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Non-projective compactifications of $\mathbf{C}^3$ III: A remark on indices

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ABSTRACT. Let (X, Y) be a non-projective Moishezon compactification of $\mathbf{C}^3$ with $b_2(X) = 1$. Then we have $K_X = -rY$ $(0 < r \in Z)$. In this paper, we prove $1 \leq r \leq 2$.

1. Introduction

This is a continuation of my previous papers [4] and [5]. Let (X, Y) be a smooth non-projective Moishezon compactification of $\mathbf{C}^3$ with the second Betti number equal to one, that is, X is a smooth non-projective Moishezon threefold and Y is an irreducible divisor on X such that $X - Y$ is biholomorphic to $\mathbf{C}^3$. It is well-known that Y is a non-normal and non-projective irreducible algebraic surface and that the canonical bundle K_X can be written as $K_X = -rY$ for $0 < r \in \mathbf{Z}$ (cf. [1], [7]). The positive interger $r = r(X, Y)$ is called the index of the compactification (X, Y). Now we have two cases (i) Y is *nef* or (ii) Y is *not-nef.* Then we obtained the following:

(i) If Y is *nef*, then we have $1 \leq r \leq 2$. When $r = 2$, the complete structure of (X, Y) is given in Theorem 0.3 in [4]. In the case when $r = 1$, we know only one example (see Theorem A in [5]).

(ii) If Y is *not-nef*, then there exist infinitely many examples with $1 \leq r \leq 2$ (see Theorem B in [5]).

In this paper, we shall prove the following:

THEOREM. *Let (X, Y) be a non-projective Moishezon compactification of $\mathbf{C}^3$ with the second Betti number $b_2(X) = 1$. Then we have $1 \leq r(X, Y) \leq 2$.*

2. Proof of Theorem

Let (X, Y) be a smooth non-projective Moishezon compactification of $\mathbf{C}^3$ with $b_2(X) = 1$. Then we have the following:

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LEMMA 1. (cf. [2], [3], [7])

(1) *Y is a non-normal irreducible Cartier divisor on X.*

(2) $H^i(X;\mathbf{Z}) \cong H^i(Y;\mathbf{Z})$, $H_i(Y;\mathbf{Z}) \cong H_i(X;\mathbf{Z})$ *for* $i > 0$.

(3) $H^1(X;\mathbf{Z}) = H^1(Y;\mathbf{Z}) = 0$.

(4) $H^2(X;\mathbf{Z}) = \mathbf{Z}c_1(\mathcal{O}_X(Y))$ *and* $H^2(Y;\mathbf{Z}) = \mathbf{Z}c_1(N_Y)$, *where* $N_Y := \mathcal{O}_Y(Y)$.

(5) $H^i(X;\mathcal{O}_X) = 0$ *for* $i > 0$.

(6) $H^0(X;\mathcal{O}_X(mK_X)) = 0$ *for* $m > 0$.

(7) $H^1(Y;\mathcal{O}_Y) = 0$, $H^2(Y;\mathcal{O}_Y) = 0$ (*resp.* $\mathbf{C}$) *if* $r \geq 2$ (*resp.* $r = 1$).

(8) $\operatorname{Pic} X \cong \mathbf{Z}\mathcal{O}_X(Y)$ *and* $\operatorname{Pic} Y \cong \mathbf{Z}N_Y$.

(9) $K_X = -rY$ *and* $K_Y = -(r-1)N_Y$, *where* $0 < r \in \mathbf{Z}$.

Let $\varphi : V \to X$ be the projectivization of X, that is, V is a smooth projective algebraic threefold and φ is a bimeromorphic holomorphic mapping. Let $\nu : \tilde{Y} \to Y$ be the normalization and $\mathcal{I}$ be the conductor ideal sheaf defining closed subscheme E on Y. Let $\mu : \hat{Y} \to \tilde{Y}$ be the minimal resolution with the exceptional divisor $\Delta = \bigcup \Delta_i$. Then $\hat{Y}$ is a projective algebraic surface. We set $\eta := \nu \circ \mu : \hat{Y} \to Y$. Since $\nu_* \omega_{\tilde{Y}} = \mathcal{I} \otimes \omega_Y$, we have $K_{\tilde{Y}} = -(r-1)\nu^* N_Y - \tilde{E}$, where $\tilde{E}$ is an effective Weil divisor on $\tilde{Y}$ (cf. p. 166 in [6]). Thus we have $K_{\hat{Y}} = -(r-1)\eta^* N_Y - \hat{E} - \sum_i m_i \Delta_i$ ($m_i \in \mathbf{Z}, m_i \geq 0$), where $\hat{E}$ is the proper transform of $\tilde{E}$ in $\hat{Y}$ (cf. [2]).

LEMMA 2. *$\hat{Y}$ is a ruled surface unless $\hat{Y} \cong \mathbf{P}^2$.*

PROOF. We have $K_{\hat{Y}} = -\hat{E} - \sum_i m_i \Delta_i$ if $r = 1$. Since $\hat{E}$ is an effective divisor, we obtain $H^0(\hat{Y};\mathcal{O}_{\hat{Y}}(kK_{\hat{Y}})) = 0$ for $k > 0$. Let A be a very ample irreducible divisor on V and put $D = \varphi_* A$. By Lemma 1-(8), there is an integer $k \in \mathbf{Z}, k > 0$ such that $D = kY$ and then the divisor $D|_Y$ consists of effective curves. Then $kK_{\hat{Y}} = -(r-1)\eta^* D|_Y - K\hat{E} - k\sum_i m_i \Delta_i$ is an effective divisor. Thus $H^0(\hat{Y};\mathcal{O}_Y(kK_{\hat{Y}})) = 0$ for $k \gg 0$. By the classification of algebraic surfaces, $\hat{Y}$ is isomorphic to either $\mathbf{P}^2$ or a ruled surface, that is, there exists a $\mathbf{P}^1$-fibration $\pi : \hat{Y} \to C$, where C is a smooth projective curve with the genus $h^1(\mathcal{O}_{\hat{Y}}) \geq 0$. □

In the case when $\hat{Y} \not\cong \mathbf{P}^2$, take a general fiber $\hat{f}$ of π. By the adjunction formula, we have

$$(*) \qquad -2 = (K_{\hat{Y}} \cdot \hat{f}) = -(r-1)(\eta^* N_Y \cdot \hat{f}) - (\hat{E} \cdot \hat{f}) - \sum_i m_i (\Delta_i \cdot \hat{f}).$$

LEMMA 3. *$(\eta^* N_Y \cdot \hat{f}) > 0$ for any general fiber $\hat{f}$ of π.*

PROOF. Let $\overline{Y}$ be the proper transform of Y in V. We set $f = \eta(\hat{f}) \subset Y$. Let $\bar{f}$ be the proper transform of f in $\overline{Y}$. Take a very ample irreducible divisor A on V with $\bar{f} \not\subset A$ and set $D = \varphi_* A$. Then we have $(D \cdot f) > 0$. Since

$D \sim kY$ for some $0 < k \in \mathbf{Z}$, we obtain $(Y \cdot f) = (N_Y \cdot f) > 0$. This proves the lemma. $\square$

LEMMA 4. *If* $\hat{Y} \cong \mathbf{P}^2$, *then* Y *is ample.*

PROOF. Since there is no exceptional curve on $\mathbf{P}^2$, one sees that $\hat{Y} \cong \tilde{Y} \cong \mathbf{P}^2$. By an argument similar to Lemma 3, one has $(v^* N_Y \cdot \ell) > 0$ for a general line ℓ on $\tilde{Y} \cong \mathbf{P}^2$. This shows that $v^* N_Y$ is ample. Since $X - Y \cong \mathbf{C}^3$, Y is ample by Kleiman's criterion. $\square$

We are in a position to prove Theorem. We have only to consider the case where Y is *not-nef* (see (i) in Introduction). Then we have $\hat{Y} \not\cong \mathbf{P}^2$ by Lemma 4. Assume that $r \geq 3$. By Lemma 3 and the relation $(*)$, we obtain that $r = 3$, $(\eta^* N_Y \cdot \hat{f}) = 1$, $(\hat{E} \cdot \hat{f}) = 0$ and $\sum_i m_i(\Delta_i \cdot \hat{f}) = 0$. This shows that $f \cap E = \varnothing$ and f passes through at worst rational double points on $Y - E$. Thus there is an integer n such that $nf \in \operatorname{Pic} Y$. Since $\operatorname{Pic} Y \cong \mathbf{Z} N_Y$, one has $nf = aN_Y$ for some $a \in \mathbf{Z}$. If f does not pass through any rational double point on $Y - E$, then $f \cong \mathbf{P}^1$ is a smooth Cartier divisor with $f^2 = 0$. Since $(N_Y \cdot f) > 0$, we have $a \neq 0$. Then we have $0 = f^2 = a(N_Y \cdot f) \neq 0$. This is a contradiction. Therefore we may assume that f passes through a rational double point y_0 on $Y - E$. Then there exists an irreducible component Δ_i of $\eta^{-1}(y_0) \subset \Delta$ such that $(\Delta_i \cdot \hat{f}) > 0$. Take a general fiber $\hat{f}_0 \neq \hat{f}$ of π such that $(N_Y \cdot f_0) > 0$ and $f_0 \cap E = \varnothing$, where $f_0 = \eta(\hat{f}_0)$. Since $(\Delta_i \cdot \hat{f}_0) = (\Delta_i \cdot f) > 0$, we have $y_0 \in f_0 \cap f$. Thus we have $0 < (nf \cdot f_0) = a(N_Y \cdot f_0)$. This implies $a > 0$. On the other hand, since Y is *not-nef*, there exists an irreducible curve B such that $(Y \cdot B) < 0$, that is, $B \subset Y$ and $(N_Y \cdot B) < 0$. Thus we have $0 \leq (nf \cdot B) = a(N_Y \cdot B) < 0$. This is a contradiction. Therefore we conclude that $r \leq 2$. This completes the proof of the theorem.

References

[1] L. Brenton, Some algebraicity criteria for singular surfaces, Invent. Math. **41** (1977), 129–147.

[2] M. Furushima, The complete classification of compactifications of $\mathbf{C}^3$ which are projective manifolds with second Betti number equal to one, Math. Ann. **297** (1993), 627–662.

[3] M. Furushima, An example of a non-projective smooth compactification of $\mathbf{C}^3$ with second Betti number equal to one, Math. Ann. **300** (1994), 89–96.

[4] M. Furushima, Non-projective compactifications of $\mathbf{C}^3$ (I), Kyushu J. Math. **50** (1996), 221–239.

[5] M. Furushima, Non-projective compactifications of $\mathbf{C}^3$ II: New Examples, Kyushu J. Math. **52** (1998), 149–162.

[6] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. Math. **116** (1982), 133–176.

[7] T. Peternell and M. Schneider, Compactifications of $\mathbf{C}^n$: A survey, In: Bedford, E (ed. et al) Several complex variables and complex geometry. (Proc. Symp. Pure Math., vol. **52**, pp. 455–466) Amer. Math. Soc., Providence, RI 1991.

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