

Morse functions with sphere fibers

*Dedicated to Professor Takao Matumoto on the occasion of
his 60th birthday*

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ABSTRACT. A smooth closed manifold is said to be an almost sphere if it admits a Morse function with exactly two critical points. In this paper, we characterize those smooth closed manifolds which admit Morse functions such that each regular fiber is a finite disjoint union of almost spheres. We will see that such manifolds coincide with those which admit Morse functions with at most three critical values. As an application, we give a new proof of the characterization theorem of those closed manifolds which admit special generic maps into the plane. We also discuss homotopy and diffeomorphism invariants of manifolds related to the minimum number of critical values of Morse functions; in particular, the Lusternik-Schnirelmann category and spherical cone length. Those closed orientable 3-manifolds which admit Morse functions with regular fibers consisting of spheres and tori are also studied.

1. Introduction

In [42, 49] Suzuoka and the author studied the topology of generic smooth maps between smooth manifolds whose regular fibers are unions of spheres or homotopy spheres. One of the major motivations for studying such maps is the fact that special generic maps have such a property, where a special generic map is a generic smooth map between smooth manifolds which admit only the simplest singularity, i.e. the definite fold point (for a precise definition, see Definition 4.1 of the present paper). In other words, our aim was to generalize those topological results which had been known for special generic maps (for example, see [5, 39, 41]) to the class of generic smooth maps whose regular fibers are unions of (homotopy) spheres. This makes sense, since the global

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topology of generic maps has not been clarified so much except for the class of special generic maps.

In this paper, we study Morse functions on smooth manifolds whose regular fibers are unions of spheres or homotopy spheres. Here, a Morse function is a smooth function whose critical points are all nondegenerate. Note that we allow two or more distinct critical points to have the same value.

In order to formulate our results, in §2 we will introduce the notion of an almost sphere. A closed connected manifold of positive dimension is called an almost sphere if it admits a Morse function with exactly two critical points. We will recall several known results about almost spheres.

In §3, we will see that the class of manifolds admitting Morse functions whose regular fibers are unions of almost spheres coincide with the class of manifolds which admit Morse functions with at most three critical values (Proposition 3.2). Furthermore, we will give a characterization theorem of such manifolds in terms of their decompositions into some simple pieces (Theorem 3.7). For low dimensions, we can completely characterize those manifolds which admit Morse functions with almost sphere fibers (Corollaries 3.14–3.16). We will also deduce several necessary conditions for such manifolds (Corollary 3.17).

In §4, we will apply the result obtained in §3 to give a new proof of the characterization theorem of those closed manifolds which admit special generic maps into the plane, originally proved in [41]. In the proof, we will show that such manifolds admit Morse functions with almost sphere fibers as well.

In §5, we will study several homotopy and diffeomorphism invariants of smooth manifolds related to the number of critical values of Morse functions. It is a classical result of Lusternik-Schnirelmann [29] that the number of critical points of a smooth function on a given closed manifold M is always bounded below by the smallest number of open sets covering M each of which is contractible in M . The smallest number of such open sets minus 1 is called the Lusternik-Schnirelmann category of M . In homotopy theory, it is known that the Lusternik-Schnirelmann category is closely related to the cone length of the space, where the cone length of a space X is, roughly speaking, the number of cones necessary to obtain X homotopically from a contractible space by attaching them successively. We will see that the number of critical values of a Morse function on a given manifold M is always bounded below by its spherical cone length plus 1, where a spherical cone length is, roughly speaking, the number of cones over a bouquet of spheres necessary to obtain M homotopically from a contractible space. Using this, we will show that the difference between the minimum number of critical values of Morse functions and the minimum number of critical points of smooth functions on a given manifold can be arbitrarily large, by giving explicit examples.

In §6, we study the invariants mentioned in §5 more in detail for manifolds of dimension 3. Furthermore, we will show that a closed connected orientable 3-manifold admits a Morse function whose regular fibers are unions of spheres and tori if and only if it is a connected sum of some copies of $S^1 \times S^2$ and/or lens spaces. Finally, we will propose a conjecture concerning Morse functions on a 3-manifold and its Heegaard genus.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. The homology and cohomology groups are with integer coefficients unless otherwise indicated. The symbol “ \cong ” denotes a diffeomorphism between smooth manifolds. The symbol “ $\#$ ” is used for a usual connected sum of manifolds, while the symbol “ \natural ” is used for a boundary connected sum.

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2. Preliminaries

Let M be a smooth manifold. A smooth function $f : M \rightarrow \mathbf{R}$ is called a *Morse function* if its critical points are all nondegenerate. We do not assume that the critical values are all distinct: distinct critical points may have the same value.

DEFINITION 2.1. A smooth closed n -dimensional manifold M with $n > 0$ is called an *almost sphere* (or *almost n -sphere*) if it admits a Morse function $f : M \rightarrow \mathbf{R}$ with exactly two critical points. For example, the standard n -sphere S^n is an almost sphere.

For a smooth closed manifold M , we denote by $\text{Crit}(M)$ the minimum number of critical points over all smooth (not necessarily Morse) functions on M . Then we have the following characterization of almost n -spheres.

PROPOSITION 2.2. *For a smooth closed n -dimensional manifold M , the following three are equivalent.*

- (1) M is an almost n -sphere.
- (2) M is diffeomorphic to the closed manifold obtained by attaching two copies of the n -dimensional disk along the boundary spheres.
- (3) For $n \leq 6$, M is diffeomorphic to the standard n -dimensional sphere S^n , and for $n \geq 7$, M is a homotopy n -sphere.

Furthermore, for $n \neq 4$, all the above three conditions are equivalent to the following.

$$(4) \quad \text{Crit}(M) = 2.$$

PROOF. The equivalence (1) \Leftrightarrow (2) is an easy exercise of the standard Morse theory. The equivalence (2) \Leftrightarrow (3) is a consequence of celebrated results due to Smale [45, 46, 47], Cerf [7], etc.

It is easy to see that (1) always implies (4). Conversely, suppose (4) holds. It is known that then M is homeomorphic to the standard n -sphere (for example, see [35], [10, Proposition 7.24], [40]). Therefore, (3) holds, except possibly for $n = 4$. \square

In [36], a manifold which satisfies Proposition 2.2 (2) is called a *twisted sphere*. For Proposition 2.2 (2), refer to [34, p. 442] as well. In fact, it is not difficult to see that for every $n \geq 1$, the set of all diffeomorphism classes of oriented almost n -spheres forms an abelian group under connected sum. Following [34], we denote this group by Γ^n . Note that $\Gamma^n = 0$ for $n \leq 6$ and that Γ^n is isomorphic to the h -cobordism group Θ_n of oriented homotopy n -spheres for $n \geq 5$.

If M is an almost n -sphere, then $M \setminus \text{Int } D^n$ is diffeomorphic to the standard n -dimensional disk for every n -dimensional disk D^n embedded in M . Note also that every almost n -sphere is homeomorphic to the standard n -sphere S^n .

3. Morse functions with almost sphere fibers

In this section, we give a characterization theorem of those closed manifolds which admit Morse functions with regular fibers consisting of almost spheres. Let us begin by the following definition.

DEFINITION 3.1. A proper smooth function $f : M \rightarrow \mathbf{R}$ is *almost spherical* (or f has *almost sphere fibers*) if every component of $f^{-1}(y)$ is an almost sphere for all regular value $y \in f(M) \subset \mathbf{R}$. If every component of $f^{-1}(y)$ is diffeomorphic to the standard sphere for all regular value $y \in f(M) \subset \mathbf{R}$, then we say that f is *purely spherical*. Note that every surface clearly admits a purely spherical Morse function.

The first result of this section is the following.

PROPOSITION 3.2. *Let M be a smooth closed connected n -dimensional manifold with $n \geq 2$. Then the following three conditions are equivalent.*

- (1) *There exists a purely spherical Morse function $f : M \rightarrow \mathbf{R}$.*
- (2) *There exists an almost spherical Morse function $f : M \rightarrow \mathbf{R}$.*
- (3) *There exists a Morse function $f : M \rightarrow \mathbf{R}$ with at most three critical values.*

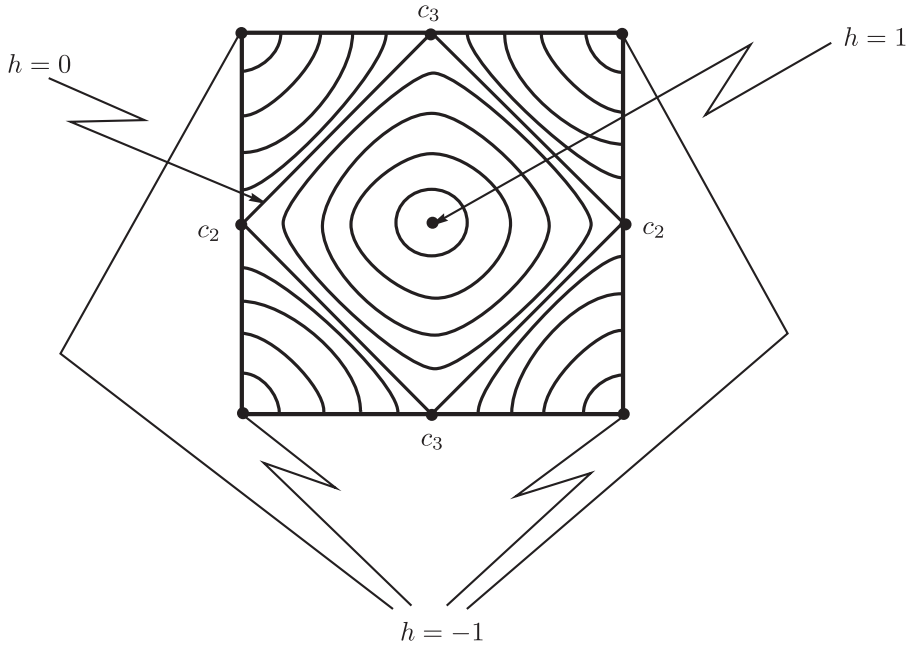


Fig. 1. Smooth function h on $J \times J$

Before proving Proposition 3.2, let us give an illustrating example of Morse functions with exactly three critical values.

EXAMPLE 3.3. Let us consider the square $J \times J$ with $J = [-1, 1]$ and a smooth function $h : J \times J \rightarrow [-1, 1]$ whose level sets are as depicted in Fig. 1. Consider the equivalence relation on $J \times J$ generated by

$$(t, -1) \sim (t, 1) \quad \text{and} \quad (-1, t) \sim (1, t) \quad (\forall t \in J).$$

Then the quotient space $(J \times J)/\sim$ is naturally identified with the 2-dimensional torus T^2 , and the function h induces a Morse function $f_1 : T^2 \rightarrow [-1, 1]$. Note that it has exactly four critical points c_1, c_2, c_3 and c_4 with $f_1(c_1) = 1, f_1(c_2) = f_1(c_3) = 0$ and $f_1(c_4) = -1$. Therefore, f_1 has exactly three critical values.

If we consider the equivalence relation on $J \times J$ generated by

$$(t, -1) \sim (-t, 1) \quad \text{and} \quad (-1, t) \sim (1, t) \quad (\forall t \in J),$$

then the quotient space $(J \times J)/\sim$ is naturally identified with the Klein bottle K^2 , and the function h induces a Morse function $f_2 : K^2 \rightarrow [-1, 1]$. This also has exactly three critical values.

PROOF OF PROPOSITION 3.2. Clearly (1) implies (2).

Let us show that (2) implies (3). Suppose that there exists an almost spherical Morse function $f : M \rightarrow \mathbf{R}$. Let $c_1 < c_2 < \cdots < c_m$ be its critical values. Take real numbers $t_i, i = 1, 2, \dots, m + 1$, such that $t_1 < c_1 < t_2 < c_2 < t_3 < \cdots < t_m < c_m < t_{m+1}$ and set $M_i = f^{-1}[t_1, t_i], i = 1, 2, \dots, m + 1$. Note that M_1 is the empty set. Since f is almost spherical, M_i is a compact n -dimensional manifold such that each connected component of ∂M_i is an almost sphere. Furthermore, $M_{i+1}, i = 1, 2, \dots, m$, is obtained from M_i by simultaneously attaching those handles which correspond to the critical points with critical value c_i .

Since $\partial M_i = f^{-1}(t_i)$ consists of almost $(n - 1)$ -spheres, for each component S of $f^{-1}(t_i), i \geq 2$, there exists an $(n - 1)$ -disk embedded in S such that the handles of index different from 0 or n are attached on the union of the $(n - 1)$ -disks. Furthermore, these $(n - 1)$ -disks can be isotoped so that it does not intersect those handles of positive index which were attached previously. Therefore, M admits a handlebody decomposition of the form

$$\left(\bigcup_{a=1}^s h_a^0 \right) \cup \left(\bigcup_{b=1}^t h_b^{r_b} \right) \cup \left(\bigcup_{c=1}^u h_c^n \right), \tag{3.1}$$

where h_*^r denotes a handle of index r , the indices of the handles in the middle satisfy $1 \leq r_b \leq n - 1, b = 1, 2, \dots, t$, and the handles $h_b^{r_b}$ are attached simultaneously to the union of the 0-handles.

Then any Morse function corresponding to such a handlebody decomposition is a desired function as in (3).

Finally a Morse function as in (3) is always purely spherical. Therefore, (3) implies (1). This completes the proof of Proposition 3.2. \square

REMARK 3.4. Suppose that a manifold M admits an almost spherical Morse function. By virtue of the above proof (see also the second paragraph of the proof of Theorem 3.7), we see easily that the minimum number of critical points over all Morse functions on M can be realized by a purely spherical Morse function.

In order to state a characterization theorem of manifolds which admit Morse functions as in Proposition 3.2, we need the following definition.

DEFINITION 3.5. (1) For $n \geq 3$ and $1 \leq r \leq n - r$, let us consider an $(n - r - 1)$ -sphere S_2 embedded in S^{n-1} . Let S_1 be the $(r - 1)$ -sphere embedded as the boundary of an r -disk fiber of the normal disk bundle to S_2 in S^{n-1} . Let W be the n -dimensional handlebody obtained from a 0-handle by attaching an r -handle and an $(n - r)$ -handle simultaneously along S_1 and S_2 respectively, where we identify S^{n-1} with the boundary of the 0-handle. Note

that W is a compact (possibly nonorientable when $r = 1$) connected n -dimensional manifold with boundary. It is easy to observe that ∂W is a homotopy $(n - 1)$ -sphere if $r < n - r$ or if $n \not\equiv 0 \pmod{4}$. If ∂W is an almost $(n - 1)$ -sphere, then W (or any manifold which is diffeomorphic to such a manifold W) is called an *elementary handlebody* with *index pair* $(r, n - r)$.

(2) For $n \geq 4$ and $n \equiv 0 \pmod{2}$, let W be an n -dimensional handlebody obtained from a 0-handle by attaching some handles of index $n/2$ simultaneously. If ∂W is an almost $(n - 1)$ -sphere, then W (or any manifold which is diffeomorphic to such a manifold W) is called a *pseudo elementary handlebody* with *index* $n/2$.

REMARK 3.6. (1) In [33, §1], Milnor constructs an elementary handlebody W_1 of dimension $m + n + 2$ with index pair $(m + 1, n + 1)$, by using the standard pair of an m -sphere and an n -sphere with linking number ± 1 disjointly embedded in S^{m+n+1} , and studies the diffeomorphism type of the boundary homotopy sphere ∂W_1 . Note that Milnor uses the “standard identifications” between spheres for attaching the handles. Therefore, even if S_2 is a standardly embedded $(n - r - 1)$ -sphere in Definition 3.5 (1), the manifold W may not be diffeomorphic to a manifold constructed by Milnor.

(2) In the stable range (i.e. when $r \geq (n/3) + 1$), the embedded $(n - r - 1)$ -sphere S_2 is standard in Definition 3.5 (1) (see, for example, [17]). In fact, in the stable range, any pair of an $(r - 1)$ -sphere and an $(n - r - 1)$ -sphere disjointly embedded in S^{n-1} with linking number ± 1 is standard by virtue of a result of Haefliger [17, 18].

Now we have the following characterization theorem.

THEOREM 3.7. *Let M be a smooth closed connected n -dimensional manifold with $n \geq 3$. Then conditions (1)–(3) of Proposition 3.2 are all equivalent to the following.*

- (4) *The manifold $M \setminus \text{Int } D^n$ is diffeomorphic to $\natural_{i=1}^k W_i$ for some $k \geq 0$, where the boundary connected sum over the empty set is assumed to be the n -dimensional disk, and each W_i is either*
- (i) *an elementary handlebody of dimension n , or*
 - (ii) *a pseudo elementary handlebody of dimension n with index $n/2$ if $n \equiv 0 \pmod{4}$.*

PROOF. Suppose that M is diffeomorphic to a manifold as in (4). Then it is easy to see that M admits a handlebody decomposition of the form

$$h^0 \cup \left(\bigcup_{j=1}^{\ell} h_j^{r_j} \right) \cup h^n$$

for some $0 \leq \ell < +\infty$ and $1 \leq r_j \leq n-1$, $j = 1, 2, \dots, \ell$, where the handles $h_1^{r_1}, h_2^{r_2}, \dots, h_\ell^{r_\ell}$ are attached simultaneously to h^0 . Therefore M admits a Morse function $f : M \rightarrow \mathbf{R}$ with $\ell + 2$ critical points such that each handle of index r corresponds to a critical point of index r and that all the critical points corresponding to the handles $h_j^{r_j}$, $j = 1, 2, \dots, \ell$, have the same value. Then f has exactly three critical values and a regular fiber is either empty or diffeomorphic to the standard $(n-1)$ -sphere. Thus (1), (2) and (3) of Proposition 3.2 hold.

Let us now show that (3) implies (4). By the proof of Proposition 3.2, M admits a handlebody decomposition as in (3.1). Since M is connected, the union of the 0-handles and the 1-handles are connected. Therefore, the union of the s 0-handles and some $s-1$ 1-handles connecting them can be considered as a 0-handle. Thus, we may assume that the number of 0-handles s is equal to 1. By dualizing the argument, we may also assume that the number of n -handles u is equal to 1.

First note that for each r with $0 \leq r \leq n$, the number of r -handles coincides with the r -th betti number of M . Therefore, for each r with $0 \leq r < n/2$, the number of r -handles coincides with that of $(n-r)$ -handles by Poincaré duality for $\mathbf{Z}/2\mathbf{Z}$ -coefficients.

For each v with $0 \leq v \leq n/2$, let M_v be the union of the 0-handle h^0 and all the handles $h_b^{r_b}$ with $1 \leq r_b \leq v$ or $n-v \leq r_b \leq n-1$. Let us show, by induction on v , that M_v , $v \geq 1$, is diffeomorphic to a boundary connected sum of (pseudo) elementary handlebodies, provided that $n \neq 5$. (For $n = 5$, we will use a different argument to prove the theorem.)

Let us begin by considering the case $v = 1$. First note that ∂M_1 must be connected, since otherwise $M \setminus \text{Int } h^n$ would have disconnected boundary, which leads to a contradiction. For an $(n-1)$ -handle h^{n-1} , its attaching sphere is an $(n-2)$ -sphere embedded in an $(n-1)$ -sphere, the boundary of the 0-handle h^0 . Therefore, by Alexander duality, the boundary of the union $h^0 \cup h^{n-1}$ has two connected components. Thus, there must be a 1-handle h^1 connecting the two connected components. Let us put $Y = h^0 \cup h^1 \cup h^{n-1}$ and show that it is an elementary handlebody.

By the differentiable Schoenflies theorem (see [4] for $n = 3$, [1] for $n = 4$, and [36, p. 112] for $n \geq 6$), the attaching sphere is isotopic to the standard $(n-2)$ -sphere embedded in an $(n-1)$ -sphere. Therefore, the union of the 0-handle h^0 and the $(n-1)$ -handle h^{n-1} is diffeomorphic to $\Sigma^{n-1} \times [0, 1]$ for some almost $(n-1)$ -sphere Σ^{n-1} . Hence, the boundary of $Y = h^0 \cup h^1 \cup h^{n-1}$ is diffeomorphic either to $\Sigma^{n-1} \# (-\Sigma^{n-1}) \cong S^{n-1}$ or to the almost $(n-1)$ -sphere $\Sigma^{n-1} \# \Sigma^{n-1}$, where $-\Sigma^{n-1}$ denotes the manifold Σ^{n-1} with the reversed orientation, and the second case occurs if and only if Y is nonorientable. Hence, Y is an elementary handlebody with index pair $(1, n-1)$.

Let C be a small open collar neighborhood of ∂Y in Y , which we identify with $\partial Y \times [0, 1)$. Note that $Y \setminus C$ is diffeomorphic to Y . Furthermore, let \mathcal{A} be a closed $(n - 1)$ -disk embedded in $\partial Y \cap \partial h^0$ such that all the handles of index 1 and $n - 1$ of M are attached outside of \mathcal{A} . Since ∂Y is an almost $(n - 1)$ -sphere, we see the following:

- (i) $Y' = (Y \setminus C) \cup (\mathcal{A} \times [0, 1))$ is diffeomorphic to Y ,
- (ii) the closure D_Y of $Y \setminus Y'$ is diffeomorphic to the n -dimensional disk, and
- (iii) $Y' \cap D_Y$ is diffeomorphic to the $(n - 1)$ -dimensional disk.

This implies that M_1 is diffeomorphic to the boundary connected sum of the elementary handlebody Y and a handlebody M'_1 obtained by simultaneously attaching 1- and $(n - 1)$ -handles to an 0-handle, where the number of 1-handles coincides with that of $(n - 1)$ -handles and is smaller than that for M_1 by one. Therefore, by an inductive argument, we see that M_1 is diffeomorphic to a boundary connected sum of elementary handlebodies with index pair $(1, n - 1)$, provided that $n \neq 5$.

Now let us proceed by induction on v . Let us assume $2 \leq v \leq n/2$ and M_{v-1} is diffeomorphic to a boundary connected sum of elementary handlebodies. Since ∂M_{v-1} is an almost $(n - 1)$ -sphere, there exists an embedded $(n - 1)$ -disk \mathcal{A}_1 in ∂M_{v-1} such that all the handles of index v and $n - v$ are attached to \mathcal{A}_1 . Therefore, M_v is diffeomorphic to the boundary connected sum of M_{v-1} and a handlebody P_v obtained by simultaneously attaching v - and $(n - v)$ -handles to a 0-handle. Let us determine the structure of P_v .

Case 1. $3 \leq v < n/2$.

Note that in this case, we have $n \geq 7$.

Let h^{n-v} be an $(n - v)$ -handle. By sliding handles of index v , we may assume that there exists a v -handle h^v such that the attaching spheres S_1 of h^v and S_2 of h^{n-v} in the boundary of the 0-handle h^0 of P_v have linking number ± 1 , by virtue of the Poincaré duality for the manifold

$$h^0 \cup \left(\bigcup_{2 \leq r_b \leq n-2} h_b^{r_b} \right), \tag{3.2}$$

which has an almost sphere boundary. Note that the boundary of the union $Y_v = h^0 \cup h^v \cup h^{n-v}$ is simply connected, since $n - 3 \geq n - v > v \geq 3$. Furthermore, it is easy to check that ∂Y_v has the same \mathbf{Z} -homology as S^{n-1} . Therefore, ∂Y_v is a homotopy $(n - 1)$ -sphere. By Proposition 2.2, ∂Y_v is an almost $(n - 1)$ -sphere, since we have assumed $n \geq 7$.

Furthermore, if $v \geq (n/3) + 1$, then the attaching spheres S_1 and S_2 form a standard pair as noted in Remark 3.6 (2). Even if $v \geq (n/3) + 1$ is not satisfied, S_1 is homotopic in $S^{n-1} \setminus S_2$ to the boundary $(v - 1)$ -sphere S_0 of a

disk fiber of the normal disk bundle to S_2 in S^{n-1} . Then by [17], we see that S_1 is always isotopic to S_0 in $S^{n-1} \setminus S_2$. Therefore, Y_v is an elementary handlebody.

Then by the same argument as for the case $v = 1$, we see that P_v is diffeomorphic to a boundary connected sum of elementary handlebodies of index pair $(v, n - v)$.

Case 2. $v = n/2$.

Since $M_v = M \setminus \text{Int } h^n$, we see that ∂M_v must be diffeomorphic to S^{n-1} . On the other hand, ∂M_v is diffeomorphic to the connected sum of ∂P_v and the almost $(n - 1)$ -sphere ∂M_{v-1} . Therefore, ∂P_v must be an almost $(n - 1)$ -sphere. Hence P_v is a pseudo elementary handlebody with index $n/2$.

Furthermore, if $n \equiv 2 \pmod{4}$ (and hence $n \geq 6$), then the intersection form

$$H_{n/2}(P_v) \times H_{n/2}(P_v) \rightarrow \mathbf{Z}$$

is unimodular and skew-symmetric and hence is isomorphic to the standard one. (Here, note that the manifold P_v is orientable and hence the intersection form is defined over the integers.) Therefore, an argument similar to that in Case 1 can be applied to show that P_v is diffeomorphic to a boundary connected sum of elementary handlebodies with index pair $(n/2, n/2)$.

Case 3. $2 = v < n/2$.

Note that in this case $n \geq 6$. Furthermore, ∂P_2 is simply connected by the following reasons.

- (1) By attaching handles of index r with $3 \leq r \leq n - 3$ to P_2 , we obtain a compact manifold with boundary homotopy equivalent to S^{n-1} .
- (2) Attaching handles of index r with $3 \leq r \leq n - 3$ does not change the fundamental group of the boundary.

Moreover, by Poincaré duality for the manifold (3.2), the intersection form

$$H_2(P_2) \times H_{n-2}(P_2) \rightarrow \mathbf{Z}$$

over the integers is unimodular, and hence ∂P_2 is a \mathbf{Z} -homology sphere. Therefore, ∂P_2 is a homotopy sphere and since $n \geq 6$, it is an almost $(n - 1)$ -sphere.

Let us show that P_2 is diffeomorphic to a boundary connected sum of elementary handlebodies of index pair $(2, n - 2)$.

Let

$$P_2 = h^0 \cup \left(\bigcup_{j=1}^w h_j^2 \right) \cup \left(\bigcup_{k=1}^w h_k^{n-2} \right)$$

be the given handlebody decomposition of P_2 . Let D be a small n -dimensional disk embedded in the interior of h^0 such that $h^0 \setminus \text{Int } D \cong \partial h^0 \times [0, 1]$. Then

there exist $(n - 2)$ -handles \bar{h}_k^{n-2} , $k = 1, 2, \dots, w$, disjointly embedded in the interior of

$$X = h^0 \cup \left(\bigcup_{k=1}^w h_k^{n-2} \right)$$

such that they are attached to D simultaneously. More precisely, \bar{h}_k^{n-2} is the union of a small closed tubular neighborhood N_k of the core $(n - 2)$ -disk of h_k^{n-2} in h_k^{n-2} and $(N_k \cap \partial h^0) \times [0, 1] \subset h^0 \setminus \text{Int } D$, $k = 1, 2, \dots, w$. Note that

$$\bar{X} = D \cup \left(\bigcup_{k=1}^w \bar{h}_k^{n-2} \right)$$

is a deformation retract of X and that the closure of $X \setminus \bar{X}$ is diffeomorphic to $\partial \bar{X} \times [0, 1]$.

Let us consider

$$Z = (X \setminus \text{Int } \bar{X}) \cup \left(\bigcup_{j=1}^w h_j^2 \right).$$

By considering the dual handlebody decomposition, we see that Z is diffeomorphic to a manifold obtained from $\partial P_2 \times [0, 1]$ by attaching w handles of index $n - 2$. Since ∂P_2 is simply connected, so is Z .

Note that $K_k = \bar{h}_k^{n-2} \cap D = \bar{h}_k^{n-2} \cap \partial D$ is diffeomorphic to $S^{n-3} \times D^2$, $k = 1, 2, \dots, w$. Let c_k be a circle embedded in $\partial D \setminus K_k$ obtained by pushing the boundary of a disk fiber $\{*\} \times D^2$ of K_k slightly toward $\partial D \setminus K_k$. Since Z is simply connected and $\dim Z \geq 6$, there exists an embedded 2-disk d_k in Z with $\partial d_k = c_k$ which intersects ∂D transversely along c_k and satisfies $d_k \cap \partial Z = c_k$. We may further assume that d_1, d_2, \dots, d_w do not mutually intersect. Then small closed tubular neighborhoods \bar{h}_k^2 of d_k in Z are 2-handles and are attached simultaneously to D .

Set

$$\bar{P}_2 = D \cup \left(\bigcup_{j=1}^w \bar{h}_j^2 \right) \cup \left(\bigcup_{k=1}^w \bar{h}_k^{n-2} \right).$$

Then by considering the intersection form of P_2 , we see that the inclusion $\bar{P}_2 \hookrightarrow P_2$ is a homotopy equivalence. Furthermore, $P_2 \setminus \text{Int } \bar{P}_2$ is homotopy equivalent to

$$Z \setminus \left(\bigcup_{k=1}^w d_k \right)$$

and is simply connected. Since $\pi_1(\partial\bar{X})$ is normally generated by the elements represented by c_1, c_2, \dots, c_w , we see that $\partial\bar{P}_2$ is simply connected. Recall that ∂P_2 is also simply connected. Therefore, $P_2 \setminus \text{Int } \bar{P}_2$ is an h -cobordism between ∂P_2 and $\partial\bar{P}_2$ of dimension $n \geq 6$ and is diffeomorphic to the product $\partial\bar{P}_2 \times [0, 1]$ by the h -cobordism theorem [36, 46]. Hence, P_2 is diffeomorphic to \bar{P}_2 .

Therefore, we may assume that the attaching circle of the 2-handle h_k^2 is the boundary of a 2-disk fiber of the normal disk bundle to the attaching $(n-3)$ -sphere of the $(n-2)$ -handle h_k^{n-2} in ∂h^0 , $k = 1, 2, \dots, w$. Then the boundary of $h^0 \cup h_k^2 \cup h_k^{n-2}$ is simply connected and hence it is an elementary handlebody of index pair $(2, n-2)$. Now the argument in Case 1 can be applied to show that P_2 is diffeomorphic to a boundary connected sum of elementary handlebodies of index pair $(2, n-2)$.

This completes the proof of Theorem 3.7 for the case $n \neq 5$.

For $n = 5$, we need the following.

LEMMA 3.8. *Let Σ^4 be a smooth closed 4-manifold homotopy equivalent to the 4-sphere S^4 . Then there exist 5-dimensional h -cobordisms W_0 and W_1 between Σ^4 and S^4 such that $W_0 \cup_{S^4} W_1$ is diffeomorphic to $\Sigma^4 \times [0, 1]$.*

The above lemma follows from [53] and [26] (see also [25]).

Let us consider the union X of the 0-handle and the 4-handles in the handlebody decomposition (3.1) of M with $s = u = 1$. It is not difficult to see that for each 4-handle, there exists a homotopy 4-sphere Σ_i embedded in $\text{Int } X$ which is the union of the core 4-disk of the 4-handle and a homotopy 4-disk properly embedded in the 0-handle. We may further assume that the embedded homotopy 4-spheres are mutually disjoint.

For each embedded homotopy 4-sphere Σ_i , its closed tubular neighborhood in X is diffeomorphic to $\Sigma_i \times [0, 1]$. Therefore, by Lemma 3.8, we can find an embedded 4-sphere S_i inside the interior of the tubular neighborhood which cuts $\Sigma_i \times [0, 1]$ into two h -cobordisms. Note that S_i is embedded in X .

There exists a 5-disk D embedded in the interior of the 0-handle such that $(D, D \cap S_i)$ is diffeomorphic to the standard disk pair for each S_i .

Now let us consider the union M_1 of X and the 1-handles. For each 1-handle, we can find a circle smoothly embedded in $\text{Int } M_1$ which is the union of the core arc of the 1-handle and a properly embedded arc in the 0-handle. We may assume that these arcs are mutually disjoint and that each circle intersects D in a properly embedded arc. Then it is not difficult to show that the union T of the 5-disk D , the 4-spheres S_i , and the circles is a deformation retract of M_1 . Furthermore, a small regular neighborhood $N(T)$ of T in M_1 is diffeomorphic to $(\#^a(S^1 \times S^4)) \# (\#^b(S^1 \tilde{\times} S^4)) \setminus \text{Int } D^5$, where $S^1 \tilde{\times} S^4$ denotes the unique nonorientable S^4 -bundle over S^1 , a is the number of orientable 1-

handles, and b is the number of nonorientable 1-handles. Therefore, the closure of $M_1 \setminus N(T)$ is an h -cobordism between S^4 and the homotopy 4-sphere ∂M_1 .

Thus, we have shown that M is diffeomorphic to

$$(\#^a(S^1 \times S^4))\#(\#^b(S^1 \tilde{\times} S^4))\#M',$$

where the smooth closed 5-manifold M' is constructed from a compact contractible 5-manifold with boundary diffeomorphic to ∂M_1 by attaching handles of index 2 and 3 simultaneously and then by attaching a 5-handle. We see easily that such a 5-manifold is simply connected and has torsion free homology groups. Then, by the classification of smooth closed simply connected 5-manifolds [2, 47], we see that M' is diffeomorphic to a connected sum of some copies of $S^2 \times S^3$ and some copies of a unique nontrivial S^3 -bundle over S^2 . Then we can show that $M' \setminus \text{Int } D^5$ is a boundary connected sum of elementary handlebodies with index pair $(2, 3)$. Thus we have the desired result, since both $(S^1 \times S^4) \setminus \text{Int } D^5$ and $(S^1 \tilde{\times} S^4) \setminus \text{Int } D^5$ are elementary handlebodies of index pair $(1, 4)$. This completes the proof of Theorem 3.7. \square

REMARK 3.9. Let W be an elementary handlebody of dimension $n \geq 3$, $n \neq 5$, with index pair $(1, n - 1)$. If W is orientable, then the manifold W is diffeomorphic to $(S^1 \times \Sigma^{n-1}) \setminus \text{Int } D^n$ for some almost sphere Σ^{n-1} of dimension $n - 1$.

In order to consider the case where W is nonorientable, let us introduce the following notation. Let Σ be an almost $(n - 1)$ -sphere which is orientation preservingly diffeomorphic to its orientation reversal $-\Sigma$. (This is equivalent to that $\Sigma \# \Sigma$ is diffeomorphic to S^{n-1} . Thus, if $2\Gamma^{n-1} = 0$, then every almost $(n - 1)$ -sphere satisfies this property.) Let $r : \Sigma \rightarrow \Sigma$ be an orientation reversing diffeomorphism. We denote by $S^1 \tilde{\times} \Sigma$ the total space of the Σ -bundle over the circle with the monodromy diffeomorphism given by r : i.e.

$$S^1 \tilde{\times} \Sigma = [0, 1] \times \Sigma / (1, x) \sim (0, r(x)).$$

Note that the diffeomorphism type of $S^1 \tilde{\times} \Sigma$ depends on the choice of r in general, but that it is uniquely determined up to a connected sum with an almost n -sphere.

Suppose that Γ^{n-1} is a $\mathbf{Z}/2\mathbf{Z}$ -module, i.e. $2\Gamma^{n-1} = 0$, and $n \geq 3$, $n \neq 5$. (For example, $n = 3, 4, 6, 7, 9, 13, 15, 17$, etc.) If W is nonorientable, then W is diffeomorphic to $S^1 \tilde{\times} \Sigma \setminus \text{Int } D^n$ for some almost $(n - 1)$ -sphere Σ . For the case $2\Gamma^{n-1} \neq 0$, see Example 3.20.

REMARK 3.10. Let W be an elementary handlebody of index pair $(2, n - 2)$ as in Definition 3.5 (1) with $n \geq 6$. Then the union of the 0-handle and the 2-handle is diffeomorphic to a D^{n-2} -bundle over S^2 . If this bundle is

trivial (or equivalently, if W is spin), then in some cases (for example, if n is odd), W is diffeomorphic to an elementary handlebody of index pair $(2, n-2)$ such that the attaching sphere of the $(n-2)$ -handle is the standard $(n-3)$ -sphere in the boundary of the 0-handle. For details, see [27, Theorem 1 and Proposition 1].

REMARK 3.11. For $n = 2s + 1 \geq 9$ with $s \neq 7$, a diffeomorphism classification of elementary handlebodies of index pair $(s, s+1)$ can be obtained from results obtained in [54].

REMARK 3.12. For n even with $n \geq 6$, pseudo elementary handlebodies of dimension n with index $n/2$ are studied in [52]. For example, it is shown that the diffeomorphism classes of such manifolds are in one-to-one correspondence with the “ n -spaces” (see [52, p. 170]).

REMARK 3.13. We see easily that a manifold M as in Theorem 3.7 is homeomorphic to a connected sum of the form $\hat{W}_1 \# \hat{W}_2 \# \dots \# \hat{W}_k$, where each \hat{W}_i is the closed topological manifold obtained from a (pseudo) elementary handlebody as appearing in Theorem 3.7 by attaching an n -dimensional disk along the boundary almost $(n-1)$ -sphere by a homeomorphism.

Suppose that W_i is an elementary handlebody of index pair $(r, n-r)$. If $r = 1$, then \hat{W}_i is homeomorphic either to $S^1 \times S^{n-1}$ or to $S^1 \tilde{\times} S^{n-1}$. If $3 \leq r \leq n-r$, then \hat{W}_i is homeomorphic to a manifold obtained from Milnor’s manifold as described in Remark 3.6 (1) by attaching an n -disk, since every $(n-r-1)$ -sphere smoothly embedded in S^{n-1} is topologically unknotted by [48, 56]. For $r = 2$, we do not know if the corresponding statement for \hat{W}_i is true or not.

As direct corollaries to Theorem 3.7, we have the following.

COROLLARY 3.14. *A smooth closed connected 3-manifold admits a purely spherical Morse function if and only if it is diffeomorphic to S^3 or to a connected sum of some copies of $S^1 \times S^2$ and/or $S^1 \tilde{\times} S^2$.*

Note that the above corollary has already been obtained in [42] under the assumption that the critical points of the Morse function have distinct critical values. In §6 we give another proof of the above corollary (see Remark 6.3). In §6 we also give a characterization theorem of 3-manifolds admitting Morse functions whose regular fibers are disjoint unions of 2-spheres and tori (see Theorem 6.5).

COROLLARY 3.15. *A smooth closed connected 4-manifold admits a purely spherical Morse function if and only if it is diffeomorphic to a manifold of the form $M_1 \# M_2$, where M_1 is S^4 or a connected sum of some copies of $S^1 \times S^3$*

and/or $S^1 \tilde{\times} S^3$, and M_2 is a closed 1-connected 4-manifold which admits a handlebody decomposition without 1- or 3-handles.

It is still an open problem whether or not every smooth closed 1-connected 4-manifold admits a handlebody decomposition without 1- or 3-handles, as far as the author knows (see also the final open question posed in [9, p. 459]). Note also that every closed 1-connected 4-manifold is known to be covered by three open 4-balls [28].

The following is a direct consequence of the proof of Theorem 3.7 for the case $n = 5$.

COROLLARY 3.16. *A smooth closed connected 5-manifold admits a purely spherical Morse function if and only if it is diffeomorphic to S^5 or to a connected sum of some copies of the following manifolds:*

- (i) $S^1 \times S^4$,
- (ii) $S^1 \tilde{\times} S^4$,
- (iii) $S^2 \times S^3$,
- (iv) nontrivial S^3 -bundle over S^2 .

As another corollary to Theorem 3.7, we have the following.

COROLLARY 3.17. *If a smooth closed connected n -dimensional manifold M admits an almost spherical Morse function, then we have the following.*

- (1) *The fundamental group $\pi_1(M)$ of M is free.*
- (2) *The homology group $H_i(M)$ is torsion free for all i if M is orientable.*
- (3) *The cup product*

$$\smile: \tilde{H}^i(M; R) \times \tilde{H}^j(M; R) \rightarrow \tilde{H}^{i+j}(M; R)$$

is nontrivial only if $i + j = n$, where $R = \mathbf{Z}$ if M is orientable and $R = \mathbf{Z}/2\mathbf{Z}$ otherwise.

- (4) *$\text{Crit}(M) \leq 3$.*
- (5) *$M \setminus \{\text{point}\}$ is homotopy equivalent to a bouquet of spheres.*

PROOF. Since $M \setminus \text{Int } D^n$ admits a handlebody decomposition consisting of a 0-handle and some handles of positive index which are attached simultaneously to the 0-handle, (5) follows immediately.

(1) and (2) are direct consequences of (5).

The cup product of a manifold is the dual of its intersection form. By virtue of the handlebody decomposition described above, the intersection of two cycles representing the natural generators of the homology groups lies in the 0-handle. Hence, the result is nontrivial only when the intersection is 0-dimensional. Therefore, (3) follows.

(4) follows from Proposition 3.2 (3) and [50, Chap. II] (see also [10, Remark 7.27]). □

Note that (3) can also be proved by using the well-known relationship between the cup-length and the Lusternik-Schnirelmann category (for the definition, see §5). See [55, Chap. X] or [21, §4], for example. (1) can also be proved by using the Lusternik-Schnirelmann category (see [12, §23] or [21, §4]).

EXAMPLE 3.18. Let us consider an $(n-1)$ -connected $2n$ -dimensional closed manifold with $n \geq 3$. By Smale [46], such a manifold admits a handlebody decomposition consisting of one 0-handle, some n -handles attached to the 0-handle simultaneously, and one $2n$ -handle. Thus, such a manifold satisfies the properties mentioned in Proposition 3.2 and Corollary 3.17. See also [52] where the diffeomorphism types of such manifolds are studied.

EXAMPLE 3.19. Let V be a compact manifold obtained from D^n by simultaneously attaching some handles of positive index. Then the double of V , $V \cup V$, satisfies the properties mentioned in Proposition 3.2 and Corollary 3.17. This construction is due to Mielke [31].

In Remark 3.9, when the manifold is nonorientable, without the hypothesis $2\Gamma^{n-1} = 0$ we cannot conclude a similar result as the following example shows.

EXAMPLE 3.20. Let A be the $2k$ -dimensional disk bundle over S^{2k} obtained as the closed tubular neighborhood of the diagonal of $S^{2k} \times S^{2k}$. Furthermore, let W be the compact manifold of dimension $4k$ obtained by plumbing eight copies of A according to the so-called E_8 diagram. Then it is known that for $k \geq 2$, $\Sigma^{4k-1} = \partial W$ is homeomorphic to the standard sphere S^{4k-1} , but is not diffeomorphic to S^{4k-1} (see [23] or [35, §2]). It is also known that $\Sigma^{4k-1} \# \Sigma^{4k-1}$ is not diffeomorphic to S^{4k-1} .

Note that W has a handlebody decomposition consisting of one 0-handle and eight $2k$ -handles attached simultaneously to the 0-handle. Note also that $W \# W$ gives a cobordism between Σ^{4k-1} and its orientation reversal $-\Sigma^{4k-1}$ and that this cobordism is obtained by simultaneously attaching sixteen $2k$ -handles to $\Sigma^{4k-1} \times [-1, 1]$ on $\Sigma^{4k-1} \times \{-1\}$.

Let us consider the following construction. We first attach a $4k$ -dimensional $(4k-1)$ -handle to a 0-handle along the standardly embedded $(4k-2)$ -sphere in the boundary of the 0-handle so that the resulting manifold X_1 is diffeomorphic to $\Sigma^{4k-1} \times [-1, 1]$. We then attach sixteen $2k$ -handles to X_1 simultaneously along $\Sigma^{4k-1} \times \{-1\}$ so that they do not intersect the $(4k-1)$ -handle and that the result has boundary oriented diffeomorphic to the union of two copies of Σ^{4k-1} . Let us denote by X_2 the resulting compact $4k$ -dimensional manifold. Then we attach a 1-handle to X_2 so that it connects the two boundary components of X_2 and that we get a *nonorientable* manifold. The resulting nonorientable manifold X_3 has boundary diffeomorphic

to $\Sigma^{4k-1}\sharp(-\Sigma^{4k-1}) \cong S^{4k-1}$. Finally, we attach a $4k$ -handle to X_3 to obtain a closed nonorientable $4k$ -dimensional manifold M .

By the above construction, M has a handlebody decomposition of the form

$$M = h^0 \cup h^1 \cup \left(\bigcup^{16} h^{2k} \right) \cup h^{4k-1} \cup h^{4k},$$

where h^r denotes a handle of index r and the handles of indices 1 , $2k$ and $4k - 1$ are attached simultaneously to the 0 -handle. Therefore, M admits a Morse function $f : M \rightarrow \mathbf{R}$ with exactly three critical values. Note that f is purely spherical.

However, the union $h^0 \cup h^1 \cup h^{4k-1}$ has boundary diffeomorphic to $\Sigma^{4k-1}\sharp\Sigma^{4k-1}$, which is not diffeomorphic to S^{4k-1} . Therefore, we cannot attach the $4k$ -disk so as to obtain a closed manifold. Note that $2\Gamma^{4k-1} \neq 0$, $k \geq 2$.

REMARK 3.21. Manifolds which admit Morse functions with exactly three critical points (and hence with three critical values) are studied in [11].

4. An application to special generic maps

As another interesting corollary to Theorem 3.7, we obtain a new proof of the characterization theorem of those closed n -dimensional manifolds which admit a certain generic smooth map into the plane. Let us first recall the following definition.

DEFINITION 4.1. Let $f : M \rightarrow N$ be a smooth map between smooth manifolds, where we assume $n = \dim M \geq \dim N = p$. A point $x \in M$ is a *singular point* of f if $\text{rank } df_x < p$. A singular point x of f is called a *definite fold singular point* if there exist local coordinates (x_1, x_2, \dots, x_n) around x and (y_1, y_2, \dots, y_p) around $f(x)$ such that f has the form

$$y_i \circ f = \begin{cases} x_i, & 1 \leq i \leq p - 1, \\ x_p^2 + x_{p+1}^2 + \dots + x_n^2, & i = p. \end{cases}$$

A smooth map $f : M \rightarrow N$ is called a *special generic map* if every singular point of f is a definite fold singular point (see [5, 41]).

Then we have the following characterization theorem, originally proved in [41] (see also [5, 39]).

COROLLARY 4.2. *A smooth closed connected n -dimensional manifold with $n \geq 2$ admits a special generic map into \mathbf{R}^2 if and only if it is diffeomorphic to a*

manifold of the form M_0 or $M_0 \sharp M_1$, where the closed manifolds M_0 and M_1 are as follows.

- (1) M_0 is an almost n -sphere.
- (2) M_1 is the connected sum

$$(\sharp_{i=1}^k (S^1 \times \Sigma_i^{n-1})) \sharp (\sharp_{j=1}^{\ell} (S^1 \tilde{\times} \tilde{\Sigma}_j^{n-1}))$$

for some $0 \leq k < +\infty$ and $\ell = 0$ or 1 with $k + \ell \geq 1$, where Σ_i^{n-1} is an almost $(n - 1)$ -sphere, $\tilde{\Sigma}_j^{n-1}$ is an almost $(n - 1)$ -sphere such that $\tilde{\Sigma}_j^{n-1} \sharp \tilde{\Sigma}_j^{n-1}$ is diffeomorphic to S^{n-1} , and $S^1 \tilde{\times} \tilde{\Sigma}_j^{n-1}$ stands for the total space of a nonorientable $\tilde{\Sigma}_j^{n-1}$ -bundle over the circle.

PROOF. It is easy to see that the closed n -dimensional manifolds as above admit special generic maps into \mathbf{R}^2 (for example, see [41]).

Conversely, let $f : M \rightarrow \mathbf{R}^2$ be a special generic map of a closed n -dimensional manifold M into the plane. Let us recall the Stein factorization of f :

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbf{R}^2 \\ & \searrow q_f & \nearrow \tilde{f} \\ & & W_f. \end{array}$$

Here, q_f is the map identifying each component of the inverse image of a point under f to a point, W_f is the quotient space, and the continuous map \tilde{f} is defined by the commutativity of the diagram (for details, see [41]). Note that W_f has the structure of a compact connected 2-dimensional smooth manifold with boundary such that \tilde{f} is a smooth immersion.

By choosing an orthogonal projection $\pi : \mathbf{R}^2 \rightarrow \mathbf{R}$ generically, we may assume that the composition $g = \pi \circ f : M \rightarrow \mathbf{R}$ is a Morse function (see [30] and [13]). Take a regular value $y \in g(M)$. Since $g = \pi \circ f = \pi \circ \tilde{f} \circ q_f$, we have $g^{-1}(y) = q_f^{-1}(\tilde{f}^{-1}(\pi^{-1}(y)))$. Note that $\pi^{-1}(y)$ is a line in \mathbf{R}^2 and that $\tilde{f} : W_f \rightarrow \mathbf{R}^2$ is an immersion of a compact surface with boundary such that $\tilde{f}|_{\partial W_f}$ is transverse to $\pi^{-1}(y)$. Therefore, each connected component γ of $\tilde{f}^{-1}(\pi^{-1}(y))$ is a properly embedded arc in W_f . Furthermore, the map $q_f|_{q_f^{-1}(\gamma)} : q_f^{-1}(\gamma) \rightarrow \gamma \cong [0, 1]$ is a Morse function with exactly one maximum and one minimum. Hence $q_f^{-1}(\gamma)$ is an almost $(n - 1)$ -sphere. Therefore, $g : M \rightarrow \mathbf{R}$ is an almost spherical Morse function (see also [42, Theorem 3.2 and Proposition 3.3]).

Furthermore, we see easily that the critical points of g have indices $0, 1, n - 1$ or n . Hence, by Theorem 3.7 and its proof together with Remark 3.9, we see that M is diffeomorphic to a closed n -dimensional manifold as described in Corollary 4.2, provided that M is orientable or $n = 5$.

When M is nonorientable and $n \neq 5$, we can take off orientable connected summands from M_1 until we have just one 1-handle left, where M_1 is as in the proof of Theorem 3.7. (For this, consider sliding handles of index 1.) The resulting manifold is a nonorientable elementary handlebody of index pair $(1, n-1)$ and is diffeomorphic to the union of $\Sigma^{n-1} \times [0, 1]$ and a nonorientable 1-handle for some almost $(n-1)$ -sphere Σ^{n-1} . Since there are no handles of index between 2 and $n-2$, this must have boundary diffeomorphic to S^{n-1} . This implies that $\Sigma^{n-1} \# \Sigma^{n-1}$ is diffeomorphic to S^{n-1} . Therefore, there exists an orientation reversing diffeomorphism $\Sigma^{n-1} \rightarrow \Sigma^{n-1}$ and the union of $\Sigma^{n-1} \times [0, 1]$ and the nonorientable 1-handle is diffeomorphic to $(S^1 \tilde{\times} \Sigma^{n-1}) \setminus \text{Int } D^n$. Thus we have the desired conclusion. This completes the proof. \square

Note that the above proof is based on the idea given in [42] (see Proposition 3.3 and the succeeding paragraph).

REMARK 4.3. Let $f : M \rightarrow \mathbf{R}^p$ be a generic smooth map of a closed n -dimensional manifold into \mathbf{R}^p with $n > p$ such that every regular fiber is a union of almost spheres (see [42]). Then by an argument similar to that in the proof of Corollary 4.2, we see that $f|_{f^{-1}(L)} : f^{-1}(L) \rightarrow L$ is an almost spherical Morse function for every generic line L in \mathbf{R}^p . Therefore, each component of the submanifold $f^{-1}(L)$ of M is always diffeomorphic to a manifold as described in Theorem 3.7.

5. Several invariants of manifolds

In this section, we study relationships among several homotopy or diffeomorphism invariants of manifolds related to smooth functions.

DEFINITION 5.1. Let M be a smooth closed manifold of dimension n . We define $\mu(M)$ to be the minimum number of critical values among all Morse functions on M (see [6, 32]).

Furthermore, we define the *cell number*, denoted by $C(M)$, to be the minimum number of open n -balls covering M (see [32, 44] or [10, §3.1]).

We say that a finite set of closed n -balls¹ $\{B_i\}$ in M is a *ball covering* of M if it covers M and $B_i \cap B_j = \partial B_i \cap \partial B_j$ is an $(n-1)$ -dimensional manifold whenever $i \neq j$. We define $b(M)$ to be the minimum number of closed n -balls among all ball coverings of M (see [6, 24]).

Note that these are diffeomorphism invariants of M .

¹Here, we mean by an “ n -ball” a combinatorial n -ball with respect to a smooth triangulation of M .

A subset U of a topological space² X is said to be *categorical* if the inclusion map $U \hookrightarrow X$ is null-homotopic. We denote by $\text{cat}(X)$ the smallest possible integer k such that there exist $k + 1$ open sets³ of X covering X each of which is categorical. If no such covering exists, then we define $\text{cat}(X)$ to be $+\infty$. This integer (possibly $+\infty$) is called the *Lusternik-Schnirelmann category* of X [29] (see also [3, 8, 9, 10, 12, 14, 16, 21, 22, 37, 43, 51, 55] etc.).

Furthermore, we denote by $\text{gcat}(X)$ the smallest possible integer k such that there exist $k + 1$ open sets of X covering X each of which is contractible in itself. Again, if there is no such covering, then we define $\text{gcat}(X) = +\infty$. We denote by $\text{Cat}(X)$ the smallest possible $\text{gcat}(Y)$ among all spaces Y homotopy equivalent to X . This integer (possibly $+\infty$) is called the *strong category* of X . Note that both $\text{cat}(X)$ and $\text{Cat}(X)$ are homotopy invariants of X .

Moreover, we denote by $\text{Cl}_S(X)$, the *spherical cone length*, the minimum number n of cofibrations

$$Z_i \rightarrow X_i \rightarrow X_{i+1}$$

such that $0 \leq i < n$, $X_0 \simeq *$, $X_n \simeq X$, and each Z_i is homotopy equivalent to a bouquet of spheres (for example, see [10, §3.5] or [43]). The *cone length*, denoted by $\text{Cl}(X)$, is the minimum number n of cofibrations

$$Z_i \rightarrow X_i \rightarrow X_{i+1}$$

such that $0 \leq i < n$, $X_0 \simeq *$ and $X_n \simeq X$. Note that both $\text{Cl}_S(X)$ and $\text{Cl}(X)$ are homotopy invariants of X . It is known that $\text{Cl}(X) = \text{Cat}(X)$ always holds [14]. Furthermore, by [51] we always have

$$\text{cat}(X) \leq \text{Cat}(X) \leq \text{cat}(X) + 1.$$

The following is known or is easy to prove. For the reader's convenience, we will give a brief proof.

PROPOSITION 5.2. *For a smooth closed connected manifold M of positive dimension, we have the following.*

(1) *We always have*

$$\begin{aligned} 2 \leq \text{cat}(M) + 1 &\leq \text{Cat}(M) + 1 = \text{Cl}(M) + 1 \\ &\leq \text{C}(M) \leq \text{Crit}(M) \leq \mu(M) \leq \dim M + 1, \\ \text{Cl}(M) + 1 &\leq \text{Cl}_S(M) + 1 \leq \mu(M), \end{aligned}$$

²In what follows, we consider only those topological spaces which have the homotopy type of a CW complex.

³In this definition, “open” can be replaced by “closed”, provided that X is a normal ANR. See [12] or [10, §1.2].

and

$$C(M) \leq b(M) \leq \mu(M).$$

(2) $\mu(M) = 2$ if and only if M is an almost sphere.

PROOF. (1) Since M is closed of positive dimension, it is not contractible, and hence $\text{cat}(M) \geq 1$. The inequalities $\text{cat}(M) \leq \text{Cat}(M) \leq C(M) - 1$ follow from the definitions. The inequality $C(M) \leq \text{Crit}(M)$ is proved in [15].

Suppose $f : M \rightarrow \mathbf{R}$ is a Morse function with the minimum number of critical values. We may assume that the critical points of index 0 (or $n = \dim M$) all have the same value. Then, since M is connected, we may assume that the number of critical points of index 0 (or n) is equal to 1. We may further assume that for any critical point p of index 1 and for any critical point $p', f(p') < f(p)$ holds only if p' has index 0. Then each critical level of f is connected, and by using Takens' technique [50, Chap. II], we can construct a smooth function g on M such that the number of critical points of g is equal to the number of critical values of f . Hence $\text{Crit}(M) \leq \mu(M)$ follows.

The inequality $\mu(M) \leq \dim M + 1$ can easily be proved by using the standard Morse theory (see [36, Theorem 4.8], for example).

The inequality $\text{Cl}(M) \leq \text{Cl}_S(M)$ follows from the very definition.

Suppose that a compact manifold W is obtained from a compact manifold V with connected boundary by simultaneously attaching some handles to its boundary. Then it is easy to see that there is a cofibration

$$Z \rightarrow V \rightarrow W$$

with Z a homotopy bouquet of spheres. (Take Z to be the union of the attaching spheres of the handles in the boundary of V and some arcs connecting them. See [9, §4].) Therefore, the inequality $\text{Cl}_S(M) + 1 \leq \mu(M)$ follows.

Finally, $C(M) \leq b(M)$ follows from the very definition. The inequality $b(M) \leq \mu(M)$ can be proved by using an argument similar to that in the proof of [24, Theorem 2.7].

(2) This also follows from the standard Morse theory. This completes the proof. \square

EXAMPLE 5.3. Let us consider a smooth S^3 -bundle E over S^4 as follows. Such a bundle has the same homotopy type as

$$S^3 \cup_{\psi} e^4 \cup_{\varphi} e^7,$$

where e^k stands for the k -dimensional cell, and $\psi : \partial e^4 \rightarrow S^3$ and $\varphi : \partial e^7 \rightarrow S^3 \cup_{\psi} e^4$ are attaching maps. It is known that if ψ represents an element of $\pi_3(S^3) \cong \mathbf{Z}$ distinct from ± 1 , then we have $\text{cat}(E) = 2$ (for example, see [3,

22])⁴. However, if $\psi \neq 0, \pm 1$, then $E - \{\text{point}\}$ is not homotopy equivalent to a bouquet of spheres, since $H_3(S^3 \cup_\psi e^4)$ has a nontrivial torsion. Hence, $\mu(E)$ cannot be less than or equal to 3 by Corollary 3.17 (5). Thus we have $\text{cat}(E) + 1 < \mu(E)$. (In fact, we can show that $3 = \text{cat}(E) + 1 = \text{Cat}(E) + 1 < \text{Cl}_S(E) + 1 = \mu(E) = 4$.)

Furthermore, since E is 2-connected, by [50, Corollary 6.6], we have $\text{Crit}(E) = 3$. Therefore, we have $3 = C(E) = \text{Crit}(E) < \mu(E) = 4$. (We do not know the exact value of $b(E)$. It should be equal to 3 or 4.)

As Scheerer and Tanré [43, §3.3] point out, the difference $\text{Cl}_S(X) - \text{cat}(X)$ can be arbitrarily large for CW complexes X . Modifying their example, we can show the following.

PROPOSITION 5.4. *For any positive integer k , there exists a smooth closed connected manifold M such that*

$$\mu(M) - \text{Crit}(M) \geq (\text{Cl}_S(M) + 1) - \text{Crit}(M) \geq k.$$

PROOF. Let us consider the complex projective space $\mathbf{C}P^\ell$, where $\ell = 2^n$ with $n \geq 1$. It is easy to see that for a generator a of $H^2(\mathbf{C}P^\ell; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, we have

$$Sq^{2^n} \circ Sq^{2^{n-1}} \circ \cdots \circ Sq^4 \circ Sq^2(a) = a^{2^n} \neq 0.$$

Therefore, we have

$$\text{Cl}_S(\mathbf{C}P^\ell) > n$$

(see [10, Example 3.25], for example). Furthermore, for the suspension $\Sigma\mathbf{C}P^\ell$, we have

$$\text{Cl}_S(\Sigma\mathbf{C}P^\ell) > n$$

as well, since the suspension operation is consistent with the squaring operation. Note that $\text{cat}(\Sigma\mathbf{C}P^\ell) = 1$ and that $\Sigma\mathbf{C}P^\ell$ is simply connected.

Now we can embed $\Sigma\mathbf{C}P^\ell$ in an Euclidean space \mathbf{R}^N of sufficiently high dimension. Let V be a regular neighborhood of $\Sigma\mathbf{C}P^\ell$ in \mathbf{R}^N . Note that V is a compact N -dimensional manifold with boundary and is homotopy equivalent to $\Sigma\mathbf{C}P^\ell$. Therefore, V is simply connected, and if N is sufficiently large, ∂V is also simply connected.

Thus by [10, §7.4] (see also [9]), there exists a smooth function f on V which is regular, constant and maximal on ∂V with the number of critical points at most $\text{cat}(V) + 2 = 3$.

⁴The inequality $\text{cat}(E) \leq 2$ can easily be seen, since $S^3 \cup_\psi e^4$ is homotopy equivalent to the suspension of $S^2 \cup_{\psi'} e^3$ for an attaching map $\psi' : \partial e^3 \rightarrow S^2$ whose suspension is homotopic to ψ .

Now let us consider the double of V : $M = V \cup V$. Then M is a smooth closed manifold of dimension N . If N is sufficiently large, then the inclusion $V \hookrightarrow M$ induces isomorphisms on cohomology groups of dimension $\leq \dim \Sigma \mathbf{C}P^\ell$. Therefore, we see that

$$Sq^{i_n} \circ Sq^{i_{n-1}} \circ \dots \circ Sq^{i_2} \circ Sq^{i_1}(\beta) \neq 0$$

for some $\beta \in H^*(M; \mathbf{Z}/2\mathbf{Z})$ and $i_1, i_2, \dots, i_{n-1}, i_n \geq 1$. Thus we have $\mu(M) \geq \text{Cl}_S(M) + 1 > n + 1$.

On the other hand, using the above constructed smooth function $f : V \rightarrow \mathbf{R}$, we can construct a smooth function on M with at most 6 critical points. Therefore, $\text{Crit}(M) \leq 6$ and we have $\mu(M) - \text{Crit}(M) \geq (\text{Cl}_S(M) + 1) - \text{Crit}(M) > n - 5$. Choosing n so that $n - 4 \geq k$, we have the desired conclusion. This completes the proof. \square

REMARK 5.5. By [10, Example 7.32], for the above constructed manifold M , we have $\text{cat}(M) \leq 2$.

REMARK 5.6. We do not know how close are $\text{Cl}_S(M) + 1$ and $\mu(M)$ (see also [43, §7]).

Recall that $\mu(M) \leq 3$ if and only if M satisfies one (and hence all) of the conditions (1), (2) and (3) of Proposition 3.2.

In fact, we have the following.

PROPOSITION 5.7. *Let M be a smooth closed manifold. Then $\mu(M) \leq 3$ if and only if there exists a Morse function $f : M \rightarrow \mathbf{R}$ such that for each regular value $y \in f(M)$, we have $\mu(f^{-1}(y)) \leq 2$.*

PROOF. If $\mu(M) \leq 3$, then there exists a Morse function $f : M \rightarrow \mathbf{R}$ with at most three critical values. Then it is easy to see that $f^{-1}(y)$ is a union of finitely many standard spheres for every regular value $y \in f(M)$ and hence $\mu(f^{-1}(y)) \leq 2$.

Conversely, if a Morse function $f : M \rightarrow \mathbf{R}$ as in the proposition exists, then $\mu(M) \leq 3$ holds by Proposition 3.2. \square

It may be conjectured that for each integer $k \geq 2$, $\mu(M) \leq k + 1$ if and only if there exists a Morse function $f : M \rightarrow \mathbf{R}$ such that for each regular value $y \in f(M)$, we have $\mu(f^{-1}(y)) \leq k$. The above proposition shows that this is true for $k = 2$.

Note that the above conjecture is true at least for $\dim M \leq 3$. The first unknown case would be that of $\dim M = 4$ and $k = 3$.

REMARK 5.8. Let $f : M \rightarrow \mathbf{R}$ be a Morse function on a closed manifold M . By applying [38, Theorem 1.1] to the quotient map $q_f : M \rightarrow W_f$ in the

Stein factorization of f , we see that if $\text{cat}(q_f^{-1}(x)) \leq m$ for all $x \in W_f$, then $\text{cat}(M) \leq 2m + 1$ holds. In particular, if $\mu(f^{-1}(y)) \leq k$ for each regular value $y \in f(M)$, then we have $\text{cat}(M) \leq 2k + 1$.

REMARK 5.9. For a smooth closed manifold M , we could consider the minimum number of critical values among all smooth (not necessarily Morse) functions on M . However, it is clear that the number is always equal to one (consider a constant function). Nevertheless, if the set of critical points has some topological restrictions, then such a number makes sense. For details, see [8].

REMARK 5.10. For smooth closed manifolds M , the author does not know if $\text{Crit}(M)$, $\text{C}(M)$, $\text{b}(M)$ or $\mu(M)$ are invariants of the homotopy type of M .

6. Further results in the 3-dimensional case

In this section, we study the invariants mentioned in the previous section more in detail for 3-dimensional manifolds. We also study Morse functions on 3-dimensional manifolds whose regular fibers are unions of 2-spheres and tori.

First we show the following.

PROPOSITION 6.1. *Let M be a smooth closed connected 3-manifold. Then we have the following.*

- (1) $\text{cat}(M) = \text{Cat}(M) = \text{Cl}(M) = \text{Cl}_S(M)$.
- (2) $\text{C}(M) = \text{b}(M) = \text{Crit}(M) = \mu(M)$.

PROOF. (1) If $\text{Cl}_S(M) \leq 1$, then we have

$$\text{cat}(M) = \text{Cat}(M) = \text{Cl}(M) = \text{Cl}_S(M) = 1 \quad (6.1)$$

and M is a homotopy 3-sphere. Furthermore, if M is a homotopy 3-sphere, then the equalities in (6.1) hold.

If $\text{Cl}_S(M) = 2$, then we have

$$\text{cat}(M) \leq \text{Cat}(M) = \text{Cl}(M) \leq \text{Cl}_S(M) = 2$$

and M has a free fundamental group ([16, 37]). If $\text{Cl}(M) \leq 1$ or $\text{cat}(M) \leq 1$, then M is a homotopy 3-sphere and $\text{Cl}_S(M) = 1$, which is a contradiction. Thus we have

$$\text{cat}(M) = \text{Cat}(M) = \text{Cl}(M) = \text{Cl}_S(M) = 2$$

and M has a nontrivial free fundamental group.

Suppose now $\text{Cl}_S(M) = 3$. If $\text{Cl}(M) \leq 2$ or $\text{cat}(M) \leq 2$, then M has a free fundamental group and is homotopy equivalent to a connected sum of

some copies of $S^1 \times S^2$ and/or $S^1 \tilde{\times} S^2$ (see [19, p. 57]). Thus we have $\text{Cl}_S(M) \leq 2$, which is a contradiction. Thus we have

$$\text{cat}(M) = \text{Cat}(M) = \text{Cl}(M) = \text{Cl}_S(M) = 3.$$

This completes the proof of (1).

(2) If $\mu(M) = 2$, then by Proposition 5.2 we have $C(M) = b(M) = \text{Crit}(M) = \mu(M)$. If $\mu(M) = 3$, then we have $C(M) \leq \text{Crit}(M) \leq 3$ and $b(M) \leq 3$. If $\text{Crit}(M) = 2$, $C(M) = 2$, or $b(M) = 2$, then M is homeomorphic to S^3 and $\mu(M) = 2$, which is a contradiction. Thus we have $C(M) = b(M) = \text{Crit}(M) = \mu(M) = 3$.

If $\mu(M) = 4$, then we have $C(M) \leq \text{Crit}(M) \leq 4$ and $b(M) \leq 4$. If $\text{Crit}(M) \leq 3$, $C(M) \leq 3$, or $b(M) \leq 3$, then by [50, Theorem (3.3)], [20] or [24, Theorem 4.3], M is diffeomorphic to a connected sum of some copies of $S^1 \times S^2$ and/or $S^1 \tilde{\times} S^2$. Therefore, we have $\mu(M) \leq 3$, which is a contradiction. This completes the proof. \square

REMARK 6.2. As the above proof shows, we have

$$\begin{aligned} \text{cat}(M) + 1 &= \text{Cat}(M) + 1 = \text{Cl}(M) + 1 = \text{Cl}_S(M) + 1 \\ &= C(M) = b(M) = \text{Crit}(M) = \mu(M) \end{aligned}$$

for all smooth closed connected 3-manifolds M if and only if the Poincaré conjecture is positive; i.e. if and only if every homotopy 3-sphere is homeomorphic to S^3 .

REMARK 6.3. Corollary 3.14 can also be proved in the following way. For a smooth closed connected 3-manifold M , if $\mu(M) \leq 3$, then we have $\text{Crit}(M) \leq 3$. Hence, by [50, Theorem (3.3)] M is diffeomorphic to a connected sum of some copies of $S^1 \times S^2$ and/or $S^1 \tilde{\times} S^2$. Conversely, if M is diffeomorphic to such a manifold, then it is easy to see that $\mu(M) \leq 3$ holds.

Let us now study Morse functions on 3-manifolds more in detail. Let M be a smooth closed orientable 3-dimensional manifold and $f : M \rightarrow \mathbf{R}$ a Morse function. Note that every regular fiber of f is a closed orientable surface.

DEFINITION 6.4. The *fiber genus* of a Morse function $f : M \rightarrow \mathbf{R}$ on a smooth closed orientable 3-manifold M is the maximum over all genera of the components of regular fibers of f . For example, a Morse function is of fiber genus 0 if and only if f is purely spherical.

In the rest of this section, we will prove the following.

THEOREM 6.5. *Let M be a smooth closed connected orientable 3-dimensional manifold. Then there exists a Morse function $f : M \rightarrow \mathbf{R}$ on M of fiber genus*

at most 1 if and only if M is diffeomorphic to a connected sum of some copies of the following 3-manifolds:

$$S^3, \quad S^1 \times S^2, \quad \text{lens space } L(p, q), \quad p \geq 2.$$

For the proof, we need the following.

LEMMA 6.6. *Let $f : M \rightarrow \mathbf{R}$ be a Morse function of fiber genus g . Then there exists a Morse function $h : M \rightarrow \mathbf{R}$ of the same fiber genus g whose critical points have distinct critical values.*

PROOF. Suppose that $c \in f(M) \subset \mathbf{R}$ is a critical value and let p_1, p_2, \dots, p_ℓ be the critical points lying in $f^{-1}(c)$. We can take a sufficiently small $\varepsilon > 0$ such that c is the unique critical value in the interval $[c - \varepsilon, c + \varepsilon]$. Then, changing f slightly on $V = f^{-1}([c - (\varepsilon/2), c + (\varepsilon/2)])$, we can construct a Morse function $f_c : M \rightarrow \mathbf{R}$ with the following properties:

- (1) $f_c = f$ outside of V ,
- (2) $f_c(V) \subset [c - (\varepsilon/2), c + (\varepsilon/2)]$,
- (3) f_c has the same critical points as f , and
- (4) $f_c(p_i) < f_c(p_j)$ if $i < j$.

By renumbering the critical points p_1, p_2, \dots, p_ℓ if necessary, we may further assume that their corresponding indices are arranged in a form

$$0, \dots, 0, 2, \dots, 2, 1, \dots, 1, 3, \dots, 3.$$

It is easy to observe that attaching a 0-handle or a 2-handle to a compact orientable 3-manifold does not increase the maximum genus of the components of the boundary surface. Dualizing the argument, we also see that attaching a 3-handle or a 1-handle does not decrease the maximum genus.

Since the maximum genus of the components of $f^{-1}(c \pm \varepsilon) = f_c^{-1}(c \pm \varepsilon)$ is smaller than or equal to g , the new Morse function f_c has the same fiber genus as f .

Repeating this procedure for each critical value of f , we obtain a desired Morse function h . This completes the proof of Lemma 6.6. \square

PROOF OF THEOREM 6.5. It is easy to see that the 3-sphere S^3 , lens spaces $L(p, q)$ and $S^1 \times S^2$ all admit a Morse function of fiber genus at most 1. Then on a 3-manifold obtained by their connected sum, we can also construct such a Morse function by the obvious connected sum construction with respect to disk neighborhoods of minimum or maximum points.

Conversely, suppose that M admits a Morse function f of fiber genus at most 1. By Lemma 6.6, we may assume that the critical points of f have distinct critical values. Let p_1, p_2, \dots, p_m be the critical points of f and we assume $f(p_1) < f(p_2) < \dots < f(p_m)$. Set $f(p_i) = c_i$, $i = 1, 2, \dots, m$, and take real numbers t_i , $i = 1, 2, \dots, m + 1$, such that $t_1 < c_1 < t_2 < c_2 < t_3 < \dots <$

$t_m < c_m < t_{m+1}$. We set $M_i = f^{-1}[t_1, t_i]$, $i = 1, 2, \dots, m + 1$. In the following, for a compact 3-dimensional manifold V , we denote by \hat{V} the compact 3-dimensional manifold obtained by attaching 3-dimensional disks to V along all the 2-sphere components of the boundary ∂V .

Let us show, by induction on i , that each component of \hat{M}_i , $i \geq 2$, is a connected sum of some copies of the following 3-manifolds:

$$S^3, \quad S^1 \times S^2, \quad \text{lens space } L(p, q), \quad S^1 \times D^2. \quad (6.2)$$

Note that the assertion for $i = m + 1$ implies the theorem.

The assertion trivially holds for $i = 2$, since M_2 is a 3-dimensional disk. Let us assume that the assertion holds for \hat{M}_i for an i with $m \geq i \geq 2$.

If the index of the critical point p_i is equal to 0 or 3, then it is easy to see that the assertion holds also for \hat{M}_{i+1} .

Suppose that the index of p_i is equal to 2. Then M_{i+1} is obtained from M_i by attaching a 2-handle. If the attaching circle bounds a 2-disk in the boundary surface, then we see easily that $\hat{M}_i \cong \hat{M}_{i+1}$. If the attaching circle does not bound a 2-disk in ∂M_i , then the boundary component along which the 2-handle is attached should be a torus. By our induction hypothesis, this implies that \hat{M}_i has $S^1 \times D^2$ as a connected summand. Then we see that \hat{M}_{i+1} is obtained from \hat{M}_i by replacing this $S^1 \times D^2$ factor with S^3 , $S^1 \times S^2$, or a lens space $L(p, q)$.

Now suppose that the index of p_i is equal to 1. Then M_{i+1} is obtained from M_i by attaching a 1-handle. If the 1-handle is attached to a connected component of ∂M_i , then this component should be a 2-sphere and we see that $\hat{M}_{i+1} \cong \hat{M}_i \# (S^1 \times D^2)$. Suppose that the 1-handle is attached to distinct connected components of ∂M_i . Note that then one of the components, say S , should be a 2-sphere. First, let us consider the case where these two components are contained in a connected component of M_i . If both of these components are 2-spheres, then we see easily that $\hat{M}_{i+1} \cong \hat{M}_i \# (S^1 \times S^2)$.

If one of the components, say T , is a torus, then we see that \hat{M}_{i+1} is diffeomorphic to the boundary connected sum of \hat{M}_i and $(S^1 \times S^2) \setminus \text{Int } D^3$, where the boundary connected sum is performed along T . (This can be proved as follows. Let B^3 be the 3-disk in \hat{M}_i attached to M_i along the 2-sphere component S , and let \tilde{B}^3 be the union of B^3 and a small closed collar neighborhood C of ∂B^3 in M_i . Furthermore, let N be a closed tubular neighborhood of a properly embedded arc in $M_i \setminus \text{Int } \tilde{B}^3$ connecting the torus component T and $\partial \tilde{B}^3$. We may assume that $N \cap T$ coincides with one of the attaching 2-disks of the 1-handle. Note that $\hat{M}_i \setminus \text{Int}(\tilde{B}^3 \cup N)$ is diffeomorphic to \hat{M}_i . Then \hat{M}_{i+1} is obtained from \hat{M}_i by removing $\text{Int}(\tilde{B}^3 \cup N)$ and by attaching the union of C , N and the 1-handle along a 2-disk. Since the latter union is diffeomorphic to $(S^1 \times S^2) \setminus \text{Int } D^3$, the above assertion follows.)

Therefore, we have $\hat{M}_{i+1} \cong \hat{M}_i \sharp (S^1 \times S^2)$ again.

Finally, suppose that the 1-handle is attached to distinct connected components of M_i . If the corresponding components of ∂M_i are 2-spheres, then \hat{M}_{i+1} is obtained from \hat{M}_i by taking the connected sum of two components. In fact, this holds even if one of the components is a torus.

Therefore, we have proved that each component of \hat{M}_{i+1} is a connected sum of some copies of the 3-manifolds as in (6.2). This completes the induction and hence the proof of Theorem 6.5. \square

REMARK 6.7. Recall that a smooth closed connected orientable 3-manifold can always be decomposed as the union of two 3-dimensional handlebodies attached along their boundaries (for example, see [19]). Such a decomposition is called a *Heegaard decomposition* of the 3-manifold, and the genus of the attaching surface is called the *genus* of the Heegaard decomposition. The *Heegaard genus* of a 3-manifold is the minimum genus of its Heegaard decompositions.

It is known that the Heegaard genus of a 3-manifold is at most 1 if and only if it is diffeomorphic to S^3 , $S^1 \times S^2$, or a lens space $L(p, q)$. Therefore, Theorem 6.5 can be interpreted as follows. For a smooth closed connected orientable 3-manifold M , there exists a Morse function $f: M \rightarrow \mathbf{R}$ of fiber genus at most 1 if and only if M is diffeomorphic to a connected sum of some closed orientable 3-manifolds of Heegaard genus at most 1. We do not know if the corresponding statement for higher genera holds or not. (Note that the corresponding statement for genus 0 does not hold.)

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