1}$$

Then $\theta_t f_0 \equiv f_0(\theta_t w) = \int_{-\infty}^{t} e^{w_\kappa(t) - w_\kappa(s)}\, ds$ and we have the following (see [7]):

$$E_w\{T_x\} = M_x = 2\int_0^x \theta_y f_0\, dy; \tag{0.2}$$

the first equality of (0.2) holds for $x \geq 0$ and the second one holds for $x \in \mathbf{R}$.

$$\mathrm{Var}_w\{T_x\} = 8\int_0^x \theta_y g\, dy \text{ for } x \geq 0 \quad \left(g(w) = \int_{-\infty}^{0} e^{-w_\kappa(t)} (\theta_t f_0)^2\, dt\right). \tag{0.3}$$

$$E\{f_0\} = \gamma^{-1}, E\{f_0^2\} = 2\gamma^{-1}(2\gamma - 1)^{-1}. \tag{0.4}$$

$$E\{\mathrm{Var}_w(T_x)\} = Ax \text{ for } x \geq 0. \tag{0.5}$$

$$\mathrm{Var}\{M_x\} = Bx + O(1), x \to \infty, (\mathrm{Var} = \text{variance}). \tag{0.6}$$

It was also observed in [7] that

$$d\theta_t f_0 = \theta_t f_0\, dw(t) - (\gamma \theta_t f_0 - 1)\, dt, \tag{0.7}$$

so $\theta_t f_0$ is a stationary diffusion process obtained as the unique stationary positive solution of the stochastic differential equation (0.7). Therefore

$$\theta_t f_0 - f_0 = \int_0^t \theta_s f_0 \, dw(s) - \lambda \int_0^t \theta_s f \, ds, \tag{0.8}$$

where $f = f_0 - \gamma^{-1}$.

1. Proof of Theorem 1

For the proof of Theorem 1 we need some lemmas.

LEMMA 1 ([7]). $t^{-1/2} \max\{\theta_s f_0 : |s| \le t\} \to 0$ as $t \to \infty$.

LEMMA 2 ([7]). *For any positive constants* c_1,

$$M_{t+u} - M_t = mu(1 + o(1)) + o(\sqrt{\lambda}), \quad |t| \le c_1 \lambda, \quad u \in \mathbf{R},$$

where $o(1)$ *represents a general term that tends to* 0 *as* $\lambda \to \infty$ *uniformly in* (t, u) *such that* $|t| \le c_1 \lambda$ *and* $u \in \mathbf{R}$, *for almost all* w; $o(\sqrt{\lambda})$ *is a term that can be expressed as* $o(1)\sqrt{\lambda}$.

To prove (i) of Theorem 1 it is enough to consider $\int_0^x \theta_y f \, dy$ by virtue of (0.2). Making habitual use of t to indicate time we write

$$\frac{1}{\sqrt{\lambda}} \int_0^{\lambda t} \theta_s f \, ds = \frac{1}{\gamma\sqrt{\lambda}} \int_0^{\lambda t} \theta_s f_0 \, dw(s) - \frac{1}{\gamma\sqrt{\lambda}} (\theta_{\lambda t} f_0 - f_0). \tag{1.1}$$

By the ergodicity of $\{\theta_t\}$ and also by (0.4) we see that the quadratic variation of the stochastic integral term in (1.1) tends to $Bt/4$ as $\lambda \to \infty$, a.s., so the stochastic integral term itself converges in law to $\{(B/4)^{1/2} w(t), t \ge 0, P\}$ as $\lambda \to \infty$. The second term of the right hand side of (1.1) is negligible by Lemma 1. Therefore $X_\lambda^+ = \{\lambda^{-1/2} \int_0^{\lambda t} \theta_s f \, ds, t \ge 0\}$ converges in law to $\{(B/4)^{1/2} w(t), t \ge 0\}$, so does $X_\lambda^- = \{\lambda^{-1/2} \int_0^{-\lambda t} \theta_s f \, ds, t \ge 0\}$ because of the reversibility of the diffusion $\theta_t f$. Now the assertion (i) of Theorem 1 follows from the fact that X_λ^+ and X_λ^- are asymptotically independent as $\lambda \to \infty$.

To proceed let $\xi = \lambda m$ and put

$$\beta_\lambda(t) = (B\lambda)^{-1/2}(M_{\lambda t} - \lambda m t), \quad \tilde{\beta}_\lambda(t) = (m^{-3} B \xi)^{-1/2}(\mu(\xi t) - \xi m^{-1} t). \tag{1.2}$$

Then the assertion of (ii) of Theorem 1 follows immediately from the following Lemma.

LEMMA 3. *For any* $t_0 > 0$ *and* $\varepsilon > 0$

$$\lim_{\lambda \to \infty} P\left\{ \sup_{|t| \le t_0} |\beta_\lambda(t) + \tilde{\beta}_\lambda(t)| > \varepsilon \right\} = 0.$$

PROOF. From the second equality of (1.2) we have $\lambda mt = M(\lambda t + \tilde{\beta}_\lambda(t)m^{-1}\sqrt{B\lambda})$ and hence

$$(B\lambda)^{-1/2}\{M(\lambda t + \tilde{\beta}_\lambda(t)m^{-1}\sqrt{B\lambda}) - M(\lambda t)\} = -\beta_\lambda(t),$$

so an application of Lemma 2 yields $\tilde{\beta}_\lambda(t)(1 + o(1)) + o(1) = -\beta_\lambda(t)$, where $o(1)$ is a term tending to 0 uniformly on each finite t-interval as $\lambda \to \infty$, a.s. This implies the lemma.

The following observation will lead to another proof of (ii) of Theorem 1. If $\nu(t)$ denotes the inverse function of $\int_0^x \theta_y f_0\,dy$, then $\nu(t) = \mu(2t)$ and the derivative $\nu'(t)$, which equals to $1/\theta_{\nu(t)} f_0$, is a stationary diffusion process obtained from $\theta_t f_0$ by changing time and scale.

2. The proof of Theorem 2

We give the proof of the part (i). Taking an arbitrary positive sequence $\{\lambda_n, n = 1, 2, \ldots\}$ tending to ∞, we denote by $P^{(n)}$ the probability law of the process $\{\lambda_n^{-1/2}(T_{\lambda_n x} - \lambda_n mx), x \geq 0, \mathscr{P}\}$. Note that $P^{(n)}$ is a probability measure on the Skorohod space $D = D[0, \infty)$. For the proof of the part (i) it is enough to show that $P^{(n)}$ converges to the probability law of the process $\{\sqrt{C}w(x), x \geq 0, P\}$ as $n \to \infty$. We first prove that the sequence $\{P^{(n)}, n \geq 1\}$ is tight. If $Q_w^{(n)}$ denotes the probability law of the process $\{\lambda_n^{-1/2}(T_{\lambda_n x} - M_{\lambda_n x}), x \geq 0, P_w\}$, then $Q_w^{(n)} \to Q_1$ (P-a.s.) by Theorem A and hence $Q^{(n)} = \int Q_w^{(n)} P(dw)$ also converges to Q_1 as $n \to \infty$ where Q_1 is the probability law (on D) of the process $\{\sqrt{A}w(x), x \geq 0, P\}$. All the convergence here is to be understood as the convergence of probability measures on D. Therefore for any $\varepsilon > 0$ there exists a compact set $K_1 \subset D$ such that $Q^{(n)}(K_1^c) < \varepsilon^2$ for all $n \geq 1$. We then have $P\{L_n\} < \varepsilon$ where $L_n = \{w : Q_w^{(n)}(K_1^c) > \varepsilon\} = \{w : Q_w^{(n)}(K_1) \leq 1 - \varepsilon\}$. We also introduce, for each fixed w, an element $\varphi_n(w)$ of Ω which is defined to be the function $\lambda_n^{1/2}(M_{\lambda_n x} - \lambda_n mx)$ of x. Then $P \circ \varphi_n^{-1} \to Q_2$ as $n \to \infty$ by Theorem 1 where Q_2 is the probability law (on Ω) of the process $\{\sqrt{B}w(x), x \geq 0, P\}$. We can thus find a compact set $K_2 \subset \Omega$ such that $P \circ \varphi_n^{-1}(K_2^c) < \varepsilon$ for all $n \geq 1$. We now put $K = \{w_1 + w_2 : w_1 \in K_1, w_2 \in K_2\}$. Then K is a compact subset of D. Since

$$\frac{T_{\lambda_n x} - \lambda_n mx}{\sqrt{\lambda_n}} = \frac{T_{\lambda_n x} - M_{\lambda_n x}}{\sqrt{\lambda_n}} + \frac{M_{\lambda_n x} - \lambda_n mx}{\sqrt{\lambda_n}},$$

we have

$$\begin{aligned} P^{(n)}(K) &= \iint 1_K(w_1 + \varphi_n(w))Q_w^{(n)}(dw_1)P(dw) \\ &\geq \iint 1_{K_1}(w_1)1_{K_2}(\varphi_n(w))Q_w^{(n)}(dw_1)P(dw) \\ &= \int_{\varphi_n^{-1}(K_2)} Q_w^{(n)}(K_1)P(dw) \\ &\geq \int_{\varphi_n^{-1}(K_2)\cap L_n^c} (1-\varepsilon)P(dw) \geq (1-\varepsilon)(1-2\varepsilon), \end{aligned}$$

which proves that $\{P^{(n)}, n \geq 1\}$ is tight. Therefore, for the proof of Theorem 2 it is enough to show that

$$\lim_{n\to\infty} \int f(w)P^{(n)}(dw) = \iint f(w_1 + w_2)Q_1(dw_1)Q_2(dw_2), \tag{2.1}$$

for any function f of the form

$$f(w) = \exp\left\{\sqrt{-1}\sum_{j=1}^{k} \alpha_j w(t_j)\right\},$$

where $\alpha_j \in \mathbf{R}$ and $t_j \geq 0$, $1 \leq j \leq k$. For such an f the left hand side of (2.1) equals

$$\begin{aligned} &\lim_{n\to\infty} \iint f(w_1 + \varphi_n(w))Q_w^{(n)}(dw_1)P(dw) \\ &= \lim_{n\to\infty} \int_W \left\{\int_D f(w_1)Q_w^{(n)}(dw_1)\right\} f(\varphi_n(w))P(dw) \\ &= \int f(w_1)Q_1(dw_1) \int f(w_2)Q_2(dw_2) \end{aligned}$$

which also equals the right hand side of (2.1). This completes the proof of (i) of Theorem 2.

The part (ii) of Theorem 2 can be proved in a way similar to the above by making use of Theorem A and Theorem 1.

3. Supplement to the proof of (i) of Theorem A

The proof of Theorem A was given in [7]; however, some details in the proof of the part (i) were omitted. It will be worth supplementing them.

The proof of Theorem A given in [7] proceeds as follows. Let $\tau_k =$

$T_k - T_{k-1}$, $\tilde{\tau}_k = \tau_k - E_w\{\tau_k\}$, $k \geq 1$. Then it was proved that, for almost all w, $\{\tilde{\tau}_k, k \geq 1, P_w\}$ is a sequence of independent random variables satisfying the Lindeberg condition. Therefore the central limit theorem holds for T_n with respect to P_w, for almost all w. Note that $E_w\{T_n\} = M_n$ and $\mathrm{Var}_w\{T_n\} \sim An$ as $n \to \infty$ (P-a.s.). Now the rest of the proof, whose detail was omitted in [7], is given as follows.

Let $t_{nk} = \mathrm{Var}_w\{\tau_k\}/\mathrm{Var}_w\{T_n\}$, $\zeta_{nk} = (An)^{-1/2}\{T_k - M_k\}$, $1 \leq k \leq n$, and $t_{n0} = \zeta_{n0} = 0$. For each fixed w we construct a piece-wise linear function $\xi_n(x)$, $0 \leq x \leq 1$, with vertexes $(\sum_{j=0}^k t_{nj}, \zeta_{nk})$, $0 \leq k \leq n$. We regard $\{\xi_n(x), 0 \leq x \leq 1, P_w\}$ as a process with time parameter x. Then by Theorem 3.1 of [5], for almost all w, the process $\{\xi_n(x), 0 \leq x \leq 1, P_w\}$ converges in law to a Brownian motion as $n \to \infty$. We now modify $\xi_n(x)$ slightly, namely, we consider a piece-wise linear function $\eta_n(x)$ with vertexes $(k/n, \zeta_{nk})$, $0 \leq k \leq n$. Then $\eta_n(x)$ can be represented as $\eta_n(x) = \xi_n(\varphi_n(x))$ where $\varphi_n(x)$ is the piece-wise linear function with vertexes $\left(k/n, \sum_{j=0}^{k} t_{nj}\right)$, $0 \leq k \leq n$. On the other hand it is easy to see that, for each fixed x, $\varphi_n \to x$ as $n \to \infty$ for almost all w. This combined with the fact that φ is increasing implies that $\varphi_n \to x$ uniformly as $n \to \infty$ (P-a.s.). Therefore

(3.1) the process $\{\eta_n(x), 0 \leq x \leq 1, P_w\}$ converges in law to a Brownian motion as $n \to \infty$ for almost all w.

We finally prove that the process $\{(A\lambda)^{-1/2}(T_{\lambda x} - M_{\lambda x}), x \in [0,1], P_w\}$ converges in law to a Brownian motion as $\lambda \to \infty$ for almost all w; the time interval $[0,1]$ can be replaced by an arbitrary interval $[0, t_0]$ with a minor modification of the proof. Given $x \in (0,1]$ and an integer $n \geq 1$ we take the integer k such that $(k-1)/n < x \leq k/n$. Then $T_{nx} - M_{nx} > T_{k-1} - M_k > \sqrt{An}\eta_n(x) - \tau_k - m_k$ where $m_k = M_k - M_{k-1}$. Similarly $T_{nx} - M_{nx} < \sqrt{An}\eta_n(x) + \tau_k + m_k$ and hence

$$\sqrt{An}\eta_n(x) - (\tau_k + m_k) < T_{nx} - M_{nx} < \sqrt{An}\eta_n(x) + (\tau_k + m_k).$$

This implies that for $x \in [0,1]$

$$\sqrt{An}\eta_n(x) - (\hat{\tau}_n + \hat{m}_n) < T_{nx} - M_{nx} < \sqrt{An}\eta_n(x) + (\hat{\tau}_n + \hat{m}_n), \tag{3.2}$$

where $\hat{\tau}_n = \max\{\tau_k : 1 \leq k \leq n\}$ and $\hat{m}_n = \max\{m_k : 1 \leq k \leq n\}$. Next, given $\lambda > 0$ let $n = n(\lambda)$ be the integer such that $n - 1 < \lambda \leq n$. Then $T_{(n-1)x} - M_{\lambda x} < T_{\lambda x} - M_{\lambda x} \leq T_{nx} - M_{\lambda x}$, which combined with (3.2) implies

$$\sqrt{A(n-1)}\eta_{n-1}(x) - (\hat{\tau}_{n-1} + \hat{m}_{n-1}) - (M_{\lambda x} - M_{(n-1)x})$$
$$< T_{\lambda x} - M_{\lambda x} < \sqrt{An}\eta_n(x) + (\hat{\tau}_n + \hat{m}_n) + (M_{nx} - M_{\lambda x}).$$

Since $M_{\lambda x} - M_{(n-1)x}$ and $M_{nx} - M_{\lambda x}$ are dominated by $2\hat{m}_n$, we obtain

$$(3.3)\qquad \left(\frac{n-1}{\lambda}\right)^{1/2}\eta_{n-1}(x) - (A\lambda)^{-1/2}(\hat{\tau}_n + 3\hat{m}_n) < (A\lambda)^{-1/2}(T_{\lambda x} - M_{\lambda x})$$
$$< \left(\frac{n}{\lambda}\right)^{1/2}\eta_n(x) + (A\lambda)^{-1/2}(\hat{\tau}_n + 3\hat{m}_n).$$

On the other hand we can prove that for almost all w

$$(3.4)\qquad P_w\left\{\lim_{n\to\infty} \hat{\tau}_n/\sqrt{n} = 0\right\} = 1,$$

$$(3.5)\qquad \lim_{n\to\infty} \hat{m}_n/\sqrt{n} = 0.$$

In fact, it is easy to see that $\{\tau_k, k \geq 1, \mathscr{P}\}$ is stationary and ergodic. Since τ_k^2 is integrable we have $n^{-1}\sum_1^n \tau_k^2 \to \text{const.}$ as $n \to \infty$ ($\mathscr{P}$-a.s.) and hence $n^{-1}\tau_n^2 \to 0$, namely, $n^{-1/2}\tau_n \to 0$ ($\mathscr{P}$-a.s.). This implies $n^{-1/2}\hat{\tau}_n \to 0$ ($\mathscr{P}$-a.s.) and hence (3.4). (3.5) can be proved in a similar manner. By virtue of (3.1), (3.4) and (3.5) the processes of the leftmost and rightmost hands of (3.3) converge in law to a Brownian motion as $\lambda \to \infty$. Therefore (3.3) implies the assertion for $(A\lambda)^{-1/2}(T_{\lambda x} - M_{\lambda x})$ that we wanted to prove.

References

[1] K. Itô and H. P. McKean, Diffusion Processes and Their Sample Paths, Springer-Verlag, New York, 1965.

[2] K. Kawazu and H. Tanaka, A diffusion process in a Brownian environment with drift, J. Math. Soc. Japan **49** (1997), 189–211.

[3] H. Kesten, M. V. Kozlov and F. Spitzer, A limit law for random walk in a random environment, Composito Math. **30** (1975), 145–168.

[4] Yu. V. Prohorov, Convergence of random processes and limit theorems in probability theory, Theor. Probab. Appl. **1** (1956), 157–214.

[5] F. Solomon, Random walks in a random environment, Ann. Probab. **3** (1975), 1–131.

[6] H. Tanaka, Diffusion processes in random environments, Proc. International Congress of Mathematicians, Zürich, 1994, 1047–1054, Birkhäuser Verlag, Basel, 1995.

[7] H. Tanaka, Environment-wise central limit theorem for a diffusion in a Brownian environment with large drift, Itô's Stochastic Calculus and Probability Theory, ed. by N. Ikeda et al., pp. 373–384, Springer, 1996.

[8] H. Tanaka, Limit theorems for a Brownian motion with drift in a white noise environment, Chaos, Solitons & Fractals, **8** (1997), 1807–1816.

K. Kawazu
Department of Mathematics
Faculty of Education
Yamaguchi University
Yosida, Yamaguchi 753-8513
Japan

H. Tanaka
Department of Mathematics
Faculty of Science and
Technology
Keio University
Yokohama 223-0061
Japan