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A splitting theorem for rank two vector bundles on projective spaces in positive characteristic

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ABSTRACT. We shall prove the following splitting theorem for rank two vector bundles E on the n-dimensional projective space $\mathbf{P}^n$ $(n \geq 4)$ in positive characteristic. Let P be a 4- or 5-dimensional projective linear subspace of $\mathbf{P}^n$ and $\bar{E} = E|P$ the restriction of E to P. Then E splits into line bundles if and only if the first cohomology of the sheaf of endomorphisms of $\bar{E}$ vanishes.

0. Introduction

Let E be a rank two vector bundle on the n-dimensional projective space $\mathbf{P}_k^n$ $(n \geq 4)$ defined over an algebraically closed field k.

In [4], H. Sumihiro showed the following theorem in the case of char $k = 0$.

THEOREM 0.1. *Let P be a 4- or 5-dimensional projective linear subspace of $\mathbf{P}_k^n$ and $\bar{E} = E|P$ the restriction of E to P. Then E splits into line bundles if and only if $H^1(P, \mathscr{E}nd(\bar{E})) = 0$.*

The aim of this article is to prove that this theorem holds also true in char $k = p > 0$. The proof is almost the same as the one for char $k = 0$, namely, it is obtained by studying some geometric structures of the Hilbert scheme of $\mathbf{P}_k^n$ at determinantal subvarieties. In char $k = p > 0$, however, since we cannot use the Kodaira vanishing theorem and the Le-Potier vanishing theorem (cf. [1], [3]), we have to observe some vanishings of cohomologies appearing in [4] carefully.

1. Preliminaries

We first recall the definition and some properties of determinantal varieties associated to rank two bundles (cf. [4]).

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1.1. Definition of determinantal varieties. Let E be a rank two vector bundle on $\mathbf{P}_k^n$ defined over an algebraically closed field k with arbitrary characteristic, $\pi : P(E) \to \mathbf{P}_k^n$ the projective bundle associated to E over $\mathbf{P}_k^n$, L_E the tautological line bundle on $P(E)$ and let $G = \mathrm{Grass}(H^0(E), m+1)$ be the Grassmann variety which parametrizes $(m+1)$-dimensional linear subspaces of $H^0(\mathbf{P}_k^n, E)$, where $n = 2m$ (resp. $n = 2m+1$). We assume that E is very ample, i.e., L_E is a very ample line bundle. Then we can take $s = \langle s_1, s_2, \ldots, s_{m+1} \rangle \in G$ $(s_i \in H^0(\mathbf{P}_k^n, E))$ satisfying the following condition

$(*)$
1) $Y_s = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a smooth closed subscheme of $P(E)$ of pure codimension $m+1$,
2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \varnothing$,

where D_i is the tautological divisor on $P(E)$ defined by s_i and $W(s_i)$ is the zero locus of s_i in $\mathbf{P}_k^n$ $(1 \le i \le m+1)$.

Let $X_s = \pi(Y_s)$. Then we can show that X_s is a closed subscheme of $\mathbf{P}_k^n$ which is isomorphic to Y_s through π with the following defining equations:

$$s_i \wedge s_j = 0 \qquad (1 \le i \le j \le m+1).$$

DEFINITION 1.1. *We call the closed subscheme X_s of $\mathbf{P}_k^n$ the determinantal variety associated to E defined by $s \in G$.*

Though X_s depends on the choice of $s \in G$, we call a closed subvariety X_s a determinantal variety associated to E.

As for determinantal varieties, we obtain the following.

THEOREM 1.1. *Let the notaion be as above.*
1) *$U = \{s \in G \mid s$ satisfies the condition $(*)\}$ is a Zariski open subset of G.*
2) *There exists a closed subscheme Ξ of $\mathbf{P}_k^n \times U$ such that the second projection $q : \Xi \subset \mathbf{P}_k^n \times U \to U$ is faithfully flat and $X_s = q^{-1}(s)$ for any $s \in U$. Thus smooth determinantal varieties associated to E form a smooth family over an open subset of G.*

When $n = 4$ or 5, let I_X be the defining ideal of a determinantal subvariety X in $\mathbf{P}^n$. Then I_X has the following resolution by vector bundles.

LEMMA 1.2. *In the above notation, there exists an exact sequence*

$$0 \to E^*(-c_1) \to \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}(-c_1) \to I_X \to 0,$$

where c_1 is the first Chern number of E and E^* is the dual bundle of E.

PROOF. Let $s = \{s_1, s_2, s_3\}$ be a set of global sections of E which defines the determinantal subvariety X. Then we can define homomorphisms

$$\alpha : \bigoplus^{3} \mathcal{O}_{\mathbf{P}^n} \ni e_i \wedge e_j \mapsto s_i \wedge s_j \in \bigwedge^{2} E \qquad (1 \le i < j \le 3),$$

$$\beta : E^* \ni f \mapsto f(s_3)e_1 \wedge e_2 - f(s_2)e_1 \wedge e_3 + f(s_1)e_2 \wedge e_3 \in \bigoplus^{3} \mathcal{O}_{\mathbf{P}^n},$$

where $\{e_i \wedge e_j\}$ is a basis of $\bigoplus^{3} \mathcal{O}_{\mathbf{P}^n}$. Then it suffices to verify locally on $\mathbf{P}^n$ that the following sequence is exact:

$$0 \to E^* \xrightarrow{\beta} \bigoplus^{3} \mathcal{O}_{\mathbf{P}^n} \xrightarrow{\alpha} I_X \otimes \mathcal{O}(c_1) \to 0.$$

□

1.2. Tangent bundles and normal bundles of determinantal varieties. In the following subsections, we consider the case $n = 4$ or 5, i.e., $m = 2$.

Let E be a very ample rank two bundle on $\mathbf{P}_k^n$ and X a determinantal variety associated to E which is isomorphic through π to the complete intersection Y in $P(E)$ of the tautological divisors $\{D_i \mid i = 1, 2, 3\}$.

Let H be the restriction of a hyperplane of $\mathbf{P}^n$ to X and D the restriction of a tautological divisor of $P(E)$ to X through the isomorphism π.

Then we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_{P(E)/\mathbf{P}^n}|Y & \xrightarrow{\sim} & \mathcal{O}_X(2D - c_1H) & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & T_Y & \longrightarrow & T_{P(E)}|Y & \longrightarrow & N_{Y/P(E)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbf{P}^n}|X & \longrightarrow & N_{X/\mathbf{P}^n} & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where α is the injection induced by the snake lemma. Since $N_{Y/P(E)} \simeq \bigoplus^{3} \mathcal{O}_X(D)$, we obtain the following.

PROPOSITION 1.3. *There exists an exact sequence*

$$0 \to \mathcal{O}_X(2D - c_1H) \to \bigoplus^{3} \mathcal{O}_X(D) \to N_{X/\mathbf{P}^n} \to 0.$$

1.3. Hilbert Schemes. Let $\mathscr{H}ilb$ be the Hilbert scheme of $\mathbf{P}^n$. Let $\varphi : U \ni s \mapsto X_s \in \mathscr{H}ilb$ be the morphism induced by Theorem 1.1. Let $\mathrm{Aut}(E)$ be the automorphism group of E. Then $\mathrm{Aut}(E)$ is a reduced connected linear algebraic group of dimension $\dim H^0(\mathscr{E}nd(E))$.

For every element $g \in \mathrm{Aut}(E)$ and $s = \langle s_1, s_2, s_3 \rangle \in G$, we define

$$g \cdot s = \langle g(s_1), g(s_2), g(s_3) \rangle,$$

where $g(s_i)$ is the composite of s_i with g. Then it defines an action of $\mathrm{Aut}(E)$ on G and we have

$$g \cdot s_i \wedge g \cdot s_j = \det(g) s_i \wedge s_j \qquad (1 \leq i \leq j \leq 3),$$

where $\det : \mathrm{Aut}(E) \ni g \mapsto \det(g) \in k^* = k \backslash \{0\}$ is the determinant character. Hence $X_{g \cdot s} = X_s$. Therefore $\mathrm{Aut}(E)$ acts on U and φ is an orbit morphism, i.e., φ is constant on any orbit $O(s) = \{g \cdot s \mid g \in \mathrm{Aut}(E)\}$.

Then we have the following.

Lemma 1.4. *The stabilizer* $\mathrm{Stab}(s)$ *of* $s \in U$ *coincides with the multiplicative group* k^*.

As a trivial corollary of the above lemma, we observe that every orbit has the same dimension $\dim \mathrm{Aut}(E)/k^*$, i.e., $\dim O(s) = \dim H^0(\mathscr{E}nd(E)) - 1$ $(s \in U)$. Hence the action of $\mathrm{Aut}(E)$ on U is closed, i.e., every orbit is closed in U.

2. Proof of the theorem

2.1. Since it is well-known that E splits into line bundles if and only if $\bar{E} = E|P$ splits into line bundles, where P is a 4- or 5-dimensional linear subspaces of $\mathbf{P}^n$, we may assume that E is a rank two vector bundle on $\mathbf{P}^n$ (n being either 4 or 5) (cf. [2]). In addition after multiplying E by a suitable ample line bundle, we may assume that E is a very ample vector bundle enjoying $H^i(E \otimes K_{\mathbf{P}^n}) = 0$ $(1 \leq i \leq 4)$, where $K_{\mathbf{P}^n}$ is the canonical line bundle of $\mathbf{P}^n$.

By Proposition 1.3, we have the following exact sequence

$$0 \to H^0(\mathscr{O}_X(2D - c_1 H)) \to \bigoplus^3 H^0(\mathscr{O}_X(D)) \to H^0(N_{X/\mathbf{P}^n})$$
$$\to H^1(\mathscr{O}_X(2D - c_1 H)) \to \bigoplus^3 H^1(\mathscr{O}_X(D)).$$

Now we recall $Y = D_1 \cap D_2 \cap D_3$. Consider the canonical exact sequence

$$(*)_1 \qquad 0 \to \mathscr{O}_{P(E)}(D - c_1 H) \to \mathscr{O}_{P(E)}(2D - c_1 H) \to \mathscr{O}_{D_1}(2D - c_1 H) \to 0,$$

from which we obtain the following exact sequence:

$$\begin{aligned}0 &\to H^0(\mathcal{O}_{P(E)}(D - c_1H)) \to H^0(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^0(\mathcal{O}_{D_1}(2D - c_1H))\\ &\to H^1(\mathcal{O}_{P(E)}(D - c_1H)) \to H^1(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^1(\mathcal{O}_{D_1}(2D - c_1H))\\ &\to H^2(\mathcal{O}_{P(E)}(D - c_1H)).\end{aligned}$$

Since $H^i(\mathcal{O}_{P(E)}(D - c_1H)) = H^i(E^*)$ $(0 \le i \le 4)$ and we can show that $H^0(E^*) = 0$ and $H^i(E^*) = H^{n-i}(E \otimes K_{\mathbf{P}^n}) = 0$ $(i = 1, 2)$ by our assumption, it turns out that $H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1}(2D - c_1H))$ $(i = 0, 1)$.

In addition considering the following exact sequences similarly

$$(*)_2 \qquad \begin{aligned}&0 \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to 0,\\ &0 \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to 0,\end{aligned}$$

$$(*)_3 \qquad \begin{aligned}&0 \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to \mathcal{O}_Y(2D - c_1H) \to 0,\\ &0 \to \mathcal{O}_{D_1}(-c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to 0,\\ &0 \to \mathcal{O}_{P(E)}(-D - c_1H) \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{D_1}(-c_1H) \to 0,\end{aligned}$$

we obtain isomorphisms $H^i(\mathcal{O}_{D_1}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H))$ and $H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H)) \simeq H^i(\mathcal{O}_Y(2D - c_1H))$ $(i = 0, 1)$ because $H^i(\mathcal{O}_{P(E)}(-D - c_1H)) = 0$ $(0 \le i \le 4)$. Summing up the above, we conclude that $H^i(\mathcal{O}_X(2D - c_1H)) \simeq H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \simeq H^i(\mathbf{P}^n, S^2(E)(-c_1))$ $(i = 0, 1)$.

On the other hand, since there exists an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{E}nd(E) \to S^2(E)(-c_1) \to 0,$$

we have a canonical isomorphism $H^1(S^2(E)(-c_1)) \simeq H^1(\mathcal{E}nd(E))$ and $\dim H^0(S^2(E)(-c_1)) = \dim H^0(\mathcal{E}nd(E)) - 1$.

Moreover we easily see that $\dim H^0(\mathcal{O}_X(D)) = \dim H^0(E) - 3$.

Summarizing the above, we get the following proposition.

Proposition 2.1. *With the above assumption, if* $H^1(\mathcal{E}nd(E)) = 0$, *then*

$$\dim H^0(N_{X/\mathbf{P}^n}) = 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1.$$

Remark 2.1. *When char* $k = 0$, *we get* $H^i(E^*) \simeq H^{n-i}(E \otimes K_{\mathbf{P}_k^n}) = 0$ *for* $0 \le i \le n - 2$ *by the Le-Potier vanishing theorem. So we do not need the assumption* $H^i(E \otimes K_{\mathbf{P}^n}) = 0$ $1 \le i \le 4$ *in Proposition* 2.1. *Also the proof itself becomes slightly simpler because we can use the vanishing theorems.*

2.2. Let $\mathscr{H}ilb^0$ be an irreducible component of $\mathscr{H}ilb$ containing the closure $\overline{\varphi(U)}$ of $\varphi(U)$ in $\mathscr{H}ilb$ and $T_{X_s,\mathscr{H}ilb}$ the Zariski tangent space of $\mathscr{H}ilb$ at X_s. Then it is known that $T_{X_s,\mathscr{H}ilb} \simeq H^0(N_{X_s/\mathbf{P}^n})$. So we have the following proposition.

PROPOSITION 2.2. *Under the same assumptions in Proposition* 2.1, *if* $H^1(\mathscr{E}nd(E)) = 0$ *then*

1) $\mathscr{H}ilb^0$ *coincides with* $\overline{\varphi(U)}$.
2) $\mathscr{H}ilb^0$ *is smooth at the determinantal subvarieties associated to E.*

PROOF. It is sufficient to prove that $\dim \overline{\varphi(U)} = \dim H^0(N_{X_s/\mathbf{P}^n})$ for any determinantal surface X_s. Using the exact sequence in Proposition 1.3, we see that $\varphi^{-1}(\varphi(s))$ $(s \in U)$ consists of finitely many orbits. Hence

$$\begin{aligned}\dim \overline{\varphi(U)} &= \dim U - \dim O(s)\\ &= \dim \mathrm{Grass}(H^0(E), 3) - \dim H^0(\mathscr{E}nd(E)) + 1\\ &= 3(\dim H^0(E) - 3) - \dim H^0(\mathscr{E}nd(E)) + 1.\end{aligned}$$

So our assertion follows by Proposition 2.1. □

2.3. Let $\mathrm{PGL}(n+1,k)$ be the automorphism group of $\mathbf{P}^n$ and let $T_\sigma : \mathbf{P}^n \ni x \mapsto \sigma x \in \mathbf{P}^n$ be the transformation of $\mathbf{P}^n$ defined by $\sigma \in \mathrm{PGL}(n+1,k)$.

Suppose that $H^1(\mathscr{E}nd(E)) = 0$. Then it follows from Proposition 2.2 that $\sigma\overline{\varphi(U)} = \overline{\varphi(U)}$ for every element $\sigma \in \mathrm{PGL}(n+1,k)$. Since $\varphi(U)$ is a constructible set, there exist two elements $s, t \in U$ satisfying $X_{\sigma^*(s)} = X_t$, where $X_{\sigma^*(s)}$ is the determinantal subvariey associated to $T_\sigma^*(E)$ defined by $\sigma^*(s) = \langle T_\sigma^*(s_1), T_\sigma^*(s_2), T_\sigma^*(s_3)\rangle$. Consider the resolutions of the defining ideal sheaves I_{X_t} of X_t and $I_{X_{\sigma^*(s)}}$ of $X_{\sigma^*(s)}$ respectively (cf. Lemma 1.2):

$$(**)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & E^* & \longrightarrow & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_t} \otimes \mathcal{O}(c_1) & \longrightarrow & 0\\ & & & & \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & T_\sigma^*(E^*) & \longrightarrow & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_{\sigma^*(s)}} \otimes \mathcal{O}(c_1) & \longrightarrow & 0.\end{array}$$

Then it is observed that there exists an isomorphism $\psi : \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \to \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}$ such that ψ makes the diagram in $(**)$ commutative and so we see that $T_\sigma^*(E)$ is isomorphic to E, i.e., E is a homogeneous vector bundle. Since every homogeneous bundle on $\mathbf{P}^n$ of rank $r < n$ is a direct sum of line bundles even if char $k = p > 0$ (cf. [2]), we can complete the proof of Theorem 0.1.

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