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N-body long-range scattering matrix

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Abstract. We give a definition of scattering matrices based on the asymptotic behaviors of generalized eigenfunctions proving existence of radial limits of the functions and show that these scattering matrices are equivalent to the ones defined by wave operator approach in long-range N-body problems including the problems of Coulomb interaction potentials. Equivalence of stationary and time-dependent definitions of the generalized Fourier transforms is also shown.

1. Introduction

Scattering matrices play an important role in the study of long-time asymptotic behaviors of the solutions to Schrödinger equations. Scattering matrices are defined in two different ways. In the time-dependent viewpoint, the scattering matrices are defined using wave operators and the Fourier transforms. On the other hand, in the stationary viewpoint, they are defined using the asymptotic behaviors of generalized eigenfunctions at infinity. In this paper we prove that both the definitions are equivalent in long-range N-body problems. We also give a definition of the generalized Fourier transforms using the asymptotic behaviors of outgoing solutions to nonhomogeneous equations. We prove that they are equivalent to the ones using wave operators.

Before we consider the N-body problems, it is instructive to recall the results for 2-body problems in which only two particles appear. In quantum mechanics, a state of a particle is represented by an element in a Hilbert space. The Hilbert space is the set of square-integrable functions. The time evolution of the state of the particle is described by unitary operators in the Hilbert space. However, in practice, functions which are not square-integrable are used as waves representing scattering processes. These functions are not square-integrable because they do not decay enough as $|x| \to \infty$, where x is the relative position of the particles. They are called generalized eigenfunctions, since they satisfy the Schrödinger equation $Hu = \lambda u$ but they are not

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eigenfunctions of H in the Hilbert space framework, where $H := -\Delta + V(x)$ is the Hamiltonian and $\lambda \in \mathbb{R}$. Here, Δ is the Laplacian and V is a real valued function called a potential. In the scattering process, we are interested in the angular distribution of the probability to find a scattered particle after particles collided and when the distance between the particles is large enough. The function representing the angular distribution is called scattering amplitude. The scattering amplitude is usually calculated considering the asymptotic behaviors of the generalized eigenfunction as $|x| \to \infty$. However, the relation between the long-time asymptotic behavior of the function in the Hilbert space and the generalized eigenfunction which is not in the Hilbert space is not obvious.

In fact, even the existence of the generalized eigenfunctions with appropriate asymptotic behaviors is not obvious. Let the potential $V(x) \in C^\infty(\mathbb{R}^\nu)$ satisfy

$$|\partial^\gamma V(x)| = \mathcal{O}(|x|^{-\mu-|\gamma|}), \tag{1}$$

for $\mu > 0$ as $|x| \to \infty$, where $\nu \in \mathbb{N}$ is the space dimension. The generalized eigenfunction used in the calculation of the scattering amplitude in the two-body problem with potentials decaying fast enough is a distorted plane wave u_ξ composed of the plane wave $e^{-ix\cdot\xi}$ and the scattered wave, where ξ is the momentum of the particle. The function u_ξ satisfies the Lippmann-Schwinger equation $u_\xi = e^{-ix\cdot\xi} - (H - \lambda - i0)^{-1} Ve^{-ix\cdot\xi}$, where $\lambda = |\xi|^2$ and $(H - \lambda - i0)^{-1}$ is the resolvent of the Hamiltonian. However, if the potential $V(x)$ does not decay enough as $|x| \to \infty$, the term $(H - \lambda - i0)^{-1} Ve^{-ix\cdot\xi}$ is not defined because $Ve^{-ix\cdot\xi}$ does not decay enough and it is not in the domain of $(H - \lambda - i0)^{-1}$. We can overcome this problem by using a spherical incident wave

$$v[g] := g(\hat{x})e^{-iK(x)}|x|^{(1-\nu)/2}, \qquad \hat{x} := x/|x|,$$

instead of the plane wave $e^{-ix\cdot\xi}$, where $K(x) = \sqrt{\lambda}|x| + o(|x|)$, $|x| \to \infty$, is a solution to the eikonal equation

$$|\nabla K(x)|^2 + V(x) = \lambda, \qquad \lambda > 0.$$

Notice that the wave $v[g]$ multiplied by $e^{-i\lambda t}$ moves toward the origin as time advances, and hence we call it incoming. Using $v[g]$ the generalized eigenfunction

$$\tilde{u}_\lambda[g] = v[g] - (H - \lambda - i0)^{-1}(H - \lambda)v[g], \tag{2}$$

is well-defined. This function $\tilde{u}_\lambda[g]$ would be regarded as the smeared function

$$\int_{\mathbb{S}^{\nu-1}} g(\omega)u_{\sqrt{\lambda}\omega}\, d\omega,$$

of the distorted plane waves $u_{\sqrt{\lambda}\omega}$ as above by $C^\infty(\mathbb{S}^{\nu-1})$-function $g(\omega)$ with respect to the angle ω of the incident wave, and the Schwartz integral kernel of the map $g \mapsto \tilde{u}_\lambda[g]$ would be the distorted plane wave $u_{\sqrt{\lambda}\omega}$. After constructing the generalized eigenfunction we also need to study the asymptotic behavior of the function. Practically, we need to prove the existence of the radial limits of the function. It is expected that the scattered wave component $-(H-\lambda-i0)^{-1}(H-\lambda)v[g]$ of $\tilde{u}_\lambda[g]$ in (2) has an asymptotic behavior as

$$-(H-\lambda-i0)^{-1}(H-\lambda)v[g] = f(\hat{x})e^{iK(x)}|x|^{(1-\nu)/2} + o(|x|^{(1-\nu)/2}),$$

as $|x| \to \infty$, where f is a function on the sphere $\mathbb{S}^{\nu-1}$. The map $\Sigma(\lambda) : g \mapsto f$ is the scattering matrix defined in the stationary way. The Schwartz integral kernel of the scattering matrix $\Sigma(\lambda)$ whose singularity due to the incident wave without being scattered removed would be the scattering amplitude.

Finally, we consider the relation between the scattering matrices defined by stationary and time-dependent way. For short-range potentials, that is, when $\mu > 1$ in (1), as time t tends to $\pm\infty$, the asymptotic behaviors of the solutions $e^{-itH}\psi, \psi \in \mathscr{H}_{ac}(H)$ to the Schrödinger equation are given by the free evolution $e^{it\Delta}\psi_\pm$ for some $\psi_\pm \in L^2(\mathbb{R}^\nu)$, where $\mathscr{H}_{ac}(H)$ is the absolutely continuous subspace of H. In other words,

$$\|e^{-itH}\psi - e^{it\Delta}\psi_\pm\| \to 0,$$

as $t \to \pm\infty$. On the contrary, for any $\psi_\pm \in L^2(\mathbb{R}^\nu)$ there exists $\psi \in \mathscr{H}_{ac}(H)$ such that

$$\|e^{it\Delta}\psi_\pm - e^{-itH}\psi\| \to 0,$$

as $t \to \pm\infty$. The wave operators $W_\pm : L^2(\mathbb{R}^\nu) \to L^2(\mathbb{R}^\nu)$ are defined by $W_\pm\psi_\pm := \psi$. The wave operators $W_\pm$ are partial isometries from $L^2(\mathbb{R}^\nu)$ to $\mathscr{H}_{ac}(H)$. The scattering operator S is defined as the map $S\psi_- := \psi_+$. Let $\mathbf{F}$ be the Fourier transform. Then, $\hat{S} := \mathbf{F}S\mathbf{F}^*$ commutes with any bounded Borel functions of $|\xi|^2$, and therefore, there exists $\hat{S}(\lambda) \in \mathscr{L}(L^2(\mathbb{S}^{\nu-1}))$, *a.e.* $\lambda > 0$ such that

$$(\hat{S}f)(\lambda, \omega) = (\hat{S}(\lambda)f(\lambda))(\omega), \qquad \xi = \sqrt{\lambda}\omega, \qquad \omega \in \mathbb{S}^{\nu-1},$$

a.e. $\lambda > 0$ for any $f(\xi) \in L^2(\mathbb{R}^\nu)$ (see e.g. Reed-Simon [18]). Here $f(\lambda) \in L^2(\mathbb{S}^{\nu-1})$ is defined by $(f(\lambda))(\omega) := f(\lambda, \omega)$. The operators $\hat{S}(\lambda)$ are scattering matrices defined in time-dependent way. Thus the scattering matrices give the map from the datum as $t \to -\infty$ to the one as $t \to \infty$. The time-dependent and stationary scattering matrices $\hat{S}(\lambda)$ and $\Sigma(\lambda)$ are equivalent in the sense that the following equation holds (see e.g. [18] and Melrose [17]).

$$\hat{S}(\lambda) = i^{\nu-1}\Sigma(\lambda)\mathscr{R},$$

where $\mathscr{R}$ is the reflection operator, i.e. $(\mathscr{R}g)(\omega) := g(-\omega)$. A similar result as above for long-range (i.e., $\mu \leq 1$) 2-body problems has been proved by Gâtel-Yafaev [5].

We now turn to N-body problems. When we consider collisions of composite particles such as atoms and molecules, we need to consider many-body problems in which the Hamiltonian consists of kinetic energy of the particles and the potentials between pairs of the particles. Even if there are seemingly only two particles before and after the collision, when at least one of the particles is a composite particle as in the scattering of an electron by an atom, rigorous treatment of the scattering needs to deal with the N-body problems. For there are interactions between particles within the same or different composite particles. We shall introduce the configuration spaces needed for N-body problems (see e.g. [3]). Let ν, N be natural numbers. We consider N particles in ν-dimensional space with masses $m_i > 0$. Let $x_i \in \mathbb{R}^\nu$ be the position of the i-th particle. The tuple $(x_1, \dots, x_N)$ of the positions of the particles is a point in $\mathbb{R}^{\nu N}$. Since the center of mass of the particles moves freely, we are interested only in the relative motion of the particles. The relative positions of the particles are indicated by a point in the center of mass configuration space

$$X := \left\{ x = (x_1, \dots, x_N) : x_i \in \mathbb{R}^\nu, \sum_{i=1}^{N} m_i x_i = 0 \right\}.$$

We need to consider the relative positions of the particles within subsets of N particles and the relative positions of these subsets. For this purpose we introduce the notion of cluster decomposition. Let $C_1, \dots, C_k$ be nonempty subsets of $\{1, 2, \dots, N\}$. Then we call the set $a := \{C_1, \dots, C_k\}$ a cluster decomposition if $C_i \cap C_j = \emptyset$ $(i \neq j)$ and $\bigcup_{i=1}^{k} C_i = \{1, 2, \dots, N\}$. A simple and important cluster decomposition (ij) is defined by

$$(ij) := \{\{i, j\}, \{1\}, \{2\}, \dots, \{\check{i}\}, \dots, \{\check{j}\}, \dots, \{N\}\},$$

where $\{\check{k}\}$ means that $\{k\}$ is absent. In (ij) only i-th and j-th particles form a cluster. The configuration space X^a of the internal coordinates of a is defined by

$$X^a := \left\{ x = (x_1, \dots, x_N) \in X : \sum_{j \in C} m_j x_j = 0 \text{ for all } C \in a \right\}.$$

The configuration space X_a of the inter-cluster coordinates of a is defined by

$$X_a := \{ x \in X : x_i = x_j \text{ if } i, j \in C \text{ for some } C \in a \}.$$

Then X_a is the orthogonal complement of X^a in X with respect to the inner product $\sum_{i=1}^N m_i(x_i \cdot y_i)$. Concerning the cluster decomposition (ij), $x_i - x_j$ can be used as a coordinate of $X^{(ij)}$. However, we do not specify particular coordinates in X^a and X_a in general, since specific coordinates for a can not be used as coordinates for all of the other cluster decompositions. Let a and b be cluster decompositions. We write $b \le a$ and call b is finer than a, if for any $D \in b$ there exists $C \in a$ such that $D \subset C$. As for (ij), it is readily confirmed that $(ij) \le a$ if $i, j \in C$ for some $C \in a$. We denote by Π^a and Π_a the orthogonal projections in X onto X^a and X_a respectively. We decompose $x = x_a \oplus x^a \in X_a \oplus X^a$. The operators $-\Delta_a$ and $-\Delta^a$ denote the Laplacians in X_a and X^a respectively.

An N-body Hamiltonian is an operator of the form

$$H := -\Delta + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j).$$

Here Δ is the Laplacian in X and V_{ij} is a real-valued function on $\mathbb{R}^\nu$ which is in this paper a sum of a compactly supported Laplacian-compact short-range part and a smooth long-range part (cf. Assumption 1). It should be emphasized that physically important Coulomb potential $V_{ij}(x) = 1/|x|$ satisfies Assumption 1. Let $\mathscr{H} := L^2(X)$, $\mathscr{H}_a := L^2(X_a)$ and $\mathscr{H}^a := L^2(X^a)$. Under this assumption H is a self-adjoint operator on $\mathscr{H}$. We will also need the subsystem Hamiltonian H^a on $\mathscr{H}^a$ defined by

$$H^a := -\Delta^a + \sum_{(ij) \le a} V_{ij}(x_i - x_j).$$

The set of thresholds is defined by

$$\mathscr{T}(H) := \bigcup_{\# a \ge 2} \sigma_{\mathrm{pp}}(H^a),$$

where $\sigma_{pp}(A)$ is the set of eigenvalues of A. We label the eigenvalues of H^a counted with multiplicities, by integers m, and we call the pairs $\alpha = (a, m)$ channels. We denote the eigenvalue of the channel α and the corresponding normalized eigenfunction by E_α and u_α respectively. We can identify a channel α with a tuple (a, E_α, u_α) as $\alpha = (a, E_\alpha, u_\alpha)$.

Time-dependent scattering matrices are defined as follows. In the following we denote channels by $\alpha = (a, E_\alpha, u_\alpha)$ and $\beta = (b, E_\beta, u_\beta)$. Set $w \in \mathscr{H}_a$. Then there exists $\psi_\alpha^\pm \in \mathscr{H}$ such that

$$\| e^{-iS_a^\pm(p_a, t) - iE_\alpha t}(u_\alpha \otimes w) - e^{-itH}\psi_\alpha^\pm \| \to 0, \tag{3}$$

as $t \to \pm\infty$, where $S_a^{\pm}(\xi_a, t)$ is a solution to a Hamilton-Jacobi equation

$$\frac{\partial S_a}{\partial t}(\xi_a, t) = |\xi_a|^2 + \tilde{I}_a(\nabla_{\xi_a} S_a(\xi_a, t)),$$

and $p_a = -i\nabla_{x_a}$. Here, $\tilde{I}_a$ is the effective inter-cluster potential (cf. Section 2). The tensor product $u_\alpha \otimes w$ is a wave in the state w with respect to inter-cluster coordinates and in the bound state u_α with respect to internal coordinates. $S_a^{\pm}$ is a generating function of the asymptotic displacement by classical trajectories, i.e. $\nabla_{\xi_a} S_a^{\pm}(\xi_a, t)$ equals asymptotically the displacement by the trajectory with the asymptotic momentum ξ_a (cf. [3, Section 2.7]). The unitary operator $e^{-iS_a^{\pm}(p_a, t) - iE_\alpha t}$ gives asymptotic time evolution. Note that in momentum space it is a multiplication by $e^{-iS_a^{\pm}(\xi_a, t) - iE_\alpha t}$ whose absolute value is 1, and therefore, it does not change the distribution of momentum. Thus w has the same distribution of momentum as the asymptotic one of $e^{-itH}\psi_\alpha^{\pm}$ which is the subject of scattering theory. The wave operator $W_\alpha^{\pm}$ is defined by $W_\alpha^{\pm} w := \psi_\alpha^{\pm}$. Let $\tilde{\psi}_\alpha^-$ be the component of ψ_α^- such that $\psi_\alpha^- = \tilde{\psi}_\alpha^- + \hat{\psi}_\alpha^-$, $\tilde{\psi}_\alpha^- \in \mathrm{Ran}\ W_\beta^+$, $\hat{\psi}_\alpha^- \in (\mathrm{Ran}\ W_\beta^+)^\perp$. Then in an opposite manner to (3), there exists $w_\beta \in \mathscr{H}_b$ such that

$$\| e^{-iS_b^+(p_b, t) - iE_\beta t}(u_\beta \otimes w_\beta) - e^{-itH}\tilde{\psi}_\alpha^- \| \to 0,$$

as $t \to \infty$ which is obvious by $\tilde{\psi}_\alpha^- \in \mathrm{Ran}\ W_\beta^+$ and the definition of W_β^+. The function w_β is given by the equation $w_\beta = (W_\beta^+)^* \psi_\alpha^-$. The scattering operator $S_{\beta\alpha}$ is defined by $S_{\beta\alpha} w := w_\beta$. Let $F_\alpha : \mathscr{H}_a \to L^2((E_\alpha, \infty), L^2(C_a))$ be the Fourier transform which maps functions in $\mathscr{H}_a$ to the functions with respect to polar coordinates, where $C_a := \mathbb{S}^{n_a - 1} \cap X_a$, $n_a := \dim X_a$. Then we can write

$$\hat{S}_{\beta\alpha} := F_\beta S_{\beta\alpha} F_\alpha^* = \int_{\max\{E_\alpha, E_\beta\}}^{\infty} \oplus\, \hat{S}_{\beta\alpha}(\lambda) d\lambda,$$

where $\hat{S}_{\beta\alpha}(\lambda) \in \mathscr{L}(L^2(C_a), L^2(C_b))$ is the fiber of $\hat{S}_{\beta\alpha}$ (see e.g. [19]). The operators $\hat{S}_{\beta\alpha}(\lambda)$ are called scattering matrices.

The other definition comes from the asymptotic behaviors of generalized eigenfunctions as in the 2-body problems. In N-body problems there are directions in which the potential $V_{ij}(x_i - x_j)$ does not decay in the configuration space X which cause singular behaviors of wave functions in those directions. Thus we restrict the function g assigning angular distributions of a incident wave to a function whose support is away from such directions i.e. $\mathrm{supp}\ g \subset C_a' := C_a \setminus \bigcup_{b \not\leq a} X_b$. If α is a channel obeying some decay condition (cf. (14)), for $\lambda \in \mathscr{E}_\alpha := (E_\alpha, \infty) \setminus (\sigma_{pp}(H) \cap \mathscr{T}(H))$ and $g \in C_c^\infty(C_a')$ there exists a generalized eigenfunction u (corresponding to $P_{\lambda,\alpha}^+[g]$ in Section 4) of H such that

$u - u_\alpha(x^a) \otimes (g(\hat{x}_a) r_a^{(1-n_a)/2} e^{-iK_a(x_a, \lambda - E_\alpha)})$ is outgoing in the sense of Section 4, where $r_a := |x_a|$, $\hat{x}_a := x_a/|x_a|$ and $K_a(x_a, \lambda)$ is a solution to an eikonal equation $|\nabla_{x_a} K_a|^2 + \tilde{I}_a = \lambda$. Here,

$$u_\alpha(x^a) \otimes (g(\hat{x}_a) r_a^{(1-n_a)/2} e^{-iK_a(x_a, \lambda - E_\alpha)}),$$

is an incident wave which is spherical with respect to the inter-cluster coordinates and a bound state with respect to the internal coordinates. The outgoing component of u is expected to have the form as

$$\sum_\beta u_\beta(x^b) \otimes (f_\beta(\hat{x}_b) r_b^{(1-n_b)/2} e^{iK_b(x_b, \lambda - E_\beta)}), \qquad f_\beta \in L^2(C_b).$$

One of the greatest challenges for the stationary definition in the long-range N-body case would be rigorous justification of this fact and to obtain the data f_β. In the present result we extract f_β from the generalized eigenfunction u as a functional on $C_c^\infty(C_b')$ using an expected equation

$$\int_{C_b} f_\beta(\hat{x}_b) h(\hat{x}_b) d\hat{x}_b$$
$$= \lim_{\rho \to \infty} \rho^{-1} \int_{C_b} \int_{r_b < \rho} r_b^{(1-n_b)/2} h(\hat{x}_b) e^{-iK_b(r_b \hat{x}_b, \lambda - E_\beta)} (\pi_\beta u)(r_b \hat{x}_b) dr_b d\hat{x}_b, \quad (4)$$

for any $h \in C_c^\infty(C_b')$, where $(\pi_\beta u)(x_b) := \int \bar{u}_\beta(x^b) u(x_b, x^b) dx^b$. (The right-hand side corresponds to $Q_{\lambda,\beta}^+(u) = Q_{\lambda,\beta}^+(P_{\lambda,\alpha}^+[g])$ in Sections 3 and 4.) Note that if we substitute $u_\beta(x^b) \otimes (f_\beta(\hat{x}_b) r_b^{(1-n_b)/2} e^{iK_b(x_b, \lambda - E_\beta)})$ into u in the right-hand side, we certainly obtain the left-hand side. Although we do not use the following fact explicitly in this paper, it deserves attention in order to understand mechanism extracting f_β. The incoming component $u_\alpha(x^a) \otimes (g(\hat{x}_a) r_a^{(1-n_a)/2} e^{-iK_a(x_a, \lambda - E_\alpha)})$ of u does not contribute to the limit in the right-hand side of (4). For if the incoming component is substituted into u in the right-hand side, for $\beta \neq \alpha$ the integrand decays somewhat fast and for $\beta = \alpha$ the integral oscillates and $\rho^{-1} \to 0$. Thus we can use the generalized eigenfunction u itself in (4) instead of its outgoing component. The scattering matrix $\Sigma_{\beta\alpha}(\lambda)$ is defined by $\Sigma_{\beta\alpha}(\lambda) g = f_\beta$.

In contrast to $\hat{S}_{\beta\alpha}(\lambda)$ the definition of $\Sigma_{\beta\alpha}(\lambda)$ does not need time evolution at all. The main result of this paper is the following relation between the two definitions of scattering matrices:

$$\hat{S}_{\beta\alpha}(\lambda) = e^{i\pi(n_a + n_b - 2)/4} \lambda_\alpha^{1/4} \lambda_\beta^{-1/4} \Sigma_{\beta\alpha}(\lambda) \mathscr{R}_a, \qquad \lambda \in \mathscr{E}_\alpha \cap \mathscr{E}_\beta,$$

where $\lambda_\alpha := \lambda - E_\alpha$ and $\mathscr{R}_a$ is the reflection operator on $L^2(C_a)$. To show this equivalence of $\hat{S}_{\beta\alpha}(\lambda)$ and $\Sigma_{\beta\alpha}(\lambda)$, an explicit representation of the radial limit

$Q^{+}_{\lambda,\beta}(u)$ is necessary. We obtain such a representation by proving a representation formula for radial limits in conic regions for long-range decaying potentials (cf. Lemma 1). Actually, the non-trivial existence of the radial limits itself is proved at the same time. The main idea of the proof is to insert a cut-off function having a linear slope in most part of its support into the representation by inner products and consider the limit as the support spreads. In order to obtain the relation between the wave operator and the generalized Fourier transform we use the representation of the asymptotic time evolution of $e^{-iS^{\pm}_{a}(p_a,t)}$ by an integral of spherical waves in Ikebe-Isozaki [11] (cf. Theorem 2). In fact, Theorem 2 is nothing but equivalence of time-dependent and stationary generalized Fourier transforms. For the proof of the relation between the resolvent of H and the Poisson operator $P^{\pm}_{\lambda,\alpha}[g]$ we employ a nontrivial equation obtained by the uniqueness theorem of outgoing and incoming solutions to nonhomogeneous equations of the form $(H-\lambda)u=f$ in Isozaki [14] (cf. Lemmas 7 and 4). Although spherical waves and their tensor products with eigenfunctions would be the simplest outgoing and incoming functions, the property has not been proved in the previous literature as far as the author knows. The property for spherical waves can be proved by pseudodifferential techniques only, but that of the tensor products of eigenfunctions and spherical waves need other techniques, because the commutator of the potential V_{ij} and a pseudodiffetential operator does not have a good decay property. We replace the pseudodifferential operator by a function of a first order differential operator B in Gérard-Isozaki-Skibsted [6] having a good commutator estimate with V_{ij} during the commutator calculus. To localize the pseudodifferential operator onto the subspace X_a we use decay of the eigenfunction. In the proof of the equivalence, there could exist other possibilities for transforming the time-dependent scattering matrix to representations using inner products and operators such as the resolvent, the Poisson operator $P^{\pm}_{\lambda,\alpha}$ and $Q^{\pm}_{\lambda,\alpha}$, but our method would be a simple one for the proof of the equivalence of the scattering matrices.

There are significant differences in difficulty in proving the results as above between short-range (i.e. $\mu>1$ in Assumption 1) and long-range (i.e. $1\geq\mu>0$) potentials and between 2-body and N-body problems. This is because the slow decay of potentials causes substantial change to both the time evolution of wave functions and asymptotic behaviors of generalized eigenfunctions and there are many difficulties in estimation of decay with respect to time or distance especially in N-body problems. Isozaki [13] and Hassell [8] proved similar results for 2-cluster to 3-cluster scattering in 3-body problems and for the free channel scattering in which all particles are separated in N-body problems respectively under rather strong decay conditions using different methods. Vasy [21] proved a similar result for short-range smooth potentials in N-body problems. Yafaev [22] also obtained a stationary representation for short-range N-body scattering matrices defined by time-dependent way for any channel. However, even a stationary definition of scattering matrices has not been obtained for long-range potentials in N-body problems and the relation to the time-dependent scattering matrices has not been known so far.

The main points of the approaches of the previous and present results are as follows. Since in [13] generalized eigenfunctions with plane incident waves are considered, the fast decay of the potentials is needed. For a decay estimate of inter-cluster potentials in the proof, the three-body structure is essential in [13]. In [8] the equivalence of the free channel scattering matrices is proved relating both the stationary and time-dependent scattering matrices for the free channel to the transition matrix, and also uses generalized eigenfunctions with plane incident waves in the proof, so that the fast decay of the potentials is needed. In [22] the stationary representation of the time-dependent scattering matrices for all channels is obtained proving new resolvent estimates, but to prove the existence of the radial limits of generalized eigenfunctions necessary for the stationary definition of scattering matrices, other resolvent estimates as those obtained for the free channel scattering in [9, Corollary 5.3] are needed. In [21] this problem is bypassed using a kind of weak radial limits. The proof of the existence of the limits and a representation of the limits by inner products in [21] depends on the short-range assumption (cf. Remark 1). In the proof of the existence of the radial limits, an ordinary differential equation with respect to r_a is used based on the fact that the phase iK_a in the asymptotic behaviors of the generalized eigenfunctions has the form $i\sqrt{\lambda - E_\alpha} r_a$. However, in long-range problems K_a depends not only on r_a but also on angular coordinates $\hat{x}_a$. The present result overcomes this problem introducing different weak radial limits (4) and obtaining a representation of the limit for Hamiltonians with long-range potentials. The limit in (4) is a weak limit associated with the little-o notation $o_{av}(|x|^{-(n-1)/2})$ in [5] introduced to study asymptotic behaviors of spherical waves, where $\varepsilon(x) \in o_{av}(|x|^{-(n-1)/2})$ if and only if

$$\lim_{\rho \to \infty} \rho^{-1} \int_{|x| \leq \rho} |\varepsilon(x)|^2 dx = 0.$$

The incident component of the Poisson operator in [21] has a form as $P^0_{a,\pm}(\lambda)g$, where $P^0_{a,\pm}(\lambda)$ is the Poissson operator for the free Laplacian. On the other hand, we use the spherical waves directly as incident waves which makes the structure of the generalized eigenfunction clear and would make the analysis of the function simple. The method to transform the time-dependent scattering matrix in the present result depends on the representation of the asymptotic time evolution for long-range two-body problems in [11] and the uniqueness theorem in [14].

Although we use a kind of weak limit in (4) in contrast to a kind of strong convergence for a short-range decaying potential (see e.g. [17]), i.e. short-range two-body problems, the stationary definition and the relation to the time dependent definition would still be useful in the study of quantities in scattering phenomena such as the scattering amplitude in N-body problems. For if we construct a generalized eigenfunction u or its approximation, the right-hand side of (4) could be calculated. Moreover, since the relation between weakly defined quantities are clear now by the present results, only the existence of stronger limits would remain as a problem in the results as above with other definitions of the limits.

The content of this paper is as follows. In Section 2 the existence of radial limits of functions in conic regions is proved. Using the limits we define radial limits for channels in Section 3. In Section 4 we introduce Poisson operators and stationary scattering matrices using the radial limits for channels. In Section 5 we introduce the well-known time-dependent definition of scattering matrices and prove the equivalence of the time-dependent and stationary scattering matrices. Equivalence of stationary and time-dependent definitions of the generalized Fourier transforms is also proved. In Appendix A proofs of outgoing and incoming properties and boundedness of functions and operators are given.

2. Preliminaries

In this section we prove the existence of radial limits of functions in conic regions under a certain condition. We assume the potentials V_{ij} obey the following.

Assumption 1. *There exists* $\mu \in (0,1]$ *such that* $V_{ij}(x) = V_{ij}^s(x) + V_{ij}^l(x)$, *where*

(1) $V_{ij}^s(x)$ *is compactly supported and* V_{ij}^s *is* $-\Delta_x$ *compact, i.e.* $V_{ij}^s(-\Delta_x + 1)^{-1}$ *is compact.*

(2) $V_{ij}^l(x) \in C^\infty(\mathbb{R}^\nu)$ *and for any* $\gamma \in \mathbb{N}^\nu$

$$|\partial^\gamma V_{ij}(x)| = \mathcal{O}(|x|^{-\mu-|\gamma|}).$$

Let $\eta \in C^\infty(\mathbb{R})$ be a function such that $\operatorname{supp} \eta \subset (1,\infty)$ and $\eta(t) = 1$ for $t > 2$. Set $I_a := \sum_{(ij)\not\leq a} V_{ij}^l$. Removing directions in which I_a does not decay, we define

$$\tilde{I}_a = \tilde{I}_a(x_a) := I_a(x_a) \prod_{b \not\leq a} \eta(|\Pi^b x_a| \ln\langle x_a\rangle / \langle x_a\rangle),$$

which is a generalization of the "free channel" potential in [9, Definition 2.2] to the general cluster decomposition a. This potential can be regarded as a one-body potential fulfilling for any $\mu' \in (0, \mu)$ the bounds

$$|\partial^\gamma \tilde{I}_a(x_a)| = \mathcal{O}(|x_a|^{-\mu'-|\gamma|}). \tag{5}$$

To confirm (5) we note that for $x_a = (x_1, \dots, x_N) \in X_a \subset X$ and $b = (ij)$ we have

$$\Pi^{(ij)} x_a = \left(0, \dots, 0, \frac{m_j}{m_i + m_j}(x_i - x_j), 0, \dots, 0, \frac{m_i}{m_i + m_j}(x_j - x_i), 0, \dots, 0\right) \in X,$$

where only i-th and j-th components are not 0. Thus by the definition of the inner product $\sum_i m_i x_i \cdot y_i$ in X, we have $|\Pi^{(ij)} x_a|^2 = \frac{m_i m_j}{m_i + m_j}|x_i - x_j|^2$. Since for $(ij) \not\leq a$ the inequality

$$|\Pi^{(ij)} x_a| > \langle x_a \rangle / \ln\langle x_a \rangle,$$

holds on supp $\eta(|\Pi^{(ij)} x_a| \ln\langle x_a \rangle / \langle x_a \rangle)$, it follows that $|x_i - x_j| > C\langle x_a \rangle / \ln\langle x_a \rangle$, where $C = \left(\frac{m_i + m_j}{m_i m_j}\right)^{1/2}$. Thus we conclude that if $(ij) \not\leq a$,

$$|\partial^\gamma V_{ij}^l(x_i - x_j)| = \mathcal{O}(\langle x_a \rangle^{-\mu-|\gamma|}(\ln\langle x_a \rangle)^{-\mu-|\gamma|}),$$

on the support from which we can see that (5) holds.

We let $K_a(\cdot, \lambda)$, $\lambda > 0$ denote the (approximate) solution to the eikonal equation

$$|\nabla_{x_a} K_a|^2 + \tilde{I}_a = \lambda \tag{6}$$

as taken from [12] and [11]. The function K_a is a C^∞-function and there exists $C > 0$ such that (6) holds for $|x_a| > C$. K_a satisfies $K_a(x_a, \lambda) = \sqrt{\lambda} r_a + Y_a(x_a, \lambda)$ and

$$|\partial^\gamma Y_a(x_a, \lambda)| = \mathcal{O}(|x_a|^{1-|\gamma|-\mu'}), \tag{7}$$

(the bounds being locally uniform in λ).

We drop for the moment the subscript a of x_a, $\tilde{I}_a$, K_a etc. and consider the operator $\tilde{H} = -\Delta + \tilde{I}$ on $L^2(\mathbb{R}^n)$ identifying $X_a = \mathbb{R}^n$. We need Besov spaces $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}^*(\mathbb{R}^n)$. We set

$$\Omega_0 := \{x \in \mathbb{R}^n : |x| < 1\},$$

$$\Omega_j := \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^j\}, \qquad (j \in \mathbb{N},\ j \geq 1).$$

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of functions u such that

$$\|u\|_{\mathcal{B}(\mathbb{R}^n)} := \sum_{j=0}^{\infty} 2^{j/2} \|u\|_{L^2(\Omega_j)} < \infty.$$

Then the dual space $\mathscr{B}^*(\mathbb{R}^n)$ of $\mathscr{B}(\mathbb{R}^n)$ is the set of functions u such that

$$\|u\|_{\mathscr{B}^*(\mathbb{R}^n)} := \sup_{j\geq 0} 2^{-j/2}\|u\|_{L^2(\Omega_j)} < \infty.$$

We can easily see that there exists a constant $C > 0$ such that

$$C^{-1}\|u\|_{\mathscr{B}^*(\mathbb{R}^n)} \leq \left(\sup_{\rho>1} \rho^{-1}\int_{|x|<\rho} |u(x)|^2 dx\right)^{1/2} \leq C\|u\|_{\mathscr{B}^*(\mathbb{R}^n)}. \tag{8}$$

The relation between $L^2_l(\mathbb{R}^n) := \langle x\rangle^{-l}L^2(\mathbb{R}^n)$, $\langle x\rangle := (1+|x|^2)^{1/2}$, $\mathscr{B}(\mathbb{R}^n)$ and $\mathscr{B}^*(\mathbb{R}^n)$ is as follows: for $l > 1/2$

$$L^2_l(\mathbb{R}^n) \subset \mathscr{B}(\mathbb{R}^n) \subset L^2_{1/2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset L^2_{-1/2}(\mathbb{R}^n) \subset \mathscr{B}^*(\mathbb{R}^n) \subset L^2_{-l}(\mathbb{R}^n).$$

For $U \subset \mathbb{R}^n$ the notation F_U stands for multiplication by 1_U. The following lemma which guarantees existence of radial limits of a function u satisfying a certain condition in conic region U is crucial to the stationary definition of scattering matrices.

Lemma 1. *Let U be an open subset of $\mathbb{R}^n$ such that $U' := U \cap \mathbb{S}^{n-1} \neq \varnothing$ and $U \cap \{|x| \geq 1\} = \{x = cx' : c \in [1,\infty),\ x' \in U'\}$. Let for any $g \in C_c^\infty(U')$ and $\lambda > 0$,*

$$v^\pm(x) = v^\pm_\lambda[g](x) := \eta(r)g(\hat{x})r^{(1-n)/2}e^{\pm iK(x,\lambda)}, \qquad r := |x|,\ \hat{x} := x/r.$$

Suppose $\tilde{u} \in \mathscr{B}^* \cap H^2_{\mathrm{loc}}$ and $F_U(\tilde{H}-\lambda)\tilde{u} \in \mathscr{B}$. Then

$$\lim_{\rho\to\infty} \rho^{-1}\int_{r<\rho} \overline{v^\pm(x)}\tilde{u}(x)dx$$
$$= \pm 2^{-1}i\lambda^{-1/2}(\langle v^\pm, (\tilde{H}-\lambda)\tilde{u}\rangle - \langle(\tilde{H}-\lambda)v^\pm, \tilde{u}\rangle), \tag{9}$$

where $\langle v, u\rangle = \int \bar{v}u\, dx$.

Proof. First, by calculation of the quantity in the middle of (8) integrating with respect to the polar coordinates, we can check $v^\pm \in \mathscr{B}^*$. To estimate $(\tilde{H}-\lambda)v^\pm$, using $|\nabla K|^2 + \tilde{I} = \lambda$ we compute

$$(\tilde{H}-\lambda)v^\pm = \mp i[(\varDelta K)\eta g r^{(1-n)/2} + 2(\nabla K)\cdot\nabla(\eta g r^{(1-n)/2})$$
$$\mp i\varDelta(\eta g r^{(1-n)/2})]e^{\pm iK} \tag{10}$$

for $|x|$ large enough. Using $\varDelta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\varLambda}{r^2}$ and $K = \sqrt{\lambda}r + Y(x,\lambda)$ by a direct calculation the right-hand side is rewritten as

$$\mp i[(\varDelta Y)\eta g r^{(1-n)/2} + 2(\nabla Y)\cdot\nabla(\eta g r^{(1-n)/2})$$
$$+ 2\sqrt{\lambda}(\partial_r\eta)g r^{(1-n)/2} \mp i\varDelta(\eta g r^{(1-n)/2})]e^{\pm iK},$$

where Λ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Since noticing

$$|\partial^\gamma Y(x,\lambda)| = \mathcal{O}(|x|^{1-|\gamma|-\mu'}),$$

we have

$$\left|\left(\frac{\partial^2}{\partial r^2}+\frac{\Lambda}{r^2}\right)Y\right| = \mathcal{O}(|x|^{-1-\mu'}), \qquad |\nabla Y| = \mathcal{O}(|x|^{-\mu'}),$$

$$|\nabla(\eta g r^{(1-n)/2})| = \mathcal{O}(|x|^{-(n+1)/2}), \qquad |\Delta(\eta g r^{(1-n)/2})| = \mathcal{O}(|x|^{-(n+3)/2}),$$

and $\partial_r\eta$ has a compact support, we obtain $(\tilde{H}-\lambda)v^\pm = \mathcal{O}(|x|^{-(n+1)/2-\mu'})$. Thus we obtain

$$(\tilde{H}-\lambda)v^\pm \in L^2_s(\mathbb{R}^n), \qquad \text{for } 1/2 < s < 1/2+\mu'. \tag{11}$$

In particular $(\tilde{H}-\lambda)v^\pm \in \mathscr{B}$, whence the right hand side of (9) is well-defined.

For any $\varepsilon \in (0, 1/9)$ choose a decreasing function $\chi_\varepsilon \in C^\infty(\mathbb{R})$ such that

$$\chi_\varepsilon(t) = \begin{cases} 1 & \text{for } t \le \varepsilon, \\ 1+\varepsilon-t & \text{for } 3\varepsilon \le t \le 1, \\ 0 & \text{for } t \ge 1+2\varepsilon, \end{cases}$$

and $\chi_\varepsilon' \ge -1$. Letting $\chi_{\varepsilon,\rho} = \chi_\varepsilon(r/\rho)$, $\rho > 1$, we compute the right-hand side of (9) as

$$\begin{aligned} &\mp 2^{-1}\lambda^{-1/2}\lim_{\rho\to\infty}\langle v^\pm, i[\tilde{H}, \chi_{\rho^{-1/2},\rho}]\tilde{u}\rangle \\ &\quad = \pm i\lambda^{-1/2}\lim_{\rho\to\infty}\rho^{-1}\left\langle v^\pm, \nabla\cdot\frac{x}{|x|}\chi'_{\rho^{-1/2}}(|\cdot|/\rho)\tilde{u}\right\rangle \\ &\quad = \mp i\lambda^{-1/2}\lim_{\rho\to\infty}\rho^{-1}\langle \partial_r v^\pm, \chi'_{\rho^{-1/2}}(|\cdot|/\rho)\tilde{u}\rangle, \end{aligned} \tag{12}$$

where in the first equality we used $\nabla\chi_{\rho^{-1/2},\rho} = \frac{x}{\rho|x|}\chi'_{\rho^{-1/2}}(|\cdot|/\rho)$ and

$$\begin{aligned} &|\langle x\rangle \Delta\chi_{\rho^{-1/2},\rho}| = \mathcal{O}(\rho^{-1}), \\ &\left|\int_{|x|<2\rho}\langle x\rangle^{-1}uv\,dx\right| = \mathcal{O}(\log\rho), \end{aligned} \tag{13}$$

as $\rho\to\infty$ for $u, v \in \mathscr{B}^*$, and the second equality follows from $\partial_r = \frac{x}{|x|}\cdot\nabla$. We can calculate $\partial_r v^\pm$ as

$$\begin{aligned} \partial_r v^\pm = {} & (\partial_r\eta) g r^{(1-n)/2} e^{\pm iK} + \frac{1-n}{2}\eta g r^{-(n+1)/2} e^{\pm iK} \pm i\partial_r Y \eta g r^{(1-n)/2} e^{\pm iK} \\ & \pm i\sqrt{\lambda}\eta g r^{(1-n)/2} e^{\pm iK}. \end{aligned}$$

Using $\rho^{-1}|x| < 2$ on $\operatorname{supp} \chi'_{\rho^{-1/2}}(|\cdot|/\rho)$ and (13), the limits including first three terms vanish. Thus the right-hand side of (12) is equal to

$$-\lim_{\rho\to\infty} \rho^{-1}\langle v^{\pm}, \chi'_{\rho^{-1/2}}(|\cdot|/\rho)\tilde{u}\rangle.$$

We can rewrite this expression as

$$\begin{aligned} -\rho^{-1}\langle v^{\pm}, \chi'_{\rho^{-1/2}}(|\cdot|/\rho)\tilde{u}\rangle &= \rho^{-1}\int_{3\rho^{-1/2}<r/\rho<1} \overline{v^{\pm}(x)}\tilde{u}(x)dx + \mathcal{O}(\rho^{-1/4}) \\ &= \rho^{-1}\int_{r/\rho<1} \overline{v^{\pm}(x)}\tilde{u}(x)dx + \mathcal{O}(\rho^{-1/4}). \end{aligned}$$

Here we used

$$\begin{aligned} &\rho^{-1}\int_{a_1\le r\le a_2} |v^{\pm}\tilde{u}|dx \\ &\le \rho^{-1/2}\left(\int_{a_1\le r\le a_2} |v^{\pm}|^2 dx\right)^{1/2}\left(\rho^{-1}\int_{a_1\le r\le a_2} |\tilde{u}|^2 dx\right)^{1/2} \\ &= \mathcal{O}(\rho^{-1/2}|a_2 - a_1|^{1/2}), \end{aligned}$$

where $a_1 = 0$, $a_2 = 3\rho^{1/2}$ or $a_1 = \rho$, $a_2 = \rho + 2\rho^{1/2}$. Whence

$$-\lim_{\rho\to\infty} \rho^{-1}\langle v^{\pm}, \chi'_{\rho^{-1/2}}(|\cdot|/\rho)\tilde{u}\rangle = \lim_{\rho\to\infty} \rho^{-1}\int_{r/\rho<1} \overline{v^{\pm}(x)}\tilde{u}(x)dx,$$

which is the left-hand side of (9) and completes the proof.

Remark 1. *A result analogous to this lemma with* $\tilde{I} = 0$ *(i.e.,* $\tilde{H} = -\Delta$*) is the "boundary pairing" in* [17, Proposition 13] *in which* $v^{\pm}$ *is replaced by a function having both outgoing and incoming components. A localized version of the boundary pairing in conic regions with* $\tilde{I} = 0$ *is obtained by* [21, Proposition 3.3]. *For short-range potentials* $|\tilde{I}| = \mathcal{O}(|x|^{-\mu})$, $\mu > 1$ *boundary pairing for* $\tilde{H} = -\Delta$ *is sufficient, because if* $\tilde{u} \in L^2_{-1/2-\varepsilon}$ *and* $(-\Delta + \tilde{I} - \lambda)\tilde{u} \in L^2_{1/2+\varepsilon}$, *then* $(-\Delta - \lambda)\tilde{u} \in L^2_{1/2+\varepsilon}$ *holds for* $\varepsilon > 0$ *small enough using* $\tilde{I}\tilde{u} \in L^2_{1/2+\varepsilon}$.

3. Radial limits for channels

Consider a channel $\alpha = (a, E_\alpha, u_\alpha)$ assuming $\|u_\alpha\|_{L^2(X^a)} = 1$ and

$$u_\alpha \in \mathcal{D}(\langle x^a\rangle^{s_0}) \qquad \text{for some } s_0 > 1. \tag{14}$$

Whence, alternatively stated, $u_\alpha \in L^2_{s_0}(X^a) := \langle x^a\rangle^{-s_0}L^2(X^a)$ for some $s_0 > 1$ which holds at least if $E_\alpha \notin \mathcal{T}(H)$ (cf. [4]). Let $\pi_\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{H}_a)$ be given by

$(\pi_\alpha v)(x_a) := \langle u_\alpha, v(\cdot, x_a)\rangle$. It is a consequence of (14) that also

$$\pi_\alpha \in \mathscr{L}(\mathscr{B}^*(X), \mathscr{B}^*(X_a)). \tag{15}$$

The proof of (15) is given in the Appendix A.

Let $C_a := X_a \cap \mathbb{S}^{n_a - 1}$, $n_a := \dim X_a$ and $C'_a := C_a \setminus \bigcup_{b \not\leq a} X_b$. Suppose $u \in \mathscr{B}^*(X)$ obeys $(H - \lambda)u \in \mathscr{B}(X)$, $\lambda > E_\alpha$. Then we can define the map $Q^{\pm}_{\lambda,\alpha}(u) : C_c^\infty(C'_a) \to \mathbb{C}$ by the following recipe: Let for any $g \in C_c^\infty(C'_a)$

$$(Q^{\pm}_{\lambda,\alpha}(u))(g) = \lim_{\rho\to\infty} \rho^{-1} \int_{r_a<\rho} \overline{v^{\pm}_{\lambda,\alpha}[g](x_a)}(\pi_\alpha u)(x_a)dx_a, \tag{16}$$

where

$$v^{\pm}_{\lambda,\alpha}[g](x_a) = \eta(r_a)g(\hat{x}_a)r_a^{(1-n_a)/2}e^{\pm iK_a(x_a,\lambda_\alpha)},$$

with $\lambda_\alpha = \lambda - E_\alpha$. The map $(Q^{\pm}_{\lambda,\alpha}(u))(g)$ is linear with respect to u and anti-linear with respect to g. Notice that Lemma 1 applies to $\tilde{u} = \pi_\alpha u$ by considering an open $U' \subseteq C_a$ with $\operatorname{supp} g \subset U' \subset \overline{U'} \subset C'_a$ so that in fact using $H^a u_\alpha = E_\alpha u_\alpha$

$$\begin{aligned}(Q^{\pm}_{\lambda,\alpha}(u))(g) &= \pm 2^{-1} i \lambda_\alpha^{-1/2}(\langle v^{\pm}_{\lambda,\alpha}[g], (-\Delta_a + \tilde{I}_a - \lambda_\alpha)\pi_\alpha u\rangle \\ &\qquad - \langle(-\Delta_a + \tilde{I}_a - \lambda_\alpha)v^{\pm}_{\lambda,\alpha}[g], \pi_\alpha u\rangle) \\ &= \pm 2^{-1} i \lambda_\alpha^{-1/2}(\langle J_\alpha v^{\pm}_{\lambda,\alpha}[g], (H-\lambda)u\rangle - \langle (H-\lambda)J_\alpha v^{\pm}_{\lambda,\alpha}[g], u\rangle),\end{aligned} \tag{17}$$

where the outgoing and incoming quasi-modes $J_\alpha v^{\pm}_{\lambda,\alpha}[g] := u_\alpha \otimes v^{\pm}_{\lambda,\alpha}[g]$ obey

$$J_\alpha v^{\pm}_{\lambda,\alpha}[g] \in L^2_{-\tilde{s}}(X), \tag{18}$$

for any $\tilde{s} > 1/2$ and

$$(H - \lambda)J_\alpha v^{\pm}_{\lambda,\alpha}[g] \in L^2_s(X) \subset \mathscr{B}(X), \tag{19}$$

for some $s = s(\mu, \alpha) > 1/2$, and the right-hand side of (17) is well-defined. (18) follows from $u_\alpha \in L^2(X^a)$ and $v^{\pm}_{\lambda,\alpha}[g] \in L^2_{-\tilde{s}}(X_a)$ for any $\tilde{s} > 1/2$. (19) is proved considering each term of

$$\begin{aligned}(H-\lambda)J_\alpha v^{\pm}_{\lambda,\alpha}[g] &= (I^s_a + I_a - \tilde{I}_a)J_\alpha v^{\pm}_{\lambda,\alpha}[g] + J_\alpha(-\Delta_a + \tilde{I}_a - \lambda_\alpha)v^{\pm}_{\lambda,\alpha}[g] \\ &= (I^s_a + I_a - \tilde{I}_a)J_\alpha v^{\pm}_{\lambda,\alpha}[g] + J_\alpha(\tilde{H}_a - \lambda_\alpha)v^{\pm}_{\lambda,\alpha}[g],\end{aligned} \tag{20}$$

where $I^s_a := \sum_{(ij)\not\leq a} V^s_{ij}$ and $\tilde{H}_a := -\Delta_a + \tilde{I}_a$. The proof is given in the Appendix A.

By the definition (16) and (15), it is easily seen that there exists $C > 0$ independent of u and g such that

$$|(Q^{\pm}_{\lambda,\alpha}(u))(g)| \leq C\|u\|_{\mathscr{B}^*(X)}\|g\|_{L^2(C_a)}. \tag{21}$$

Therefore, by Riesz theorem there exists $h \in L^2(C'_a) = L^2(C_a)$ such that

$$(Q^{\pm}_{\lambda,\alpha}(u))(g) = \langle g, h \rangle,$$

for any $g \in C^{\infty}_c(C'_a)$. We denote this h also by the same notation $Q^{\pm}_{\lambda,\alpha}(u)$. With these notations we can write

$$(Q^{\pm}_{\lambda,\alpha}(u))(g) = \langle g, Q^{\pm}_{\lambda,\alpha}(u) \rangle. \tag{22}$$

We defined $Q^{\pm}_{\lambda,\alpha}(u)$ in (16) as antilinear functional to make $Q^{\pm}_{\lambda,\alpha}$ linear with respect to u and in order not to make complex conjugate $\bar{g}$ appear in (22) at the same time.

Summarizing the results above we have the following theorem.

Theorem 1. *For any channel* (a, E_α, u_α) *obeying* (14) *and any* $u \in \mathscr{B}^*(X)$ *obeying* $(H - \lambda)u \in \mathscr{B}(X)$ *for some* $\lambda > E_\alpha$ *there exist week limits*

$$Q^{\pm}_{\lambda,\alpha}(u) = \text{w-}L^2(C_a)\text{-}\lim_{\rho \to \infty} \rho^{-1} \int_{r_a < \rho} r_a^{(n_a-1)/2} e^{\mp i K_a(r_a \hat{x}_a, \lambda_\alpha)} (\pi_\alpha u)(r_a \hat{x}_a) dr_a. \tag{23}$$

A useful example is given by $u = R(\lambda \pm i0)f$, where $f \in \mathscr{B}(X)$, $R(\lambda \pm i0) := (H - \lambda \mp i0)^{-1}$ and

$$\lambda \in \mathscr{E}_\alpha := (E_\alpha, \infty) \backslash (\sigma_{pp}(H) \cup \mathscr{T}(H)).$$

This function u is defined by familiar limiting absorption principle (LAP cf. [1, Theorem 9.4.19]):

$$R(\lambda \pm i0) \in \mathscr{L}(\mathscr{B}(X), \mathscr{B}^*(X)). \tag{24}$$

Notice that $\mathscr{H}^{-1}_{1/2,1}(X)$ and $\mathscr{H}^{1}_{-1/2,\infty}(X)$ in [1, Theorem 9.4.19] correspond to

$$\langle -i\nabla \rangle \mathscr{B}(X) \qquad \text{and} \qquad \langle -i\nabla \rangle^{-1} \mathscr{B}^*(X),$$

respectively.

As we see in the proof of Lemma 4 2. below, we can show

$$Q^{\pm}_{\lambda,\alpha}(R(\lambda \mp i0)f) = 0.$$

On the other hand the function $Q^{\pm}_{\lambda,\alpha}(R(\lambda \pm i0)f)$ is in general nonzero, see Lemmas 4 2. and 5 below.

4. Poisson operators and geometric scattering matrix

Consider a channel $\alpha = (a, E_\alpha, u_\alpha)$ obeying (14), and consider the quasi-modes

$$J_\alpha v^{\pm}_{\lambda,\alpha}[g] := u_\alpha \otimes v^{\pm}_{\lambda,\alpha}[g], \qquad g \in C^{\infty}_c(C'_a).$$

The assertions (19) and (24) allow us to define for $\lambda \in \mathscr{E}_\alpha$ the exact generalized eigenfunctions in $\mathscr{B}^*(X)$,

$$P^{\mp}_{\lambda,\alpha}[g] := J_\alpha v^{\mp}_{\lambda,\alpha}[g] - R(\lambda \pm i0)(H-\lambda)J_\alpha v^{\mp}_{\lambda,\alpha}[g], \tag{25}$$

$$\breve{P}^{\mp}_{\lambda,\alpha}[g] := J_\alpha v^{\mp}_{\lambda,\alpha}[g] - R(\lambda \mp i0)(H-\lambda)J_\alpha v^{\mp}_{\lambda,\alpha}[g]. \tag{26}$$

Remark 2. *Since the function*

$$v^{\pm}_{\lambda,\alpha}[g](x_a) = \eta(r_a)g(\hat{x}_a)r_a^{(1-n_a)/2}e^{\pm iK_a(x_a,\lambda_\alpha)},$$

is a spherical wave, $P^{\mp}_{\lambda,\alpha}[g]$ is a generalized eigenfunction with a spherical incoming or outgoing wave. It is plausible that $P^{\mp}_{\lambda,\alpha}[g]$ is a smeared distorted plane wave and the Schwartz integral kernel of the map $g \mapsto P^{\mp}_{\lambda,\alpha}[g]$ forms a family of distorted plane waves. In other words, there would exist a family of generalized eigenfunction $u^{\mp}_{\sqrt{\lambda_\alpha}\omega_a}$, $\omega_a \in C'_a$ such that $u^{\mp}_{\sqrt{\lambda_\alpha}\omega_a} - u_\alpha \otimes e^{\mp i(x_a\cdot(\sqrt{\lambda_\alpha}\omega_a)+Y_a(x_a,\lambda_\alpha))}$ is an outgoing or incoming spherical wave, and the following equation holds.

$$P^{\mp}_{\lambda,\alpha}[g] = \int_{C'_a} g(\omega_a)u^{\mp}_{\sqrt{\lambda_\alpha}\omega_a}\,d\omega_a. \tag{27}$$

Here $e^{\mp i(x_a\cdot(\sqrt{\lambda_\alpha}\omega_a)+Y_a(x_a,\lambda_\alpha))}$ is the "plane wave". Note that we need $Y_a(x_a,\lambda_\alpha)$ in the exponent as a modification from the true plane wave in the scattering by long-range potentials which holds even in the two-body scattering by a Coulomb potential (see [20, Section 21]*). It would be rather difficult to obtain the asymptotic behavior of $u^{\mp}_{\sqrt{\lambda_\alpha}\omega_a}$, because the radial distribution of the scattered wave is known to be singular even in the two-body Coulomb problem (see* [20, Section 21]*). The equation* (27) *has been proved by* [8, page 3808] *for the free channel scattering in which all particles are separated in the N-body problem with rapidly decreasing potentials. For a general channel with long-range potentials it is an open problem to construct $u^{\mp}_{\sqrt{\lambda_\alpha}\omega_a}$ and prove* (27).

Lemma 4 below is based on the uniqueness of solutions to nonhomogeneous equations under outgoing (incoming) condition below. For any $k, s \in \mathbb{R}$ we let $\mathscr{R}^{k,s}$ be a class of functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|\partial_x^\gamma \partial_\xi^{\gamma'} p(x,\xi)| \le C_{\gamma\gamma'}\langle x\rangle^{s-|\gamma|}\langle\xi\rangle^{-k},$$

for any $\gamma, \gamma' \in \mathbb{N}^n$. We set

$$\mathscr{R}^s := \bigcap_{k\in\mathbb{R}} \mathscr{R}^{k,s}.$$

We define the pseudodifferential operator $Op(p)$ corresponding to $p \in \mathscr{R}^{k,s}$ by

$$Op(p)f := (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(y,\xi)f(y)dyd\xi,$$

for any $f \in \mathbb{S}(\mathbb{R}^n)$. We let $\mathscr{R}_+(\tau)$ (resp., $\mathscr{R}_-(\tau)$) denote the class of functions $p \in \mathscr{R}^0$ such that on supp $p(x,\xi)$

$$\inf_{x,\xi} \hat{x}\cdot\xi > \tau \qquad \left(\text{resp., } \sup_{x,\xi} \hat{x}\cdot\xi < \tau\right).$$

We call $u \in L^2_{-s}(\mathbb{R}^n)$, $s > 1/2$ is outgoing (resp., incoming), if there exist $0 < s' < 1/2$ and $\varepsilon > 0$ such that $Op(p_-)u \in L^2_{-s'}(\mathbb{R}^n)$ (resp., $Op(p_+)u \in L^2_{-s'}(\mathbb{R}^n)$) for any $p_-(x,\xi) \in \mathscr{R}_-(\varepsilon)$ (resp., $p_+(x,\xi) \in \mathscr{R}_+(-\varepsilon)$). The physical meaning of outgoing (resp., incoming) properties would be that as time advances, the wave moves toward the infinity (resp., the origin). The function $v_\lambda^+[g]$ (resp., $v_\lambda^-[g]$) defined in Lemma 1 should be outgoing (resp., incoming), since multiplied by $e^{-i\lambda t}$, $\lambda > 0$ they are spherical incoming and outgoing waves. We can see the definition of outgoing and incoming properties as above is suitable by the following lemma.

Lemma 2. *$v_\lambda^+[g]$ (resp., $v_\lambda^-[g]$) is outgoing (resp., incoming).*

Considering the physical meaning of outgoing and incoming properties as above, we expect that the tensor product $J_\alpha v^\pm_{\lambda,\alpha}[g] = u_\alpha \otimes v^\pm_{\lambda,\alpha}[g]$ of an eigenfunction and an outgoing or incoming wave also satisfies the condition, which is indeed true.

Lemma 3. *$J_\alpha v^+_{\lambda,\alpha}[g]$ (resp., $J_\alpha v^-_{\lambda,\alpha}[g]$) is outgoing (resp., incoming).*

We defer the proofs of Lemmas 2 and 3 to the Appendix A.

In the following lemma we make use of the many-body version of Sommerfeld uniqueness result in [14, Theorem 1.3], that is, uniqueness of the outgoing and incoming solution $u \in L^2_{-s}(X)$, $s > 1/2$ to the equation $(H-\lambda)u = f$, $f \in L^2_{\tilde{s}}(X)$, $\tilde{s} > 1/2$.

Lemma 4. *For any channel $\alpha = (a, E_\alpha, u_\alpha)$ obeying* (14), *$\lambda \in \mathscr{E}_\alpha$ and $g \in C_c^\infty(C'_a)$*

1. $\breve{P}^\mp_{\lambda,\alpha}[g] = 0$,
2. $g = Q^\mp_{\lambda,\alpha}(P^\mp_{\lambda,\alpha}[g])$,
3. $g = Q^\mp_{\lambda,\alpha}(R(\lambda \mp i0)(H-\lambda)J_\alpha v^\mp_{\lambda,\alpha}[g])$.

Proof. 1. By the definition (26) of $\breve{P}^\mp_{\lambda,\alpha}[g]$ we have only to prove

$$J_\alpha v^\mp_{\lambda,\alpha}[g] = R(\lambda \mp i0)(H-\lambda)J_\alpha v^\mp_{\lambda,\alpha}[g]. \tag{28}$$

We note that $J_\alpha v^\mp_{\lambda,\alpha}[g]$ and $R(\lambda \mp i0)(H-\lambda)J_\alpha v^\mp_{\lambda,\alpha}[g]$ are solutions to the equation

$$(H-\lambda)u = (H-\lambda)J_\alpha v^\mp_{\lambda,\alpha}[g],$$

for u. Thus combining Lemma 3 and the fact that $R(\lambda + i0)f$ (resp., $R(\lambda - i0)f$), $f \in L_s^2(X)$, $s > 1/2$ is outgoing (resp., incoming) (cf. [6, Theorem 2.12]), (28) follows from the uniqueness of the outgoing and incoming solution stated above the lemma.

2. By (17), (26) and 1, we have

$$\begin{aligned}
&(Q_{\lambda,\alpha}^{\pm}(R(\lambda \mp i0)f))(g) \\
&\quad = \pm 2^{-1} i\lambda_\alpha^{-1/2}(\langle J_\alpha v_{\lambda,\alpha}^{\pm}[g], f\rangle - \langle (H-\lambda)J_\alpha v_{\lambda,\alpha}^{\pm}[g], R(\lambda \mp i0)f\rangle) \\
&\quad = \pm 2^{-1} i\lambda_\alpha^{-1/2}(\langle J_\alpha v_{\lambda,\alpha}^{\pm}[g], f\rangle - \langle R(\lambda \pm i0)(H-\lambda)J_\alpha v_{\lambda,\alpha}^{\pm}[g], f\rangle) \\
&\quad = \pm 2^{-1} i\lambda_\alpha^{-1/2}\langle \check{P}_{\lambda,\alpha}^{\pm}[g], f\rangle \\
&\quad = 0.
\end{aligned}$$

Thus we can see that

$$Q_{\lambda,\alpha}^{\pm}(R(\lambda \mp i0)f) = 0,$$

and therefore, by (25)

$$Q_{\lambda,\alpha}^{\mp}(P_{\lambda,\alpha}^{\mp}[g]) = Q_{\lambda,\alpha}^{\mp}(J_\alpha v_{\lambda,\alpha}^{\mp}[g]).$$

By (23) we can readily check that the right-hand side is equal to g.

3. By (28) we have

$$Q_{\lambda,\alpha}^{\mp}(R(\lambda \mp i0)(H-\lambda)J_\alpha v_{\lambda,\alpha}^{\mp}[g]) = Q_{\lambda,\alpha}^{\mp}(J_\alpha v_{\lambda,\alpha}^{\mp}[g]).$$

By (23) the right-hand side is equal to g.

LEMMA 5. *For any channel $\alpha = (a, E_\alpha, u_\alpha)$ obeying* (14), *$g \in C_c^\infty(C_a')$ and any $f \in \mathscr{B}(X)$,*

$$\pm 2^{-1} i\lambda_\alpha^{-1/2}\langle P_{\lambda,\alpha}^{\pm}[g], f\rangle = \langle g, Q_{\lambda,\alpha}^{\pm}(R(\lambda \pm i0)f)\rangle. \tag{29}$$

In particular, $P_{\lambda,\alpha}^{\pm} \in \mathscr{L}(L^2(C_a), \mathscr{B}^*(X))$ with a strongly continuous dependence on $\lambda \in \mathscr{E}_\alpha$ with respect to weak-$*$ topology.

PROOF. Applying (17) to $u = R(\lambda \pm i0)f$ we obtain

$$\begin{aligned}
&\langle g, Q_{\lambda,\alpha}^{\pm}(R(\lambda \pm i0)f)\rangle \\
&\quad = \pm 2^{-1} i\lambda_\alpha^{-1/2}(\langle J_\alpha v_{\lambda,\alpha}^{\pm}[g], f\rangle - \langle (H-\lambda)J_\alpha v_{\lambda,\alpha}^{\pm}[g], R(\lambda \pm i0)f\rangle) \\
&\quad = \pm 2^{-1} i\lambda_\alpha^{-1/2}\langle P_{\lambda,\alpha}^{\pm}[g], f\rangle,
\end{aligned} \tag{30}$$

which is (29). By (21) and (24) we have

$$Q^{\pm}_{\lambda,\alpha}R(\lambda \pm i0) \in \mathscr{L}(\mathscr{B}(X), L^2(C_a)). \tag{31}$$

(31) and (29) imply

$$P^{\pm}_{\lambda,\alpha} \in \mathscr{L}(L^2(C_a), \mathscr{B}^*(X)).$$

The continuous dependence on λ follows from the second expression in (30), (20) and (10) combined with regularity of the function K_a and continuity of the boundary values of the resolvent (cf. [1, Theorem 9.4.19]).

For two channels $\alpha = (a, E_\alpha, u_\alpha)$ and $\beta = (b, E_\beta, u_\beta)$ with the decay condition (14) fulfilled for u_α as well as for u_β the geometric scattering matrix, or rather the component given by considering α as incoming and β as outgoing, is given by

$$\Sigma_{\beta\alpha}(\lambda)g := Q^{+}_{\lambda,\beta}(P^{-}_{\lambda,\alpha}[g]), \qquad \lambda \in \mathscr{E}_\alpha \cap \mathscr{E}_\beta,\ g \in C_c^\infty(C'_a).$$

Alternatively, this quantity is given also by

$$\Sigma_{\beta\alpha}(\lambda)g = -Q^{+}_{\lambda,\beta}(R(\lambda + i0)(H - \lambda)J_\alpha v^{-}_{\lambda,\alpha}[g]). \tag{32}$$

Using (29), (32), (17), $(H - \lambda)P^{+}_{\lambda,\beta}[g] = 0$ and (22) we compute the adjoint

$$\Sigma^{*}_{\beta\alpha}(\lambda)g = Q^{-}_{\lambda,\alpha}(P^{+}_{\lambda,\beta}[g]), \qquad \lambda \in \mathscr{E}_\alpha \cap \mathscr{E}_\beta,\ g \in C_c^\infty(C'_b). \tag{33}$$

Lemma 6. *The component $\Sigma_{\beta\alpha}(\lambda)$ of the geometric scattering matrix extends to an operator in $\mathscr{L}(L^2(C_a), L^2(C_b))$ with weakly continuous dependence on $\lambda \in \mathscr{E}_\alpha \cap \mathscr{E}_\beta$.*

Proof. The assertion $\Sigma_{\beta\alpha}(\lambda) \in \mathscr{L}(L^2(C_a), L^2(C_b))$ follows from Lemma 5 and (21). By (32) (22) and (17) we have

$$\begin{aligned}\langle h, \Sigma_{\beta\alpha}(\lambda)g\rangle = -2^{-1}i\lambda_\beta^{-1/2}(&\langle J_\beta v^{+}_{\lambda,\beta}[h], (H - \lambda)J_\alpha v^{-}_{\lambda,\alpha}[g]\rangle \\ &- \langle (H - \lambda)J_\beta v^{+}_{\lambda,\beta}[h], R(\lambda + i0)(H - \lambda)J_\alpha v^{-}_{\lambda,\alpha}[g]\rangle),\end{aligned}$$

for any $g \in C_c^\infty(C'_a)$ and $h \in C_c^\infty(C'_b)$. Thus the continuity follows from (20), (10) combined with regularity of K_a and continuity of $R(\lambda + i0)$.

5. Time-dependent scattering theory and equivalence to stationary definitions

Let us first remember the standard time-dependent definitions. Let $S_a(\xi_a, t)$ be a solution to the Hamilton-Jacobi equation

$$\frac{\partial S_a}{\partial t}(\xi_a, t) = |\xi_a|^2 + \tilde{I}_a(\nabla_{\xi_a} S_a(\xi_a, t)), \tag{34}$$

defined in terms of the Legendre transform of the function $K_a(\cdot,\lambda)$ (cf. [11, Section 6]). More precisely, S_a is defined as follows. There exist $x_a(\xi_a,t) \in C^\infty(X_a^*\backslash\{0\} \times \mathbb{R}; X_a)$ and $\lambda(\xi_a,t) \in C^\infty(X_a^*\backslash\{0\} \times \mathbb{R}; \mathbb{R})$ satisfying the following condition: For any compact set $\Lambda \subset X_a^*\backslash\{0\}$ there exists a positive constant T such that for $\xi_a \in \Lambda$ and $t > T$ we have

$$\xi_a = \nabla_a K_a(x_a(\xi_a,t),\lambda(\xi_a,t)), \qquad t = \frac{\partial K_a}{\partial \lambda}(x_a(\xi_a,t),\lambda(\xi_a,t)).$$

We define S_a by

$$S_a(\xi_a,t) = x_a(\xi_a,t)\cdot\xi_a + \lambda(\xi_a,t)t - K_a(x_a(\xi_a,t),\lambda(\xi_a,t)).$$

Then for any compact set $\Lambda \subset X_a^*\backslash\{0\}$ there exists $T > 0$ such that (34) holds for $\xi_a \in \Lambda$ and $t > T$. For any channel $\alpha = (a, E_\alpha, u_\alpha)$ obeying (14) we can easily show the existence of channel wave operators by Cook criterion. Whence we introduce

$$W_\alpha^\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J_\alpha e^{-i(S_a^\pm(p_a,t)+E_\alpha t)},$$

where $p_a = -i\nabla_{x_a}$ and $S_a^\pm(\xi_a,\pm|t|) = \pm S_a(\pm\xi_a,|t|)$.

We combine the Fourier transformation and unitary transformations of the change of the variables $(\lambda_\alpha,\omega) = \left(|\xi_a|^2, \frac{\xi_a}{|\xi_a|}\right)$ and $\lambda = \lambda_\alpha + E_\alpha$. The combined transformation is denoted by $F_\alpha : \mathscr{H}_a \to L^2((E_\alpha,\infty), L^2(C_a))$ and for $w \in \mathscr{H}_a$ explicitly written as

$$(F_\alpha w)(\lambda,\omega) = (2\pi)^{-n_a/2} 2^{-1/2} (\lambda - E_\alpha)^{(n_a-2)/4} \int e^{-i(\lambda-E_\alpha)^{1/2}\omega\cdot x_a} w(x_a)\,dx_a.$$

The adjoint operator F_α^* of F_α is obtained by combining the inverse transformations, and the transformation $F_\alpha^* f$ of $f(\lambda,\omega)$ is explicitly given by

$$(F_\alpha^* f)(x_a) = (2\pi)^{-n_a/2} 2^{1/2} \int e^{ix_a\cdot\xi_a} |\xi_a|^{-(n_a-2)/2} f(|\xi_a|^2 + E_\alpha, \hat{\xi}_a)\,d\xi_a, \qquad (35)$$

where $\hat{\xi}_a := \xi_a/|\xi_a|$. By the intertwining property $HW_\alpha^\pm \supset W_\alpha^\pm(p_a^2 + E_\alpha)$ and the fact that $p_a^2 + E_\alpha$ is diagonalized by the unitary map F_α, we can write

$$\hat{S}_{\beta\alpha} := F_\beta (W_\beta^+)^* W_\alpha^- F_\alpha^* = \int_{\max\{E_\alpha,E_\beta\}}^\infty \oplus\, \hat{S}_{\beta\alpha}(\lambda)\,d\lambda. \qquad (36)$$

Here the fiber operator $\hat{S}_{\beta\alpha}(\lambda) \in \mathscr{L}(L^2(C_a), L^2(C_b))$ is defined for *a.e.* $\lambda > \max\{E_\alpha, E_\beta\}$.

Similarly the restriction of the maps $F_\alpha(W_\alpha^\pm)^*$ has strong almost everywhere interpretations, at least formally in this case,

$$F_\alpha(W_\alpha^\pm)^* = \int_{E_\alpha}^\infty \oplus (F_\alpha(W_\alpha^\pm)^*)(\lambda)d\lambda.$$

When applied to $f \in \mathscr{B}(X) \subset L^2(X)$, we have the assertion below.

Let $\mathscr{R}_a$ denote the reflection operator on $L^2(C_a)$, and let

$$c_\alpha^\pm(\lambda) := e^{\pm(n_a-3)\pi i/4}\pi^{-1/2}\lambda_\alpha^{1/4}, \qquad \lambda > E_\alpha.$$

For $\lambda \in \mathscr{E}_\alpha$ let us define stationary generalized Fourier transforms by

$$\begin{aligned} \mathscr{G}_\alpha^+(\lambda) &:= c_\alpha^+(\lambda)Q_{\lambda,\alpha}^+ R(\lambda + i0) \in \mathscr{L}(\mathscr{B}(X), L^2(C_a)), \\ \mathscr{G}_\alpha^-(\lambda) &:= c_\alpha^-(\lambda)\mathscr{R}_a Q_{\lambda,\alpha}^- R(\lambda - i0) \in \mathscr{L}(\mathscr{B}(X), L^2(C_a)). \end{aligned} \tag{37}$$

Theorem 2. *For any channel* $\alpha = (a, E_\alpha, u_\alpha)$ *obeying* (14) *and any* $f \in \mathscr{B}(X) \subset \mathscr{H}$ *the restrictions* $(F_\alpha(W_\alpha^\pm)^* f)(\cdot) \in L^2(C_a)$ *are weakly continuous in* $\mathscr{E}_\alpha$. *In fact for any* $f \in \mathscr{B}(X)$

$$(F_\alpha(W_\alpha^\pm)^* f)(\lambda) = \mathscr{G}_\alpha^\pm(\lambda)f, \qquad \lambda \in \mathscr{E}_\alpha.$$

Proof. We mimic the proof of [15, Lemma 3.8] using as input [11, Lemma 6.4]. Only a simplified version of the proof of [15, Lemma 3.8] is needed. We can assume that $f \in L_1^2(X)$, and we will consider the plus case only. For any given $g \in C_c^\infty(\mathscr{E}_\alpha \times C_a') \subset L^2((E_\alpha, \infty), L^2(C_a))$ we have

$$\begin{aligned} \langle F_\alpha(W_\alpha^+)^* f, g\rangle &= \langle f, W_\alpha^+ F_\alpha^* g\rangle \\ &= \lim_{t\to\infty} \langle f, e^{itH} J_\alpha e^{-i(S_a^+(p_a,t)+E_\alpha t)} F_\alpha^* g\rangle \\ &= \lim_{t\to\infty} \left\langle f, e^{itH} J_\alpha \int_0^\infty e^{-i(\lambda_\alpha+E_\alpha)t}(2\pi i b(\lambda_\alpha))^{-1} v_{\lambda,\alpha}^+[g(\lambda_\alpha + E_\alpha, \cdot)]d\lambda_\alpha \right\rangle \\ &= \lim_{t\to\infty} \left\langle f, e^{itH} J_\alpha \int_{E_\alpha}^\infty e^{-i\lambda t}(2\pi i b(\lambda_\alpha))^{-1} v_{\lambda,\alpha}^+[g(\lambda, \cdot)]d\lambda \right\rangle, \end{aligned} \tag{38}$$

where $b(\lambda_\alpha) := e^{(n_a-3)\pi i/4}\pi^{-1/2}\lambda_\alpha^{1/4}$. The third equality follows substituting the definition

$$v_{\lambda,\alpha}^+[g(\lambda_\alpha + E_\alpha, \cdot)](x_a) = \eta(r_a)g(\lambda_\alpha + E_\alpha, \hat{x}_a)r_a^{(1-n_a)/2}e^{iK_a(x_a,\lambda_\alpha)},$$

and a formula

$$e^{-iS_a^+(p_a,t)}F_\alpha^* g = (2\pi)^{-n_a/2}2^{1/2}\int e^{i(x_a\cdot\xi_a - S_a^+(\xi_a,t))}|\xi_a|^{-(n_a-2)/2}g(|\xi_a|^2 + E_\alpha, \hat{\xi}_a)d\xi_a,$$

obtained by (35) into

$$\|(2\pi)^{-n_a/2}2^{1/2}\int e^{i(x_a\cdot\xi_a - S_a^+(\xi_a,t))}|\xi_a|^{-(n_a-2)/2}\phi(|\xi_a|^2,\hat{\xi}_a)d\xi_a$$

$$-\int_0^\infty e^{-i(\lambda_\alpha t - K_a(x_a,\lambda_\alpha))}(2\pi ib(\lambda_\alpha))^{-1}\eta(r_a)r_a^{-(n_a-1)/2}\phi(\lambda_\alpha,\hat{x}_a)d\lambda_\alpha\| \to 0, \qquad (39)$$

as $t\to\infty$, which holds for any $\phi\in C_c^\infty(\mathbb{R}_+\times C_a)$. The formula (39) has been obtained in [11, Lemma 6.4] applying the Fourier transformation to the both terms, inserting cut-off functions for x_a/t and ξ_a into the second term, changing the variable as $x_a = ty_a$ and applying the stationary phase theorem. The index $(n_a-1)\pi i/4$ of

$$ib(\lambda_\alpha) = e^{(n_a-1)\pi i/4}\pi^{-1/2}\lambda_\alpha^{1/4},$$

comes from the factor $e^{i\pi\sigma/4}$ in the stationary phase theorem, where σ is the signature of the Hessian matrix of $y_a\cdot\xi_a + \lambda_\alpha - \sqrt{\lambda_\alpha}|y_a|$ at $(y_a,\lambda_\alpha) = (2\xi_a,|\xi_a|^2)$. Here, $y_a\cdot\xi_a + \lambda_\alpha - \sqrt{\lambda_\alpha}|y_a|$ appears as the main term of $(x_a\cdot\xi_a + \lambda_\alpha t - K_a(x_a,\lambda_\alpha))/t$ after the change of the variable $x_a = ty_a$.

Inserting $e^{-s\varepsilon}$ to the last expression of (38) we obtain

$$\begin{aligned}\langle F_\alpha(W_\alpha^+)^*f,g\rangle &= \lim_{t\to\infty}\left\langle f, e^{itH}J_\alpha\int_{E_\alpha}^\infty e^{-it\lambda}(2\pi ib(\lambda_\alpha))^{-1}v_{\lambda,\alpha}^+[g(\lambda,\cdot)]d\lambda\right\rangle \\ &= \lim_{\varepsilon\downarrow 0}\varepsilon\int_0^\infty e^{-\varepsilon t}\left\langle f, e^{itH}J_\alpha\int_{E_\alpha}^\infty e^{-it\lambda}(2\pi ib(\lambda_\alpha))^{-1}v_{\lambda,\alpha}^+[g(\lambda,\cdot)]d\lambda\right\rangle dt \\ &= \lim_{\varepsilon\downarrow 0}\varepsilon\int_{E_\alpha}^\infty\left\langle f, \int_0^\infty e^{it(H-\lambda)-t\varepsilon}J_\alpha(2\pi ib(\lambda_\alpha))^{-1}v_{\lambda,\alpha}^+[g(\lambda,\cdot)]dt\right\rangle d\lambda.\end{aligned}$$

The second equality follows from Abel's theorem. Integrating with respect to t gives

$$\begin{aligned}&\langle F_\alpha(W_\alpha^+)^*f,g\rangle \\ &\quad = \lim_{\varepsilon\downarrow 0} i\varepsilon\int_{E_\alpha}^\infty\langle f, R(\lambda-i\varepsilon)J_\alpha(2\pi ib(\lambda_\alpha))^{-1}v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle d\lambda \\ &\quad = \lim_{\varepsilon\downarrow 0}\int_{E_\alpha}^\infty\langle f, (1-R(\lambda-i\varepsilon)(H-\lambda))J_\alpha(2\pi ib(\lambda_\alpha))^{-1}v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle d\lambda \\ &\quad = \int_{E_\alpha}^\infty(2\pi ib(\lambda_\alpha))^{-1}\langle f, (1-R(\lambda-i0)(H-\lambda))J_\alpha v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle d\lambda. \qquad (40)\end{aligned}$$

On the other hand, by (22) and (17) we have

$$\begin{aligned}
&\langle \mathscr{G}_\alpha^+(\lambda) f, g(\lambda,\cdot)\rangle \\
&\quad = \overline{c_\alpha^+(\lambda)} \langle Q_{\lambda,\alpha}^+ R(\lambda+i0) f, g(\lambda,\cdot)\rangle \\
&\quad = -\overline{c_\alpha^+(\lambda)}(2^{-1} i \lambda_\alpha^{-1/2})(\langle f, J_\alpha v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle \\
&\quad\quad - \langle R(\lambda+i0) f, (H-\lambda) J_\alpha v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle) \\
&\quad = -\overline{c_\alpha^+(\lambda)}(2^{-1} i \lambda_\alpha^{-1/2})\langle f, (1 - R(\lambda - i0)(H-\lambda)) J_\alpha v_{\lambda,\alpha}^+[g(\lambda,\cdot)]\rangle. \quad (41)
\end{aligned}$$

Since $(2\pi i b(\lambda_\alpha))^{-1} = -\overline{c_\alpha^+(\lambda)}(2^{-1} i \lambda_\alpha^{-1/2})$, by (40) and (41) we can check

$$\langle f, W_\alpha^+ F_\alpha^* g\rangle = \int_{E_\alpha}^{\infty} \langle \mathscr{G}_\alpha^+(\lambda) f, g(\lambda,\cdot)\rangle d\lambda.$$

Noticing

$$\langle f, W_\alpha^+ F_\alpha^* g\rangle = \int_{E_\alpha}^{\infty} \langle (F_\alpha (W_\alpha^+)^* f)(\lambda,\cdot), g(\lambda,\cdot)\rangle d\lambda,$$

we obtain the result.

Under the condition of asymptotic completeness and with (14) fulfilled for all (open) channels we can write, using Theorem 2,

$$\mathscr{G}_\beta^+(\lambda) = \sum_\alpha \hat{S}_{\beta\alpha}(\lambda) \mathscr{G}_\alpha^-(\lambda).$$

Applying this formula to $f = (H-\lambda) J_\alpha v_{\lambda,\alpha}^-[g]$ leads with Lemma 4 3. and (32) to the identification of $\Sigma_{\beta\alpha}(\lambda)$ and $\hat{S}_{\beta\alpha}(\lambda)$. However, we will do the identification (stated precisely below) under weaker conditions.

LEMMA 7. *Let $g \in C_c^\infty(\mathscr{E}_\alpha \times C_a')$, where (a, E_α, u_α) is any channel obeying* (14). *Letting $f_{\lambda,g}^\pm := (H-\lambda) J_\alpha v_{\lambda,\alpha}^\pm[g(\lambda,\cdot)]$ the map $\mathbb{R} \ni \lambda \mapsto f_{\lambda,g}^\pm$ is a continuous $L_s^2(X)$-valued function for some $s > 1/2$ and*

$$\int P_{\lambda,\alpha}^\mp[g(\lambda,\cdot)] d\lambda = \pm \operatorname*{w\text{-}\mathscr{H}\text{-}lim}_{\varepsilon\downarrow 0} \int (R(\lambda - i\varepsilon) - R(\lambda + i\varepsilon)) f_{\lambda,g}^\mp \, d\lambda. \quad (42)$$

PROOF. **I.** Thanks to (20), (10) and regularity of the function K_a the map $\lambda \mapsto f_{\lambda,g}^\pm$ is checked to be a continuous $L_s^2(X)$-valued function for some $s > 1/2$.

II. Write $i2^{-1}(R(\lambda - i\varepsilon) - R(\lambda + i\varepsilon)) = P_\varepsilon(\lambda) \geq 0$ and note the familiar

$$\int_{-\infty}^{\infty} \langle \varphi, P_\varepsilon(\lambda)\varphi\rangle d\lambda = \pi \|\varphi\|^2, \qquad \varphi \in \mathscr{H},$$

which follows from Stone's formula and $\int_{a_1}^{a_2} R(\lambda \pm i\mu)\varphi\, d\mu \to 0$ as $\lambda \to \pm\infty$ uniformly with respect to $0 < a_j < 1$, $j = 1, 2$. By the support properties of g, familiar LAP bounds (cf. [1]) and Step I we can see that there exists $C > 0$ such that $\int_{-\infty}^{\infty} \langle f_{\lambda,g}^{\pm}, P_\varepsilon(\lambda) f_{\lambda,g}^{\pm}\rangle d\lambda \le C \sup_{\lambda \in \operatorname{supp} g} \|f_{\lambda,g}^{\pm}\|_{L_s^2(X)}$. Thus by Cauchy-Schwarz inequality, for any $\varphi \in \mathscr{H}$ we have

$$\left|\left\langle \varphi, \int_{-\infty}^{\infty} P_\varepsilon(\lambda) f_{\lambda,g}^{\pm}\, d\lambda \right\rangle\right| \le \int_{-\infty}^{\infty} 2^{-1}(\langle \varphi, P_\varepsilon(\lambda)\varphi\rangle + \langle f_{\lambda,g}^{\pm}, P_\varepsilon(\lambda) f_{\lambda,g}^{\pm}\rangle) d\lambda$$

$$\le 2^{-1}\left(\pi\|\varphi\|^2 + C \sup_{\lambda \in \operatorname{supp} g} \|f_{\lambda,g}^{\pm}\|_{L_s^2(X)}\right).$$

Since $\|\psi\| = \sup_{\|\varphi\|=1} |\langle \varphi, \psi\rangle|$, we can see that the $\mathscr{H}$-valued function given by the integral to the right in (42) is bounded in $\varepsilon \in (0, 1)$.

III. We will consider the minus case only. Taking any $f \in \mathscr{B}(X)$ we compute, using in the second step Lemma 4 1.,

$$\lim_{\varepsilon \to 0} \int \langle f, (R(\lambda - i\varepsilon) - R(\lambda + i\varepsilon)) f_{\lambda,g}^{-}\rangle d\lambda$$

$$= \int \langle f, (R(\lambda - i0) - R(\lambda + i0)) f_{\lambda,g}^{-}\rangle d\lambda$$

$$= \int \langle f, J_\alpha v_{\lambda,\alpha}^{-}[g(\lambda, \cdot)] - R(\lambda + i0) f_{\lambda,g}^{-}\rangle d\lambda$$

$$= \int \langle f, P_{\lambda,\alpha}^{-}[g(\lambda, \cdot)]\rangle d\lambda.$$

Since $\mathscr{B}(X)$ is dense in $\mathscr{H}$, by Step II we obtain the result.

The following theorem is our main result.

Theorem 3. *Let two channels $\alpha = (a, E_\alpha, u_\alpha)$ and $\beta = (b, E_\beta, u_\beta)$ with the decay condition* (14) *fulfilled for u_α as well as u_β be given. Then*

$$\hat{S}_{\beta\alpha}(\lambda) = e^{i\pi(n_a + n_b - 2)/4} \lambda_\alpha^{1/4} \lambda_\beta^{-1/4} \Sigma_{\beta\alpha}(\lambda) \mathscr{R}_a, \qquad \lambda \in \mathscr{E}_\alpha \cap \mathscr{E}_\beta. \tag{43}$$

In particular, the map

$$\mathscr{E}_\alpha \cap \mathscr{E}_\beta \ni \lambda \mapsto \hat{S}_{\beta\alpha}(\lambda) \in \mathscr{L}(L^2(C_a), L^2(C_b)),$$

is weakly continuous.

Proof. Let $g_\alpha \in C_c^\infty(\mathscr{E}_\alpha \times C_a')$ and $g_\beta \in C_c^\infty(\mathscr{E}_\beta \times C_b')$. By Lemma 7 we have

$$\int P_{\lambda',\beta}^{+}[g_\beta(\lambda', \cdot)] d\lambda', \qquad \int P_{\lambda,\alpha}^{-}[g_\alpha(\lambda, \cdot)] d\lambda \in \mathscr{H}.$$

We compute using Lemma 7,

$$\left\langle \int P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)]d\lambda', \int P^-_{\lambda,\alpha}[g_\alpha(\lambda,\cdot)]d\lambda \right\rangle$$

$$= \lim_{\varepsilon\downarrow 0} \int \left\langle \int P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)]d\lambda', (R(\lambda - i\varepsilon) - R(\lambda + i\varepsilon)) f^-_{\lambda,g_\alpha} \right\rangle d\lambda$$

$$= \lim_{\varepsilon\downarrow 0} \iint \left\langle \frac{2i\varepsilon}{(\lambda'-\lambda)^2+\varepsilon^2} P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)], f^-_{\lambda,g_\alpha} \right\rangle d\lambda' d\lambda. \tag{44}$$

In the second step we used

$$\langle P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)], R(\lambda \mp i\varepsilon) f\rangle$$

$$= \langle P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)], (\lambda' - \lambda \pm i\varepsilon)^{-1} f\rangle, \qquad f \in \mathscr{B}(X),$$

which follows from $(H-\lambda')P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)] = 0$. Changing the variable as $\lambda' = \lambda + \varepsilon t$ we can rewrite the last expression in (44) as

$$\lim_{\varepsilon\downarrow 0} \iint \left\langle \frac{2i\varepsilon}{(\lambda'-\lambda)^2+\varepsilon^2} P^+_{\lambda',\beta}[g_\beta(\lambda',\cdot)], f^-_{\lambda,g_\alpha} \right\rangle d\lambda' d\lambda$$

$$= \lim_{\varepsilon\downarrow 0} \iint_{-\infty}^{\infty} \left\langle \frac{2i}{t^2+1} P^+_{\lambda+\varepsilon t,\beta}[g_\beta(\lambda+\varepsilon t,\cdot)], f^-_{\lambda,g_\alpha} \right\rangle dt d\lambda$$

$$= \lim_{C\to\infty} \lim_{\varepsilon\downarrow 0} \iint_{-C}^{C} \left\langle \frac{2i}{t^2+1} P^+_{\lambda+\varepsilon t,\beta}[g_\beta(\lambda+\varepsilon t,\cdot)], f^-_{\lambda,g_\alpha} \right\rangle dt d\lambda$$

$$= \lim_{C\to\infty} \int_{-C}^{C} \frac{(-2i)}{t^2+1} dt \int \langle P^+_{\lambda,\beta}[g_\beta(\lambda,\cdot)], f^-_{\lambda,g_\alpha}\rangle d\lambda$$

$$= -2\pi i \int \langle P^+_{\lambda,\beta}[g_\beta(\lambda,\cdot)], f^-_{\lambda,g_\alpha}\rangle d\lambda,$$

where we used that $\lambda \mapsto P^+_{\lambda,\beta}[g_\beta(\lambda,\cdot)]$ is a continuous $L^2_{-s}(X)$-valued function for any $s > 1/2$ (cf. the proof of Lemma 5). By (22), (17), $(H-\lambda)P^+_{\lambda,\beta}[g_\beta(\lambda,\cdot)] = 0$ and (33) the last expression is equal to

$$4\pi \int \lambda_\alpha^{1/2} \langle Q^-_{\lambda,\alpha} P^+_{\lambda,\beta}[g_\beta(\lambda,\cdot)], g_\alpha(\lambda,\cdot)\rangle d\lambda$$

$$= 4\pi \int \lambda_\alpha^{1/2} \langle g_\beta(\lambda,\cdot), \Sigma_{\beta\alpha}(\lambda) g_\alpha(\lambda,\cdot)\rangle d\lambda.$$

Summarizing the calculations above we have

$$\left\langle \int P^{+}_{\lambda',\beta}[g_\beta(\lambda',\cdot)]d\lambda', \int P^{-}_{\lambda,\alpha}[g_\alpha(\lambda,\cdot)]d\lambda \right\rangle$$

$$= 4\pi \int \lambda_\alpha^{1/2} \langle g_\beta(\lambda,\cdot), \Sigma_{\beta\alpha}(\lambda) g_\alpha(\lambda,\cdot)\rangle d\lambda. \tag{45}$$

On the other hand, applying Lemma 5, the definition (37) of $\mathscr{G}_\alpha^{\pm}(\lambda)$ and Theorem 2 we can see that for any $f \in \mathscr{H}$ the following holds.

$$\left\langle \int P^{+}_{\lambda',\beta}[g_\beta(\lambda',\cdot)]d\lambda', f \right\rangle = \langle W_\beta^{+} F_\beta^{*} \tilde{g}_\beta, f\rangle, \tag{46}$$

$$\left\langle f, \int P^{-}_{\lambda,\alpha}[g_\alpha(\lambda,\cdot)]d\lambda \right\rangle = \langle f, W_\alpha^{-} F_\alpha^{*} \tilde{g}_\alpha\rangle, \tag{47}$$

where

$$\tilde{g}_\beta(\lambda) = (-2^{-1} i \lambda_\beta^{-1/2})^{-1} \overline{c_\beta^{+}(\lambda)^{-1}} g_\beta(\lambda),$$

$$\tilde{g}_\alpha(\lambda) = (2^{-1} i \lambda_\alpha^{-1/2})^{-1} \overline{c_\alpha^{-}(\lambda)^{-1}} \mathscr{R}_a g_\alpha(\lambda).$$

By (45)–(47) and (36) we obtain (43). The second assertion follows from the first and Lemma 6.

Corollary 1. *Under the condition of asymptotic completeness and with* (14) *fulfilled for all channels (at least for all open channels) the map*

$$\mathscr{E}_\alpha \cap \mathscr{E}_\beta \ni \lambda \mapsto \hat{S}_{\beta\alpha}(\lambda) \in \mathscr{L}(L^2(C_a), L^2(C_b)),$$

is strongly continuous.

Proof. We mimic [22, Theorem 6.7]. By asymptotic completeness $\hat{S}_{\beta\alpha}(\lambda)$ is a component of a unitary operator

$$\hat{S}(\lambda) : \bigoplus_\alpha L^2(C_a) \to \bigoplus_\alpha L^2(C_a).$$

The strong continuity of a unitary operator follows from weak continuity. Thus, $\hat{S}(\lambda)$ is strongly continuous, and therefore, $\hat{S}_{\beta\alpha}(\lambda)$ is also strongly continuous.

Remark 3. *It is an open problem to show the strong (or weak) continuity of Corollary* 1 *for long-range potentials without imposing the decay condition* (14). *Note that* (14) *is not needed for asymptotic completeness for a class of long-range potentials* [2]. *For short-range potentials the condition is not needed for the conclusion of Corollary* 1 *(see* [22]*). However, this conclusion is not* *known for example for Schrödinger operators with Coulomb pair-potentials without (implicit) decay assumptions on threshold bound states.*

A. Outgoing and incoming properties and the boundedness of functions and operators

In this appendix we prove (15), (19), Lemmas 2 and 3.

Proof of (15). (15) can be proved under the weaker assumption $s_0 > 1/2$ in (14). $\|\pi_\alpha u\|^2_{B^*(X_a)}$ can be estimated as

$$\|\pi_\alpha u\|^2_{B^*(X_a)} \le C_1 \sup_{\rho>1} \rho^{-1} \int_{|x_a|<\rho} |(\pi_\alpha u)(x_a)|^2 dx_a$$
$$\le C_2 \sup_{\rho>1} \rho^{-1} \int_{|x_a|<\rho} \int_{X^a} \langle x^a \rangle^{-2s_0} |u(x)|^2 dx^a dx_a,$$

because of $\|\langle x^a \rangle^{s_0} u_\alpha\|^2_{L^2(X^a)} < \infty$ by (14). Now for $\rho > 1$, let us take $J_\rho \in \mathbb{N}$ such that $2^{J_\rho - 1} \le \rho < 2^{J_\rho}$. Then we have

$$\int_{|x_a|<\rho} \int_{X^a} \langle x^a \rangle^{-2s_0} |u(x)|^2 dx^a dx_a$$
$$\le \iint_{|x|<2^{J_\rho+1}} \langle x^a \rangle^{-2s_0} |u(x)|^2 dx^a dx_a$$
$$+ \sum_{j=J_\rho+2}^{\infty} \iint_{\substack{2^{j-1}<|x|<2^j \\ |x_a|<\rho}} \langle x^a \rangle^{-2s_0} |u(x)|^2 dx^a dx_a$$
$$\le \iint_{|x|<2^{J_\rho+1}} |u(x)|^2 dx^a dx_a$$
$$+ \sum_{j=J_\rho+2}^{\infty} \iint_{\substack{2^{j-1}<|x|<2^j \\ |x_a|<\rho}} (1 + (2^{j-1})^2 - \rho^2)^{-s_0} |u(x)|^2 dx^a dx_a$$
$$\le \sum_{j=0}^{J_\rho+1} \|u\|^2_{L^2(\Omega_j)} + \sum_{j=J_\rho+2}^{\infty} (1 + (2^{2(j-1-J_\rho)} - 1) \cdot 2^{2J_\rho})^{-s_0} \|u\|^2_{L^2(\Omega_j)}$$
$$\le \sum_{j=0}^{J_\rho+1} 2^j \|u\|^2_{B^*(X)} + \sum_{j=J_\rho+2}^{\infty} (1 + 2^{2(j-1-J_\rho)-1} \cdot 2^{2J_\rho})^{-s_0} 2^j \|u\|^2_{B^*(X)}$$
$$\le \sum_{j=0}^{J_\rho+1} 2^j \|u\|^2_{B^*(X)} + \sum_{j=J_\rho+2}^{\infty} 2^{-s_0(2j-3)+j} \|u\|^2_{B^*(X)}$$

$$= \left(2^{J_\rho+2} - 1 + \frac{2^{3s_0+(J_\rho+2)(1-2s_0)}}{1-2^{1-2s_0}}\right)\|u\|^2_{B^*(X)}$$

$$= \left(8\rho - 1 + \frac{2^{2-s_0}\rho^{1-2s_0}}{1-2^{1-2s_0}}\right)\|u\|^2_{B^*(X)}.$$

Thus

$$\|\pi_\alpha u\|^2_{B^*(X_a)} \le C_2 \sup_{\rho>1} \rho^{-1}\left(8\rho - 1 + \frac{2^{2-s_0}\rho^{1-2s_0}}{1-2^{1-2s_0}}\right)\|u\|^2_{B^*(X)} = C_3\|u\|^2_{B^*(X)},$$

which completes the proof.

Proof of (19). Thanks to (11) the second term in the right-hand side of (20) belongs to $L^2_s(X)$ for some $s > 1/2$. The first term is decomposed as

$$\begin{aligned}(I^s_a(x) + I_a(x) - \tilde{I}_a(x_a))J_\alpha v^\pm_{\lambda,\alpha}[g] \\ = \kappa(x)(I^s_a(x) + I_a(x) - \tilde{I}_a(x_a))J_\alpha v^\pm_{\lambda,\alpha}[g] \\ + (1-\kappa(x))(I^s_a(x) + I_a(x) - \tilde{I}_a(x_a))J_\alpha v^\pm_{\lambda,\alpha}[g],\end{aligned} \tag{48}$$

where $\operatorname{supp}\kappa \subset \{x : \varepsilon|x_a| > |x^a|\}$ and $\kappa(x) = 1$ in $\{x : 2^{-1}\varepsilon|x_a| > |x^a|\}$ with some $\varepsilon > 0$. Here, we find $I_a(x) - I_a(x_a) = \int_0^1 x^a \nabla^a I_a(tx^a, x_a)dt$. On $\operatorname{supp}\kappa$ we have

$$|x| \le |x_a| + |x^a| \le |x_a| + \varepsilon|x_a|,$$

and thus

$$|x_a| \ge c|x|, \tag{49}$$

where $c := 1/(1+\varepsilon)$. Remembering $\operatorname{supp} g \subset C_a \backslash \bigcup_{b\not\le a} X_b$ and $X_{(ij)} = \{x : x_i = x_j\}$, we deduce that for $(ij) \not\le a$ there exists $\tilde{c} > 0$ such that $\frac{|x_i - x_j|}{|x_a|} > \tilde{c}$ on $\operatorname{supp} g \cap \{x : |x_a| > 1\}$. Combining these facts yields $|x_i - x_j| > c\tilde{c}|x|$ on $\operatorname{supp}\kappa g$ for $(ij) \not\le a$ and $|x|$ sufficiently large. Therefore, we have $|\nabla^a I_a(tx^a, x_a)| = \mathcal{O}(|x|^{-1-\mu})$ uniformly with respect to $0 < t < 1$ on $\operatorname{supp}\kappa g$. Hence by using (18),

$$\kappa(x)(I_a(x) - I_a(x_a))J_\alpha v^\pm_{\lambda,\alpha}[g] \in L^2_{1/2+\mu-\varepsilon'}(X), \tag{50}$$

for any $\varepsilon' > 0$. We can also see that by Assumption 1,

$$I^s_a(x) = 0, \tag{51}$$

for $|x|$ large enough on $\operatorname{supp}\kappa g$. Since $g = 0$ near $C_a \backslash C'_a = C_a \cap (\bigcup_{b\not\le a} X_b) = \bigcup_{b\not\le a}\{\hat{x}_a \in C_a : \Pi^b \hat{x}_a = 0\}$, by $\operatorname{supp}(I_a(x_a) - \tilde{I}_a(x_a)) \subset \bigcup_{b\not\le a}\{x_a : |\Pi^b x_a| \le$

$2(\ln\langle x_a\rangle)^{-1}\langle x_a\rangle\}$, and (49) we have

$$I_a(x_a) - \tilde{I}_a(x_a) = 0, \tag{52}$$

for $|x|$ large enough on supp κg. Thanks to (14) and $J_\alpha v^{\pm}_{\lambda,\alpha}[g] = u_\alpha \otimes v^{\pm}_{\lambda,\alpha}[g]$ we can see that

$$(1-\kappa(x))(I_a^s(x) + I_a(x) - \tilde{I}_a(x_a))J_\alpha v^{\pm}_{\lambda,\alpha}[g] \in L^2_{s_0-1/2-\varepsilon'}(X), \tag{53}$$

for any $\varepsilon' > 0$. By (50)–(53) the right-hand side of (48) belongs to $L^2_s(X)$ for some $s > 1/2$ which completes the proof.

For the proofs of outgoing and incoming properties in Lemmas 2 and 3, we collect standard results of pseudodifferential operators. For $p_1 \in \mathscr{R}^{k,s}$ and $p_2 \in \mathscr{R}^{k',s'}$ by the theorem of compositions of pseudodifferential operators (see e.g., [10, Section 18]), there exists $q \in \mathscr{R}^{k+k',s+s'}$ such that

$$Op(p_1)Op(p_2) = Op(q). \tag{54}$$

Moreover q has an expansion (see e.g., [16, Theorem 2.7.4])

$$q \sim \sum_{l=0}^{\infty} \frac{1}{i^l l!}[(-\nabla_\xi \cdot \nabla_u)^l \{p_1(u,\eta)p_2(x,\xi)\}]_{\substack{u=x\\ \eta=\xi}}. \tag{55}$$

The meaning of the expansion (55) is that the following holds:

$$q - \sum_{l=0}^{m} \frac{1}{i^l l!}[(-\nabla_\xi \cdot \nabla_u)^l \{p_1(u,\eta)p_2(x,\xi)\}]_{\substack{u=x\\ \eta=\xi}} \in \mathscr{R}^{k+k',s+s'-m-1}.$$

If $p \in \mathscr{R}^{0,0}$, by theorem of L^2 continuity of pseudodifferential operators (see e.g., [10, Section 18]) we have

$$Op(p) \in \mathscr{L}(L^2(\mathbb{R}^n)). \tag{56}$$

We frequently use the following: since $\langle x\rangle^s \in \mathscr{R}^{0,s}$ for any $s \in \mathbb{R}$, by (54) and (56), we have $\langle x\rangle^s Op(p)\langle x\rangle^{-s} \in \mathscr{L}(L^2(\mathbb{R}^n))$ and therefore,

$$Op(p) \in \mathscr{L}(L^2_s(\mathbb{R}^n)), \tag{57}$$

for any $s \in \mathbb{R}$.

Proof of Lemma 2. Let $p_\mp \in \mathscr{R}_\mp(\pm\varepsilon)$, and $\chi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $\chi(x) = 1$ for $|x| < 1$. Then we can write with the convergence in the sense of distribution

$$Op(p_\mp)v^{\pm}_{\lambda}[g] = \lim_{\delta\downarrow 0}\int \chi(\delta y)e^{i(x-y)\cdot\xi \pm iK(y,\lambda)}p_\mp(y,\xi)g(\hat{y})\eta(y)|y|^{-(n-1)/2}dyd\xi.$$

Choosing $\varepsilon < \sqrt{\lambda}/2$ in the definition of $p_{\pm}$, we have

$$|\xi \mp \sqrt{\lambda}\hat{y}| > (|\xi| + \sqrt{\lambda})/2, \tag{58}$$

for $(y, \xi) \in \operatorname{supp} p_{\mp}(y, \xi)$, and we obtain

$$i|\xi \mp \sqrt{\lambda}\hat{y}|^{-2}(\xi \mp \sqrt{\lambda}\hat{y}) \cdot \nabla_y e^{-iy\cdot\xi \pm i\sqrt{\lambda}|y|} = e^{-iy\cdot\xi \pm i\sqrt{\lambda}|y|}.$$

Integration by parts yields

$$\begin{aligned} Op(p_{\mp})v_{\lambda}^{\pm}[g] &= \lim_{\delta\downarrow 0} \int e^{i(x-y)\cdot\xi \pm i\sqrt{\lambda}|y|}\{\delta\nabla\chi(\delta y)\cdot \tilde{p}_{\mp}w(y) + \chi(\delta y)\cdot \nabla(\tilde{p}_{\mp}w)(y)\}dyd\xi \\ &= \lim_{\delta\downarrow 0} \int e^{i(x-y)\cdot\xi \pm i\sqrt{\lambda}|y|}\chi(\delta y)\cdot \nabla(\tilde{p}_{\mp}w)(y)dyd\xi \\ &= Op(\tilde{p}_{\mp})\cdot(e^{\pm i\sqrt{\lambda}|x|}\nabla w) + Op((\nabla\cdot\tilde{p}_{\mp})\langle y\rangle^{\mu'})\langle x\rangle^{-\mu'}e^{\pm i\sqrt{\lambda}|x|}w, \end{aligned} \tag{59}$$

where

$$\tilde{p}_{\mp}(y, \xi) := i|\xi \mp \sqrt{\lambda}\hat{y}|^{-2}(\xi \mp \sqrt{\lambda}\hat{y})p_{\mp}(y, \xi),$$
$$w(x) := -e^{\pm iY(x,\lambda)}g(\hat{x})\eta(x)|x|^{-(n-1)/2}.$$

Using the estimate (7) we have

$$e^{\pm i\sqrt{\lambda}|x|}\nabla w,\ \langle x\rangle^{-\mu'}e^{\pm i\sqrt{\lambda}|x|}w \in L^2_{-s'}(\mathbb{R}^n), \tag{60}$$

for $1/2 > s' > 1/2 - \mu'$ and $\tilde{p}_{\mp}, (\nabla\cdot\tilde{p}_{\mp})\langle y\rangle^{\mu'} \in \mathscr{R}^0$. Thus by (57) we have $Op(\tilde{p}_{\mp}), Op((\nabla\cdot\tilde{p}_{\mp})\langle y\rangle^{\mu'}) \in \mathscr{L}(L^2_{-s'}(\mathbb{R}^n))$. Combining this continuity and (60) we can see that the right hand side of (59) belongs to $L^2_{-s'}(\mathbb{R}^n)$ which shows the outgoing or incoming property.

The following lemma used in the proof of Lemma 3 is proved in the similar way as Lemma 2.

Lemma 8. *If $\phi \in C^\infty(\mathbb{R})$ satisfies $\phi(t) = 0$ near λ and $\phi(t) = 1$ for t large enough, we have $\phi(-\Delta)v_{\lambda}^{\pm}[g] \in L^2_{-s'}(\mathbb{R}^n)$ for some $0 < s' < 1/2$.*

Proof. Instead of (58) we have

$$|\xi \mp \sqrt{\lambda}\hat{y}| > c,$$

for some constant $c > 0$ on $\operatorname{supp} \phi(|\xi|^2)$. Thus we can show the assertion in the same way as in the proof of Lemma 2 with $p_{\mp}(x, \xi)$ replaced by $\phi(|\xi|^2)$.

Lemma 3 is concerned with $J_\alpha v_{\lambda,\alpha}^{\pm}[g] = u_\alpha \otimes v_{\lambda,\alpha}^{\pm}[g]$. Since the energy of $v_{\lambda,\alpha}^{\pm}[g]$ is λ_α and the one of u_α is E_α, the energy of $J_\alpha v_{\lambda,\alpha}^{\pm}[g]$ is finite and we can insert a cut-off function with respect to the energy (Hamiltonian) $H_a :=$

$-\Delta_a + H^a$ in the proof of Lemma 3 below. Because in pseudodifferential calculus we need localization with respect to momentum, we need a relation between the cut-off functions of the energy and those of momentum. Lemma 9 below provides such a relation. Let a be a cluster decomposition and $\psi \in C_c^\infty(\mathbb{R})$. Let $\varphi \in C^\infty(\mathbb{R})$ be a function such that $\operatorname{supp} \varphi \subset (1, \infty)$ and $\varphi(t) = 1$ for $t > 2$. We define

$$K_C := \varphi(-\Delta/C)\psi(H_a). \tag{61}$$

Lemma 9. *$\|K_C\|_{\mathscr{L}(L^2(X))} < 1/2$, for sufficiently large C.*

Proof. Set

$$G(t, C) := e^{t\Delta}\varphi(-\Delta/C)(\psi(H_a))^2\varphi(-\Delta/C)e^{t\Delta}.$$

Then we compute for any $u \in L^2(X)$

$$-\frac{d}{dt}\langle u, G(t, C)u\rangle = 2\operatorname{Re}\langle u, e^{t\Delta}\varphi(-\Delta/C)(-\Delta)(\psi(H_a))^2\varphi(-\Delta/C)e^{t\Delta}u\rangle.$$

Since $(-\Delta)(H_a - i)^{-1}$ and $(H_a - i)\psi(H_a)$ are bounded operators and

$$\|\varphi(-\Delta/C)e^{t\Delta}u\| \leq e^{-tC}\|u\|,$$

we can estimate as

$$-\frac{d}{dt}\langle u, G(t, C)u\rangle \leq C_1\|\varphi(-\Delta/C)e^{t\Delta}u\|^2 \leq C_1 e^{-2tC}\|u\|^2, \tag{62}$$

where $C_1 := 2\|(-\Delta)(\psi(H_a))^2\|_{\mathscr{L}(L^2(X))}$.

Noticing $G(0, C) = \varphi(-\Delta/C)(\psi(H_a))^2\varphi(-\Delta/C)$ and integrating (62) with respect to t, we obtain

$$\|\psi(H_a)\varphi(-\Delta/C)u\|^2 \leq C_1(2C)^{-1}\|u\|^2.$$

Hence

$$\begin{aligned}\|K_C\|_{\mathscr{L}(L^2(X))} = \|K_C^*\|_{\mathscr{L}(L^2(X))} &= \|\psi(H_a)\varphi(-\Delta/C)\|_{\mathscr{L}(L^2(X))} \\ &\leq C_1^{1/2}(2C)^{-1/2}.\end{aligned}$$

Thus we have $\|K_C\|_{\mathscr{L}(L^2(X))} < 1/2$, for sufficiently large C.

By Lemma 9 the operator $(1 - K_C)^{-1}$ is well-defined. Here we note that

$$\langle x\rangle^{-s}(1 - K_C)^{-1}\langle x\rangle^s \in \mathscr{L}(L^2(X)), \tag{63}$$

holds for any $s \in \mathbb{R}$ (cf. [6, Lemma 2.3]).

In the proof of Lemma 3 we insert a cut-off function of momentum using Lemma 9. However, the operator K_C remains in the expression, and the commutator of K_C and a pseudodifferential operator does not have a good decay property due to the potential V_{ij} which does not decay in all directions. We can overcome this problem by considering functions of a kind of first order differential operator instead of the pseudodifferential operator. Since the commutator of V_{ij} and the operator such as $\frac{1}{2}\left(\frac{x}{\langle x\rangle}\cdot(-i\nabla)+(-i\nabla)\cdot\frac{x}{\langle x\rangle}\right)$ has a good decay property ignoring the singularity of V_{ij}, in the proof of Lemma 3 we replace psudodifferential operators by such a differential operator. More precisely, to allow the singularity of V_{ij} near the origin we use the Graf vector field w (see [7]) whose norm is bounded and the operator

$$B := (w\cdot(-i\nabla)+(-i\nabla)\cdot w)/2,$$

as in [6, page 138]. The commutator of B and $(1-K_C)^{-1}$ decays and we have

$$\langle x\rangle^{t}[B,(1-K_C)^{-1}]\langle x\rangle^{t'} \in \mathscr{L}(L^2(X)), \tag{64}$$

for any $t, t' \in \mathbb{R}$ such that $t + t' \le 1$ (cf. [6, Lemma 2.3]). To insert a function of B we need Lemma 10 below. For any $\tau > 0$ we let $\mathscr{F}_+(\tau)$ (resp., $\mathscr{F}_-(\tau)$) denote the class of functions $f \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} f \subset (\tau,\infty)$ and $f(t) = 1$ for $t > 2\tau$ (resp., $\operatorname{supp} f \subset (-\infty,\tau)$ and $f(t) = 1$ for $t < \tau/2$). As in the proof of [6, Theorem 2.12], we have the following lemma which is convincing by the expansion of a product of pseudodifferential operators (55) considering the product $Op(p_\mp)F_\pm(B)$ as if B were a psudodifferential operator with the symbol $\frac{x}{\langle x\rangle}\cdot\xi$ and $F_\pm(B)$ were that with $F_\pm\left(\frac{x}{\langle x\rangle}\cdot\xi\right)$.

Lemma 10. *Let ε be a positive constant.*

(1) *If $p_- \in \mathscr{R}_-(\varepsilon)$ and $F_+ \in \mathscr{F}_+(\varepsilon)$, then*

$$\langle x\rangle^{r} Op(p_-)F_+(B)\langle x\rangle^{r} \in \mathscr{L}(L^2(X)) \tag{65}$$

for any $r \in \mathbb{R}$.

(2) *If $p_+ \in \mathscr{R}_+(\varepsilon)$ and $F_- \in \mathscr{F}_-(\varepsilon)$, then*

$$\langle x\rangle^{r} Op(p_+)F_-(B)\langle x\rangle^{r} \in \mathscr{L}(L^2(X)) \tag{66}$$

for any $r \in \mathbb{R}$.

The proof of (65) is exactly the same as the proof for the boundedness of A_2 in the proof of [6, Theorem 2.12], except that we replace the constants $\rho_j\sqrt{a(\lambda)}$, $j = 1,2$ and $\rho\sqrt{a(\lambda)}$ with $\rho, \rho_2 < \rho_1$ there by constants $\varepsilon, \varepsilon_2 < \varepsilon_1$. The proof of (66) is similar.

Remark 4. *Our definition of* $\mathscr{F}_{\pm}$ *is customized to have a convenient form in the proof of Lemma* 11 *below. We have* $1 - F_+ \in \mathscr{F}_-(2\varepsilon)$ *by our definition. Lemma* 10 *holds for more general* $F_{\pm}$ *and* $\varepsilon \in \mathbb{R}$. *In fact, only the support property and* $|F_{\pm}^{(k)}(t)| \le C_k(1+|t|)^{-k}$, *are needed.*

Using Lemma 10 and taking a commutator of a function of B and $(1-K_C)^{-1}$ we can move a pseudodifferential operator from the left of $(1-K_C)^{-1}$ to the right as in the following lemma. We shall prove only the outgoing case of Lemma 3, since the proof for the incoming case is completely analogous. Thus the lemmas below are used for the outgoing case.

Lemma 11. *Let* $p_- \in \mathscr{R}_-(\varepsilon)$, $\varepsilon > 0$, $\tilde{\varphi} \in C_c^{\infty}(\mathbb{R})$ *identifying* $X = \mathbb{R}^n$ *and* K_C *be the operator in* (61). *Then there exist* $\tilde{p}_- \in \mathscr{R}_-(6\varepsilon)$ *and operators* T_1, T_2 *such that*

$$Op(p_-)(1-K_C)^{-1}\tilde{\varphi}(-\Delta) = T_1 Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) + T_2,$$

and $T_1 \in \mathscr{L}(L_s^2(X))$, $\langle x\rangle^t T_2 \langle x\rangle^{t'} \in \mathscr{L}(L^2(X))$ *for any* $s \in \mathbb{R}$ *and* $t + t' \le 1$.

Proof. Let $F_+ \in \mathscr{F}_+(\varepsilon)$. We decompose the operator as

$$\begin{aligned} Op(p_-)(1-K_C)^{-1}\tilde{\varphi}(-\Delta) = {} & Op(p_-)(1-F_+(B))(1-K_C)^{-1}\tilde{\varphi}(-\Delta) \\ & + Op(p_-)F_+(B)(1-K_C)^{-1}\tilde{\varphi}(-\Delta). \end{aligned} \tag{67}$$

As for the second term, by (65), (63) and continuity (57) of $\tilde{\varphi}(-\Delta)$ we have

$$\langle x\rangle^r Op(p_-)F_+(B)(1-K_C)^{-1}\tilde{\varphi}(-\Delta)\langle x\rangle^r \in \mathscr{L}(L^2(X)),$$

for any $r \in \mathbb{R}$. As for the first term, commuting $1 - F_+(B)$ and $(1-K_C)^{-1}$, we obtain

$$\begin{aligned} (1-F_+(B))(1-K_C)^{-1}\tilde{\varphi}(-\Delta) = {} & (1-K_C)^{-1}(1-F_+(B))\tilde{\varphi}(-\Delta) \\ & - [F_+(B),(1-K_C)^{-1}]\tilde{\varphi}(-\Delta). \end{aligned} \tag{68}$$

By (64) and (57) the second term in the right-hand side satisfies

$$\langle x\rangle^t [F_+(B),(1-K_C)^{-1}]\tilde{\varphi}(-\Delta)\langle x\rangle^{t'} \in \mathscr{L}(L^2(X)),$$

for $t + t' \le 1$.

Let us consider the first term in the right-hand side of (68). Let $f_- \in \mathscr{F}_-(6\varepsilon)$ and $\hat{\varphi} \in C_c^{\infty}(\mathbb{R})$ be a function such that $\hat{\varphi}(t) = 1$ on $\operatorname{supp}\tilde{\varphi}$ and set

$$q_-(x,\xi) := f_-(\hat{x}\cdot\xi)\hat{\varphi}(|\xi|^2),$$

$$q_+(x,\xi) := (1 - f_-(\hat{x}\cdot\xi))\hat{\varphi}(|\xi|^2).$$

Then we can easily see that $q_- \in \mathscr{R}_-(6\varepsilon)$ and $q_+ \in \mathscr{R}_+(2\varepsilon)$. We can decompose the operator as

$$(1 - F_+(B))\tilde{\varphi}(-\Delta) = (1 - F_+(B))Op(q_-)\tilde{\varphi}(-\Delta) + (1 - F_+(B))Op(q_+)\tilde{\varphi}(-\Delta). \tag{69}$$

As for the second term, since $(1 - F_+) \in \mathscr{F}_-(2\varepsilon)$ and $\tilde{p}_+ \in \mathscr{R}_+(2\varepsilon)$, by (66) and (57) we obtain

$$\langle x \rangle^r (1 - F_+(B))Op(q_+)\tilde{\varphi}(-\Delta)\langle x \rangle^r \in \mathscr{L}(L^2(X)),$$

for any $r \in \mathbb{R}$.

By (67)–(69), (63) and the estimates of the operators as above we obtain

$$Op(p_-)(1 - K_C)^{-1}\tilde{\varphi}(-\Delta) = T_1 Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) + T_2,$$

where $T_1 := (1 - K_C)^{-1}(1 - F_+(B))(-\Delta - i)^{-1}$, $\tilde{p}_-(x, \xi) := (|\xi|^2 - i)q_-(x, \xi)$ and T_2 satisfying the condition in the lemma. Since we have

$$\langle x \rangle^{-s}(1 - F_+(B))(-\Delta - i)^{-1}\langle x \rangle^s \in \mathscr{L}(L^2(X)),$$

for any $s \in \mathbb{R}$ (cf. [6, Lemma 2.3]), by (63) T_1 satisfies the condition in the lemma. It is easy to see that the condition $\tilde{p}_-(x, \xi) \in \mathscr{R}_-(6\varepsilon)$ holds.

In order to utilize the outgoing property of $v^+_{\lambda,\alpha}[g]$, we need to replace the pseudodifferential operator in X by that in X_a. We achieve this aim by inserting cut-off functions. In the region where $|x^a| > \delta|x|$ holds for some constant $\delta > 0$, using the decay of u_α we have a good decay estimate of $J_\alpha v^+_{\lambda,\alpha}[g]$. The other region is close to X_a and we can introduce the pseudodifferential operator on X_a in such region.

Lemma 12. *Let ε be a positive constant, $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ and $\tilde{p}_-(x, \xi) \in \mathscr{R}_-(\varepsilon)$. Then there exist $p^a_-(x_a, \xi_a) \in \mathscr{R}_-(4\varepsilon)$, $\zeta(x) \in C^\infty(X)$ and operators $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$ such that*

$$Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) = \tilde{T}_1 Op(p^a_-) + \tilde{T}_2\zeta(x) + \tilde{T}_3,$$

ζ is homogeneous of degree 0 for $|x| > 2$, $\operatorname{supp}\zeta \cap \{x : |x| > 1\} \subset \{x : |x^a| \geq \delta|x|\}$ for some $\delta > 0$, $\tilde{T}_1, \tilde{T}_2 \in \mathscr{L}(L^2_s(X))$ and $\langle x \rangle^t \tilde{T}_3 \langle x \rangle^{t'} \in \mathscr{L}(L^2(X))$ for any $s \in \mathbb{R}$ and $t + t' \leq 1$.

Proof. We decompose the operator as

$$\begin{aligned} Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) &= Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)\tilde{\zeta}(x) + Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)(1 - \tilde{\zeta}(x)) \\ &= Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)\tilde{\zeta}(x) + \tilde{T}_2\zeta(x), \end{aligned} \tag{70}$$

where $\tilde{\zeta}(x) \in C^\infty(X)$ satisfies $\operatorname{supp} \tilde{\zeta} \subset \{x : |x^a| < 2\delta|x|, |x| > 1\}$, $\tilde{\zeta}(x) = 1$ in $\{x : |x^a| < \delta|x|, |x| > 1\}$ for some $\delta > 0$ and homogeneous of degree 0 for $|x| > 2$, $\tilde{T}_2 := Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)$ and $\zeta := 1 - \tilde{\zeta}$. By the continuity (57) of the pseudodifferential operator $Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)$ and the support property of $\tilde{\zeta}$, $\tilde{T}_2$ and ζ satisfy the conditions in the lemma.

Set a symbol $\hat{p}_-(x,\xi) := \tilde{p}_-(x,\xi)\tilde{\varphi}(|\xi|^2)\zeta(x)$. Then by expansion (55) of a product of pseudodifferential operators, it follows that the first term in the right-hand side of (70) is decomposed as

$$Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)\tilde{\zeta}(x) = Op(\hat{p}_-) + Op(q), \tag{71}$$

where $q \in \mathscr{R}^{-1}$, so that

$$\langle x\rangle^t Op(q)\langle x\rangle^{t'} \in \mathscr{L}(L^2(X)), \tag{72}$$

for any $t + t' \le 1$. Choosing sufficiently small δ, on $\operatorname{supp} \hat{p}_-$ we have $|\hat{x}^a| < \tilde{\varepsilon}$ and $|\xi^a|^2 < 2C$, with $\tilde{\varepsilon} > 0$ arbitrarily small, so that choosing sufficiently small $\tilde{\varepsilon}$

$$\hat{x}_a \cdot \xi_a < \hat{x} \cdot \xi + \varepsilon < 2\varepsilon, \tag{73}$$

on $\operatorname{supp} \hat{p}_-$, where we used $\tilde{p}_- \in \mathscr{R}_-(\varepsilon)$ in the second inequality.

Let $\varphi_a \in C_c^\infty(\mathbb{R})$ be a function such that $\varphi_a(t) = 1$ on a interval $[-C, C] \supset \operatorname{supp} \tilde{\varphi}(t)$ and $\tilde{f}_- \in \mathscr{F}_-(4\varepsilon)$, and set a symbol $p_-^a(x_a, \xi_a) := \tilde{f}_-(\hat{x}_a \cdot \xi_a)\varphi_a(|\xi_a|^2)$. Then it is easily seen that $p_-^a \in \mathscr{R}_-(4\varepsilon)$. Moreover, by $|\xi_a|^2 \le |\xi|^2$ we have $1 - \varphi_a(|\xi_a|^2) = 0$ on $\operatorname{supp} \tilde{\varphi}(|\xi|^2)$, and by (73), $1 - \tilde{f}_-(\hat{x}_a \cdot \xi_a) = 0$ on $\operatorname{supp} \hat{p}_-$. Combining these support properties we obtain

$$1 - p_-^a(x_a, \xi_a) = 0 \qquad \text{on } \operatorname{supp} \hat{p}_-(x,\xi). \tag{74}$$

We decompose the first term of (71) as

$$Op(\hat{p}_-) = Op(\hat{p}_-)Op(p_-^a) + Op(\hat{p}_-)(1 - Op(p_-^a)). \tag{75}$$

By (74), the expansion formula (55) and (56) we have

$$\langle x\rangle^r Op(\hat{p}_-)(1 - Op(p_-^a))\langle x\rangle^r \in \mathscr{L}(L^2(X)), \tag{76}$$

for any $r \in \mathbb{R}$.

Combining (70), (71) and (75) we obtain

$$Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) = \tilde{T}_1 Op(p_-^a) + \tilde{T}_2\zeta(x) + \tilde{T}_3,$$

where $\tilde{T}_1 := Op(\hat{p}_-)$ and $\tilde{T}_3 := Op(q) + Op(\hat{p}_-)(1 - Op(p_-^a))$. It follows from the continuity (57) of the pseudodifferential operator $Op(\hat{p}_-)$, (72) and (76) that the $\tilde{T}_1$ and $\tilde{T}_3$ satisfy the conditions in the lemma which completes the proof.

With the above preliminaries we can now prove Lemma 3.

Proof of Lemma 3. Our goal is to prove

$$Op(p_{\mp})J_\alpha v^{\pm}_{\lambda,\alpha}[g] \in L^2_{-s'}(X),$$

for some $0 < s' < 1/2$, where $p_- \in \mathscr{R}_-(\varepsilon)$ and $p_+ \in \mathscr{R}_+(-\varepsilon)$ for some $\varepsilon > 0$ identifying $X = \mathbb{R}^n$. We shall consider the outgoing case. We fix ε in the following and will choose sufficiently small ε near the end of the proof. Let $\psi_0 \in C_c^\infty(\mathbb{R})$ be a function such that $\psi_0(t) = 1$ near 0. Then by Lemma 8 we have $(1 - \psi_0(-\Delta_a - \lambda_\alpha))v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X_a)$, and thus,

$$(1 - \psi_0(-\Delta_a - \lambda_\alpha))J_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X), \tag{77}$$

for some $0 < s' < 1/2$. By (77) and (57) we have

$$Op(p_-)(1 - \psi_0(-\Delta_a - \lambda_\alpha))J_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X).$$

Thus we only need to prove

$$Op(p_-)\psi_0(-\Delta_a - \lambda_\alpha)J_\alpha v^+_{\lambda,\alpha}[g] = Op(p_-)\psi_0(-\Delta_a - \lambda_\alpha)u_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X). \tag{78}$$

Since $H^a u_\alpha = E_\alpha u_\alpha$ and $\lambda_\alpha = \lambda - E_\alpha$, we have

$$\psi_0(-\Delta_a - \lambda_\alpha)u_\alpha v^+_{\lambda,\alpha}[g] = \psi_0(H_a - \lambda)u_\alpha v^+_{\lambda,\alpha}[g], \tag{79}$$

where $H_a := -\Delta_a + H^a = -\Delta + \sum_{(ij)\le a} V_{ij}$. Using $\tilde{\psi} \in C_c^\infty(\mathbb{R})$ satisfying $\tilde{\psi} = 1$ near λ, the operator $\psi_0(H_a - \lambda)$ can be rewritten as $\tilde{\psi}(H_a)$. Let $\varphi \in C^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\varphi \subset (1, \infty)$ and $\varphi(t) = 1$ for $t > 2$, and $\psi \in C_c^\infty(\mathbb{R})$ be a function such that $\psi = 1$ on $\operatorname{supp}\tilde{\psi}$. Setting $\varphi_0 := 1 - \varphi$ we have

$$\tilde{\psi}(H_a) = (K_C + \varphi_0(-\Delta/C))\tilde{\psi}(H_a), \tag{80}$$

where $C > 0$ and $K_C := \varphi(-\Delta/C)\psi(H_a)$. By Lemma 9, $\|K_C\|_{\mathscr{L}(L^2(X))} < 1/2$, for sufficiently large C. Therefore, by (80) we can write

$$\tilde{\psi}(H_a) = (1 - K_C)^{-1}\varphi_0(-\Delta/C)\tilde{\psi}(H_a),$$

and therefore,

$$\tilde{\psi}(H_a)u_\alpha v^+_{\lambda,\alpha}[g] = (1 - K_C)^{-1}\varphi_0(-\Delta/C)\tilde{\psi}(H_a)u_\alpha v^+_{\lambda,\alpha}[g]. \tag{81}$$

Since by (77) we have

$$(1 - \tilde{\psi}(H_a))u_\alpha v^+_{\lambda,\alpha}[g] = (1 - \psi_0(-\Delta_a - \lambda_\alpha))u_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X),$$

for some $0 < s' < 1/2$, by (63) and (57) we obtain

$$Op(p_-)(1 - K_C)^{-1}\varphi_0(-\Delta/C)(1 - \tilde{\psi}(H_a))u_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X). \tag{82}$$

By (79), $\psi_0(H_a - \lambda) = \tilde{\psi}(H_a)$, (81) and (82) we can see that (78) holds if we show

$$Op(p_-)(1 - K_C)^{-1}\varphi_0(-\Delta/C)u_\alpha v^+_{\lambda,\alpha}[g], \tag{83}$$

belongs to $L^2_{-s'}(X)$ for some $0 < s' < 1/2$. Defining functions $\tilde{\varphi}$ and $\Psi_{\alpha,\lambda}[g]$ by $\tilde{\varphi}(t) := \varphi_0(t/C)$ and $\Psi_{\alpha,\lambda}[g] := u_\alpha v^+_{\lambda,\alpha}[g] \in L^2_{-s}(X)$, $\forall s > 1/2$ we can rewrite (83) as

$$Op(p_-)(1 - K_C)^{-1}\tilde{\varphi}(-\Delta)\Psi_{\alpha,\lambda}[g]. \tag{84}$$

By Lemma 11, (84) is equal to

$$T_1 Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)\Psi_{\alpha,\lambda}[g] + T_2\Psi_{\alpha,\lambda}[g],$$

where T_1, T_2 and $\tilde{p}_-$ satisfies the conditions in Lemma 11. As for the second term, choosing $1/2 < s < 1$ we have

$$T_2\Psi_{\alpha,\lambda}[g] = \{T_2\langle x\rangle^s\}\{\langle x\rangle^{-s}\Psi_{\alpha,\lambda}[g]\} \in L^2(X).$$

Therefore, we have only to show first term belongs to $L^2_{-s'}(X)$ for some $0 < s' < 1/2$. By Lemma 12 there exist $p^a_-(x_a, \xi_a) \in \mathscr{R}_-(24\varepsilon)$, $\zeta \in C^\infty(X)$ and operators $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$ such that

$$T_1 Op(\tilde{p}_-)\tilde{\varphi}(-\Delta) = T_1\tilde{T}_1 Op(p^a_-) + T_1\tilde{T}_2\zeta(x) + T_1\tilde{T}_3, \tag{85}$$

and satisfying the conditions in the lemma. Using the properties of T_1 and $\tilde{T}_3$, we can easily see that the third term satisfies

$$T_1\tilde{T}_3\Psi_{\alpha,\lambda}[g] \in L^2(X). \tag{86}$$

As for the second term, it follows from the support property of ζ that there exists $C > 0$ such that $|\langle x\rangle^{s_0}\zeta(x)| \le C|\langle x^a\rangle^{s_0}\zeta(x)|$. Thus by the decay assumption (14) of u_α and $v^+_{\lambda,\alpha}[g] \in L^2_{-s}(X_a)$, $\forall s > 1/2$ we have

$$\zeta(x)\Psi_{\lambda,\alpha}[g] = \zeta(x)u_\alpha v^+_{\lambda,\alpha}[g] = \langle x\rangle^{-s_0}\langle x\rangle^{s_0}\zeta(x)u_\alpha v^+_{\lambda,\alpha}[g] \in L^2(X). \tag{87}$$

As for the first term, the outgoing property of $v^+_{\lambda,\alpha}[g]$ implies

$$Op(p^a_-)v^+_{\lambda,\alpha}[g] \in L^2_{-s'}(X_a), \tag{88}$$

for some $0 < s' < 1/2$ and sufficiently small ε. Since u_α is obviously in $L^2(X^a)$ and $\Psi_{\alpha,\lambda}[g] = u_\alpha v^+_{\lambda,\alpha}[g]$, by (88) it follows that

$$Op(p^a_-)\Psi_{\alpha,\lambda}[g] \in L^2_{-s'}(X). \tag{89}$$

Combining (85)–(87) (89) and the continuity of T_1, $\tilde{T}_1$, $\tilde{T}_2$ we obtain

$$T_1 Op(\tilde{p}_-)\tilde{\varphi}(-\Delta)\Psi_{\alpha,\lambda}[g] \in L^2_{-s'}(X),$$

for some $0 < s' < 1/2$ which completes the proof. The proof for the incoming case is similar.

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