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Dwork hypersurfaces of degree six and Greene's hypergeometric function

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Abstract. In this paper, we give a formula for the number of rational points on the Dwork hypersurfaces of degree six over finite fields by using Greene's finite-field hypergeometric function, which is a generalization of Goodson's formula for the Dwork hypersurfaces of degree four. Our formula is also a higher-dimensional and a finite field analogue of Matsumoto-Terasoma-Yamazaki's formula. Furthermore, we also explain the relation between our formula and Miyatani's formula.

1. Introduction

It is an interesting problem to express the number of rational points on certain varieties over finite fields by using finite-field hypergeometric functions. Finite-field hypergeometric functions were introduced independently by Greene [8], Katz [14], Koblitz [3] and McCarthy [4]. For example, in [5], McCarthy gave a formula for the Dwork hypersurfaces over finite fields by using his hypergeometric functions. In [15], Salerno gave a formula for diagonal hypersurfaces, which are generalizations of the Dwork hypersurfaces, by using Katz's hypergeometric functions.

In [1, Theorem 1.1], Goodson gave a formula for the number of rational points on the Dwork hypersurfaces of degree four over finite fields by using Greene's hypergeometric functions and Jacobi sums. Furthermore, in [2, Theorem 1.2], she also gave a similar formula in the case of odd degrees by Greene's hypergeometric functions and Gauss sums. The purpose of this paper is to extend Goodson's result to the Dwork hypersurfaces of degree six. We give the formula by using Greene's hypergeometric functions and Jacobi sums. The formula in [1, Theorem 1.1] and our formula are higher-dimensional and finite field analogues of the formula of Matsumoto-Terasoma-Yamazaki [9, Theorem 1], for the complex periods of a Hesse cubic curve, that is, the Dwork hypersurface of degree three. (For more details, see Remark 5.)

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Now, we explain our formula precisely. First, we recall Goodson's results. Let $\mathbb{F}_q$ be the finite field with $q = p^e$ elements, where p is a prime number. Let d be a positive integer. For $\lambda \in \mathbb{F}_q$, we define the Dwork hypersurface X_λ^d by the projective equation

$$x_1^d + x_2^d + \cdots + x_d^d = d\lambda x_1 x_2 \cdots x_d$$

over $\mathbb{F}_q$. Let $\hat{\mathbb{F}}_q^\times$ be the group of characters on $\mathbb{F}_q^\times$ in $\mathbb{C}^\times$. For a character $\chi \in \hat{\mathbb{F}}_q^\times$, we extend it by putting $\chi(0) = 0$. We define the trivial character $\epsilon \in \hat{\mathbb{F}}_q^\times$ by putting $\epsilon(x) = 1$ for any $x \in \mathbb{F}_q^\times$ and extend it by putting $\epsilon(0) = 0$. Then for $\chi \in \hat{\mathbb{F}}_q^\times$, we define the Gauss sum $g(\chi)$ by

$$g(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \exp\left(\frac{2\pi\sqrt{-1} \cdot \mathrm{tr}(x)}{p}\right),$$

where tr is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. Note that we obtain $g(\epsilon) = -1$ from $\epsilon(0) = 0$. Furthermore, for characters $\chi, \psi \in \hat{\mathbb{F}}_q^\times$, we define the Jacobi sum by

$$J(\chi, \psi) := \sum_{x \in \mathbb{F}_q} \chi(x)\psi(1 - x) = \sum_{x+y=1} \chi(x)\psi(y).$$

More generally, for characters $\chi_1, \chi_2, \ldots, \chi_n \in \hat{\mathbb{F}}_q^\times$, we define the Jacobi sum by

$$J(\chi_1, \chi_2, \ldots, \chi_n) = \sum_{\substack{x_1, \ldots x_n \in \mathbb{F}_q \\ x_1 + \cdots + x_n = 1}} \chi_1(x_1) \cdots \chi_n(x_n).$$

Next, we define Greene's hypergeometric function. For $A, B \in \hat{\mathbb{F}}_q^\times$, we define the normalized Jacobi sum by

$$\binom{A}{B} := \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\bar{B}(1 - x) = \frac{B(-1)}{q} J(A, \bar{B}),$$

where $\bar{B}$ is the complex conjugate of B. Then for $n \geq 1$, $A_0, A_1, \ldots, A_n$, $B_1, B_2, \ldots, B_n \in \hat{\mathbb{F}}_q^\times$ and $x \in \mathbb{F}_q$, we define Greene's hypergeometric function ${}_{n+1}F_n$ by

$$\begin{aligned} &{}_{n+1}F_n\left(\begin{matrix} A_0 & A_1 & \ldots & A_n \\ & B_1 & \ldots & B_n \end{matrix} \,\middle|\, x\right)_q \\ &:= \begin{cases} \dfrac{q}{q-1} \displaystyle\sum_{\chi \in \hat{\mathbb{F}}_q^\times} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi}\chi(x) & (n \geq 2) \\ \epsilon(x)\dfrac{A_1B_1(-1)}{q} \displaystyle\sum_{y \in \mathbb{F}_q} A_1(y)\overline{A_1}B_1(1 - y)\overline{A_0}(1 - xy) & (n = 1). \end{cases} \end{aligned}$$

Then, Goodson obtained the following results.

THEOREM 1 ([1, Theorem 1.1]). *Let $q = p^e$ be a power of a prime number such that q is congruent to 1 modulo 4 and ω a generator of $\hat{\mathbb{F}}_q^{\times}$. We put $t = (q-1)/4$. For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^4 \neq 1$, we have*

$$\#X_\lambda^4(\mathbb{F}_q) = \frac{q^3-1}{q-1} + 12q\omega^t(-1)\omega^{2t}(1-\lambda^4) + q^2{}_3F_2\left(\begin{matrix}\omega^t & \omega^{2t} & \omega^{3t} \\ & \epsilon & \epsilon\end{matrix} \middle| \frac{1}{\lambda^4}\right)_q$$
$$+ 3q^2\binom{\omega^{3t}}{\omega^t}{}_2F_1\left(\begin{matrix}\omega^{3t} & \omega^t \\ & \omega^{2t}\end{matrix} \middle| \frac{1}{\lambda^4}\right)_q.$$

THEOREM 2 ([2, Theorem 1.4]). *Let $q = p^e$ be a power of a prime number such that q is congruent to 1 modulo 5 and ω a generator of $\hat{\mathbb{F}}_q^{\times}$. We put $t = (q-1)/5$. For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^5 \neq 1$, we have*

$$\#X_\lambda^5(\mathbb{F}_q) = \frac{q^4-1}{q-1} + q^3{}_4F_3\left(\begin{matrix}\omega^t & \omega^{2t} & \omega^{3t} & \omega^{4t} \\ & \epsilon & \epsilon & \epsilon\end{matrix} \middle| \frac{1}{\lambda^5}\right)_q$$
$$+ 20q^2{}_2F_1\left(\begin{matrix}\omega^{2t} & \omega^{3t} \\ & \epsilon\end{matrix} \middle| \frac{1}{\lambda^5}\right)_q + 20q^2{}_2F_1\left(\begin{matrix}\omega^{t} & \omega^{4t} \\ & \epsilon\end{matrix} \middle| \frac{1}{\lambda^5}\right)_q$$
$$+ 30q^2{}_2F_1\left(\begin{matrix}\omega^{t} & \omega^{3t} \\ & \omega^{4t}\end{matrix} \middle| \frac{1}{\lambda^5}\right)_q + 30q^2{}_2F_1\left(\begin{matrix}\omega^{t} & \omega^{2t} \\ & \omega^{3t}\end{matrix} \middle| \frac{1}{\lambda^5}\right)_q.$$

In [2, Theorem 1.2], she also explained the formula for the Dwork hypersurfaces of odd degrees in terms of Greene's hypergeometric function. We remark that the coefficients of Greene's hypergeometric functions in her formula are products of Gauss sums. From a comparison with Matsumoto-Terasoma-Yamazaki's formula for the periods of the Hesse cubic curve over $\mathbb{C}$, the author considers that their coefficients should be written by Jacobi sums. (See also Remark 5.)

In this paper, we consider the Dwork hypersurfaces of degree six. For a generator ω of $\hat{\mathbb{F}}_q^{\times}$, we put $t := (q-1)/6$, $\omega_6 := \omega^t$, $\omega_3 := \omega^{2t}$, and $\omega_2 := \omega^{3t}$. The main result of this paper is the following.

THEOREM 3. *Let $q = p^e$ be a power of a prime number such that q is congruent to 1 modulo 6. For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^6 \neq 1$, we have*

$$\#X_\lambda^6(\mathbb{F}_q) = \frac{q^5-1}{q-1} + 360q^2\omega_2(1-\lambda^6) + q^4{}_5F_4\left(\begin{matrix}\omega_6 & \omega_3 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon & \epsilon & \epsilon & \epsilon\end{matrix} \middle| \frac{1}{\lambda^6}\right)_q$$
$$+ 30q^3\omega_6(-1){}_3F_2\left(\begin{matrix}\omega_3 & \omega_2 & \bar{\omega}_3 \\ & \epsilon & \epsilon\end{matrix} \middle| \frac{1}{\lambda^6}\right)_q$$
$$+ 30q^3{}_3F_2\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_6 \\ & \epsilon & \epsilon\end{matrix} \middle| \frac{1}{\lambda^6}\right)_q$$

$$
\begin{aligned}
&- 15q^3\omega_6(-1)J(\omega_2, \bar{\omega}_3, \bar{\omega}_6)\,{}_4F_3\left(\begin{matrix}\omega_6 & \bar{\omega}_6 & \bar{\omega}_3 & \omega_3 \\ & \epsilon & \epsilon & \omega_2\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 20q^3\omega_6(-1)J(\omega_6, \omega_3, \omega_2)\,{}_4F_3\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon & \omega_3 & \omega_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&+ 60q^2\omega_6(-1)J(\omega_6, \omega_6, \bar{\omega}_3)J(\omega_2, \bar{\omega}_3, \bar{\omega}_6)\,{}_3F_2\left(\begin{matrix}\omega_6 & \bar{\omega}_3 & \omega_2 \\ & \epsilon & \bar{\omega}_6\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&+ 60q^2J(\omega_3, \omega_3, \omega_3)J(\omega_2, \bar{\omega}_3, \bar{\omega}_6)\,{}_3F_2\left(\begin{matrix}\omega_3 & \bar{\omega}_6 & \omega_2 \\ & \epsilon & \omega_6\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&+ 90q^3\,{}_3F_2\left(\begin{matrix}\omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ & \omega_6 & \omega_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 30q^2J(\omega_6, \omega_6)J(\omega_6, \omega_3, \omega_2)\,{}_3F_2\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_6 \\ & \omega_3 & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 120q^2J(\omega_6, \omega_3, \omega_2)\,{}_2F_1\left(\begin{matrix}\omega_6 & \omega_3 \\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 120q^2J(\omega_2, \bar{\omega}_3, \bar{\omega}_6)\,{}_2F_1\left(\begin{matrix}\bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 180q^2J(\omega_6, \omega_3, \omega_2)\,{}_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_3 \\ & \omega_2\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&- 180q^2J(\omega_6, \omega_3, \omega_2)\,{}_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_6 \\ & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q.
\end{aligned}
$$

REMARK 4. The right hand side of this formula does not depend on the choice of ω. However, each term of the right hand side of this formula may depend on the choice of ω.

REMARK 5. One of the novelties of the above result is an expression by using the Jacobi sum. In [9, Theorem 1], Matsumoto-Terasoma-Yamazaki gave the formula for the periods of the Hesse cubic curve over $\mathbb{C}$ by the hypergeometric series and the beta functions. The Jacobi sum is an analogue of the beta function. Hence, Theorem 3 and [1, Theorem 1.1] are higher-dimensional and finite field analogues of the formula due to Matsumoto-Terasoma-Yamazaki. Furthermore, Theorem 3 and [1, Theorem 1.1] suggest that [2, Theorem 1.2] can be rewritten by using the Jacobi sum.

Finally, we explain the proof of our formula. In the same way as the proof of Goodson's, our proof is based on Koblitz's formula for the number of rational points on diagonal hypersurfaces. In [2, page 145, lines 2 to 4], Goodson pointed out a possibility to deduce her formula from Miyatani's formula in terms of McCarthy's hypergeometric functions since the relation between Greene's hypergeometric functions and McCarthy's is known in this case. (We remark that the coefficients of hypergeometric functions in his formula are not Jacobi sums but products and quotients of Gauss sums. (Cf. Theorem 21 in Appendix A.) In Appendix A, we give another simpler proof of our formula based on Miyatani's formula. We consider that our proof based on Koblitz's formula also has its own value since it is more elementary and self-contained.

2. Example

As a special case of Theorem 3, we give the following example.

Example 6. *We define the character* $\omega \in \hat{\mathbb{F}}_{13}^{\times}$ *by* $\omega(\overline{2}^k) = \exp(k\pi\sqrt{-1}/6)$. (*Note that* $\overline{2} \in \mathbb{F}_{13}$ *is a generator of* $\mathbb{F}_{13}^{\times}$.) *Then* ω *is a generator of* $\hat{\mathbb{F}}_{13}^{\times}$. *We put* $\zeta = \exp(2\pi\sqrt{-1}/12)$. *For* $\lambda \in \mathbb{F}_q$ *with* $\lambda \neq 0$ *and* $\lambda^6 \neq 1$, *we obtain*

$$\begin{aligned}
\#X_\lambda^6(\mathbb{F}_{13}) = {} & \frac{13^5-1}{13-1} + 360 \cdot 13^2 \omega_2(1-\lambda^6) \\
& + 13^4 \cdot {}_5F_4\left(\begin{matrix} \omega_6 & \omega_3 & \omega_2 & \overline{\omega}_3 & \overline{\omega}_6 \\ & \epsilon & \epsilon & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& + 30 \cdot 13^3 \cdot {}_3F_2\left(\begin{matrix} \omega_3 & \omega_2 & \overline{\omega}_3 \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& + 30 \cdot 13^3 \cdot {}_3F_2\left(\begin{matrix} \omega_6 & \omega_2 & \overline{\omega}_6 \\ & \epsilon & \epsilon \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& - 15 \cdot 13^3(4\zeta^2-1)\,{}_4F_3\left(\begin{matrix} \omega_6 & \overline{\omega}_6 & \overline{\omega}_3 & \omega_3 \\ & \epsilon & \epsilon & \omega_2 \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& - 20 \cdot 13^3(-4\zeta^2+3)\,{}_4F_3\left(\begin{matrix} \omega_6 & \omega_2 & \overline{\omega}_3 & \overline{\omega}_6 \\ & \epsilon & \omega_3 & \omega_3 \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& + 60 \cdot 13^2(\zeta^2-4)(4\zeta^2-1)\,{}_3F_2\left(\begin{matrix} \omega_6 & \overline{\omega}_3 & \omega_2 \\ & \epsilon & \overline{\omega}_6 \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& + 60 \cdot 13^2(3\zeta^2+1)(4\zeta^2-1)\,{}_3F_2\left(\begin{matrix} \omega_3 & \overline{\omega}_6 & \omega_2 \\ & \epsilon & \omega_6 \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13} \\
& + 90 \cdot 13^3 \cdot {}_3F_2\left(\begin{matrix} \omega_2 & \overline{\omega}_3 & \overline{\omega}_6 \\ & \omega_6 & \omega_3 \end{matrix} \,\middle|\, \frac{1}{\lambda^6}\right)_{13}
\end{aligned}$$

$$- 30 \cdot 13^2(-\zeta^2+4)(-4\zeta^2+3)_3F_2\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_6 \\ & \omega_3 & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_{13}$$

$$- 120 \cdot 13^2(-4\zeta^2+3)_2F_1\left(\begin{matrix}\omega_6 & \omega_3 \\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_{13}$$

$$- 120 \cdot 13^2(4\zeta^2-1)_2F_1\left(\begin{matrix}\bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_{13}$$

$$- 180 \cdot 13^2(-4\zeta^2+3)_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_3 \\ & \omega_2\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_{13}$$

$$- 180 \cdot 13^2(-4\zeta^2+3)_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_6 \\ & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_{13}.$$

3. The proof by Koblitz's formula

3.1. Identities for the Gauss sums. In this subsection, we give identities for the Gauss sums. Let ω be a generator of $\hat{\mathbb{F}}_q^\times$, which we fix throughout the rest of this paper. First, we recall the Hasse-Davenport product relation.

Theorem 7 ([11, Theorem 10.1]). *Let m be a positive integer and let q be a power of a prime number such that q is congruent to 1 modulo m. For a character $\chi \in \hat{\mathbb{F}}_q^\times$ of order m and a character $\psi \in \hat{\mathbb{F}}_q^\times$, we have*

$$\prod_{i=0}^{m-1} g(\chi^i\psi) = -g(\psi^m)\psi^{-m}(m)\prod_{i=0}^{m-1} g(\chi^i).$$

Corollary 8. *Let q be a power of a prime number such that q is congruent to 1 modulo 6. For $j \in \mathbb{Z}$ and $t = (q-1)/6$, we have*

$$g(\omega^{6j}) = \frac{\prod_{i=0}^{5} g(\omega^{it+j})}{\omega^{-6j}(6)\prod_{i=1}^{5} g(\omega^{it})}.$$

Proof. This follows from Theorem 7 applied to $m = 6$, $\chi = \omega^t$, and $\psi = \omega^j$. □

We use the following lemma to prove Theorem 3.

Lemma 1. *Let a, b be multiples of t. Then, we have*

$$\sum_{j=0}^{q-2} g(\omega^{j+a})g(\omega^{-j+b})\omega^j(-1)\omega^{6j}(\lambda) = (q-1)g(\omega^{a+b})\omega^b(-1)\omega^{-(a+b)}(1-\lambda^6).$$

PROOF. We can prove this result similarly as in [2, Proposition 3.1]. (Just replace 4 with 6 everywhere.) □

3.2. Koblitz's formula for diagonal hypersurfaces. In this subsection, we recall the general formula by Koblitz. Koblitz gave a formula for the number of $\mathbb{F}_q$-rational points on diagonal hypersurfaces

$$D_\lambda : x_1^d + x_2^d + \cdots + x_n^d - d\lambda x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n} = 0,$$

where $d \mid q-1$, $h_1 + \cdots + h_n = d$ and $\gcd(d, h_1, \dots, h_n) = 1$. Let W be the set of all n-tuples $\boldsymbol{w} = (w_1, \dots, w_n)$ of the elements of $\mathbb{Z}/d\mathbb{Z}$ satisfying $\sum_i w_i = 0$, that is, we put

$$W := \left\{ \boldsymbol{w} = (w_1, \dots, w_n) \in (\mathbb{Z}/d\mathbb{Z})^n \;\middle|\; \sum_{i=1}^n w_i = 0 \right\}.$$

We put $t := (q-1)/d$. Then it is known that the number of $\mathbb{F}_q$-rational points on the projective diagonal hypersurface

$$x_1^d + \cdots + x_n^d = 0$$

is given by $\sum_{\boldsymbol{w} \in W} N_q(0, \boldsymbol{w})$, where

$$N_q(0, \boldsymbol{w}) := \begin{cases} (q^{n-1} - 1)/(q-1) & (w_i = 0 \text{ for all } i) \\ (1/q) \prod_{i=1}^n g(\omega^{w_i t}) & (w_i \neq 0 \text{ for all } i) \\ 0 & (\text{otherwise}). \end{cases} \tag{1}$$

(For example, see [6, (2.12)] and [12].) We define an equivalence relation $\sim$ on W by

$$\boldsymbol{w} \sim \boldsymbol{w}' \qquad \text{if } \boldsymbol{w} - \boldsymbol{w}' \text{ is a multiple of } (h_1, \dots, h_n).$$

We denote an equivalence class of $\boldsymbol{w}$ by $[\boldsymbol{w}]$. Then, we have the following theorem.

THEOREM 9 ([6, Theorem 2]). *We put $t := (q-1)/d$. For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^d \neq (h_1^{h_1} \cdots h_n^{h_n})^{-1}$, we have*

$$\#D_\lambda(\mathbb{F}_q) = \sum_{\boldsymbol{w} \in W} N_q(0, \boldsymbol{w}) + \frac{1}{q-1} \sum_{[\boldsymbol{w}] \in W/\sim} \sum_{j=0}^{q-2} \frac{\prod_{i=1}^n g(\omega^{w_i t + h_i j})}{g(\omega^{dj})} \omega^{dj}(d\lambda). \tag{2}$$

REMARK 10. The assumptions that λ is not equal to zero and λ^d is not equal to $(h_1^{h_1} \cdots h_n^{h_n})^{-1}$ imply that D_λ is smooth.

REMARK 11. Note that $g(\omega^{w_i t + h_i j})$ itself is not well-defined, and only $\prod_{i=1}^n g(\omega^{w_i t + h_i j})$ is well-defined since we assume that $\sum_i w_i$ and $\sum_i h_i$ are equal to zero in $\mathbb{Z}/d\mathbb{Z}$.

REMARK 12. The notation in Theorem 9 differs from that in [6, Theorem 2]. In [6, Theorem 2], the summation of the second term is over $s \in (d/(q-1))\mathbb{Z}/\mathbb{Z}$ and $w \in W$. We obtain an expression that is equivalent to the identity (2) by replacing s with ds and summing over $s \in (1/(q-1))\mathbb{Z}/\mathbb{Z}$.

3.3. An application of Koblitz's formula. We explain our strategy for proving the main theorem. As a first step, we apply Koblitz's formula to the Dwork hypersurfaces of degree six and we list up all of the elements of $W/\sim$. Next, we calculate the right hand side of the identity (3), which is given in this subsection. From $\sum_{\boldsymbol{w} \in W} N_q(0, \boldsymbol{w}) = \sum_{[\boldsymbol{w}] \in W/\sim} \sum_{\boldsymbol{w} \in [\boldsymbol{w}]} N_q(0, \boldsymbol{w})$, we calculate the right hand side of the identity (3) for each $[\boldsymbol{w}] \in W/\sim$.

By (2) with $d = n = 6$ and $(h_1, \dots, h_n) = (1, 1, 1, 1, 1, 1)$, we have the following.

COROLLARY 13. *For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^6 \neq 1$, we have*

$$\#X_\lambda^6(\mathbb{F}_q) = \sum_{\boldsymbol{w} \in W} N_q(0, \boldsymbol{w}) + \frac{1}{q-1} \sum_{[\boldsymbol{w}] \in W/\sim} \sum_{j=0}^{q-2} \frac{\prod_{i=1}^{6} g(\omega^{w_i t + j})}{g(\omega^{6j})} \omega^{6j}(6\lambda). \tag{3}$$

Let S_6 be the symmetric group of degree six. An action of S_6 on $W/\sim$ is naturally defined.

DEFINITION 14. We let

$$\langle [w_1, \dots, w_6] \rangle^k = \left\{ [x_1, \dots, x_6] \in W/\sim \,\middle|\, \begin{array}{l} \text{There exists } \sigma \in S_6 \text{ such that} \\ [x_{\sigma(1)}, \dots, x_{\sigma(6)}] = [w_1, \dots, w_6] \end{array} \right\},$$

where

$$k = \#\left\{ [x_1, \dots, x_6] \in W/\sim \,\middle|\, \begin{array}{l} \text{There exists } \sigma \in S_6 \text{ such that} \\ [x_{\sigma(1)}, \dots, x_{\sigma(6)}] = [w_1, \dots, w_6] \end{array} \right\}.$$

By abuse of notation, we denote $\bar{a} = a$ for $\bar{a} \in \mathbb{Z}/6\mathbb{Z}$. Then, we have

$$\begin{aligned} W/\sim = {} & \langle [0,0,0,0,0,0] \rangle^1 \cup \langle [0,0,0,0,1,5] \rangle^{30} \cup \langle [0,0,0,0,2,4] \rangle^{30} \\ & \cup \langle [0,0,0,0,3,3] \rangle^{15} \cup \langle [0,0,0,1,1,4] \rangle^{60} \cup \langle [0,0,0,1,2,3] \rangle^{120} \\ & \cup \langle [0,0,0,2,2,2] \rangle^{20} \cup \langle [0,0,0,2,5,5] \rangle^{60} \cup \langle [0,0,0,3,4,5] \rangle^{120} \\ & \cup \langle [0,0,1,1,2,2] \rangle^{90} \cup \langle [0,0,2,2,4,4] \rangle^{30} \cup \langle [0,0,2,2,3,5] \rangle^{180} \\ & \cup \langle [0,0,1,3,3,5] \rangle^{180} \cup \langle [0,0,1,2,4,5] \rangle^{360}. \end{aligned}$$

We put

$$S_{[\boldsymbol{w}]} := (q-1)^{-1}\sum_{j=0}^{q-2}\left(\prod_{i=1}^{6} g(\omega^{w_i t+j})/g(\omega^{6j})\right)\omega^{6j}(6\lambda).$$

PROPOSITION 15. *We obtain the following identities:*

$$\sum_{\boldsymbol{w}\in[0,0,0,0,0,0]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,0,0,0]}$$
$$= \frac{q^5-1}{q-1} + q^4 {}_5F_4\left(\begin{matrix}\omega_6 & \omega_3 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6\\ & \epsilon & \epsilon & \epsilon & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{4}$$

$$\sum_{\boldsymbol{w}\in[0,0,0,0,1,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,0,1,5]} = q^3\omega_6(-1)_3F_2\left(\begin{matrix}\omega_3 & \omega_2 & \bar{\omega}_3\\ & \epsilon & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{5}$$

$$\sum_{\boldsymbol{w}\in[0,0,0,0,2,4]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,0,2,4]} = q^3{}_3F_2\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_6\\ & \epsilon & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{6}$$

$$\sum_{\boldsymbol{w}\in[0,0,1,1,2,2]} N_q(0,\boldsymbol{w}) + S_{[0,0,1,1,2,2]} = q^3{}_3F_2\left(\begin{matrix}\omega_2 & \bar{\omega}_3 & \bar{\omega}_6\\ & \omega_6 & \omega_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q. \tag{7}$$

PROOF. We give the proof of the identity (4). First, we calculate $S_{[0,\dots,0]}$. We obtain

$$\begin{aligned}
S_{[0,0,0,0,0,0]} &= \frac{1}{q-1}\sum_{j=0}^{q-2}\frac{g(\omega^j)^6}{g(\omega^{6j})}\omega^{6j}(6\lambda)\\
&= \frac{1}{q-1}\left\{\frac{(-1)^6}{-1} + \frac{\sum_{i=1}^{5} g(\omega^{it})^6}{-1} + \sum_{\substack{j=0\\ j\neq 0,t,\dots,5t}}\frac{g(\omega^j)^6 g(\omega^{-6j})}{q}\omega^{6j}(6\lambda)\right\}\\
&= \frac{1}{q-1}\Bigg\{-1 - \sum_{i=1}^{5} g(\omega^{it})^6 + \frac{1}{q}\sum_{j=0}^{q-2} g(\omega^j)^6 g(\omega^{-6j})\omega^{6j}(6\lambda)\\
&\qquad - \frac{1}{q}(-1)^6\cdot(-1) - \frac{1}{q}\sum_{i=1}^{5} g(\omega^{it})^6\cdot(-1)\Bigg\}\\
&= \frac{1}{q-1}\left\{-\frac{q-1}{q} - \frac{q-1}{q}\sum_{i=1}^{5} g(\omega^{it})^6 + \frac{1}{q}\sum_{j=0}^{q-2} g(\omega^j)^6 g(\omega^{-6j})\omega^{6j}(6\lambda)\right\}\\
&= -\frac{1}{q} - \sum_{\boldsymbol{w}\in[0,0,0,0,0,0]} N_q(0,\boldsymbol{w}) + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g(\omega^j)^6 g(\omega^{-6j})\omega^{6j}(6\lambda).
\end{aligned}$$

Second, we calculate the hypergeometric function. We have

$$
\begin{aligned}
&{}_5F_4\left(\begin{matrix}\omega_6 & \omega_3 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon & \epsilon & \epsilon & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \\
&= \frac{q}{q-1}\sum_{j=0}^{q-2}\left(\frac{\omega^j(-1)}{q^5}J(\omega^{t+j},\overline{\omega^j})J(\omega^{2t+j},\overline{\omega^j})J(\omega^{3t+j},\overline{\omega^j})\right. \\
&\qquad\qquad \left.\times J(\omega^{4t+j},\overline{\omega^j})J(\omega^{5t+j},\overline{\omega^j})\omega^j\left(\frac{1}{\lambda^6}\right)\right) \\
&= \frac{1}{q^4(q-1)}\sum_{j=0}^{q-2}\frac{g(\omega^{-j})^5\omega^j\left(\frac{-1}{\lambda^6}\right)\prod_{i=1}^5 g(\omega^{it+j})}{\prod_{i=1}^5 g(\omega^{it})} \\
&= \frac{1}{q^4(q-1)}\sum_{j=0}^{q-2}\frac{g(\omega^{6j})\omega^{-6j}(6)g(\omega^{-j})^5\omega^j\left(\frac{-1}{\lambda^6}\right)}{g(\omega^j)} \\
&= \frac{1}{q^4(q-1)}\left\{\frac{-1\cdot(-1)^5}{-1}+\sum_{j=1}^{q-2}\frac{g(\omega^{6j})g(\omega^{-j})^6\omega^{-6j}(-6\lambda)}{q\cdot\omega^j(-1)}\right\} \\
&= \frac{1}{q^4(q-1)}\left\{-1-\frac{1}{q}(-1)\cdot(-1)^6+\frac{1}{q}\sum_{j=0}^{q-2}g(\omega^{6j})g(\omega^{-j})^6\omega^{-6j}(6\lambda)\right\} \\
&= \frac{-1}{q^5}+\frac{1}{q^5(q-1)}\sum_{j=0}^{q-2}g(\omega^{6j})g(\omega^{-j})^6\omega^{-6j}(6\lambda) \\
&= \frac{-1}{q^5}+\frac{1}{q^5(q-1)}\sum_{j=0}^{q-2}g(\omega^{-6j})g(\omega^{j})^6\omega^{6j}(6\lambda).
\end{aligned}
$$

Here, the first equality follows from the definition of the hypergeometric function. The second equality follows from the definition of the Jacobi sum. The third equality is obtained by using Corollary 8. The fourth equality is obtained by using the identity $g(\omega^j)g(\omega^{-j}) = q\cdot\omega^j(-1)$ for $j \neq 0$.

This completes the proof of the identity (4). Similarly, we can show the identities (5), (6) and (7). $\square$

Proposition 16. *We obtain the following identities:*

$$
\begin{aligned}
&\sum_{\boldsymbol{w}\in[0,0,0,0,3,3]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,0,3,3]} \\
&\quad = -q^3\omega_6(-1)J(\omega_2,\bar{\omega}_3,\bar{\omega}_6)\,{}_4F_3\left(\begin{matrix}\omega_6 & \bar{\omega}_6 & \bar{\omega}_3 & \omega_3 \\ & \epsilon & \epsilon & \omega_2\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \qquad (8)
\end{aligned}
$$

$$\sum_{\boldsymbol{w}\in[0,0,2,2,4,4]} N_q(0,\boldsymbol{w}) + S_{[0,0,2,2,4,4]}$$

$$= -q^2 J(\omega_6,\omega_6)J(\omega_6,\omega_3,\omega_2)\,{}_3F_2\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_6 \\ & \omega_3 & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{9}$$

$$\sum_{\boldsymbol{w}\in[0,0,0,2,2,2]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,2,2,2]}$$

$$= q^3\omega_6(-1)J(\omega_6,\omega_3,\omega_2)\,{}_4F_3\left(\begin{matrix}\omega_6 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ & \epsilon & \omega_3 & \omega_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q. \tag{10}$$

PROOF. We give the proof of only the identity (8) since we can show the identities (9) and (10) similarly. From a calculation similar to $S_{[0,0,0,0,0,0]}$, we obtain

$$\begin{aligned}
S_{[0,0,0,0,3,3]} &= \frac{1}{q-1}\sum_{j=0}^{q-2}\frac{g(\omega^j)^4 g(\omega^{3t+j})^2}{g(\omega^{6j})}\omega^{6j}(6\lambda) \\
&= \frac{1}{q-1}\cdot\frac{(-1)^4 g(\omega^{3t})^2}{-1} + \frac{1}{q-1}\,\frac{\sum_{i=1}^5 g(\omega^{it})^4 g(\omega^{3+it})^2}{-1} \\
&\quad + \frac{1}{q-1}\sum_{\substack{j=0\\ j\neq 0,t,\dots,5t}}^{q-2}\frac{g(\omega^j)^4 g(\omega^{3t+j})^2 g(\omega^{-6j})\omega^{6j}(6\lambda)}{q} \\
&= -\frac{1}{q-1}g(\omega^{3t})^2 - \frac{1}{q-1}\sum_{i=1}^5 g(\omega^{it})^4 g(\omega^{3t+it})^2 \\
&\quad + \frac{1}{q(q-1)}\Biggl(-(-1)^4 g(\omega^{3t})^2\cdot(-1) - \sum_{i=1}^5 g(\omega^{it})^4 g(\omega^{3t+it})^2\cdot(-1) \\
&\qquad + \sum_{j=0}^{q-2} g(\omega^j)^4 g(\omega^{3t+j})^2 g(\omega^{-6j})\omega^{6j}(6\lambda)\Biggr) \\
&= -\frac{1}{q-1}g(\omega^{3t})^2 - \frac{1}{q-1}\sum_{i=1}^5 g(\omega^{it})^4 g(\omega^{3t+it})^2 \\
&\quad + \frac{1}{q(q-1)}g(\omega^{3t})^2 + \frac{1}{q(q-1)}\sum_{i=1}^5 g(\omega^{it})^4 g(\omega^{3t+it})^2 \\
&\quad + \frac{1}{q(q-1)}\sum_{j=0}^{q-2} g(\omega^j)^4 g(\omega^{3t+j})^2 g(\omega^{-6j})\omega^{6j}(6\lambda)
\end{aligned}$$

$$= -\frac{1}{q} g(\omega^{3t})^2 - \frac{1}{q} \sum_{i=1}^{5} g(\omega^{it})^4 g(\omega^{3t+it})^2$$

$$+ \frac{1}{q(q-1)} \sum_{j=0}^{q-2} g(\omega^j)^4 g(\omega^{3t+j})^2 g(\omega^{-6j}) \omega^{6j}(6\lambda)$$

$$= -\omega^t(-1) - q - N_q(0, [0,0,0,0,3,3])$$

$$+ \frac{1}{q(q-1)} \sum_{j=0}^{q-2} g(\omega^j)^4 g(\omega^{3t+j})^2 g(\omega^{-6j}) \omega^{6j}(6\lambda).$$

Next, we calculate the hypergeometric function. In a similar way to the proof of the identity (4) in Proposition 15, we obtain

$${}_4F_3\left(\begin{matrix} \omega_6 & \bar{\omega}_6 & \bar{\omega}_3 & \omega_3 \\ & \epsilon & \epsilon & \omega_2 \end{matrix} \middle| \frac{1}{\lambda^6}\right)_q$$

$$= \frac{q}{q-1} \sum_{j=0}^{q-2} \frac{\omega^t(-1)}{q^4} J(\omega^{t+j}, \overline{\omega^j}) J(\omega^{5t+j}, \overline{\omega^j}) J(\omega^{4t+j}, \overline{\omega^j}) J(\omega^{2t+j}, \overline{\omega^{3t+j}}) \omega^j\left(\frac{1}{\lambda^6}\right)$$

$$= \frac{\omega^t(-1)}{q^3(q-1)} \sum_{j=0}^{q-2} \frac{g(\omega^{t+j}) g(\omega^{5t+j}) g(\omega^{4t+j}) g(\omega^{2t+j}) g(\overline{\omega^j})^3 g(\overline{\omega^{3t+j}}) \omega^{-6j}(\lambda)}{g(\omega^t) g(\omega^{5t}) g(\omega^{4t}) g(\omega^{-t})}$$

$$= \frac{\omega^t(-1) J(\omega^{2t}, \omega^{3t})}{q^3(q-1)} \sum_{j=0}^{q-2} \frac{g(\omega^{6j}) g(\omega^{-j})^3 g(\omega^{-3t-j}) \omega^{-6j}(6\lambda)}{g(\omega^j) g(\omega^{3t+j})}$$

$$= \frac{\omega_6(-1) J(\omega_3, \omega_2)}{q^3(q-1)} \left(\frac{(-1)(-1)^3 g(\omega^{3t})}{(-1) g(\omega^{3t})} + \frac{(-1) g(\omega^{-3t})^3 (-1)}{g(\omega^{3t})(-1)} \right.$$

$$\left. + \sum_{\substack{j=0 \\ j \neq 0, 3t}}^{q-2} \frac{g(\omega^{6j}) g(\omega^{-j})^4 g(\omega^{-3t-j})^2 \omega^{-6j}(6\lambda)}{q^2 \omega^{-j-3t-j}(-1)} \right)$$

$$= \frac{\omega_6(-1) J(\omega_3, \omega_2)}{q^3(q-1)} \left(-1 - g(\omega^{3t})^2 + \frac{1}{q} + \frac{g(\omega^{3t})^2}{q} \right.$$

$$\left. + \sum_{j=0}^{q-2} \frac{g(\omega^{6j}) g(\omega^{-j})^4 g(\omega^{-3t-j})^2 \omega^{-6j}(6\lambda)}{q^2 \omega^t(-1)} \right)$$

$$
\begin{aligned}
&= \frac{\omega_6(-1)J(\omega_3,\omega_2)}{q^3(q-1)}\left(\frac{-(q-1)}{q} - \frac{(q-1)g(\omega^{3t})^2}{q}\right.\\
&\qquad\qquad\left. + \sum_{j=0}^{q-2}\frac{g(\omega^{6j})g(\omega^{-j})^4 g(\omega^{-3t-j})^2\omega^{-6j}(6\lambda)}{q^2\omega^t(-1)}\right)\\
&= \frac{\omega_6(-1)J(\omega_3,\omega_2)}{q^3}\left(\frac{-1}{q} - \frac{g(\omega^{3t})^2}{q}\right)\\
&\quad + \frac{J(\omega_3,\omega_2)}{q^5(q-1)}\sum_{j=0}^{q-2} g(\omega^{6j})g(\omega^{-j})^4 g(\omega^{-3t-j})^2\omega^{-6j}(6\lambda)\\
&= \frac{J(\omega_3,\omega_2)}{q^3}\left(\frac{-\omega_6(-1)}{q} - 1\right)\\
&\quad + \frac{J(\omega_3,\omega_2)}{q^5(q-1)}\sum_{j=0}^{q-2} g(\omega^{6j})g(\omega^{-j})^4 g(\omega^{-3t-j})^2\omega^{-6j}(6\lambda).
\end{aligned}
$$

Then, the identity (8) follows from the identity

$$
\frac{q^4}{J(\omega_3,\omega_2)} = -q^3\omega_6(-1)J(\omega_2,\bar{\omega}_3,\bar{\omega}_6). \qquad \square
$$

Proposition 17. *We obtain*

$$
\begin{aligned}
&\sum_{\boldsymbol{w}\in[0,0,0,1,1,4]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,1,1,4]} + \sum_{\boldsymbol{w}\in[0,0,0,2,5,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,2,5,5]}\\
&\quad = q^2\omega_6(-1)J(\omega_6,\omega_6,\bar{\omega}_3)J(\omega_2,\bar{\omega}_3,\bar{\omega}_6)\,{}_3F_2\left(\left.\begin{matrix}\omega_6 & \bar{\omega}_3 & \omega_2\\ & \epsilon & \bar{\omega}_6\end{matrix}\,\right|\,\frac{1}{\lambda^6}\right)_q\\
&\qquad + q^2 J(\omega_3,\omega_3,\omega_3)J(\omega_2,\overline{\omega_3},\bar{\omega}_6)\,{}_3F_2\left(\left.\begin{matrix}\omega_3 & \bar{\omega}_6 & \omega_2\\ & \epsilon & \omega_6\end{matrix}\,\right|\,\frac{1}{\lambda^6}\right)_q.
\end{aligned}
$$

Proof. By an argument similar to that in Proposition 16, we have

$$
\begin{aligned}
&\sum_{\boldsymbol{w}\in[0,0,0,1,1,4]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,1,1,4]}\\
&\qquad = q^2\cdot J(\omega_3,\omega_3,\omega_3)J(\omega_2,\overline{\omega_3},\bar{\omega}_6)\cdot {}_3F_2\left(\left.\begin{matrix}\omega_3 & \bar{\omega}_6 & \omega_2\\ & \epsilon & \omega_6\end{matrix}\,\right|\,\frac{1}{\lambda^6}\right)_q\\
&\qquad\quad + \frac{1}{q}g(\omega^t)^2 g(\omega^{4t}) - \frac{1}{q}g(\omega^{5t})^3 g(\omega^{3t}) + \omega^t(-1)g(\omega^{2t})^3\\
&\qquad\quad - \frac{1}{q}g(\omega^{2t})g(\omega^{5t})^2 + \frac{1}{q}g(\omega^t)^3 g(\omega^{3t}) - \omega^t(-1)g(\omega^{4t})^3,
\end{aligned}
$$

and

$$\sum_{w\in[0,0,0,2,5,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,2,5,5]}$$

$$= q^2\omega_6(-1)J(\omega_6,\omega_6,\bar{\omega}_3)J(\omega_2,\bar{\omega}_3,\bar{\omega}_6)\cdot {}_3F_2\left(\begin{matrix}\omega_6 & \bar{\omega}_3 & \omega_2\\ & \epsilon & \bar{\omega}_6\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q$$

$$+\frac{1}{q}g(\omega^{5t})^2g(\omega^{2t})+\omega^t(-1)g(\omega^{4t})^3-\frac{1}{q}g(\omega^t)^3g(\omega^{3t})$$

$$-\frac{1}{q}g(\omega^t)^2g(\omega^{4t})-\omega^t(-1)g(\omega^{2t})^3+\frac{1}{q}g(\omega^{5t})^3g(\omega^{3t}).$$

This completes the proof of Proposition 17. □

In the same fashion as the proof of [1, Proposition 4.6], we can obtain the following results by using Lemma 8.

PROPOSITION 18. *We obtain the following identities:*

$$\sum_{w\in[0,0,0,1,2,3]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,1,2,3]} = -q^2J(\omega_6,\omega_3,\omega_2)_2F_1\left(\begin{matrix}\omega_6 & \omega_3\\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{11}$$

$$\sum_{w\in[0,0,0,3,4,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,3,4,5]} = -q^2J(\omega_2,\bar{\omega}_3,\bar{\omega}_6)_2F_1\left(\begin{matrix}\bar{\omega}_3 & \bar{\omega}_6\\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{12}$$

$$\sum_{w\in[0,0,1,3,3,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,1,3,3,5]} = -q^2J(\omega_6,\omega_3,\omega_2)_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_3\\ & \omega_2\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q \tag{13}$$

$$\sum_{w\in[0,0,2,2,3,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,2,2,3,5]} = -q^2J(\omega_6,\omega_3,\omega_2)_2F_1\left(\begin{matrix}\omega_3 & \bar{\omega}_6\\ & \bar{\omega}_3\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q. \tag{14}$$

PROOF. In the same way as the proof of [1, Proposition 4.6], we have

$$\sum_{w\in[0,0,0,1,2,3]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,1,2,3]}$$

$$= -q^2\omega^t(-1)J(\omega^t,\omega^{2t},\omega^{3t})\cdot\frac{\omega^t(-1)}{q}\sum_{y\in\mathbb{F}_q}\omega^{2t}(y)\omega^{5t}(1-y)\omega^{4t}(1-\lambda^6 y)$$

$$\left(= -q^2\omega_6(-1)J(\omega_6,\omega_3,\omega_2)\cdot {}_2F_1\left(\begin{matrix}\omega_3 & \omega_3\\ & \omega_6\end{matrix}\,\middle|\,\lambda^6\right)_q\right).$$

Then, the substitution $y\mapsto y/\lambda^6$ implies

$$\sum_{w\in[0,0,0,1,2,3]} N_q(0,\boldsymbol{w}) + S_{[0,0,0,1,2,3]}$$

$$= -q^2\omega^t(-1)J(\omega^t,\omega^{2t},\omega^{3t})\cdot\frac{\omega^t(-1)}{q}\sum_{y\in\mathbb{F}_q}\omega^{2t}\left(\frac{y}{\lambda^6}\right)\omega^{5t}\left(1-\frac{y}{\lambda^6}\right)\omega^{4t}(1-y)$$

$$= -q^2\omega^t(-1)J(\omega^t,\omega^{2t},\omega^{3t})\cdot\frac{\omega^t(-1)}{q}\sum_{y\in\mathbb{F}_q}\omega^{2t}(y)\omega^{4t}(1-y)\omega^{5t}\left(1-\frac{y}{\lambda^6}\right)$$

$$= -q^2J(\omega_6,\omega_3,\omega_2)\cdot{}_2F_1\left(\begin{matrix}\omega_6 & \omega_3\\ & \epsilon\end{matrix}\,\middle|\,\frac{1}{\lambda^6}\right)_q. \qquad \square$$

Finally, we calculate in the case of $[\boldsymbol{w}]=[0,0,1,2,4,5]$.

Proposition 19. *We obtain*

$$\sum_{w\in[0,0,1,2,4,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,1,2,4,5]} = q^2\omega_2(1-\lambda^6).$$

Proof. First, we obtain

$$\sum_{w\in[0,0,1,2,4,5]} N_q(0,\boldsymbol{w}) = N_q(0,(3,3,4,5,1,2))$$

$$= \frac{1}{q}g(\omega^{3t})^2g(\omega^{4t})g(\omega^{5t})g(\omega^t)g(\omega^{2t}) = q^2.$$

Second, we calculate $S_{[0,0,1,2,4,5]}$. From Corollary 8, we have

$$S_{[0,0,1,2,4,5]} = \frac{1}{q-1}\prod_{k=1}^{5}g(\omega^{kt})\sum_{j=0}^{q-2}\frac{g(\omega^j)^2g(\omega^{t+j})g(\omega^{2t+j})g(\omega^{4t+j})g(\omega^{5t+j})}{g(\omega^j)\cdots g(\omega^{5t+j})}\omega^{6j}(\lambda)$$

$$= \frac{q^2\omega^t(-1)g(\omega^{3t})}{q-1}\sum_{j=0}^{q-2}\frac{g(\omega^j)}{g(\omega^{3t+j})}\omega^{6j}(\lambda).$$

We consider $g(\omega^j)/g(\omega^{3t+j})$. If $j\neq 3t$, we have

$$\frac{g(\omega^j)}{g(\omega^{3t+j})} = \frac{g(\omega^j)g(\omega^{-3t-j})}{g(\omega^{3t+j})g(\omega^{-3t-j})} = \frac{1}{q}\omega^{t+j}(-1)g(\omega^j)g(\omega^{-3t-j}).$$

If $j=3t$, we have

$$\frac{g(\omega^j)}{g(\omega^{3t+j})}\omega^{6j}(\lambda) = \frac{g(\omega^{3t})}{-1}\cdot 1 = -g(\omega^{3t}).$$

By Lemma 1, we obtain

$$\sum_{j=0}^{q-2}\frac{g(\omega^j)}{g(\omega^{3t+j})}\omega^{6j}(\lambda) = \sum_{j=0,\,j\neq 3t}^{q-2}\frac{g(\omega^j)}{g(\omega^{3t+j})}\omega^{6j}(\lambda) + \frac{g(\omega^{3t})}{g(\omega^{3t+3t})}\omega^{6\cdot 3t}(\lambda)$$

$$= \frac{1}{q}\sum_{j=0,\,j\neq 3t}^{q-2}\omega^{t+j}(-1)g(\omega^j)g(\omega^{-3t-j})\omega^{6j}(\lambda) - g(\omega^{3t})$$

$$= \frac{1}{q}\left\{\sum_{j=0}^{q-2}\omega^{t+j}(-1)g(\omega^j)g(\omega^{-3t-j})\omega^{6j}(\lambda)\right.$$

$$\left. - \omega^{t+3t}(-1)g(\omega^{3t})g(\omega^{-3t-3t})\omega^{6\cdot 3t}(\lambda)\right\} - g(\omega^{3t})$$

$$= \frac{1}{q}\{\omega^t(-1)\cdot(q-1)g(\omega^{-3t})\omega^{-3t}(-1)\omega^{3t}(1-\lambda^6)$$

$$- g(\omega^{3t})\cdot(-1)\} - g(\omega^{3t})$$

$$= \frac{1}{q}\{(q-1)g(\omega^{3t})\omega^{3t}(1-\lambda^6) + g(\omega^{3t})\} - g(\omega^{3t}).$$

Hence, we conclude

$$\sum_{\boldsymbol{w}\in[0,0,1,2,4,5]} N_q(0,\boldsymbol{w}) + S_{[0,0,1,2,4,5]}$$

$$= q^2 + \frac{1}{q-1}q^2\omega^t(-1)g(\omega^{3t})\left[\frac{1}{q}\{(q-1)g(\omega^{3t})\omega^{3t}(1-\lambda^6) + g(\omega^{3t})\} - g(\omega^{3t})\right]$$

$$= q^2 + q^2\omega^{3t}(1-\lambda^6) + \frac{q^2-q^3}{q-1}$$

$$= q^2\omega^{3t}(1-\lambda^6). \qquad \square$$

Let us prove Theorem 3.

PROOF. From Equation (3), we obtain

$$\#X_\lambda^6(\mathbb{F}_q) = \sum_{\boldsymbol{w}\in W} N_q(0,\boldsymbol{w}) + \frac{1}{q-1}\sum_{[\boldsymbol{w}]\in W/\sim}\sum_{j=0}^{q-2}\frac{\prod_{i=1}^6 g(\omega^{w_i t+j})}{g(\omega^{6j})}\omega^{6j}(6\lambda)$$

$$= \sum_{[\boldsymbol{w}]\in W/\sim}\sum_{\boldsymbol{w}\in[\boldsymbol{w}]} N_q(0,\boldsymbol{w}) + \frac{1}{q-1}\sum_{[\boldsymbol{w}]\in W/\sim}\sum_{j=0}^{q-2}\frac{\prod_{i=1}^6 g(\omega^{w_i t+j})}{g(\omega^{6j})}\omega^{6j}(6\lambda)$$

$$= \sum_{[\boldsymbol{w}] \in W/\sim} \left\{ \sum_{\boldsymbol{w} \in [\boldsymbol{w}]} N_q(0, \boldsymbol{w}) + \frac{1}{q-1} \sum_{j=0}^{q-2} \frac{\prod_{i=1}^{6} g(\omega^{w_i t + j})}{g(\omega^{6j})} \omega^{6j}(6\lambda) \right\}$$

$$= \sum_{[\boldsymbol{w}] \in W/\sim} \left\{ \sum_{\boldsymbol{w} \in [\boldsymbol{w}]} N_q(0, \boldsymbol{w}) + S_{[\boldsymbol{w}]} \right\}.$$

Then, Propositions 15 through 19 complete the proof of Theorem 3. □

Appendix A. The proof by Miyatani's formula

In this appendix, we give a proof of Theorem 3 by Miyatani's formula which is expressed in terms of McCarthy's finite-field hypergeometric functions. In [13, Proposition 3.9], Miyatani expressed the number of rational points on hypersurfaces

$$X_\lambda : c_1 X^{a_1} + \cdots + c_{n+1} X^{a_{n+1}} = \lambda X_1 \cdots X_{n+1},$$

where $\lambda \in \mathbb{F}_q^\times$ such that X_λ is smooth, $c_1, \dots, c_{n+1} \in \mathbb{F}_q^\times$, and $a_i := {}^t(a_{1,i}, \dots, a_{n+1,i}) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $a_{1,i} + \cdots + a_{n+1,i} = n+1$ and none of a_i's being equal to ${}^t(1, \dots, 1)$ (for $i = 1, \dots, n+1$). Note that the notation X^{a_i} means the monomial $X_1^{a_{1,i}} \cdots X_{n+1}^{a_{n+1,i}}$ for $a_i = {}^t(a_{1,i}, \dots, a_{n+1,i})$. In our case, McCarthy's hypergeometric function can be expressed as a product of the normalized Jacobi sums and Greene's hypergeometric function. (See Proposition 25.) Thus, we can also obtain Theorem 3, [1, Theorem 1.1] and [2, Theorem 1.2] by Miyatani's formula.

A.1. McCarthy's finite-field hypergeometric functions. In this subsection, we introduce the finite-field hypergeometric function defined by McCarthy in [4].

For $A_1, \dots, A_{n+1}$, $B_1, \dots, B_{n+1} \in \hat{\mathbb{F}}_q^\times$, we define McCarthy's finite-field hypergeometric function ${}_{n+1}\tilde{F}_{n+1}$ by

$${}_{n+1}\tilde{F}_{n+1}\left(\begin{matrix} A_1 & \cdots & A_{n+1} \\ B_1 & \cdots & B_{n+1} \end{matrix}; x\right)_{\mathbb{F}_q} := \frac{-1}{q-1} \sum_{\chi \in \hat{\mathbb{F}}_q^\times} \prod_{i=1}^{n+1} \frac{g(A_i \chi)}{g(A_i)} \frac{g(\overline{B_i \chi})}{g(\overline{B_i})} \chi(-1)^{n+1} \chi(x).$$

Furthermore, we use the following notation.

Definition 20. Let $A_1, \dots, A_{n+1}$, $B_1, \dots, B_{n+1}$ be characters on $\mathbb{F}_q^\times$ in $\mathbb{C}^\times$. By sorting index, we assume that $\{A_1, \dots, A_{n'+1}\}$ and $\{B_1, \dots, B_{n'+1}\}$ have no intersection and that $\{A_{n'+2}, \dots, A_{n+1}\}$ and $\{B_{n'+2}, \dots, B_{n+1}\}$ are equal as multisets. Then we define the hypergeometric function with reduced parameters over $\mathbb{F}_q$ by

$$\bullet\tilde{F}_\bullet\ \mathrm{Red}\begin{pmatrix} A_1 & \cdots & A_{n+1} \\ B_1 & \cdots & B_{n+1} \end{pmatrix}; x\Bigg)_{\mathbb{F}_q} := {}_{n'+1}\tilde{F}_{n'+1}\begin{pmatrix} A_1 & \cdots & A_{n'+1} \\ B_1 & \cdots & B_{n'+1} \end{pmatrix}; x\Bigg)_{\mathbb{F}_q}.$$

A.2. Miyatani's formula. In this subsection, we recall Miyatani's formula. To state his formula, we introduce some notations. For the matrix $A' := (a_{i,j} - 1)_{1 \le i,j \le n+1}$, the kernel Δ of the homomorphism $\mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ defined by A' is generated by a uniquely determined vector ${}^t(\alpha_1 \dots, \alpha_{n+1})$ with all $\alpha_i > 0$. (See [13, Proposition 2.2].) We put $\alpha := \sum_{i=1}^{n+1} \alpha_i$. Let N be a positive integer divisible by all α_i and α, and we put $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. Let

$$f_N : (\mathbb{Z}_N)^{n+1}/\Delta \to (\mathbb{Z}_N)^{n+1}$$

be the morphism induced by the endomorphism of $(\mathbb{Z}_N)^{n+1}$ defined by the matrix $A' \bmod N$ and let $d_1, \dots, d_n$ be non-zero elementary divisors of A'. By an isomorphism $\mathrm{Im}(f_N) \simeq \bigoplus_{i=1}^n d_i\mathbb{Z}/N\mathbb{Z}$, we see that the kernel of f_N consists of $d := d_1 \cdots d_n$ elements. We fix $s_0 := {}^t(0, \dots, 0)$, $s_1, \dots, s_{d-1} \in \{0, \dots, q-2\}^{n+1}$ that represent $\mathrm{Ker}(f_{q-1})$. For $s_j := {}^t(s_{1,j}, \dots, s_{n+1,j})$, we put $|s_j| := \sum_i s_{i,j}$, $t_{i,j} := s_{i,j}/\alpha_i$ and $t_j := (\sum_i s_{i,j})/\alpha$. To simplify notations, we put $\omega_\beta := \omega^{(q-1)/\beta}$ for a positive integer β. Note that ω_β is well-defined if q is congruent to 1 modulo β. For each $j = 0, \dots, d-1$, we put

$$F(s_j) := \begin{cases} \bullet\tilde{F}_\bullet\ \mathrm{Red}\left(\begin{matrix} & [\omega_\alpha] & \\ [\omega_{\alpha_1}] & \cdots & [\omega_{\alpha_{n+1}}] \end{matrix}; C\lambda^{-\alpha}\right)_{\mathbb{F}_q} & (j = 0) \\ q^{\delta_{s_j}-1}\bullet\tilde{F}_\bullet\ \mathrm{Red}\left(\begin{matrix} & [\omega_\alpha] & \\ \omega^{t_{1,j}}[\omega_{\alpha_1}] & \cdots & \omega^{t_{n+1,j}}[\omega_{\alpha_{n+1}}] \end{matrix}; C\lambda^{-\alpha}\right)_{\mathbb{F}_q} & (j \neq 0), \end{cases}$$

where $[\omega_\beta]$ and $\omega^{t_{i,j}}[\omega_\beta]$ are respectively the sequences $\varepsilon, \omega_\beta, \omega_\beta^2, \dots, \omega_\beta^{\beta-1}$ and $\omega^{t_{i,j}}, \omega^{t_{i,j}} \cdot \omega_\beta, \omega^{t_{i,j}} \cdot \omega_\beta^2, \dots, \omega^{t_{i,j}} \cdot \omega_\beta^{\beta-1}$, and where $C := \alpha^\alpha \cdot \prod_{i=1}^{n+1} (c_i/\alpha_i)^{\alpha_i}$ and

$$\delta_{s_j} := \begin{cases} 1 & (|s_j| \equiv 0 \bmod q-1) \\ 0 & (|s_j| \not\equiv 0 \bmod q-1). \end{cases}$$

For each $j = 0, \dots, d-1$, we put

$$\gamma(s_j) := \prod_{i=1}^{n+1} \omega^{s_{i,j}}(\alpha_i^{-1}c_i)\omega^{|s_j|}((-\lambda)^{-1}\alpha)$$
$$\times \prod_{i=1}^{n+1}\left(g(\omega^{-t_{i,j}}) \prod_{b_i=1}^{\alpha_i-1} \frac{g(\omega^{-t_{i,j}}\omega_{\alpha_i}^{b_i})}{g(\omega_{\alpha_i}^{b_i})}\right) g(\omega^{t_j}) \prod_{b=1}^{\alpha-1} \frac{g(\omega^{t_j}\omega_\alpha^b)}{g(\omega_\alpha^b)}.$$

For the matrix $A := (a_{i,j})$, we define $(n+2) \times (n+1)$ matrix $\tilde{A}$ as $\tilde{A} := \left(\frac{A}{1 \cdots 1}\right)$. For a $k \times l$ matrix $M := (m_{i,j})_{i,j}$ with coefficients in $\mathbb{Z}$, we define the morphism $\varphi(M) : (\hat{\mathbb{F}}_q^\times)^l \to (\hat{\mathbb{F}}_q^\times)^k$ by $\varphi(M)((\chi_i)_{i=1,\dots,l}) = (\chi_1^{m_{j,1}} \cdots \chi_n^{m_{j,l}})_{j=1,\dots,k}$. Let $J := \{j_1, \dots, j_t\}$ be an arbitrary subset of $\{1, \dots, n+1\}$ with $t \geq (n+1)/2$ elements, let $\sigma(J)$ be the number of indices $i \in \{1, \dots, n+1\}$ with $a_{i,j} = 0$ for all $j \notin J$ and let $i_1, \dots, i_{\sigma(J)}$ be all such indices. (We may assume that $i_1, \dots, i_{\sigma(J)}$ are elements of J. See [13, Proposition 2.1].) Then we put

$$u := \sum_{\substack{J \subset \{1,\dots,n+1\} \\ (n+1)/2 \leq \#J \leq n}} \sum_{i=0}^{\#J-\sigma(J)} (-1)^{\#J-\sigma(J)-i} q^{i-1} \sum \prod_{j=1}^{\sigma(J)} g(\chi_j^{-1})\chi_j(c_j),$$

where the most inner sum runs through all elements ${}^t(\chi_1, \dots, \chi_{\sigma(J)}) \in \mathrm{Ker}(\varphi(\tilde{A}))$ such that exactly $n - 2i + 1$ components are non-trivial. Then Miyatani's formula is the following.

Theorem 21 ([13, Proposition 3.9]). *Suppose that the following conditions hold:*

(1) *$q - 1$ is divisible by all α_i's and by α.*
(2) *Each $s_{i,j}$ is divisible by α_i and $|s_j| (= \sum_i s_{i,j})$ is divisible by α.*
(3) *All elementary divisors of the $(t+1) \times \sigma(J)$ matrix*

$$\begin{pmatrix} a_{j_1,i_1} & \cdots & a_{j_1,i_{\sigma(J)}} \\ & \vdots & \\ a_{j_t,i_1} & \cdots & a_{j_t,i_{\sigma(J)}} \\ 1 & \cdots & 1 \end{pmatrix}$$

divide $q - 1$.

Then for $\lambda \in \mathbb{F}_q^\times$ such that X_λ is smooth and $\lambda^\alpha \neq C (= \alpha^\alpha \prod_{i=1}^{n+1} (c_i/\alpha_i)^{\alpha_i})$, we have

$$\#X_\lambda(\mathbb{F}_q) = \sum_{i=1}^{n-1} q^i + u + q^{(n-1)/2} \cdot D + (-1)^n \sum_{j=0}^{d-1} \gamma(s_j) F(s_j),$$

where D is defined to be the number of subsets $J \subset \{1, 2, \dots, n+1\}$ such that $\#J$ is equal to $(n+1)/2$ and that for all $i = 1, \dots, n+1$ there exists $j \notin J$ such that $a_{i,j} \geq 1$.

A.3. The proof of the main theorem. First, we give a lemma to apply Theorem 21 to the Dwork hypersurfaces. From the Hasse-Davenport product relation (Theorem 7), we have the following.

Lemma 2. *We have*

$$\gamma(s_j) = g(\omega^{t_j\alpha})\omega^{-t_j\alpha}(\alpha) \prod_{i=1}^{n+1} \omega^{s_{i,j}}(\alpha_i^{-1}c_i)\omega^{|s_j|}((-\lambda)^{-1}\alpha) \prod_{i=1}^{n+1} g(\omega^{-t_{i,j}\alpha_i})\omega^{t_{i,j}\alpha_i}(\alpha_i).$$

Next, we apply Miyatani's formula to the Dwork hypersurfaces of degree six. By definition, u and D in Theorem 21 are equal to zero, and the matrix A' is of size 6×6 and is given by

$$A' = \begin{pmatrix} 5 & -1 & \cdots & -1 \\ -1 & 5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & 5 \end{pmatrix}.$$

Δ is the group generated by ${}^t(1,1,1,1,1,1)$. (See [13, Example 2.5].) Then, we have

$$\begin{aligned} &\mathrm{Ker}(f_{q-1}) \\ &= \left\{ {}^t(x_1,\dots,x_6) \in (\mathbb{Z}_{q-1})^6/\Delta \;\middle|\; \begin{array}{l} 6x_1 = \cdots = 6x_6, \\ x_1 + \cdots + x_5 - 5x_6 = 0 \;\; in \;\; \mathbb{Z}_{q-1} \end{array} \right\} \\ &= \left\{ {}^t(w_1 t \pmod{q-1}, \dots, w_6 t \pmod{q-1}) \;\middle|\; \begin{array}{l} w_1,\dots,w_6 \in \mathbb{Z},\; t = \frac{q-1}{6} \\ w_1 t + \cdots + w_6 t = 0 \;\; in \;\; \mathbb{Z}_{q-1} \end{array} \right\}. \end{aligned}$$

Note that $\mathrm{Ker}(f_{q-1})$ has 6^4 elements since the elementary divisors of A' are $1,6,6,6,6,0$. (See also [13, Example 3.3].) From Theorem 21 and Lemma 2, we have the following.

Corollary 22. *Let $q = p^e$ be a power of a prime number such that q is congruent to 1 modulo 6. For $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$ and $\lambda^6 \neq 1$, we have*

$$\# X_\lambda^6(\mathbb{F}_q) = \frac{q^5 - 1}{q - 1} - \sum_{s_j = {}^t(s_{1,j},\dots,s_{6,j}) \in \{s_0,\dots,s_{6^4-1}\}} \gamma(s_j) F(s_j),$$

where

$$\gamma(s_j) = -\prod_{i=1}^{6} g(\omega^{-s_{i,j}})$$

and

$$F(s_j) = \begin{cases} {}_\bullet\tilde{F}_\bullet \, \mathrm{Red}\begin{pmatrix} \epsilon & \omega_6 & \omega_3 & \omega_2 & \bar{\omega}_3 & \bar{\omega}_6 \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \end{pmatrix}; \frac{1}{\lambda^6}\Big)_{\mathbb{F}_q} & (j = 0) \\ {}_\bullet\tilde{F}_\bullet \, \mathrm{Red}\begin{pmatrix} \omega^{|s_j|/6} & \omega^{t+|s_j|/6} & \omega^{2t+|s_j|/6} & \omega^{3t+|s_j|/6} & \omega^{4t+|s_j|/6} & \omega^{5t+|s_j|/6} \\ \omega^{s_{1,j}} & \omega^{s_{2,j}} & \omega^{s_{3,j}} & \omega^{s_{4,j}} & \omega^{s_{5,j}} & \omega^{s_{6,j}} \end{pmatrix}; \frac{1}{\lambda^6}\Big)_{\mathbb{F}_q} & (j \neq 0). \end{cases}$$

Remark 23. For the Dwork hypersurfaces of degree six, condition (1) in Theorem 21 is equivalent to $q \equiv 1 \bmod 6$, and conditions (2) and (3) are automatically satisfied. (See [13, Example 3.3].)

From the definition of $\mathrm{Ker}(f_{q-1})$, an action of the symmetric group of degree six S_6 on $\mathrm{Ker}(f_{q-1})$ is naturally defined. We put

$$\langle {}^t(s_1, \ldots, s_6) \rangle^k := \left\{ {}^t(v_1, \ldots, v_6) \in \{s_0, \ldots, s_{6^4-1}\} \,\middle|\, \begin{array}{l} \text{There exists a permutation } \sigma \in S_6 \\ \text{such that } {}^t(v_{\sigma(1)}, \ldots, v_{\sigma(6)}) = {}^t(s_1, \ldots, s_6) \end{array} \right\},$$

where

$$k = \#\left\{ {}^t(v_1, \ldots, v_6) \in \{s_0, \ldots, s_{6^4-1}\} \,\middle|\, \begin{array}{l} \text{There exists a permutation } \sigma \in S_6 \\ \text{such that } {}^t(v_{\sigma(1)}, \ldots, v_{\sigma(6)}) = {}^t(s_1, \ldots, s_6) \end{array} \right\}.$$

Then, we have

$$\begin{aligned} \{s_0 \ldots, s_{6^4-1}\} = {} & \langle {}^t(0,0,0,0,0,0) \rangle^1 \cup \langle {}^t(0,0,0,0,t,5t) \rangle^{30} \cup \langle {}^t(0,0,0,0,2t,4t) \rangle^{30} \\ & \cup \langle {}^t(0,0,0,0,3t,3t) \rangle^{15} \cup \langle {}^t(0,0,0,t,t,4t) \rangle^{60} \\ & \cup \langle {}^t(0,0,0,t,2t,3t) \rangle^{120} \cup \langle {}^t(0,0,0,2t,2t,2t) \rangle^{20} \\ & \cup \langle {}^t(0,0,0,2t,5t,5t) \rangle^{60} \cup \langle {}^t(0,0,0,3t,4t,5t) \rangle^{120} \\ & \cup \langle {}^t(0,0,t,t,2t,2t) \rangle^{90} \cup \langle {}^t(0,0,2t,2t,4t,4t) \rangle^{30} \\ & \cup \langle {}^t(0,0,t,3t,4t,4t) \rangle^{180} \cup \langle {}^t(0,0,t,3t,3t,5t) \rangle^{180} \\ & \cup \langle {}^t(0,0,t,2t,4t,5t) \rangle^{360}. \end{aligned}$$

Next, we calculate $A(s) := \gamma(s)F(s)$ for

$$s = {}^t(0,0,0,0,0,0), {}^t(0,0,0,0,t,5t), \ldots, {}^t(0,0,t,2t,4t,5t)$$

by using Theorem 22.

Proposition 24. *We have the following identities:*

$$A({}^t(0,0,0,0,0,0)) = -{}_5\tilde{F}_5\left(\begin{matrix} \omega^t & \omega^{2t} & \omega^{3t} & \omega^{4t} & \omega^{5t} \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \end{matrix} ; \frac{1}{\lambda^6} \right)_{\mathbb{F}_q} \tag{15}$$

$$A({}^t(0,0,0,0,t,5t)) = -q\omega^t(-1)\,{}_3\tilde{F}_3\left(\begin{matrix} \omega^{2t} & \omega^{3t} & \omega^{4t} \\ \epsilon & \epsilon & \epsilon \end{matrix} ; \frac{1}{\lambda^6} \right)_{\mathbb{F}_q} \tag{16}$$

$$A({}^t(0,0,0,0,2t,4t)) = -q\,{}_3\tilde{F}_3\left(\begin{matrix} \omega^t & \omega^{3t} & \omega^{5t} \\ \epsilon & \epsilon & \epsilon \end{matrix} ; \frac{1}{\lambda^6} \right)_{\mathbb{F}_q} \tag{17}$$

$$A({}^t(0,0,0,0,3t,3t)) = q\omega^t(-1)_4\tilde{F}_4\begin{pmatrix} \omega^t & \omega^{2t} & \omega^{4t} & \omega^{5t} \\ \epsilon & \epsilon & \epsilon & \omega^{3t} \end{pmatrix}; \frac{1}{\lambda^6}\Bigg)_{\mathbb{F}_q} \tag{18}$$

$$A({}^t(0,0,0,t,t,4t)) = -qJ(\omega^{2t},\omega^{5t},\omega^{5t})_3\tilde{F}_3\left(\begin{matrix} \omega^{2t} & \omega^{3t} & \omega^{5t} \\ \epsilon & \epsilon & \omega^{t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{19}$$

$$A({}^t(0,0,0,2t,5t,5t)) = -qJ(\omega^{t},\omega^{t},\omega^{4t})_3\tilde{F}_3\left(\begin{matrix} \omega^{3t} & \omega^{4t} & \omega^{t} \\ \epsilon & \epsilon & \omega^{5t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{20}$$

$$A({}^t(0,0,0,2t,2t,2t)) = -qJ(\omega^{4t},\omega^{4t},\omega^{4t})_4\tilde{F}_4\left(\begin{matrix} \omega^{t} & \omega^{3t} & \omega^{4t} & \omega^{5t} \\ \epsilon & \epsilon & \omega^{2t} & \omega^{2t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{21}$$

$$A({}^t(0,0,0,3t,4t,5t)) = -qJ(\omega^{t},\omega^{2t},\omega^{3t})_2\tilde{F}_2\left(\begin{matrix} \omega^{2t} & \omega^{t} \\ \epsilon & \epsilon \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{22}$$

$$A({}^t(0,0,0,t,2t,3t)) = -qJ(\omega^{3t},\omega^{4t},\omega^{5t})_2\tilde{F}_2\left(\begin{matrix} \omega^{4t} & \omega^{5t} \\ \epsilon & \epsilon \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{23}$$

$$A({}^t(0,0,t,t,2t,2t)) = -qJ(\omega^{4t},\omega^{4t},\omega^{5t},\omega^{5t})_3\tilde{F}_3\left(\begin{matrix} \omega^{3t} & \omega^{4t} & \omega^{5t} \\ \epsilon & \omega^{t} & \omega^{2t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{24}$$

$$A({}^t(0,0,2t,2t,4t,4t)) = -q^2{}_3\tilde{F}_3\left(\begin{matrix} \omega^{3t} & \omega^{5t} & \omega^{t} \\ \epsilon & \omega^{2t} & \omega^{4t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{25}$$

$$A({}^t(0,0,t,3t,4t,4t)) = qJ(\omega^{2t},\omega^{2t},\omega^{3t},\omega^{5t})_2\tilde{F}_2\left(\begin{matrix} \omega^{2t} & \omega^{5t} \\ \epsilon & \omega^{4t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{26}$$

$$A({}^t(0,0,t,3t,3t,5t)) = -q^2{}_2\tilde{F}_2\left(\begin{matrix} \omega^{2t} & \omega^{4t} \\ \epsilon & \omega^{3t} \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q} \tag{27}$$

$$A({}^t(0,0,t,2t,4t,5t)) = -q^2\omega^t(-1)_1\tilde{F}_1\left(\begin{matrix} \omega^{3t} \\ \epsilon \end{matrix}; \frac{1}{\lambda^6}\right)_{\mathbb{F}_q}. \tag{28}$$

Finally, we rewrite Proposition 24 by using Greene's hypergeometric function. McCarthy gave the relation between his hypergeometric function and Greene's hypergeometric function.

Proposition 25 ([4, Proposition 2.5]). *For characters $A_0,\dots,A_n$, $B_1,\dots,B_n$ with $A_0 \neq \epsilon$ and $A_i \neq B_i$ $(i=1,\dots,n)$, we have*

$$\begin{aligned} &{}_{n+1}\tilde{F}_{n+1}\left(\begin{matrix} A_0 & A_1 & \cdots & A_n \\ \epsilon & B_1 & \cdots & B_n \end{matrix}; x\right)_{\mathbb{F}_q} \\ &\quad = \left(\prod_{i=1}^{n}\binom{A_i}{B_i}^{-1}\right){}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots & A_n \\ & B_1, & \dots & B_n \end{matrix}\,\middle|\, x\right)_q. \end{aligned}$$

Theorem 3 follows from using Propositions 24 and 25 and applying the identity

$$ {}_1F_0\left(\begin{matrix} \omega_\alpha \end{matrix} \middle| \, x \right)_q = \varepsilon(x)\bar{\omega}_\alpha(1-x) $$

(see [8, (3.11)]) to the identity (28).

Remark 26. We can also prove [1, Theorem 1.1] and [2, Theorem 1.2] similarly.

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