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Hardy's theorem for the Jacobi transform

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Abstract. Let $\alpha\beta = \frac{1}{4}$ for positive constants α, β. Hardy's theorem states that the function $f(x) = e^{-\alpha x^2}$ is the only function (modulo constants) satisfying the decay conditions $f(x) = O(e^{-\alpha x^2})$ and $\hat{f}(x) = O(e^{-\beta x^2})$, where $\hat{f}$ denotes the Fourier transform of f. We generalise this theorem and its L^p analogues to the Jacobi transform. We then consider the Fourier transform on the real hyperbolic spaces $SO_o(m,n)/SO_o(m-1,n)$, $m,n \in \mathbf{N}$, and show, as an application of our results for the Jacobi transform, that Hardy's theorem only can be generalised to the Riemannian $(m = 1)$ case. It can, in particular, not be generalised to $SL(2,\mathbf{R}) \simeq SU(1,1) \simeq SO_o(2,2)/SO_o(1,2)$.

1. Introduction

Let f be a measurable function on $\mathbf{R}$ and let $\hat{f}$ be its Fourier transform. Assume that $|f(t)| \le Ce^{-\alpha|t|^2}$ and $|\hat{f}(\lambda)| \le Ce^{-\beta|\lambda|^2}$, where C, α, β are positive constants. Hardy's theorem, [11], states that if:

(1) $\alpha\beta > \frac{1}{4}$, then $f = 0$.

(2) $\alpha\beta = \frac{1}{4}$, then $f(t) = const.\ e^{-\alpha t^2}$.

(3) $\alpha\beta < \frac{1}{4}$, then there are infinitely many linearly independent solutions.

We note that (2) implies (1) and (3). The central part of Hardy's theorem, the $\alpha\beta = \frac{1}{4}$ case, can be reformulated in terms of the Heat kernel: $h_t(x) := (4\pi t)^{-1/2} e^{-x^2/4t}$, $t > 0$. We note that $\hat{h}_t(x) = e^{-tx^2}$, and thus the only functions satisfying (2) are constant multiples of h_β, with $\beta = 1/4\alpha$. The $\alpha\beta > \frac{1}{4}$ case is also known as Hardy's uncertainty principle: f and $\hat{f}$ cannot both be *very* rapidly decreasing. A Generalisation of Hardy's theorem with L^p growth conditions was furthermore given by Cowling and Price in [6].

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Analogues of Hardy's uncertainty principle and its L^p versions for the Fourier transforms on (semisimple) Lie groups and Riemannian symmetric spaces of the non-compact type have now been studied in several papers, see [7], [17], [21], [24] and references therein.

The Heat kernel on a Riemannian symmetric space is also a well-defined and much studied object, in particular its decay properties. It was very recently shown that the Heat kernel in this set-up also characterises the functions satisfying a natural analogue of the decay conditions above, with the Helgason–Fourier transform replacing the Fourier transform, for $\alpha\beta = \frac{1}{4}$, see [18] and [23]. See also [19] for generalisations of the Cowling–Price results.

Consider the Jacobi transform $f \mapsto \hat{f}^{a,b}$ of order (a,b), where f is an even function and a, b are complex numbers. We remark that the spherical Helgason–Fourier transform for Riemannian symmetric spaces of rank 1 can be viewed as the Jacobi transform for certain half-integer values of a and b, but that in general the notion of the Heat kernel is not defined. However, we can show the following generalisation of Hardy's theorem:

Theorem 1.1. *Let $a, b \in \mathbf{C}$, $a \notin -\mathbf{N}$ and $\rho := a + b + 1$. Assume that f is an even measurable function on $\mathbf{R}$ satisfying:*

$$|f(t)| \leq C(1+|t|)^M e^{-\Re\rho|t|} e^{-\alpha|t|^2}, \quad t \in \mathbf{R}$$

and

$$|\hat{f}^{a,b}(\lambda)| \leq C e^{-\beta|\lambda|^2}, \quad \lambda \in \mathbf{R},$$

for non-negative constants C, M, α, β, with $M \geq \Re a + \frac{1}{2}$ and $\alpha\beta = \frac{1}{4}$, then $\hat{f}^{a,b}(\lambda) = const.\ e^{-\beta\lambda^2}$.

We remark that the (Jacobi) inverse of the function $e^{-\beta\lambda^2}$ is a non-zero even C^∞ function on $\mathbf{R}$ satisfying the left-hand-side growth estimate, and that we also prove injectivity for the Jacobi transform on the appropriate subspaces of even functions, for all a, b with $a \notin -\mathbf{N}$. The Jacobi transform reduces to the cosine–Fourier transform when $a = b = -\frac{1}{2}$, in which case Theorem 1.1 is a slight modification of Hardy's classical theorem.

The above theorem is part of our main result for the Jacobi transform, Theorem 3.5, which is a L^p version of Hardy's theorem for the Jacobi transform. We use a Cowling–Price approach to prove this Theorem. The $\alpha\beta > \frac{1}{4}$ and $\alpha\beta < \frac{1}{4}$ cases again follow as corollaries, see also [2] and [3] for different proofs of Hardy's uncertainty principle and its L^p versions for the Jacobi transform.

As an application of our results for the Jacobi transform, we consider the Fourier transform on the real hyperbolic spaces $\mathbf{X} = SO_o(m,n)/SO_o(m-1,n)$, $m,n \in \mathbf{N}$. We first give a (different) proof of Hardy's theorem in the Riemannian ($m = 1$) case, using explicit expressions of the matrix coefficients in terms of modified Jacobi functions. We stress that a function satisfying the natural decay properties necessarily is spherical (bi-K-invariant), being a scalar multiple of the (spherical) Heat kernel.

We then show that a similar result does not hold in the pseudo-Riemannian case. The K-types on $\mathbf{X}$ can be identified with integers (r,s), where r is identically zero when $m = 1$. It turns out that the natural decay properties only imply a restriction on the second of the K-type variables and we can construct an infinite, albeit countable (indexed by the K-types $(r,0)$), family of linearly independent functions on $\mathbf{X}$ satisfying them.

We note that Hardy's Uncertainty Principle and its L^p-versions (the $\alpha\beta \geq (>) \dfrac{1}{4}$ cases) still hold and that there are infinitely (uncountably) many linearly independent functions satisfying the natural decay conditions with $\alpha\beta < \dfrac{1}{4}$; this also follows as corollaries of the results for the Jacobi transform. Hardy's uncertainty principle for $\mathbf{X}$ was also proved in [22], as a corollary of the similar result for the Heckman–Opdam transform (of which the Jacobi transform is a special case).

We end the paper by discussing the $SL(2,\mathbf{R}) \simeq SU(1,1) \simeq SO_o(2,2)/SO_o(1,2)$ case in more detail.

2. Jacobi functions and the Jacobi transform

Let $a,b,\lambda \in \mathbf{C}$ and $0 < t < \infty$. We consider the differential equation

$$(\Delta^{a,b}(t))^{-1}\frac{d}{dt}\left(\Delta^{a,b}(t)\frac{du(t)}{dt}\right) = -(\lambda^2+\rho^2)u(t), \tag{1}$$

where $\rho := a+b+1$ and $\Delta^{a,b}(t) = (2\sinh(t))^{2a+1}(2\cosh(t))^{2b+1}$. Using the substitution $x = -\sinh^2(t)$, we can rewrite (1) as a hypergeometric differential equation with parameters $\dfrac{1}{2}(\rho+i\lambda)$, $\dfrac{1}{2}(\rho-i\lambda)$ and $a+1$ (see [8, 2.1.1]). Let ${}_2F_1$ denote the Gauß hypergeometric function. The Jacobi function (of order (a,b)),

$$\varphi_\lambda^{a,b}(t) := {}_2F_1\left(\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda), a+1; -\sinh^2(t)\right),$$

is for $a \notin -\mathbf{N}$ the unique solution to (1) satisfying $\varphi_\lambda^{a,b}(0) = 1$ and $\left.\frac{d}{dt}\right|_{t=0} \varphi_\lambda^{a,b} = 0$. The Jacobi functions satisfy the following growth estimates:

Lemma 2.1. *There exists for each $a, b \in \mathbf{C}$ a constant $C > 0$ such that:*

$$|\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)| \leq C(1+|\lambda|)^k(1+t)e^{(|\Im\lambda| - \Re\rho)t},$$

for all $t \geq 0$, where $k = 0$ if $\Re a > -\frac{1}{2}$ and $k = \left[\frac{1}{2} - \Re a\right]$ if $\Re a \leq -\frac{1}{2}$.

Proof. See [15, Lemma 2.3]. □

Here $[\cdot]$ denotes integer part. We note that $\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)$ is an entire function in the variables a, b and $\lambda \in \mathbf{C}$ (also for $a \in -\mathbf{N}$). The Jacobi transform (of order (a,b)) is defined by:

$$\hat{f}^{a,b}(\lambda) := \int_{\mathbf{R}_+} f(t)\varphi_\lambda^{a,b}(t)\Delta^{a,b}(t)dt,$$

for all even functions f and all complex numbers λ for which the right hand side is well-defined. The Paley–Wiener theorem for the Jacobi transform, [15, Theorem 3.4], states that the (normalised) application $f \mapsto \Gamma(a+1)^{-1}\hat{f}^{a,b}$ is a bijection from $C_c^\infty(\mathbf{R})_{\text{even}}$ onto $\mathscr{H}(\mathbf{C})_{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type, for all $a, b \in \mathbf{C}$.

The Jacobi functions of the second kind:

$$\phi_\lambda^{a,b}(t) = (2\cosh(t))^{i\lambda-\rho}{}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(a-b+1-i\lambda), 1-i\lambda; \cosh^{-2}(t)\right),$$

defines for $\lambda \notin -i\mathbf{N}$ another solution of (1), characterised by the property that $\phi_\lambda^{a,b}(t) \sim e^{(i\lambda-\rho)t}$ for $t \to \infty$. We also remark that $\phi_\lambda^{a,b}$ is singular if, and only if, $\lambda \in -i\mathbf{N}$, with simple poles. Define the meromorphic Jacobi c-functions as:

$$c^{a,b}(\lambda) := 2^{\rho-i\lambda}\frac{\Gamma(a+1)\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(i\lambda+\rho)\right)\Gamma\left(\frac{1}{2}(i\lambda+a-b+1)\right)}, \tag{2}$$

then

$$\varphi_\lambda^{a,b} = c^{a,b}(\lambda)\phi_\lambda^{a,b} + c^{a,b}(-\lambda)\phi_{-\lambda}^{a,b}, \tag{3}$$

as a meromorphic identity, see [16, (2.15–18)].

Lemma 2.2. *Let $0 \leq \eta < \frac{1}{2}$. There exists for $\Im\lambda \geq -\eta$ a converging series such that:*

$$\phi_\lambda^{a,b}(t) = e^{(i\lambda-\rho)t}\sum_{n=0}^\infty \Gamma_n^{a,b}(\lambda)e^{-nt}, \qquad (t > 0),$$

with $\Gamma_n^{a,b}(\lambda) \in \mathbf{C}$, $\Gamma_0^{a,b} \equiv 1$. *There furthermore exist positive constants* C *and* d *(depending on* a *and* b*) such that:*

$$|\Gamma_n^{a,b}(\lambda)| < C(1+n)^d,$$

for $\Im\lambda \geq -\eta$ *and all* $n \in \mathbf{N}$. *Fix* $\delta > 0$. *There exists a constant* C_δ *such that:*

$$|\phi_\lambda^{a,b}(t)| \leq C_\delta e^{-(\Im\lambda + \Re\rho)t},$$

for $\Im\lambda \geq -\eta$ *and all* $t \in]\delta, \infty[$.

Proof. The lemma follows by extending [9, Lemma 7] to complex a, b. See also [5] for a more general set-up. □

The polynomial estimates on $|c^{a,b}(-\lambda)^{-1}|$ away from the poles given by [15, Lemma 2.2] can also be extended to $\Im\lambda \geq -\eta$.

The inversion formula for the Jacobi transform can be written as (with $\mu \geq 0$, $\mu > -\Re(a \pm b + 1)$):

$$f(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{f}^{a,b}(\lambda + i\mu)\phi_{\lambda+i\mu}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda - i\mu)}, \qquad (t > 0), \tag{4}$$

for $f \in C_c^\infty(\mathbf{R})_{\text{even}}$, see [16, Theorem 2.2]. Using residual calculus we can rewrite (4) as follows:

Theorem 2.3. *Assume that* $a \notin -\mathbf{N}$. *Let* $D_{a,b}$ *denote the finite set of zeroes for* $c^{a,b}(-\lambda)$ *with* $\Im\lambda \geq 0$. *Let* $\eta = 0$ *if* $D_{a,b} \cap \mathbf{R} = \{\varnothing\}$ *and otherwise choose* $0 < \eta < \frac{1}{2}$ *such that* $c^{a,b}(\pm\lambda) \neq 0$ *for* $\Im\lambda \in [-\eta, \eta]\backslash\{0\}$. *Then:*

$$f(t) = \frac{1}{4\pi}\int_{\mathbf{R}} \frac{\hat{f}^{a,b}(\lambda + i\eta)\varphi_{\lambda+i\eta}^{a,b}(t)}{c^{a,b}(-\lambda - i\eta)c^{a,b}(\lambda + i\eta)}\,d\lambda - \sum_{\nu \in D_{a,b}} ik_\nu \operatorname{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},$$

for $f \in C_c^\infty(\mathbf{R})_{even}$, *where* $k_\nu := 1/2$ *if* $\nu \in i\mathbf{N} \cup \mathbf{R}$, *and* $k_\nu := 1$ *otherwise.*

Proof. The set $D_{a,b}$ is determined by the poles of the Γ-functions of (2). It follows that $D_{a,b}$ consists of those elements $\nu \neq 0$, with $\Im\nu \geq 0$, which are of the form: $\nu = i(\pm b - a - 1 - 2m)$, $m \in \mathbf{N} \cup \{0\}$.

Let $\nu \in D_{a,b}$, that is, $c^{a,b}(-\nu) = 0$. Assume first that $\nu \notin i\mathbf{N}$, then $c^{a,b}(\nu) \neq 0$ by the condition $a \notin -\mathbf{N}$, and:

$$\begin{aligned}\operatorname{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)}\right\} &= \operatorname{Res}_{\lambda=\nu}\left\{\hat{f}^{a,b}(\lambda)\left(\frac{\phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)} + \frac{\phi_{-\lambda}^{a,b}(t)}{c^{a,b}(\lambda)}\right)\right\} \\ &= \operatorname{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},\end{aligned}$$

by (3), since $\phi_{-\lambda}^{a,b}(t)/c^{a,b}(\lambda)$ is regular at $\lambda = \nu$.

Now assume $\nu \in i\mathbf{N} \cap D_{a,b}$. Then ν is a zero for $c^{a,b}(-\lambda)$ of order 1; a double pole in the denominator of $c^{a,b}(-\lambda)$ at $\nu \in i\mathbf{N}$ would imply $a \in -\mathbf{N}$, which we have excluded. The c-function $c^{a,b}(\lambda)$ is regular and non-zero at $\lambda = \nu$, as the poles arising from the Γ-functions in (2) cancel each other (we have excluded the cases with double poles in the denominator). We also note that $\phi_\lambda^{a,b}(t)$ is regular at $\lambda = \nu$.

Write $\nu = i(\pm b - a - 1 - 2m)$. Fix a and m, and define, for λ in some small neighbourhood of ν, a continuous function $b(\lambda)$ by the condition: $\lambda = i(\pm b(\lambda) - a - 1 - 2m)$. It follows that $c^{a,b(\lambda)}(-\lambda) = 0$ and $\varphi_\lambda^{a,b(\lambda)}(t) = c^{a,b(\lambda)}(\lambda)\phi_\lambda^{a,b(\lambda)}(t)$, for $\lambda \neq \nu$, by (3), and $b(\nu) = b$. Since $\lim_{\lambda\to n} \dfrac{\Gamma((\lambda - n)/2)}{\Gamma(\lambda)} = 2$ for $n \in -\mathbf{N} \cup \{0\}$, it can be seen from (2), that $\lim_{\lambda\to\nu} c^{a,b(\lambda)}(\lambda) = 2\lim_{\lambda\to\nu} c^{a,b}(\lambda)$, and thus, by continuity of the Jacobi functions in all the variables:

$$\mathrm{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)}\right\} = \frac{1}{2}\ \mathrm{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},$$

since $2\phi_\nu^{a,b}(t) = 2\lim_{\lambda\to\nu} \phi_\lambda^{a,b(\lambda)}(t) = 2\lim_{\lambda\to\nu} \dfrac{\varphi_\lambda^{a,b(\lambda)}(t)}{c^{a,b(\lambda)}(\lambda)} = \lim_{\lambda\to\nu} \dfrac{\varphi_\lambda^{a,b(\lambda)}(t)}{c^{a,b}(\lambda)} = \dfrac{\varphi_\nu^{a,b}(t)}{c^{a,b}(\nu)}$.

Now choose η as in the theorem. Using the estimates from Lemma 2.2, polynomial estimates on $c^{a,b}(\lambda)^{-1}$ and since $\hat{f}^{a,b}$ satisfies the usual Paley–Wiener growth estimates, we can shift the contour toward the real axis, and (4) becomes:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{f}^{a,b}(\lambda + i\eta)\phi_{\lambda+i\eta}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda - i\eta)} + \text{Residual terms} \\ &= \frac{1}{4\pi}\int_{\mathbf{R}} \hat{f}^{a,b}(\lambda + i\eta)\phi_{\lambda+i\eta}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda - i\eta)} \\ &\quad + \frac{1}{4\pi}\int_{\mathbf{R}} \hat{f}^{a,b}(-\lambda - i\eta)\phi_{-\lambda-i\eta}^{a,b}(t)\frac{d\lambda}{c^{a,b}(\lambda + i\eta)} + \text{Residual terms}, \end{aligned}$$

where we have moved half the integral across the real axis if $D_{a,b} \cap \mathbf{R} \neq \emptyset$ and made a sign change $\lambda \mapsto -\lambda$ in the integral over the line $\Im\lambda = -\eta$. Since $\hat{f}^{a,b}$ is even, we get our inversion formula from the identity (3). □

As a corollary we get injectivity of the Jacobi transform for *nice* functions:

COROLLARY 2.4. *Let $a, b \in \mathbf{C}$, $a \notin -\mathbf{N}$. Assume that f is an even measurable function on $\mathbf{R}$ satisfying $|f(t)| \le Ce^{-\alpha|t|^2}$, $t \in \mathbf{R}$, for positive constants C and α. Then $\hat{f}^{a,b} = 0$ implies $f = 0$ almost everywhere.*

Proof. The very rapid decay implies that $f \in L^1(\mathbf{R}_+, |\Delta^{a,b}(t)|dt) \cap L^2(\mathbf{R}_+, |\Delta^{a,b}(t)|dt)$ and that $\hat{f}^{a,b}(\lambda)$ defines an analytic function in $\lambda \in \mathbf{C}$ for all $a, b \in \mathbf{C}$. Using (the proof of) Theorem 2.3, we see that:

$$\begin{aligned}
\int_{\mathbf{R}_+} f(t)h(t)\Delta^{a,b}(t)dt &= \frac{1}{2\pi}\int_{\mathbf{R}_+}\int_{\mathbf{R}} f(t)\hat{h}^{a,b}(\lambda + i\mu)\phi^{a,b}_{\lambda+i\mu}(t)\,\frac{d\lambda\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda - i\mu)} \\
&= \frac{1}{4\pi}\int_{\mathbf{R}}\int_{\mathbf{R}_+} \frac{f(t)\varphi^{a,b}_{\lambda+i\eta}(t)\hat{g}^{a,b}(\lambda + i\eta)}{c^{a,b}(-\lambda - i\eta)c^{a,b}(\lambda + i\eta)}\Delta^{a,b}(t)dtd\lambda \\
&\quad - \sum_{\nu \in D_{a,b}} ik_\nu \operatorname{Res}_{\lambda=\nu}\left\{\frac{\int_{\mathbf{R}_+} f(t)\varphi^{a,b}_{\lambda}(t)\hat{h}^{a,b}(\lambda)\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\} \\
&= \frac{1}{4\pi}\int_{\mathbf{R}} \frac{\hat{f}^{a,b}(\lambda + i\eta)\hat{h}^{a,b}(\lambda + i\eta)}{c^{a,b}(-\lambda - i\eta)c^{a,b}(\lambda + i\eta)}\,d\lambda \\
&\quad - \sum_{\nu \in D_{a,b}} ik_\nu \operatorname{Res}_{\lambda=\nu}\left\{\frac{\hat{f}^{a,b}(\lambda)\hat{h}^{a,b}(\lambda)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},
\end{aligned}$$

is identically zero for any $h \in C_c^\infty(\mathbf{R})_{\text{even}}$, and we conclude that f is zero almost everywhere. □

Remark 2.5. *Theorem* 2.3 *and its proof was communicated to us by H. Schlichtkrull. For* $a > -1$, $b \in \mathbf{R}$ *(which implies* $\eta = 0$*), it is due to* [10, *Appendix* 1] *(a minor error has been corrected with the introduction of the constant* k_ν*).*

3. Hardy's theorem for the Jacobi transform

Our approach to Hardy's Theorem for the Jacobi transform is inspired by [17] and [19], which in turn are heavily inspired by the Cowling–Price approach. The following lemma from [6] is crucial:

Lemma 3.1. *Let* $1 \le q < \infty$. *Let* $Q = \left\{\sigma e^{i\theta} \mid \sigma > 0, \theta \in \left(0, \frac{\pi}{2}\right)\right\}$. *Suppose that h is analytic on Q, continuous on the closure* $\bar{Q}$ *of Q, and that h satisfies the following growth conditions:*

$$|h(\lambda)| \le Ce^{\gamma|\Re\lambda|^2}, \quad \lambda \in \bar{Q} \qquad \text{and} \qquad \int_{\mathbf{R}_+} |h(\lambda)|^q d\lambda \le C^q < \infty,$$

for positive constants C and γ. *Then*

$$\int_{\eta}^{\eta+1} |h(\sigma e^{i\theta})| d\sigma \leq C \max\{e^{\gamma}, (\eta+1)^{1/q}\},$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$ *and* $\eta \in \mathbf{R}_+$.

Lemma 3.2. *Let* $1 \leq q < \infty$. *Assume that* h *is an entire function on* $\mathbf{C}$ *such that:*

$$|h(\lambda)| \leq C(1+|\Im\lambda|)^M e^{\gamma|\Re\lambda|^2}, \quad \lambda \in \mathbf{C} \qquad and \qquad \int_{\mathbf{R}} ((1+|\lambda|)^{-N}|h(\lambda)|)^q d\lambda < \infty,$$

for positive constants C, γ, M *and* N. *Then* h *is a polynomial with* $\deg P \leq M$ *and* $\deg P < N - 1$.

Proof. The bounds on the degrees are obvious as soon as we have proved that h is a polynomial. Define the function:

$$H(\lambda) := \frac{h(\lambda)}{(i+\lambda)^{M+N}}, \qquad \lambda \in \bar{Q}.$$

The function H satisfies the conditions of the previous lemma, whence:

$$\int_{\eta}^{\eta+1} |H(\sigma e^{i\theta})| d\sigma \leq C \max\{e^{\gamma}, (\eta+1)^{1/q}\},$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\eta \in \mathbf{R}_+$, where C here and in the following denotes some positive constant, and

$$\int_{\eta}^{\eta+1} |h(\sigma e^{i\theta})| d\sigma \leq C \max\{e^{\gamma}, (\eta+1)^{1/q}\}(\eta+2)^{M+N},$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\eta \in \mathbf{R}_+$. Applying the same procedure to $H_1(\lambda) := \overline{h(\bar{\lambda})}/(i+\lambda)^N$, $H_2(\lambda) := \overline{h(-\bar{\lambda})}/(i+\lambda)^N$ and $H_3(\lambda) := h(-\lambda)/(i+\lambda)^N$ for $\lambda \in Q$, implies that:

$$\int_{\eta}^{\eta+1} |h(\sigma e^{i\theta})| d\sigma \leq C(\eta+2)^{M+N+1/q},$$

for $\theta \in [0, 2\pi]$ and large η. Cauchy's integral formula:

$$h^{(n)}(0) = \frac{n!}{2\pi}\int_0^{2\pi} h(\sigma e^{i\theta})(\sigma e^{i\theta})^{-n} d\theta,$$

yields the estimate:

$$\begin{aligned}
|h^{(n)}(0)| &\le n! \int_\eta^{\eta+1} \int_0^{2\pi} |h(\sigma e^{i\theta})|\sigma^{-n} d\theta d\sigma \\
&\le n!\eta^{-n} \int_0^{2\pi} \int_\eta^{\eta+1} |h(\sigma e^{i\theta})| d\sigma d\theta \\
&\le Cn!\eta^{-n}(\eta+2)^{M+N+1/q}.
\end{aligned}$$

We conclude that $h^{(n)}(0) = 0$ for $n > M + N + 1/q$, that is, h is a polynomial. $\square$

In the following, we define: $L^\infty(\mathbf{R}, |\Delta^{a,b}(t)|dt) := L^\infty(\mathbf{R})$, and otherwise define $L^p(\mathbf{R}, |\Delta^{a,b}(t)|dt)$ for $0 < p < \infty$ as usual.

Theorem 3.3. *Let $a, b \in \mathbf{C}$, $a \notin -\mathbf{N}$, and $1 \le p \le \infty$, $1 \le q \le \infty$. Assume that f is an even measurable function on $\mathbf{R}$ satisfying:*

$$(1 + |\cdot|)^{-M} e^{(1-2/p)\Re\rho|\cdot|} e^{\alpha|\cdot|^2} f \in L^p(\mathbf{R}, |\Delta^{a,b}(t)|dt)$$

and

$$(1 + |\cdot|)^{-N} e^{\beta|\cdot|^2} \hat{f}^{a,b} \in L^q(\mathbf{R}),$$

for positive constants M, N, α, β such that $\alpha\beta = \dfrac{1}{4}$. Then $\hat{f}^{a,b}(\lambda) = P(\lambda)e^{-\beta\lambda^2}$ for some polynomial P, with $\deg P \le \min\{k + M + 1, N\}$, *and* $\deg P < N - 1$ *if $q < \infty$.*

For $p = \infty$ and $q = \infty$, we can rewrite the above decay properties as

$$|f(t)| \le C(1 + |t|)^M e^{-\Re\rho|t|} e^{-\alpha|t|^2}, \quad t \in \mathbf{R}$$

and

$$|\hat{f}^{a,b}(\lambda)| \le C(1 + |\lambda|)^N e^{-\beta|\lambda|^2}, \quad \lambda \in \mathbf{R},$$

for some positive constant C.

Proof. Let f be an even measurable function satisfying the above growth conditions. Then, as before, we have $f \in L^1(\mathbf{R}_+, |\Delta^{a,b}(t)|dt) \cap L^2(\mathbf{R}_+, |\Delta^{a,b}(t)|dt)$ and $\hat{f}^{a,b}(\lambda)$ defines an analytic function in $\lambda \in \mathbf{C}$ for all $a, b \in \mathbf{C}$.

Let first $p < \infty$. Using Lemma 2.1, we get the following estimates on $\hat{f}^{a,b}(\lambda)$ (for different positive constants C):

$$
\begin{aligned}
|\hat{f}^{a,b}(\lambda)| &\le C\int_{\mathbf{R}_+} |f(t)|(1+|\lambda|)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t}|\Delta^{a,b}(t)|dt \\
&\le C(1+|\lambda|)^k\int_{\mathbf{R}_+} |f(t)|e^{\alpha t^2}e^{(1-2/p)\Re\rho t}(1+t)e^{|\Im\lambda|t}e^{-\alpha t^2}e^{(2/p-2)\Re\rho t}|\Delta^{a,b}(t)|dt \\
&\le C(1+|\lambda|)^k\left(\int_{\mathbf{R}_+} ((1+t)^{M+1}e^{|\Im\lambda|t}e^{-\alpha t^2}e^{-(2/p')\Re\rho t})^{p'}|\Delta^{a,b}(t)|dt\right)^{1/p'} \\
&\le C(1+|\lambda|)^k\left(\int_{\mathbf{R}_+} (1+t)^{p'(M+1)}e^{p'|\Im\lambda|t}e^{-p'\alpha t^2}\,dt\right)^{1/p'} \\
&= C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\left(\int_{\mathbf{R}_+} (1+t)^{p'(M+1)}e^{-p'\alpha(t-|\Im\lambda|/2\alpha)^2}\,dt\right)^{1/p'} \\
&= C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\left(\int_{-|\Im\lambda|/2\alpha}^{\infty} (1+t+|\Im\lambda|/2\alpha)^{p'(M+1)}e^{-p'\alpha t^2}\,dt\right)^{1/p'} \\
&\le C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\left(\int_{\mathbf{R}} (1+|t|+|\Im\lambda|/2\alpha)^{p'(M+1)}e^{-p'\alpha t^2}\,dt\right)^{1/p'} \\
&\le C(1+|\lambda|)^{k+M+1} e^{|\Im\lambda|^2/4\alpha}\left(\int_{\mathbf{R}} (1+|t|)^{p'(M+1)}e^{-p'\alpha t^2}\,dt\right)^{1/p'} \\
&\le C(1+|\lambda|)^{k+M+1} e^{|\Im\lambda|^2/4\alpha},
\end{aligned}
$$

for $\lambda \in \mathbf{C}$, using translation invariance of dt, the Hölder inequality (with $\dfrac{1}{p}+\dfrac{1}{p'}=1$) and the inequality $|\Im\lambda| \le |\lambda|$. For $p=\infty$, we have:

$$
\begin{aligned}
|\hat{f}^{a,b}(\lambda)| &\le C\int_{\mathbf{R}_+} e^{-\alpha t^2}e^{-\Re\rho t}(1+|\lambda|)^k(1+t)^{M+1}e^{(|\Im\lambda|-\Re\rho)t}|\Delta^{a,b}(t)|dt \\
&\le C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\int_{\mathbf{R}_+} (1+t)^{M+1}e^{-\alpha(t-|\Im\lambda|/2\alpha)^2}\,dt \\
&= C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\int_{-|\Im\lambda|/2\alpha}^{\infty} (1+t+|\Im\lambda|/2\alpha)^{M+1}e^{-\alpha t^2}\,dt \\
&\le C(1+|\lambda|)^{k+M+1}e^{|\Im\lambda|^2/4\alpha},
\end{aligned}
$$

for $\lambda \in \mathbf{C}$.

Define $g(\lambda) := \hat{f}^{a,b}(\lambda)e^{\lambda^2/4\alpha} = \hat{f}^{a,b}(\lambda)e^{\beta\lambda^2}$. Then g is an entire function, and:

$$
|g(\lambda)| \le C(1+|\lambda|)^{k+M+1}e^{\beta|\Re\lambda|^2} \le C(1+|\Im\lambda|)^{k+M+1}e^{\beta'|\Re\lambda|^2},
$$

for some $\beta' > \beta$. Let $q < \infty$, then:

$$\int_{\mathbf{R}} ((1+|\lambda|)^{-N}|g(\lambda)|)^q d\lambda = \int_{\mathbf{R}} ((1+|\lambda|)^{-N} e^{\beta|\lambda|^2}|\hat{f}^{a,b}(\lambda)|)^q d\lambda < \infty,$$

so Lemma 3.2 implies that g is a polynomial, with $\deg g \leq k+M+1$ and $\deg g < N-1$. Let now $q = \infty$, then:

$$|g(\lambda)| \leq C(1+|\lambda|)^N, \qquad \lambda \in \mathbf{R},$$

which implies that g is a polynomial, with $\deg g \leq \min\{k+M+1, N\}$. We conclude the result since $\hat{f}^{a,b}(\lambda) = g(\lambda)e^{-\beta\lambda^2}$. □

As a corollary of Theorem 3.3, we get a L^p version of Hardy's Uncertainty Theorem for the Jacobi transform, see also [3, Theorem 2.3] for a different approach:

Corollary 3.4. *Let $a, b \in \mathbf{C}$, $a \notin -\mathbf{N}$, and $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that f is an even measurable function on $\mathbf{R}$ satisfies:*

$$e^{\alpha|\cdot|^2} e^{(1-2/p)\Re\rho|\cdot|} f \in L^p(\mathbf{R}, |\Delta^{a,b}(t)|dt) \qquad \textit{and} \qquad e^{\beta|\cdot|^2}\hat{f}^{a,b} \in L^q(\mathbf{R}),$$

for positive constants α, β such that $\alpha\beta \geq \dfrac{1}{4}$. Then $f = 0$ almost everywhere.

Proof. It suffices to prove the theorem for $\alpha\beta = \dfrac{1}{4}$. Put $M = N = 1$, then the function f above satisfy the decay conditions in Theorem 3.3, whence $\hat{f}^{a,b}(\lambda) = 0$ as $\deg P < 0$, and $f = 0$ by Corollary 2.4. □

Let $\beta > 0$. Inspired by the (definition of) the Heat kernel, we define the function $h_\beta^{a,b}$ as the inverse of $e^{-\beta(\lambda^2+\rho^2)}$, that is, by the inversion formula (4):

$$h_\beta^{a,b}(t) := \frac{1}{2\pi}\int_{\mathbf{R}} e^{-\beta((\lambda+i\mu)^2+\rho^2)}\phi_{\lambda+i\mu}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda-i\mu)}, \qquad (t > 0).$$

Using residual calculus as before, it can be seen that $h_\beta^{a,b}$ extends to an even C^∞ function on $\mathbf{R}$. The function $h_\beta^{a,b}$ is for certain half integers a, b exactly the heat kernel (with index β) for some Riemannian symmetric space of rank 1, see [16, §3] for details. We finally sketch a proof of the very important fact that $\widehat{h_\beta^{a,b}}^{a,b}(\lambda) = e^{-\beta(\lambda^2+\rho^2)}(*)$:

The application $(a,b) \mapsto \widehat{h_\beta^{a,b}}^{a,b}(\lambda)$ is an entire function in a and b. Following [15, §4] we can show that $(*)$ holds for $\Re a > -\dfrac{1}{2}$ and $|\Re b| < \Re(a+1)$ (writing the Jacobi transform as a composition of the "Abel" transform and the cosine transform). Then $(*)$ holds for all a, b by holomorphy. Note that we also have used the growth estimates deduced below.

As for the Heat kernel, we can prove nice growth estimates for $h_\beta^{a,b}$: Fix $\delta > 0$. Using Lemma 2.2 we get, for $t \geq \delta$:

$$\begin{aligned} h_\beta^{a,b}(t) &= \frac{1}{2\pi}\sum_{n=0}^{\infty} e^{-nt+\beta\mu^2-\beta\rho^2-\mu t-\rho t}\int_{\mathbf{R}} e^{-\beta\lambda^2}e^{-2i\beta\mu\lambda}e^{i\lambda t}\,\frac{\Gamma_n^{a,b}(\lambda+i\mu)}{c^{a,b}(-\lambda-i\mu)}\,d\lambda \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} e^{-nt+\beta\mu^2-\beta\rho^2-\mu t-\rho t-t^2/4\beta}\int_{\mathbf{R}} e^{-\beta(\lambda-it/2\beta)^2}e^{-2i\beta\mu\lambda}\,\frac{\Gamma_n^{a,b}(\lambda+i\mu)}{c^{a,b}(-\lambda-i\mu)}\,d\lambda \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} e^{-nt+\beta\mu^2-\beta\rho^2-\mu t-\rho t-t^2/4\beta} \\ &\quad\times\int_{\mathbf{R}} e^{-\beta\lambda^2}e^{-2i\beta\mu(\lambda+it/2\beta)}\,\frac{\Gamma_n^{a,b}(\lambda+i\mu+it/2\beta)}{c^{a,b}(-\lambda-i\mu-it/2\beta)}\,d\lambda \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} e^{-nt+\beta\mu^2-\beta\rho^2-\rho t-t^2/4\beta}\int_{\mathbf{R}} e^{-\beta\lambda^2}e^{-2i\beta\mu\lambda}\,\frac{\Gamma_n^{a,b}(\lambda+i\mu+it/2\beta)}{c^{a,b}(-\lambda-i\mu-it/2\beta)}\,d\lambda. \end{aligned}$$

We have the following estimates of the c-function:

$$\begin{aligned} |c^{a,b}(-\lambda-i\mu-it/2\beta)|^{-1} &\leq C(1+|\lambda+i\mu+it/2\beta|)^{a+1/2} \\ &\leq C(1+|\lambda|)^{a+1/2}(1+t/2\beta)^{a+1/2}, \end{aligned}$$

for some positive constant C. Together with the estimates of $\Gamma_n^{a,b}(\lambda)$ from Lemma 2.2, we thus have, for some positive constant C:

$$|h_\beta^{a,b}(t)| \leq C(1+t)^{a+1/2}e^{-\Re\rho t-t^2/4\beta}, \tag{5}$$

for $t \in \mathbf{R}_+$. We actually have the following sharp estimate for $a, 2b \in \mathbf{N}\cup\{0\}$ and $a \geq b$:

$$h_\beta^{a,b}(t) \asymp \beta^{-3/2}(1+t)(1+(1+t)/\beta)^{a-1/2}e^{-\rho t-\beta\rho^2-t^2/4\beta},$$

for $t \geq 0$, see [4, Theorem 5.9].

Let $\alpha = 1/4\beta$. Then:

$$\int_{\mathbf{R}}((1+t)^{-M}e^{(1-2/p)\Re\rho t}e^{\alpha t^2}h_\beta^{a,b}(t))^p|\Delta^{a,b}(t)|dt < \infty,$$

if $M > \Re a + \dfrac{1}{2} + \dfrac{1}{p}$. Putting all the above together, we can formulate Hardy's theorem for the Jacobi transform:

Theorem 3.5. *Let $a, b \in \mathbf{C}$, $a \notin -\mathbf{N}$, and $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Let $1 < N \leq 2$ if $q < \infty$ and $0 \leq N < 1$ if $q = \infty$. Assume that f is an even measurable function on $\mathbf{R}$ satisfying:*

$$(1+|\cdot|)^{-M}e^{(1-2/p)\Re\rho|\cdot|}e^{\alpha|\cdot|^2}f \in L^p(\mathbf{R}, |\Delta^{a,b}(t)|dt)$$

and

$$(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}\hat{f}^{a,b} \in L^q(\mathbf{R}),$$

with M a positive constant such that $M > \Re a + \frac{1}{2} + \frac{1}{p}$, and $\alpha\beta = \frac{1}{4}$ for positive α, β. Then $f = \hat{f}^{a,b}(i\rho)h_\beta^{a,b}$.

Proof. Theorem 3.3 implies that $\hat{f}^{a,b} = const.\ \hat{h}_\beta^{a,b}$, so $f = const.\ h_\beta^{a,b}$ by Corollary 2.4. We finally note that $\hat{f}^{a,b}(\pm i\rho) = \int_{\mathbf{R}_+} f(t)\Delta^{a,b}(t)dt$. $\square$

For $p = \infty$, it is easily seen that the decay condition on f can be reformulated as:

$$|f(t)| \leq C(1+|t|)^M e^{-\Re\rho|t|}e^{-\alpha|t|^2}, \qquad t \in \mathbf{R},$$

for a non-negative constant M such that $M \geq \Re a + \frac{1}{2}$, and Theorem 1.1 in the introduction follows with $N = 0$.

For completeness, we finally consider the $\alpha\beta > \frac{1}{4}$ and $\alpha\beta < \frac{1}{4}$ cases:

Corollary 3.6. *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Assume that f is an even measurable function on $\mathbf{R}$ satisfying:*

$$e^{\alpha|\cdot|^2}f \in L^p(\mathbf{R}, |\Delta^{a,b}(t)|dt) \qquad \text{and} \qquad e^{\beta|\cdot|^2}\hat{f}^{a,b} \in L^q(\mathbf{R}),$$

for positive constants α and β. If

(1) *$\alpha\beta > \frac{1}{4}$, then $f = 0$.*

(2) *$\alpha\beta < \frac{1}{4}$, then there are infinitely many linearly independent solutions.*

Proof. Let $\alpha\beta > \frac{1}{4}$. Choose $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$ such that $\alpha'\beta' = \frac{1}{4}$. Then f satisfy the conditions in Theorem 3.3 with α, β replaced with α', β', whence $\hat{f}^{a,b}(\lambda) = P(\lambda)h_{\beta'}(\lambda)$ for some polynomial P. But $Ph_{\beta'}$ *does not* satisfy $(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}Ph_{\beta'} \in L^q(\mathbf{R})$, that is, $\hat{f}^{a,b} = 0$ almost everywhere and $f = 0$ by Corollary 2.4.

Let $\alpha\beta < \frac{1}{4}$. Choose any $\beta' > \beta$ such that $\alpha\beta' < \frac{1}{4}$ still holds. It follows that $h_{\beta'}$ satisfies the above conditions. $\square$

The $p = q = \infty$, $\alpha\beta > \frac{1}{4}$ case is Hardy's Uncertainty Principle for the Jacobi transform, see also [2, Theorem 2.3] for a different proof.

4. The Fourier transform on real hyperbolic spaces

Let $m \geq 1$ and $n \geq 2$ be two integers and consider the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^{m+n}$ given by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_m y_m - x_{m+1} y_{m+1} - \cdots - x_{m+n} y_{m+n}, \qquad x, y \in \mathbf{R}^{m+n}.$$

Let $G = SO_o(m,n)$ denote the connected group of $(m+n) \times (m+n)$ matrices preserving $\langle \cdot, \cdot \rangle$ and let $H = SO_o(m-1,n) \subset G$ denote the isotropy subgroup of the point $(1,0,\dots,0) \in \mathbf{R}^{m+n}$. Let $K = SO(m) \times SO(n) \subset G$ be the (maximal compact) subgroup of elements fixed by the classical Cartan involution on G: $\theta(g) = (g^*)^{-1}$.

The space $\mathbf{X} := G/H$ is a semisimple symmetric space (an involution τ of G fixing H is given by $\tau(g) = JgJ$, where J is the diagonal matrix with entries $(1,-1,\dots,-1)$). The map $g \mapsto g \cdot (1,0,\dots,0)$ induces an embedding of $\mathbf{X}$ in $\mathbf{R}^{m+n}$ as the hypersurface (with $x_1 > 0$ if $m = 1$):

$$\mathbf{X} = \{x \in \mathbf{R}^{m+n} \mid \langle x, x \rangle = 1\}.$$

Let $\mathbf{Y} := \mathbf{S}^{m-1} \times \mathbf{S}^{n-1}$. We introduce spherical coordinates on $\mathbf{X}$ as:

$$x(t,y) = (v \cosh(t), w \sinh(t)), \qquad t \in \mathbf{R}_+, \, y = (v,w) \in \mathbf{Y}.$$

The map is injective, continuous and maps onto a dense subset of $\mathbf{X}$. The (K-invariant) metric distance from $x \in \mathbf{X}$ to the origin is given by $|x| = |x(t,y)| = |t|$.

The unique (up to a constant) G-invariant measure on $\mathbf{X}$ is in spherical coordinates given by:

$$\int_{\mathbf{X}} f(x)dx = \int_{\mathbf{R}_+ \times \mathbf{Y}} f(x(t,y))J(t)dtdy,$$

see [12, Part II, Example 2.3], where $J(t) = \cosh^{m-1}(t) \sinh^{n-1}(t)$ is the Jacobian, dt the Lebesgue measure on $\mathbf{R}$ and dy an invariant measure on $\mathbf{Y}$, normalised such that $\int_{\mathbf{Y}} 1 \, dy = 1$.

The action of $SO(m)$ on $C^\infty(\mathbf{S}^{m-1})$ decomposes into irreducible representations $\mathscr{H}^r$ of spherical harmonics of degree $|r|$, see [13, Introduction], characterised as the eigenfunctions of the Laplace–Beltrami operator Δ_m on $\mathbf{S}^{m-1}$ with eigenvalue $-r(r+m-2)$. Here $r = 0$ if $m = 1$, $r \in \mathbf{Z}$ for $m = 2$ and $r \in \mathbf{N} \cup \{0\}$ for $m > 2$.

Let $\mathscr{H}^{r,s} = \mathscr{H}^r \otimes \mathscr{H}^s$ and denote the representation of K on $\mathscr{H}^{r,s}$ by $\delta_{r,s}$. Let $d_{r,s} = \dim \mathscr{H}^{r,s}$ and $\chi_{r,s}$ denote the dimension and the character of $\delta_{r,s}$. A function in $L^2(\mathbf{X})$ is said to be of K-type (r,s) if its translations under the left regular action of K span a vector space which is equivalent to $\mathscr{H}^{r,s}$ as a K-module. We write $L^2(\mathbf{X})^{r,s}$ for the collection of functions of K-type (r,s). The projection $\mathbf{P}^{r,s}$ of $L^2(\mathbf{X})$ onto $L^2(\mathbf{X})^{r,s}$ is given by:

$$\mathbf{P}^{r,s}f(x) = d_{r,s}\int_K \chi_{r,s}(k^{-1})f(k\cdot x)dk, \qquad f \in L^2(\mathbf{X}),$$

for $x \in \mathbf{X}$, see [13, Chapter V, §3] and [14, Chapter III, §5]. There are similar definitions and results for functions in $L^2(\mathbf{Y})$ and also for functions in $C^\infty(\mathbf{X})$ and $C^\infty(\mathbf{Y})$.

The algebra of left-G-invariant differential operators on $\mathbf{X}$ is generated by the Laplace–Beltrami operator $\Delta_{\mathbf{X}}$, see [12, Part II, Example 4.1], which in spherical coordinates is given by:

$$\Delta_{\mathbf{X}}f = \frac{1}{J(t)}\frac{\partial}{\partial t}\left(J(t)\frac{\partial f}{\partial t}\right) - \frac{1}{\cosh^2(t)}\Delta_m f + \frac{1}{\sinh^2(t)}\Delta_n f, \qquad f \in C^\infty(\mathbf{X}),$$

see [20, p. 455]. It reduces to a differential operator $\Delta_{\mathbf{X}}^{r,s}$ in the t-variable when acting on functions of K-type (r,s):

$$\Delta_{\mathbf{X}}^{r,s}f = \Delta_{\mathbf{X}}f = \frac{1}{J(t)}\frac{\partial}{\partial t}\left(J(t)\frac{\partial f}{\partial t}\right) + \frac{r(r+m-2)}{\cosh^2(t)}f - \frac{s(s+n-2)}{\sinh^2(t)}f, \quad f \in C^\infty(\mathbf{X})^{r,s}.$$

Consider the differential equation:

$$\Delta_{\mathbf{X}}f = \Delta_{\mathbf{X}}^{r,s}f = (\lambda^2-\rho^2)f, \qquad f \in C^\infty(\mathbf{X})^{r,s}, \tag{6}$$

where $\rho = \frac{1}{2}(m+n-2)$. Altering the proof of [14, Chapter I, Proposition 2.7] to fit our setup, we see that we can write any function $f \in C^\infty(\mathbf{X})^{r,s}$ in spherical coordinates as:

$$f(x(t,y)) = \sum_i f_i(t)\phi_i^{r,s}(y), \tag{7}$$

where $\{\phi_i^{r,s}\} = \{\phi^r \otimes \phi^s\}_i$ is a (finite) basis for $\mathscr{H}^{r,s}$, and f_i is a function of the form $f_i(t) = t^{|s|}f_{i,o}(t)$, with $f_{i,o}$ even. Let $x = -\sinh^2(t)$ and $g = (1-x)^{-|r|/2}(-x)^{-|s|/2}f_i$. Then g is a solution to the hypergeometric differential equation with parameters $1/2(\lambda+\rho+|r|+|s|)$, $1/2(-\lambda+\rho+|r|+|s|)$ and $q/2+|s|$. Let $\Phi_o^{r,s}(\lambda,\cdot)$ denote the regular (for generic λ) solution to this hypergeometric differential equation satisfying the asymptotic condition $\Phi_o^{r,s}(\lambda,t) \sim e^{(\lambda-\rho)t}$ for $t \to \infty$ (for $\Re\lambda > 0$ and when defined), then

$$\begin{aligned}\Phi_o^{r,s}(\lambda,t) &= 2^{\lambda-\rho-|r|-|s|}\cosh^{|r|}(t)\sinh^{|s|}(t)\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}(\lambda+\rho+|r|+|s|)\right)\Gamma\left(\frac{1}{2}(\lambda-\rho+n-|r|+|s|)\right)}{\Gamma(\lambda)\Gamma\left(\frac{n}{2}+|s|\right)}\\ &\quad\times {}_2F_1\left(\frac{1}{2}(\lambda+\rho+|r|+|s|),\frac{1}{2}(-\lambda+\rho+|r|+|s|);\frac{n}{2}+|s|;-\sinh^2(t)\right),\end{aligned}$$

for $\Re\lambda > 0$, see [1, pp. 72 and 76]. We also note that the function $x(t, y) \mapsto \Phi_o^{r,s}(\lambda, t)\phi(y)$ extends to a solution of (6) on $\mathbf{X}$ for any $\phi \in \mathscr{H}^{r,s}$.

Let $\varepsilon \in \{0, 1\}$ and define $C_\varepsilon^\infty(\mathbf{Y}) := \{\phi \in C^\infty(\mathbf{Y}) \mid \phi(-y) = (-1)^\varepsilon \phi(y)\}$. The Poisson transform, $F_{\varepsilon,\lambda} : C_\varepsilon^\infty(\mathbf{Y}) \to C^\infty(\mathbf{X})$, is defined as:

$$F_{\varepsilon,\lambda}\phi(x) = \int_{\mathbf{Y}} |\langle x, y\rangle|^{(-\lambda-\rho)} \operatorname{sign}^\varepsilon\langle x, y\rangle \phi(y)dy, \qquad \phi \in C_\varepsilon^\infty(\mathbf{Y}), \tag{8}$$

when $-\Re\lambda \geq \rho$.

Lemma 4.1. *Let $\phi \in C_\varepsilon^\infty(\mathbf{Y})$. The (meromorphic extension of the) function $F_{\varepsilon,\lambda}\phi$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbf{X}}$ with eigenvalue $\lambda^2 - \rho^2$ (when defined), i.e.,:*

$$\Delta_{\mathbf{X}} F_{\varepsilon,\lambda}\phi = (\lambda^2 - \rho^2) F_{\varepsilon,\lambda}\phi.$$

The asymptotic behaviour of $F_{\varepsilon,\lambda}\phi$ for $t \to \infty$ is given by (when defined):

$$F_{\varepsilon,\lambda}\phi(x(t, y)) \sim e^{(\lambda-\rho)t} c(\varepsilon, \lambda)\phi(y),$$

for $\Re\lambda > 0$, where $c(\varepsilon, \lambda)$ is the so-called c-function for $\mathbf{X}$ given by:

$$c(\varepsilon, \lambda) = \frac{2^{2\rho-1}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\pi} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)} \begin{cases} \tan\left(\frac{\pi}{2}(\lambda+\rho+\varepsilon)\right) & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \tag{9}$$

Proof. The function $F_{\varepsilon,\lambda}\phi$ extends meromorphically to $\mathbf{C}$ by distribution theory, see [20, Lemma 5(a)]. Differentiating under the integral sign for $\Re(\lambda + \rho)$ very negative and then using meromorphic continuation shows that it is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbf{X}}$ with eigenvalue $\lambda^2 - \rho^2$. The asymptotic behaviour is computed in [20, Appendix A], see also [20, Lemma 4 and Lemma 5]. □

Let $\phi \in \mathscr{H}^{r,s}$. Using Schur's Lemma and properties of the Poisson transform, see [1, pp. 74–76] for details, it can be seen that (with $\varepsilon \equiv r + s \mod 2$):

$$F_{\varepsilon,\lambda}\phi(x(t, y)) = c(\varepsilon, \lambda)\Phi_o^{r,s}(\lambda, t)\phi(y) = \Phi^{r,s}(\lambda, t)\phi(y), \qquad ((t, y) \in \mathbf{R} \times \mathbf{Y}), \tag{10}$$

where $\Phi^{r,s}(\lambda, \cdot) := c(\varepsilon, \lambda)\Phi_o^{r,s}(\lambda, \cdot)$.

We define the Fourier transform $\mathscr{F}f$ of any function $f \in C_c^\infty(\mathbf{X})$ as:

$$\mathscr{F}f(\varepsilon, \lambda, y) := \int_{\mathbf{X}} |\langle x, y\rangle|^{(\lambda-\rho)} \operatorname{sign}^\varepsilon\langle x, y\rangle f(x)dx, \tag{11}$$

for $\varepsilon \in \{0, 1\}$, $\Re\lambda \geq \rho$ and $y \in \mathbf{Y}$. Let now $f \in C_c^\infty(\mathbf{X})^{r,s}$ for some fixed K-type (r, s). Using spherical coordinates and (10), we can (re)write the Fourier transform of f as:

$$\mathscr{F}f(\varepsilon,\lambda,y)=\int_{\mathbf{R}_+}\Phi^{r,s}(-\lambda,t)f(x(t,y))J(t)dt.$$

We see that $\mathscr{F}f(\varepsilon,\lambda,y)$ extends to a meromorphic function in the λ-variable, with zeros and poles completely determined by the above expressions of $\Phi_o^{r,s}$ and (9).

We first consider the Riemannian case, that is $m=1$ $(\Rightarrow r=0)$. We note that $\langle x,y\rangle>0$ for all $x\in\mathbf{X}$, $y\in\mathbf{Y}$. The Fourier transform (11) is thus the Helgason–Fourier transform on $SO_o(1,n)/SO_o(n)$, see [14, Chapter 3], and we can formulate Hardy's theorem in this case as follows:

Theorem 4.2. *Let $1\le p\le\infty$, $1\le q\le\infty$. Let $1<N\le 2$ if $q<\infty$ and $0\le N<1$ if $q=\infty$. Assume that f is a measurable function on $\mathbf{X}=SO_o(1,n)/SO_o(n)$ satisfying:*

$$(1+|\cdot|)^{-M}e^{(1-2/p)\rho|\cdot|}e^{\alpha|\cdot|^2}f\in L^p(\mathbf{X})\quad\textit{and}\quad(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}\mathscr{F}f\in L^q(i\mathbf{R}\times\mathbf{Y}),$$

for positive constants M,α,β, with $M>\rho+\dfrac{1}{p}$, and $\alpha\beta=\dfrac{1}{4}$. Then f is a constant multiple of the Heat kernel $h_\beta=h_\beta^{n/2-1,-1/2}$ on $\mathbf{X}$, i.e., f is in particular a spherical (bi-K-invariant) function.

Proof. Let f be a measurable function satisfying the above growth conditions, whence as before $f\in L^1(\mathbf{X})\cap L^2(\mathbf{X})$, and we see that the Fourier transform $\mathscr{F}f$ is well-defined.

Define $\tilde\rho=\rho+|s|$, $a=|s|+\dfrac{n}{2}-1$ and $b=-\dfrac{1}{2}$ (i.e., $\tilde\rho=a+b+1$), then (modulo constants):

$$\Phi^{0,s}(\lambda,t)=\sinh^{|s|}(t)\frac{\Gamma(\lambda+\rho+|s|)}{\Gamma(\lambda+\rho)}\varphi_{-i\lambda}^{a,b}(t)=\sinh^{|s|}(t)P_s(\lambda)\varphi_{-i\lambda}^{a,b}(t),$$

where $P_s(\lambda):=(\lambda+\rho)(\lambda+\rho+1)\,\dots\,(\lambda+\rho+|s|-1)$. Let $f_{0,s}(t,y):=\mathbf{P}^{0,s}f(x(t,y))/\sinh^{|s|}(t)$. By (7) and continuity of the projection $\mathbf{P}^{0,s}$ we see that $f_{0,s}$ is a measurable function on $\mathbf{R}\times\mathbf{Y}$, even in the t-variable. With these identifications, we get:

$$\hat f_{0,s}^{a,b}(i\lambda,y):=\int_{\mathbf{R}_+}f_{0,s}(t,y)\varphi_{i\lambda}^{a,b}(t)\Delta^{a,b}(t)dt=P_s(\lambda)^{-1}\mathscr{F}\mathbf{P}^{0,s}f(\lambda,y).$$

We note that $\hat f_{r,s}^{a,b}(\lambda,y)$ is well-defined for all $\lambda\in\mathbf{C}$. Using spherical coordinates and the definition of $\mathbf{P}^{0,s}$, we get the following estimates of $f_{0,s}$ and $\hat f_{0,s}^{a,b}$:

$$(1+|\cdot|)^{-M}e^{(1-2/p)\tilde\rho|\cdot|}e^{\alpha|\cdot|^2}f_{0,s}(\cdot\,,y)\in L^p(\mathbf{R},|\Delta^{a,b}(t)|dt)$$

and

$$|P_s(i\,\cdot)|(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}\hat f_{0,s}^{a,b}(\cdot\,,y)\in L^q(\mathbf{R}),$$

for $y \in \mathbf{Y}$. It follows from (the proof of) Theorem 3.3 (and Lemma 3.2), that $\hat{f}^{a,b}_{0,0}(\lambda, y) = const.\ e^{-\beta\lambda^2}$ and that $\hat{f}^{a,b}_{0,s} = 0$ for $s \neq 0$. We conclude that $\mathscr{F}f = \mathscr{F}\mathbf{P}^{0,0}f$ and thus $f = \mathbf{P}^{0,0}f$, that is, f is a spherical (bi-K-invariant) function and $\mathscr{F}f = const.\ \mathscr{F}h_\beta$, implying that $f = const.\ h_\beta$. □

The above theorem has for general Riemannian symmetric spaces of the non-compact type been proved in [19] for the $p = q = \infty$ case and in [18] for the $p, q < \infty$ case. Note however that our proof is different, in particular the conclusion that the contribution from the K-types $(0, s)$ is zero for $s \neq 0$. Let us sketch their argument for this: It follows from (8) and (10) that $|\Phi^{0,s}(\lambda, t)| \leq \Phi^{0,0}(\Re\lambda, t)$, for all s, whence also $|\mathscr{F}\mathbf{P}^{0,s}f(\lambda, y)| \leq |\mathscr{F}\mathbf{P}^{0,0}f(\Re\lambda, y)|$. Assume that f and $\mathscr{F}f$ satisfy the natural decay conditions. Arguing as in the proof of theorem 3.3, it follows that $\mathscr{F}\mathbf{P}^{0,s}f(\lambda, y) = \phi^{0,s}(y)e^{-\beta\lambda^2}$, for some function $\phi^{0,s}$ on $\mathbf{Y}$. But $\Phi^{0,s}(-\rho, t) = 0$ since $F_{-\rho}\phi(x) = \int_{\mathbf{Y}} \phi(y)dy = 0$ for $\phi \in \mathscr{H}^{0,s}$, $s \neq 0$, and we conclude that $\mathscr{F}\mathbf{P}^{0,s}f = 0$ for $s \neq 0$.

We now turn to the pseudo-Riemannian case, that is, $m > 1$. It is in this case more convenient to consider a normalised Fourier transform: $\mathscr{F}_o f(\varepsilon, \lambda, y) := c(\varepsilon, -\lambda)^{-1}\mathscr{F}f(\varepsilon, \lambda, y)$; in particular:

$$\mathscr{F}_o f(\varepsilon, \lambda, y) = \int_{\mathbf{R}_+} \Phi_o^{r,s}(-\lambda, t)f(x(t, y))J(t)dt,$$

for $f \in C_c^\infty(\mathbf{X})^{r,s}$.

It is remarkable that the decay conditions in the Riemannian case force the function f to be spherical (bi-K-invariant). More so, because this is not the case in the pseudo-Riemannian case. In fact, we will show that there are infinitely (albeit countably) many linearly independent non-zero functions f on $\mathbf{X}$ satisfying the natural decay conditions with $\alpha\beta = \frac{1}{4}$, namely the pseudo-Heat kernels defined below: Let $a = \frac{n}{2} - 1$ and $b = |r| + \frac{m}{2} - 1$ and define the pseudo-Heat kernel $h_\beta^{r,0}(\phi)$ with index $(r, 0)$ on $\mathbf{X}$ by:

$$h_\beta^{r,0}(\phi)(x(t, y)) := \cosh^{|r|}(t)h_\beta^{a,b}(t)\phi(y), \qquad ((t, y) \in \mathbf{R}_+ \times \mathbf{Y}),$$

for any $\phi \in \mathscr{H}^{r,0}$. It can be seen that $h_\beta^{r,0}(\phi)$ defines a function in $C^\infty(\mathbf{X})^{r,0}$, see [1, p. 71], and (5) yields the following estimates:

$$|h_\beta^{r,0}(\phi)(x(t, y))| \leq C(1 + t)^{(1/2)(n-1)}e^{-\rho t - t^2/4\beta}|\phi(y)|, \tag{12}$$

for all $(t, y) \in \mathbf{R}_+ \times \mathbf{Y}$, where $C > 0$ is a positive constant.

THEOREM 4.3. *Let $m \geq 2$. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Let $\frac{1}{2}(n+1) < N \leq \frac{1}{2}(n+3)$ if $q < \infty$ and $\frac{1}{2}(n-1) \leq N < \frac{1}{2}(n+1)$ if $q = \infty$. Assume that f is a measurable function on $\mathbf{X}$ satisfying:*

$$(1+|\cdot|)^{-M}e^{(1-2/p)\rho|\cdot|}e^{\alpha|\cdot|^2}f\in L^p(\mathbf{X})$$

and

$$(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}\mathscr{F}_o f\in L^q(\{0,1\}\times i\mathbf{R}\times\mathbf{Y}),$$

for positive constants M,α,β, *with* $M>\frac{1}{2}(n-1)+\frac{1}{p}$, *and* $\alpha\beta=\frac{1}{4}$. *Then* $f=\sum_r \mathbf{P}^{r,0}f$. *The pseudo-Heat kernels* $h_\beta^{r,0}(\phi)$ *satisfy the above decay conditions for any* r *and any* $\phi\in\mathscr{H}^{r,0}$.

Proof. Let f be a measurable function satisfying the above growth conditions, then $f\in L^1(\mathbf{X})\cap L^2(\mathbf{X})$ and the Fourier transform $\mathscr{F}f$ is well-defined.

Define $\tilde{\rho}=\rho+|r|+|s|$, $a=|s|+\frac{n}{2}-1$ and $b=|r|+\frac{m}{2}-1$ (i.e., $\tilde{\rho}=a+b+1$), then:

$$\Phi_o^{r,s}(\lambda,t)=2^{\lambda-\tilde{\rho}}\cosh^{|r|}(t)\sinh^{|s|}(t)\frac{\Gamma\left(\frac{1}{2}(\lambda+\tilde{\rho})\right)\Gamma\left(\frac{1}{2}(\lambda-\tilde{\rho}+n+2|s|)\right)}{\Gamma\left(\frac{n}{2}+|s|\right)\Gamma(\lambda)}\varphi_{-i\lambda}^{a,b}(t).$$

Let $f_{r,s}(t,y):=\mathbf{P}^{r,s}f(x(t,y))/\cosh^{|r|}(t)\sinh^{|s|}(t)$. By (7) and continuity of the projection $\mathbf{P}^{r,s}$, we see that $f_{r,s}$ is a measurable function on $\mathbf{R}\times\mathbf{Y}$, even in the t-variable. Let also

$$Q_{r,s}(\lambda):=2^{\lambda-3\tilde{\rho}}\frac{\Gamma\left(\frac{1}{2}(\lambda+\tilde{\rho})\right)\Gamma\left(\frac{1}{2}(\lambda-\tilde{\rho}+n+2|s|)\right)}{\Gamma\left(\frac{n}{2}+|s|\right)\Gamma(\lambda)}.$$

We note that $|Q_{r,s}(i\lambda)|\sim\text{const.}\ |\lambda|^{|s|+(1/2)(n-1)}$ for $|\lambda|\to\infty$, see [8, 1.18(6)]. From the above we can write:

$$\hat{f}_{r,s}^{a,b}(i\lambda,y):=\int_{\mathbf{R}_+}f_{r,s}(t,y)\varphi_{i\lambda}^{a,b}(t)\Delta^{a,b}(t)dt=Q_{r,s}(\lambda)^{-1}\mathscr{F}_o\mathbf{P}^{r,s}f(\varepsilon,\lambda,y).$$

We note that $\hat{f}_{r,s}^{a,b}(\lambda,y)$ is well-defined for all $\lambda\in\mathbf{C}$. Using spherical coordinates and the definition of $\mathbf{P}^{r,s}$, we get the following estimates of $f_{r,s}$ and $\hat{f}_{r,s}^{a,b}$:

$$(1+|\cdot|)^{-M}e^{(1-2/p)\tilde{\rho}|\cdot|}e^{\alpha|\cdot|^2}f_{r,s}(\cdot,y)\in L^p(\mathbf{R},|\Delta^{a,b}(t)|dt)$$

and

$$|Q_{r,s}(i\cdot)|(1+|\cdot|)^{-N}e^{\beta|\cdot|^2}\hat{f}_{r,s}^{a,b}(\cdot,y)\in L^q(\mathbf{R}),$$

for $y\in\mathbf{Y}$. It follows from (the proof of) Theorem 3.3 (and Lemma 3.2), that $\hat{f}_{r,0}^{a,b}(\lambda,y)=\mathit{const.}\ e^{-\beta\lambda^2}$ for $y\in\mathbf{Y}$, and that $\hat{f}_{r,s}^{a,b}=0$ for $s\neq 0$.

We finally note that $\mathscr{F}_o h_\beta^{r,0}(\phi)(\varepsilon,\lambda,y) = Q_{r,0}(\lambda)e^{-\beta(-\lambda^2+(\rho+|r|)^2)}\phi(y)$, which together with the estimates (12) show that the pseudo-Heat kernels $h_\beta^{r,0}(\phi)$ satisfy the decay conditions. $\square$

In other words, we cannot generalise the main part of Hardy's theorem, the $\alpha\beta = \dfrac{1}{4}$ case, to pseudo-Riemannian symmetric spaces: there is *not* a unique (modulo constants) function satisfying the natural decay conditions—unless we fix the index r in the K-types $(r,0)$.

For completeness, we state Hardy's Uncertainty Principle, and its L^p versions, for the Fourier transform on $\mathbf{X}$, see also [2, Theorem 3.2] and [3, Theorem 3.2] for other proofs.

Corollary 4.4. *Let $1 \le p \le \infty$, $1 \le q \le \infty$. Assume that f is a measurable function on $\mathbf{X}$ satisfying:*

$$e^{(1-2/p)\rho|\cdot|}e^{\alpha|\cdot|^2}f \in L^p(\mathbf{X}) \qquad \textit{and} \qquad e^{\beta|\cdot|^2}\mathscr{F}_o f \in L^q(\{0,1\}\times i\mathbf{R}\times\mathbf{Y}),$$

for positive constants α and β. If

(1) $\alpha\beta > \dfrac{1}{4}$, *then* $f = 0$.

(2) $\alpha\beta = \dfrac{1}{4}$ *and* $q < \infty$, *then* $f = 0$.

(3) $\alpha\beta < \dfrac{1}{4}$, *then there are infinitely many linearly independent solutions.*

Proof. Follows as above from the similar results (or their proofs) for the Jacobi transform. $\square$

5. Remarks and further results

It is well-known that $SO_o(2,2)/SO_o(1,2) \simeq SL(2,\mathbf{R}) \simeq SU(1,1)$. We established in [1, Chapter 5] a link between the Fourier transform on $SO_o(2,2)/SO_o(1,2)$ and the group Fourier transform on $SL(2,\mathbf{R})$, and this allows us to transfer the results in §4 to $SL(2,\mathbf{R})$. A function f of K-type $(r,0)$ on $SO_o(2,2)/SO_o(1,2)$ corresponds to a spherical function f of type (r,r) on $SL(2,\mathbf{R})$, i.e., $f(k_1xk_2) = e_r(k_1)f(x)e_r(k_2)$, for all $k_1,k_2 \in SO(2)$, $x \in SL(2,\mathbf{R})$, where the e_r's are the usual characters on $SO(2)$. So, in the $SL(2,\mathbf{R})$ picture, the condition $s = 0$ implies that a function f on $SL(2,\mathbf{R})$ has the same K-dependence from the left and from the right.

Let us consider the group $G = SU(1,1)$ in more detail. We use [16, §4.3] as reference. Let $G = KAN$ denote an Iwasawa decomposition of G, where in particular:

$$K = \left\{ u_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \mid \theta \in [0, 4\pi[\right\} \qquad \text{and}$$

$$A = \left\{ a_t = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbf{R} \right\}.$$

Let also $M = \{\pm I\}$. The irreducible representations $\hat{K} \simeq \mathbf{Z}/2$ of K are given by: $\delta_r(u_\theta) = e^{ir\theta}$, and $\hat{M} \simeq \left\{0, \frac{1}{2}\right\}$ in the same identification. The principal series representation $(\pi_{\xi,\lambda}, \mathscr{H}_{\xi,\lambda})$, $\lambda \in \mathbf{C}$, $\xi \in \left\{0, \frac{1}{2}\right\}$, of G is induced by the representation $ma_tn \mapsto e^{-i\lambda t}\delta_\xi(m)$ of MAN. Let $\{e_r\}_{r \in \mathbf{Z}+\xi}$ be an orthonormal basis of $\mathscr{H}_{\xi,\lambda}$ with $e_r(u_\theta) = e^{ir\theta}$. The matrix coefficients $\pi_{\xi,\lambda,r,s}(x) = \langle \pi_{\xi,\lambda}(x)e_s, e_r\rangle$, $x \in G$, $\xi \in \left\{0, \frac{1}{2}\right\}$, $r, s \in \mathbf{Z} + \xi$, of the principal series representation of G can be written in terms of Jacobi functions as:

$$\pi_{\xi,\lambda,r,s}(a_t) = P_{|r-s|}(\lambda) \sinh^{|r-s|}(t) \cosh^{r+s}(t)\varphi_\lambda^{|r-s|,r+s}(t),$$

where $P_{|r-s|}$ is a polynomial of degree $|r-s|$, with $P_0 = 1$. This explicit expression of the matrix coefficients on G yields another path to "Hardy's theorem" for $SU(1,1) \simeq SL(2,\mathbf{R})$. We note in particular that the matrix coefficients:

$$\pi_{\xi,\lambda,r,r}(a_t) = \cosh^{2r}(t)\varphi_\lambda^{0,2r}(t),$$

for $t \in \mathbf{R}_+$, satisfy the "same" growth estimates and that they do not have any zeroes.

The Fourier transform $\mathscr{F}_G$ is defined as:

$$\mathscr{F}_G f(\pi_{\xi,\lambda}) := \int_G f(x)\pi_{\xi,\lambda}(x)dx,$$

for a nice function f on G. Let now f be an even function on $\mathbf{R}$ and define a spherical function f^r of type (r,r) on G by: $f^r(u_{\theta_1}a_tu_{\theta_2}) := \cosh^{2r}(t)f(t)e^{ir(\theta_1+\theta_2)}$. Using the Cartan decomposition of G, we compute the matrix coefficients of $\mathscr{F}_G f^r r(\pi_{\xi,\lambda})$:

$$\begin{aligned}
\langle \mathscr{F}_G f^r(\pi_{\xi,\lambda})e_r, e_r\rangle &= \int_{\mathbf{R}} f^r(a_t)\langle \pi_{\xi,\lambda}(a_t)e_r, e_r\rangle \sinh(t)\cosh(t)dt \\
&= \int_{\mathbf{R}} f^r(a_t)\cosh^{2r}(t)\varphi_\lambda^{0,2r}(t)\sinh(t)\cosh(t)dt \\
&= \int_{\mathbf{R}} f(t)\varphi_\lambda^{0,2r}(t)\sinh(t)\cosh^{4r+1}(t)dt = 2^{(-4r-1)}\hat{f}^{0,2r}(\lambda).
\end{aligned}$$

Consider in particular the functions $h^r_\beta(u_{\theta_1}a_t u_{\theta_2}) := \cosh^{2r}(t) h^{0,2r}_\beta(t) e^{ir(\theta_1+\theta_2)}$, for $r \in \mathbf{Z}/2$, then $\langle \mathscr{F}_G h^r_\beta(\pi_{\xi,\lambda}) e_r, e_r \rangle = ce^{-\beta\lambda^2}$ and $|h^r_\beta(u_{\theta_1}a_t u_{\theta_2})| \leq C(1+t)^{1/2} \cdot e^{-t-t^2/4\beta}$, for positive constants c and C.

Let $\mathbf{F}$ be either $\mathbf{C}$ or $\mathbf{H}$ and let $x \mapsto \bar{x}$ be the standard (anti)-involution of $\mathbf{F}$. Let m and n be two positive integers and let $[\,,]$ be the Hermitian form on $\mathbf{F}^{m+n}$ given by

$$[x, y] = x_1\bar{y}_1 + \cdots + x_m\bar{y}_m - x_{m+1}\bar{y}_{m+1} - \cdots - x_{m+n}\bar{y}_{m+n},$$

for $x, y \in \mathbf{F}^{m+n}$. Let $G = U(m, n; \mathbf{F})$ denote the group of all $(m+n) \times (m+n)$ matrices over $\mathbf{F}$ preserving $[\,,]$. Thus $U(m, n; \mathbf{C}) = U(m, n)$ and $U(m, n; \mathbf{H}) = Sp(m, n)$ in standard notation. Let H be the subgroup of G stabilising the line $\mathbf{F}(1, 0, \ldots, 0)$ in $\mathbf{F}^{m+n}$. We can identify H with $U(1, 0; \mathbf{F}) \times U(m-1, n; \mathbf{F})$ and the homogeneous space G/H (which is a reductive symmetric space) with the projective image of the space $\{z \in \mathbf{F}^{m+n} \,|\, [z, z] = 1\}$. The statement and proofs in the previous chapter also hold for the Fourier transform on G/H. This is seen either by embedding G/H into $SO_o(dm, dn)/SO_o(dm-1, dn)$, with $d = \dim_{\mathbf{R}} \mathbf{F}$, or again by expressing the Fourier transform of K-finite functions using modified Jacobi functions. See [1, p. 117] for more details.

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