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The moduli space of points in quaternionic projective space

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Abstract. Let $\mathscr{M}(n,m;\mathbb{FP}^n)$ be the configuration space of m-tuples of pairwise distinct points in $\mathbb{FP}^n$, that is, the quotient of the set of m-tuples of pairwise distinct points in $\mathbb{FP}^n$ with respect to the diagonal action of $\mathrm{PU}(1,n;\mathbb{F})$ equipped with the quotient topology. In this paper, by mainly using the rotation-normalized and the block-normalized algorithms, we construct the parameter spaces of both $\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}})$ and $\mathscr{M}(n,m;\mathbb{P}(V_+))$, respectively.

1. Introduction

Let $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ be respectively the set of real numbers, the set of complex numbers or the set of quaternions, and $\langle\mathbf{z},\mathbf{w}\rangle=\mathbf{w}^*J\mathbf{z}$ a Hermitian product in $(n+1)$-dimensional $\mathbb{F}$-vector space $\mathbb{F}^{n,1}$ of signature $(n,1)$. The group of transformations of $\mathbb{F}^{n+1}$ preserving this Hermitian product is the noncompact Lie group $U(1,n;\mathbb{F})$. That is

$$U(1,n;\mathbb{F})=\{g\in \mathrm{GL}(n+1,\mathbb{F}): g^*Jg=J\}.$$

These groups are traditionally denoted by

$$\mathrm{O}(n,1)=\mathrm{U}(1,n;\mathbb{R}),\qquad \mathrm{U}(n,1)=\mathrm{U}(1,n;\mathbb{C})\qquad \text{and}\qquad \mathrm{Sp}(n,1)=\mathrm{U}(1,n;\mathbb{H}).$$

Denote by $\mathbb{P}$ the natural right projection from $\mathbb{F}^{n,1}-\{0\}$ to projective space $\mathbb{FP}^n$. Let V_-, V_0, V_+ be the subsets of $\mathbb{F}^{n,1}-\{0\}$ consisting of vectors where $\langle\mathbf{z},\mathbf{z}\rangle$ is negative, zero, or positive, respectively. Their projections to $\mathbb{FP}^n$ are called isotropic, negative, and positive points, respectively. Conventionally, we denote $\mathbf{H}^n_{\mathbb{F}}=\mathbb{P}(V_-)$, $\partial\mathbf{H}^n_{\mathbb{F}}=\mathbb{P}(V_0)$ and $\overline{\mathbf{H}^n_{\mathbb{F}}}=\mathbf{H}^n_{\mathbb{F}}\cup\partial\mathbf{H}^n_{\mathbb{F}}$. The Bergman metric on $\mathbf{H}^n_{\mathbb{F}}$ is given by the distance formula

$$\cosh^2\frac{\rho(z,w)}{2}=\frac{\langle\mathbf{z},\mathbf{w}\rangle\langle\mathbf{w},\mathbf{z}\rangle}{\langle\mathbf{z},\mathbf{z}\rangle\langle\mathbf{w},\mathbf{w}\rangle},\qquad \mathbf{z}\in\mathbb{P}^{-1}(z),\ \mathbf{w}\in\mathbb{P}^{-1}(w).\tag{1}$$

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The center $\mathrm{Z}(1,n;\mathbb{F})$ of $\mathrm{U}(1,n;\mathbb{F})$ is $\{\pm I_{n+1}\}$ if $\mathbb{F}=\mathbb{R},\mathbb{H}$, and is the circle group $\{e^{\mathbf{i}\theta}I_{n+1}\}$ if $\mathbb{F}=\mathbb{C}$. We mention that $g\in U(1,n;\mathbb{F})$ acts on $\mathbb{F}\mathbb{P}^n$ as $g(z)=\mathbb{P}g\mathbb{P}^{-1}(z)$. Therefore the holomorphic isometry group $\mathrm{Isom}(\mathbf{H}_{\mathbb{F}}^n)$ of $\mathbf{H}_{\mathbb{F}}^n$ is actually the quotient $\mathrm{PU}(1,n;\mathbb{F})=\mathrm{U}(1,n;\mathbb{F})/\mathrm{Z}(1,n;\mathbb{F})$. We refer to [1, 8, 10, 17] for further details.

Let $\mathscr{M}(n,m;\mathbb{F}\mathbb{P}^n)$ be the configuration space of m-tuples of pairwise distinct points in $\mathbb{F}\mathbb{P}^n$, or equivalently, the quotient of the set of m-tuples of pairwise distinct points in $\mathbb{F}\mathbb{P}^n$ with respect to the diagonal action of $\mathrm{PU}(1,n;\mathbb{F})$ equipped with the quotient topology. It is an important problem in hyperbolic geometry to parameterize the space $\mathscr{M}(n,m;\mathbb{F}\mathbb{P}^n)$ and study the geometric and topological structures on the associated parameter space. Such a problem is called the moduli problem on $\mathbb{F}\mathbb{P}^n$ in what follows.

The moduli problems of the cases $m=1,2$ on $\partial\mathbf{H}_{\mathbb{F}}^n$ are trivial because $\mathrm{U}(1,n;\mathbb{F})$ acts doubly transitively on $\partial\mathbf{H}_{\mathbb{F}}^n$ when $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$. It is well known that $\mathrm{O}(n,1)$ acts triply transitively on the boundary. To handle the cases of $m\geq 3$, one need to develop some geometric invariants or geometric tools, such as the distance formula, Cartan's angular invariant [9, 17], and cross-ratio [21] etc.

The moduli problem of $\mathscr{M}(2,4;\partial\mathbf{H}_{\mathbb{C}}^2)$ was considered by Falbel, Parker and Platis [14, 15, 22, 23]. The main tool is the complex cross-ratio variety determined by three complex cross-ratios.

The moduli problem of $\mathscr{M}(n,m;\mathbf{H}_{\mathbb{C}}^n)$ was solved by Brehm and Et-Taoui [3, 4]. Using Bruhat decomposition, Hakim and Sandler considered the arrangement of n points in certain standard position on $\mathbb{R}\mathbb{P}^{n-1}$ [19] and the moduli problem on $\overline{\mathbf{H}_{\mathbb{C}}^n}$ [20].

We need to introduce the concept of Gram matrices of m-tuples in $\mathbb{F}\mathbb{P}^n$ for further discussion.

Definition 1.1. *Given an m-tuple $\mathfrak{p}=(p_1,\dots,p_m)$ of pairwise distinct points in $\mathbb{F}\mathbb{P}^n$ with lift $\mathbf{p}=(\mathbf{p}_1,\dots,\mathbf{p}_m)$. The following Hermitian matrix*

$$G(\mathbf{p})=(g_{ij})=(\mathbf{p}_i^*J\mathbf{p}_j)=(\langle\mathbf{p}_j,\mathbf{p}_i\rangle)$$

is called the Gram matrix associated to $\mathfrak{p}$.

For simplicity, we also say that $\mathbf{p}$ is an m-tuple of pairwise distinct points in $\mathbb{F}\mathbb{P}^n$ and $\mathbf{p}\in\mathbb{F}_{n+1,m}$, the set of $(n+1)\times m$ matrices over $\mathbb{F}$. The action of $f\in\mathrm{U}(1,n;\mathbb{F})$ on $\mathbf{p}$ is

$$f\mathbf{p}=(f\mathbf{p}_1,\dots,f\mathbf{p}_m).$$

By noting that $f^*Jf=J$, we have the following proposition.

Proposition 1.1.

$$G(\mathbf{p}) = \mathbf{p}^*J\mathbf{p} = \mathbf{p}^*f^*Jf\mathbf{p} = G(f\mathbf{p}), \qquad \forall f \in \mathrm{U}(1,n;\mathbb{F}). \tag{2}$$

Given two m-tuples $\mathfrak{p} = (p_1, \dots, p_m)$ and $\mathfrak{q} = (q_1, \dots, q_m)$ in $\mathbb{FP}^n$ with arbitrary lifts $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$. We say that $\mathfrak{p}$ and $\mathfrak{q}$ are $\mathrm{PU}(1,n;\mathbb{F})$-congruent if there exists an $f \in \mathrm{U}(1,n;\mathbb{F})$ such that

$$f(\mathbf{p}_i) = \mathbf{q}_i\lambda_i, \qquad \lambda_i \neq 0, \qquad i = 1, \dots, m,$$

in language of matrix algebra, that is,

$$f\mathbf{p} = \mathbf{q}D, \qquad D = \mathrm{diag}(\lambda_1, \dots, \lambda_m), \qquad \lambda_i \in \mathbb{F} - \{0\}.$$

Therefore

$$G(\mathbf{p}) = \mathbf{p}^*J\mathbf{p} = \mathbf{p}^*f^*Jf\mathbf{p} = D^*\mathbf{q}^*J\mathbf{q}D = D^*G(\mathbf{q})D. \tag{3}$$

Observe that an arbitrary lift of $\mathfrak{p}$ can be represented by $(\mathbf{p}_1\lambda_1, \dots, \mathbf{p}_m\lambda_m) = \mathbf{p}D$ and

$$G(\mathbf{p}D) = D^*\mathbf{p}^*J\mathbf{p}D = D^*G(\mathbf{p})D. \tag{4}$$

The formulas (3) and (4) imply that Gram matrices contain the information of the diagonal action of $\mathrm{U}(1,n;\mathbb{F})$ on $\mathbf{p}$. Moreover, a Gram matrix contains entries $\langle \mathbf{p}_i, \mathbf{p}_j \rangle$, which are base material to construct the corresponding Hermitian geometric invariants. Hence a Gram matrix is a desired tool in handling moduli problems.

The moduli problem on $\partial\mathbf{H}^n_{\mathbb{C}}$ was solved by Cunha and Gusevskii [11, 12] mainly by Gram matrices. It is interesting to consider these moduli problems in quaternionic hyperbolic geometry. However, besides the noncommutativity of quaternions, another essential difference between complex and quaternionic hyperbolic geometry is the existence of elliptic elements of forms μI_{n+1} in $\mathrm{Sp}(n,1)$, where $\mu \in \mathrm{Sp}(1)$. These properties make it difficult to define geometric invariants and determine the representative Gram matrix in its equivalent class.

Brehn and Et-Taoui [5] are the poineers on researching the congruence classes of m-tuple points in $\mathbf{H}^n_{\mathbb{H}}$. By mainly use of Gram matrices, they gave a congruence criteria on such tuples.

By using quaternionic Cartan's angular invariant and quaternionic cross-ratio in $\overline{\mathbf{H}^n_{\mathbb{H}}}$, Cao [8] solved the moduli problems of $\mathscr{M}(n,3;\overline{\mathbf{H}^n_{\mathbb{H}}})$ and $\mathscr{M}(n,4;\partial\mathbf{H}^n_{\mathbb{H}})$.

We will continue the research in this direction. In this paper we concentrate on the moduli problems of $\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}})$ with $m > 4$ and $\mathscr{M}(n,m;\mathbb{P}(V_+))$.

We need several notations to illustrate our strategies for overcoming the difficulties mentioned above.

Let $v = (v_1, \dots, v_t)$ be a row vector in $\mathbb{H}^t$ and

$$\mathbf{O}_v = \{\mu v \mu^{-1} = (\mu v_1 \mu^{-1}, \dots, \mu v_t \mu^{-1}) : \forall \mu \in \mathrm{Sp}(1)\}.$$

The set $\mathbf{O}_v$ can be thought of as the orbit of v under the action of $\mathrm{Sp}(1)/\pm 1$. The procedure of giving a coordinate to the orbit $\mathbf{O}_v$ is termed by *the rotation-normalized algorithm* in this paper. We mention that the rotation-normalized algorithm stems both from the noncommutativity of quaternions and the existence of isometries of the form μI_{n+1} in $\mathrm{Sp}(n,1)$. Such an algorithm is indigenous in quaternionic hyperbolic geometry, while obviously vacuous in complex hyperbolic geometry.

Let $i(G(\mathbf{p})) = (n_+, n_-, n_0)$ be the signature of Hermitian matrix $i(G(\mathbf{p}))$ and $V = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ be of dimension $k+1$. There are two different cases of the moduli problem on $\mathbb{P}(V_+)$ according to $n_+ + n_- = k+1$ or $n_+ + n_- = k$ (see Theorem 2.2). V is called parabolic provided that $n_+ + n_- = k$. The two cases are termed by regular and non regular cases in complex hyperbolic plane [13]. We still use this terminology in quaternionic setting.

When V is parabolic, the Gram matrix $G(\mathbf{p})$ loses the information of the configuration and only carries the information of the strati-form structure (see Example 4.1 and Proposition 4.3). This strati-form structure will help us to break down the space $V = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ into finite 2-dimensional subspaces. We mention that there exist at most $n-1$ such 2-dimensional subspaces in $\mathbb{H}^{n,1}$. These 2-dimensional subspaces share a common basis which is a fibre in V_0. In each subspace containing more than three points of the m-tuple, we need to introduce new invariants (the cross-ratios in $\mathbb{H} \cup \infty$) to parameterize their congruence classes. Of particular interest will be the harmonious coexistence of these 2-dimensional subspaces (see Proposition 5.4).

When V is not parabolic, the Gram matrix $G(\mathbf{p})$ contains the full information of the congruence class of $\mathbf{p}$. The moduli problem on $\mathbb{P}(V_0)$ is tractable for each entry in Gram matrix $G(\mathbf{p})$ being nonzero. On handling the moduli problem on $\mathbb{P}(V_+)$, the pivotal point is to find a partition of $S(m) = \{1, \dots, m\}$ to perform the rotation-normalized algorithm in each block independently. This will help us to tackle the difficulty caused by orthogonality. Such a method is termed by *the block-normalized algorithm*.

In our perspective, the parameter of the $\mathrm{PSp}(n,1)$-congruence class of $\mathbf{p}$ is independent entries of a unique representative Gram matrix when V is not parabolic. For example, the $\mathrm{PSp}(n,1)$-congruence class of three points in $\partial \mathbf{H}^2_{\mathbb{H}}$ is its quaternionic Cartan's angular invariant [1, 8]. We mainly rely on the rotation-normalized and the block-normalized algorithms to construct such a moduli space in this paper. Our approaches sound natural and elementary.

As should be apparent, our ideas and exposition owe a great deal to the works of the references cited above, especially to those of [12, 13].

The paper is organized as follows. Section 2 contains properties of quaternions, the some basic facts in quaternionic hyperbolic geometry and the inertia of Gram matrices. These properties provide us with the tool to execute the rotation-normalized algorithm and initiate the idea of the block-normalized algorithm. The parameter space of $\mathscr{M}(n,1;\mathbb{P}(V_+))$ is a single point (Theorem 2.1). Section 3 describes the parameter space of $\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}})$ for $m > 4$ (Theorem 3.2). This may be thought of as a generalization of those of [8, 12]. The application of rotation-normalized algorithm is fully described. This method will be mimicked in more complicated cases in succeeding sections. In Section 4, we mainly refine the structure of Gram matrices. These refined structures are crucial in introducing new invariants in non regular case and the block-normalized algorithm in regular case. The parameter space of $\mathscr{M}(n,2;\mathbb{P}(V_+))$ is also constructed (Theorem 4.2). In Section 5, the parameter space of $\mathscr{M}(n,m;\mathbb{P}(V_+))$ with $m \geq 3$ is constructed (Theorem 5.2) when V is parabolic. In Section 6, the parameter space of $\mathscr{M}(n,m;\mathbb{P}(V_+))$ with $m \geq 3$ is constructed (Theorem 6.2) when V is not parabolic.

Remark 1.1. *The referee kindly informed the author that Gongopadhyay and Gou etc.* [16, 18] *also considered the moduli problems of* $\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}})$. *Roughly speaking, the methods used by them can be thought of as choosing a special form of equivalent Gram matrices and figuring out some numerical invariants of this Gram matrix. Some data of this Gram matrix and these numerical invariants are used to present the parameterization of the moduli space. The method we use is the rotation-normalized algorithm. Such an algorithm attempts to parameterize the equivalent Gram matrices directly. All these methods originate from the idea of these papers* [4, 5, 8, 12, 13] *etc. and share a common spirit in dealing with the noncommutativity of quaternions.*

2. The inertia of Gram matrices

In this section, we will recall some properties of quaternions and obtain some properties of the inertia of Gram matrices.

2.1. Properties of quaternions. Recall that a quaternion is of the form $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$ where $a_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\bar{a} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and $|a| = \sqrt{\bar{a}a} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$ be the conjugate and modulus of a, respectively. We define $\Re(a) = (a+\bar{a})/2$ and $\Im(a) = (a-\bar{a})/2$. Two quaternions a and b are similar if there exists nonzero $\lambda \in \mathbb{H}$ such that

$b = \lambda a \lambda^{-1}$. It is useful to view $\mathbb{H}$ as $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$. In this way, each quaternion $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ can be uniquely expressed as

$$a = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\mathbf{j} = c_1 + c_2\mathbf{j} = c_1 + \mathbf{j}\overline{c_2}.$$

It is well-known that the action of $\mathrm{Sp}(1)/\pm 1$ on $\mathbb{H}$ coincides with the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. We recall it as the following proposition.

PROPOSITION 2.1. *Denote* $\vec{v} = (x, y, z)^T$ *for* $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{H}$, *where* A^T *is the transpose of a matrix* A. *For a unit quaternion* $\mu = u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, *we define*

$$M_\mu = \begin{pmatrix} u_1^2 + u_0^2 - u_3^2 - u_2^2 & 2u_1u_2 + 2u_0u_3 & 2u_1u_3 - 2u_0u_2 \\ 2u_1u_2 - 2u_0u_3 & u_2^2 - u_3^2 + u_0^2 - u_1^2 & 2u_2u_3 + 2u_0u_1 \\ 2u_1u_3 + 2u_0u_2 & 2u_2u_3 - 2u_0u_1 & u_3^2 - u_2^2 - u_1^2 + u_0^2 \end{pmatrix}.$$

Then $M_\mu \in \mathrm{SO}(3)$ *and*

$$\overrightarrow{\bar{\mu} v \mu} = M_\mu \vec{v}.$$

LEMMA 2.1. *Let* $v_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ *and* $v_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ *such that* $\overrightarrow{v_1}$ *and* $\overrightarrow{v_2}$ *are linear independent. Let* $v_1 \cdot v_2 = \overrightarrow{v_2}^T \overrightarrow{v_1}$. *Then there exists a unique element* $\mu \in \mathrm{Sp}(1)/\pm 1$ *such that*

$$\bar{\mu} v_1 \mu = |v_1|\mathbf{i}, \qquad \bar{\mu} v_2 \mu = \frac{v_1 \cdot v_2}{|v_1|}\mathbf{i} + \frac{\sqrt{(|v_1|\,|v_2|)^2 - (v_1 \cdot v_2)^2}}{|v_1|}\mathbf{j}. \tag{5}$$

PROOF. Let $v_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $v_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and θ the angle between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Identify $\Im(\mathbb{H})$ with the 3-dimensional real space $\mathbf{xyz}$. Geometrically, by rotating the plane spanned by v_1 and v_2 to $\mathbf{xy}$ plane and then rotating around the $\mathbf{z}$-axis or $\mathbf{x}$-axis if necessary, we can obtain a μ such that formulas (5) hold. It is helpful to regard these formulas as

$$\bar{\mu} v_1 \mu = |v_1|\mathbf{i}, \qquad \bar{\mu} v_2 \mu = |v_2| \cos\theta \mathbf{i} + |v_2| \sin\theta \mathbf{j}.$$

Suppose that there exists another unit quaternion ν satisfying the above equalities. Then we have $\nu^{-1}\mu|v_1|\mathbf{i}\bar{\mu}\bar{\nu}^{-1} = |v_1|\mathbf{i}$ and therefore $\nu^{-1}\mu$ is a unit complex number. Similarly we get $\nu^{-1}\mu\mathbf{j}\bar{\mu}\bar{\nu}^{-1} = \mathbf{j}$ which implies that $\nu^{-1}\mu = \pm 1$. Therefore $\nu = \mu$ or $\nu = -\mu$. $\square$

Lemma 2.1 is the foundation of the rotation-normalized algorithm. We give an explicit formula of such a unique μ by the following process. Note that

$$-(|v_1|\mathbf{i} + v_1)v_1(|v_1|\mathbf{i} + v_1) = |\,|v_1|\mathbf{i} + v_1|^2 |v_1|\mathbf{i}.$$

Let

$$v = v(v_1) = \begin{cases} \mathbf{j}, & \text{provided } x_1 < 0,\ y_1^2 + z_1^2 = 0; \\ \frac{|v_1|\mathbf{i} + v_1}{\sqrt{2|v_1|(|v_1| + x_1)}}, & \text{otherwise.} \end{cases} \tag{6}$$

Then

$$|v| = 1, \qquad \bar{v} v_1 v = |v_1|\mathbf{i}.$$

Let $\bar{v} v_2 v = c_1 + c_2 \mathbf{j}$, where c_1, c_2 are complex numbers. Since $c_2 \neq 0$, we have $e^{-2\mathbf{i}\alpha} c_2 = |c_2|$ with $e^{\mathbf{i}\alpha} = \sqrt{\frac{c_2}{|c_2|}}$. Therefore $\mu = \pm v e^{\mathbf{i}\alpha}$ is the desired unit quaternion. By finding the corresponding c_2 and (6), we obtain the following formula:

$$\mu = \mu(v_1, v_2) = \begin{cases} \pm \sqrt{\frac{y_2 + z_2 \mathbf{i}}{\sqrt{y_2^2 + z_2^2}}}\,\mathbf{j}, & \text{provided } x_1 < 0,\ y_1^2 + z_1^2 = 0; \\ \pm \frac{|v_1|\mathbf{i} + v_1}{\sqrt{2|v_1|(|v_1| + x_1)}} \sqrt{\frac{F}{|F|}}, & \text{otherwise,} \end{cases} \tag{7}$$

where

$$F = 2x_2(|v_1| + x_1)(y_1 + z_1\mathbf{i}) - (|v_1| + x_1)^2 (y_2 + z_2\mathbf{i}) + (y_2 - z_2\mathbf{i})(y_1 + z_1\mathbf{i})^2.$$

2.2. The inertia of Gram matrices. In this paper, the J in quaternionic Hermitian product $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z}$ given in Section 1 will be taken one of the following forms:

$$J_b = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{or} \qquad J_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The corresponding quaternionic hyperbolic spaces are usually termed by *ball model* and *Siegel domain model*, respectively. Let C be the Cayley transformation mapping the ball to the Siegel domain. Then the relation of the two models can be mainly expressed by the following two equations:

$$\mathbf{w}^* J_b \mathbf{z} = (C\mathbf{w})^* J_s (C\mathbf{z}), \qquad g^* J_b g = J_b = C^{-1} J_s C = C^{-1} (CgC^{-1})^* J_s (CgC^{-1}) C.$$

Each model has its own advantage in certain situations. Basically we work on Siegel domain model only in Section 5.

Note that $g^* J_b g = J_b$ with $g = (g_1, \dots, g_{n+1})$, that is,

$$\langle g_i, g_j \rangle = 0, \quad i \neq j, \qquad \langle g_i, g_i \rangle = 1, \quad i = 1, \dots, n, \qquad \langle g_{n+1}, g_{n+1} \rangle = -1. \tag{8}$$

In terms of Gram matrices given by Definition 1.1, we have

$$G(g) = J_b, \qquad \forall g \in \mathrm{Sp}(n, 1).$$

Based on this observation, we have the following proposition.

PROPOSITION 2.2. *Let* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ *and* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$ *such that* $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = \langle \mathbf{q}_i, \mathbf{q}_i \rangle = 1$ *and* $\langle \mathbf{p}_i, \mathbf{p}_j \rangle = \langle \mathbf{q}_i, \mathbf{q}_j \rangle = 0$, $i \neq j$. *Then there is a* $g \in \mathrm{Sp}(n,1)$ *such that*

$$g\mathbf{p}_i = \mathbf{q}_i, \qquad i = 1, \dots, m.$$

PROOF. By the signature restriction, we have $m \leq n$. We can extend $\mathbf{p}$ and $\mathbf{q}$ to $f = (\mathbf{p}, \mathbf{p}_{m+1}, \dots, \mathbf{p}_{n+1})$ and $h = (\mathbf{q}, \mathbf{q}_{m+1}, \dots, \mathbf{q}_{n+1})$ such that $f, h \in \mathrm{Sp}(n,1)$. Then $g = hf^{-1}$ is the desired isometry. □

Proposition 2.2 implies the following simple result.

THEOREM 2.1. $\mathrm{PSp}(n,1)$ *acts transitively on* $\mathbb{P}(V_+)$.

Let $\mathbf{z}^\perp = \{\mathbf{w} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{w} \rangle = 0\}$ be the orthogonal complement of the fibre $\mathbf{z}\mathbb{H}$ in $\mathbb{H}^{n,1}$ and $\dim_q(V)$ the quaternionic dimension of a subspace V of $\mathbb{H}^{n,1}$.

PROPOSITION 2.3. *We have the following statements concerning the orthogonal complements on* $\mathbb{H}^{n,1}$.

(i) *If* $\mathbf{z} \in V_-$, *then* $\mathbf{z}^\perp \subset V_+$. *There exists an orthogonal basis*

$$\{\mathbf{p}_2, \dots, \mathbf{p}_{n+1}\} \subset \mathbf{z}^\perp,$$

$\dim_q(\mathbf{z}^\perp) = n$ *and* $\{\mathbf{z}, \mathbf{p}_2, \dots, \mathbf{p}_{n+1}\}$ *is a basis of* $\mathbb{H}^{n,1}$.

(ii) *If* $\mathbf{z} \in V_0$, *then* $\mathbf{z}^\perp \subset V_+ \cup V_0$ *and* $\mathbf{z}^\perp \cap V_0 = \mathbf{z}\mathbb{H}$. *There exist mutually orthogonal vectors* $\{\mathbf{p}_2, \dots, \mathbf{p}_n\}$ *in* V_+ *and*

$$\mathbf{z}^\perp = \mathrm{span}\{\mathbf{z}, \mathbf{p}_2, \dots, \mathbf{p}_n\}.$$

(iii) *If* $\mathbf{z} \in V_+$, *then*

$$\mathbf{z}^\perp \cap V_+ \neq \varnothing, \qquad \mathbf{z}^\perp \cap V_0 \neq \varnothing, \qquad \mathbf{z}^\perp \cap V_- \neq \varnothing.$$

There exist mutually orthogonal vectors $\{\mathbf{p}_2, \dots, \mathbf{p}_n, \mathbf{p}_{n+1}\}$ *such that*

$$\mathrm{span}\{\mathbf{z}, \mathbf{p}_2, \dots, \mathbf{p}_n\} \subset V_+, \qquad \mathbf{p}_{n+1} \in V_-$$

and $\{\mathbf{z}, \mathbf{p}_2, \dots, \mathbf{p}_{n+1}\}$ *is a basis of* $\mathbb{H}^{n,1}$.

PROOF. Let $\mathbf{z} \in V_-$. Then $\mathbf{z}^\perp \subset V_+$. By (8), there exists an orthogonal basis $\{\mathbf{p}_2, \dots, \mathbf{p}_{n+1}\}$ in $\mathbf{z}^\perp$. Hence $\dim_q(\mathbf{z}^\perp) = n$ and $\{\mathbf{z}, \mathbf{p}_2, \dots, \mathbf{p}_{n+1}\}$ is a basis of $\mathbb{H}^{n,1}$. Therefore case (i) holds. Case (iii) follows similarly.

Let $\mathbf{z} \in V_0$. We may assume that $\mathbf{z} = (1, 0, \dots, 0, 1)^T$. It is obvious that $\mathbf{w} \in \mathbf{z}^\perp$ is of the form $\mathbf{w} = (q_1, q_2, \dots, q_n, q_1)^T$. Let $\mathbf{e}_i$ be the standard basis of $\mathbb{H}^{n,1}$. Then $\mathbf{e}_i$, $i = 2, \dots, n$ belong to $\mathbf{z}^\perp$ and

$$\mathbf{z}^\perp = \mathrm{span}\{\mathbf{z}, \mathbf{e}_2, \dots, \mathbf{e}_n\}. \qquad \square$$

Recall that $A \in \mathbb{H}_{n,n}$ is called Hermitian if and only if $A = A^*$. Let $\mathrm{H}_n(\mathbb{H})$ be the collection of $n \times n$ Hermitian matrices. It is well-known that the right eigenvalues of $A \in \mathrm{H}_n(\mathbb{H})$ are real and there exists an invertible matrix $B \in \mathbb{H}_{n,n}$ such that B^*AB is a diagonal matrix which has only entries $+1$, -1, 0 along the diagonal. The numbers of $+1$s, -1s and 0s are denoted by n_+, n_- and n_0, respectively. We denote the signature of A by

$$i(A) = (n_+, n_-, n_0).$$

PROPOSITION 2.4 ([8, Proposition 1.1]). *If $\mathbf{z}, \mathbf{w} \in \mathbb{H}^{n,1} - \{0\}$ with $\langle \mathbf{z}, \mathbf{z} \rangle \le 0$ and $\langle \mathbf{w}, \mathbf{w} \rangle \le 0$, then either $\mathbf{w} = \mathbf{z}\lambda$ for some $\lambda \in \mathbb{H}$ or $\langle \mathbf{z}, \mathbf{w} \rangle \ne 0$.*

PROPOSITION 2.5. *Let $\mathfrak{p} = (p_1, \dots, p_m)$ be an m-tuple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$ with lift $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ and $m \ge 2$. Then $G(\mathbf{p})$ has a negative eigenvalue.*

PROOF. Let $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2\mu$ with $\mu = -\langle \mathbf{p}_1, \mathbf{p}_2 \rangle$. By Proposition 2.4,

$$\langle \mathbf{q}, \mathbf{q} \rangle = -2|\langle \mathbf{p}_1, \mathbf{p}_2 \rangle| < 0. \tag{9}$$

Suppose that the eigenvalues of $G(\mathbf{p})$ are all non-negative. Then there exists an invertible matrix $S \in \mathbb{H}_{m,m}$ such that

$$S^* G(\mathbf{p}) S = \mathrm{diag}(1, \dots, 1, 0, \dots, 0).$$

Then $x^* S^* \mathbf{p}^* J \mathbf{p} S x \ge 0$, $\forall x \in \mathbb{H}^m$. This contradicts (9) when $x = S^{-1} l$ and $l = (1, \mu, 0, \dots, 0)^T \in \mathbb{H}^m$. □

The following proposition is obvious.

PROPOSITION 2.6. *Let S be an invertible matrix. Then $i(A) = i(S^*AS)$. Furthermore assume that $S^*AS = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Then*

$$i(A) = i(A_1) + i(A_2).$$

Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_l)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_t)$ such that $\langle \mathbf{p}_i, \mathbf{q}_j \rangle = 0$ for all i, j. Then we have

$$(\mathbf{p}, \mathbf{q})^* J (\mathbf{p}, \mathbf{q}) = \begin{pmatrix} G(\mathbf{p}) & 0 \\ 0 & G(\mathbf{q}) \end{pmatrix}. \tag{10}$$

We can now prove the following crucial result.

THEOREM 2.2. *Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m) \in \mathbb{H}_{n+1,m}$, $V = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ and*

$$\dim_q V = k + 1, \qquad i(G(\mathbf{p})) = i(\mathbf{p}^* J \mathbf{p}) = (n_+, n_-, n_0).$$

Then

$$k \le n_+ + n_- \le k+1, \quad n_+ \le n, \quad n_- \le 1, \qquad n_+ + n_- + n_0 = m.$$

In particular, we have the following statements.

(1) *If* $\mathbf{p}_i \in V_0$, $i = 1, \dots, m$, *then* $n_+ = k$, $n_- = 1$.

(2) *If* $\mathbf{p}_i \in V_+$, $i = 1, \dots, m$, *then there are three cases:*

  (i) $n_+ = k$, $n_- = 1$, *in this case* V *is hyperbolic;*

  (ii) $n_+ = k+1$, $n_- = 0$, *in this case* V *is elliptic;*

  (iii) $n_+ = k$, $n_- = 0$, *in this case* V *is parabolic.*

Proof. Let $t = k+1$. Without loss of generality, we assume that $\mathbf{p}_1, \dots, \mathbf{p}_t$ are linearly independent and

$$\mathbf{p}_j = \mathbf{p}_1 \lambda_{1j} + \cdots + \mathbf{p}_t \lambda_{tj}, \qquad j = t+1, \dots, m.$$

Let $\mathbf{q} = (\mathbf{p}_1, \dots, \mathbf{p}_t) \in \mathbb{H}_{n+1,t}$. Then $\mathbf{p} = \mathbf{q}(I_t, \Lambda)$, where $\Lambda = (\lambda_{ij})$, $i = 1, \dots, t$, $j = t+1, \dots, m$. Let $S = \begin{pmatrix} I_t & -\Lambda \\ 0 & I_{m-t} \end{pmatrix}$. Direction computation shows that

$$S^* G(\mathbf{p}) S = S^* \mathbf{p}^* J \mathbf{p} S = \begin{pmatrix} \mathbf{q}^* J \mathbf{q} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, by Proposition 2.6, we have that

$$i(\mathbf{p}^* J \mathbf{p}) = i(\mathbf{q}^* J \mathbf{q}).$$

This implies that $n_+ + n_- \le k+1$.

If $V \cap V_- \ne \varnothing$, then there exists a $\mathbf{z} \in V_-$ such that $V = \mathbf{z}\mathbb{H} \oplus (\mathbf{z}^\perp \cap V)$. In the space $\mathbf{z}^\perp \cap V$, there exist k mutually orthogonal positive lines $\mathbf{q}_1, \dots, \mathbf{q}_k$ such that $V = \mathrm{span}\{\mathbf{z}, \mathbf{q}_1, \dots, \mathbf{q}_k\}$. By (10), we have $n_+ = k$, $n_- = 1$ and V is hyperbolic in this case.

By Proposition 2.5, a space with two different null lines must contain negative lines. If $V \cap V_- = \varnothing$ and $V \cap V_0 \ne \varnothing$, then there exists a unique $\mathbf{z}\mathbb{H} \in V_0$. The space $\mathbf{z}^\perp \cap V$ contains only k mutually orthogonal positive lines $\mathbf{q}_1, \dots, \mathbf{q}_k$. In this case $n_+ = k$, $n_- = 0$ and V is parabolic.

If $V \subset V_+$, then V contains $k+1$ mutually orthogonal positive lines $\mathbf{q}_1, \dots, \mathbf{q}_{k+1}$. In this case $n_+ = k+1$, $n_- = 0$ and V is elliptic.

It follows from Proposition 2.3 and 2.5 that the statements of (1) and (2) hold. □

3. Moduli problem on $\mathbb{P}(V_0)$

In this section, we will consider the moduli problem on $\mathbb{P}(V_0)$ for $m > 4$. The application of the rotation-normalized algorithm is fully described. This method will be mimicked conceptually to more complicated cases in Sections 5 and 6.

3.1. Semi-normalized Gram matrix. We recall the following definition in [1, 8].

Definition 3.1. *The quaternionic Cartan's angular invariant of a triple* $\mathfrak{p} = (p_1, p_2, p_3)$ *of pairwise distinct points in* $\overline{\mathbf{H}^n_{\mathbb{H}}}$ *is the angular invariant* $\mathbb{A}_{\mathbb{H}}(\mathfrak{p})$, $0 \le \mathbb{A}_{\mathbb{H}}(\mathfrak{p}) \le \frac{\pi}{2}$, *given by*

$$\mathbb{A}_{\mathbb{H}}(\mathfrak{p}) = \mathbb{A}_{\mathbb{H}}(p_1, p_2, p_3) := \arccos \frac{\Re(-\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle)}{|\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle|}, \tag{11}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ *are lifts of* p_1, p_2, p_3, *respectively.*

Proposition 3.1. *Let* $\mathfrak{p} = (p_1, \dots, p_m)$ *be an m-tuple of pairwise distinct points in* $\partial \mathbf{H}^n_{\mathbb{H}}$. *Then the equivalence class of Gram matrices associated to* $\mathfrak{p}$ *contains a matrix* $G = (g_{ij})$ *with*

$$g_{ii} = 0, \quad i = 1, \dots, m, \qquad g_{i-1,i} = 1, \quad i = 2, \dots, m, \qquad g_{13} = -e^{-\mathbf{i}\mathbb{A}},$$

where $\mathbb{A} = \mathbb{A}_{\mathbb{H}}((p_1, p_2, p_3))$.

Proof. Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ be an arbitrary lift of $\mathfrak{p}$. We want to obtain a diagonal matrix D such that $G(\mathbf{p}D)$ is the desired Gram matrix.

Note that $\langle \mathbf{p}_i, \mathbf{p}_j \rangle \neq 0$ for $i \neq j$. Firstly we obtain the solutions λ_i, $i = 2, \dots, m$ of the equations below:

$$\langle \mathbf{p}_1, \mathbf{p}_2\lambda_2 \rangle = 1, \qquad \langle \mathbf{p}_2\lambda_2, \mathbf{p}_3\lambda_3 \rangle = 1, \dots, \langle \mathbf{p}_{m-1}\lambda_{m-1}, \mathbf{p}_m\lambda_m \rangle = 1. \tag{12}$$

Next, by (6), we let

$$\lambda_1 = \frac{\nu(\langle \mathbf{p}_1, \mathbf{p}_3\lambda_3 \rangle)}{\sqrt{|\langle \mathbf{p}_1, \mathbf{p}_3\lambda_3 \rangle|}} = \frac{\nu(\langle \mathbf{p}_2, \mathbf{p}_1 \rangle \langle \mathbf{p}_2, \mathbf{p}_3 \rangle^{-1} \langle \mathbf{p}_1, \mathbf{p}_3 \rangle)}{\sqrt{|\langle \mathbf{p}_2, \mathbf{p}_1 \rangle \langle \mathbf{p}_2, \mathbf{p}_3 \rangle^{-1} \langle \mathbf{p}_1, \mathbf{p}_3 \rangle|}}. \tag{13}$$

By the property of quaternionic Cartan's angular invariant, $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_3\lambda_3\lambda_1 \rangle$ is a unit complex with negative real part and therefore

$$\langle \mathbf{p}_1\lambda_1, \mathbf{p}_3\lambda_3\lambda_1 \rangle = -e^{-\mathbf{i}\mathbb{A}}.$$

Let $\mu_1 = \lambda_1$; for $i \ge 2$, $\mu_i = \lambda_i\lambda_1$ when i is odd, and $\mu_i = \lambda_i\overline{\lambda_1}^{-1}$ when i is even. Then $G(\mathbf{p}D)$ is the desired Gram matrix with

$$D = \mathrm{diag}(\mu_1, \dots, \mu_m). \qquad \square$$

Definition 3.2. *The Gram matrix* G *as in Proposition* 3.1 *of the form*

$$G(\mathbf{n}) = (g_{ij}) = \begin{pmatrix} 0 & 1 & g_{13} & g_{14} & \cdots & g_{1m} \\ 1 & 0 & 1 & g_{24} & \cdots & g_{2m} \\ \overline{g_{13}} & 1 & 0 & 1 & \cdots & g_{3m} \\ \overline{g_{14}} & \overline{g_{24}} & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ \overline{g_{1m}} & \overline{g_{2m}} & \overline{g_{3m}} & \cdots & 1 & 0 \end{pmatrix} \tag{14}$$

is called the semi-normalized Gram matrix.

By repeating almost verbatim the arguments used for the complex case in Theorems 2.1, 2.2 of [12], we obtain the following proposition.

Proposition 3.2. *Let $G = (g_{ij})$ be a Hermitian $m \times m$-matrix, $m > 2$ with*

$$g_{ii} = 0, \quad i = 1, \dots, m, \qquad g_{i-1,i} = 1, \quad i = 2, \dots, m, \qquad g_{13} = -e^{-\mathbf{i}\mathbb{A}},$$

where $\mathbb{A} \in [0, \pi/2]$. Let $i(G) = (n_+, n_-, n_0)$. Then G is a semi-normalized Gram matrix associated with some ordered m-tuple $\mathfrak{p} = (p_1, \dots, p_m)$ of pairwise distinct isotropic points in $\partial \mathbf{H}^n_{\mathbb{H}}$ if and only if

$$n_+ \le n, \qquad n_- = 1, \qquad n_+ + n_- + n_0 = m. \tag{15}$$

3.2. The parameter space of $\mathscr{M}(n, m; \partial \mathbf{H}^n_{\mathbb{H}})$. The following lemma shows that a semi-normalized Gram matrix is just an equivalent class, and also indicates the necessity of performing the rotation-normalized algorithm.

Lemma 3.1. *Suppose that the Gram matrix $G(\mathbf{p})$ is a semi-normalized Gram matrix for $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$. Then $G(\mathbf{p}D)$ is still a semi-normalized Gram matrix with $D = \operatorname{diag}(\mu_1, \dots, \mu_m)$ if and only if*

$$D = \mu I_m = \operatorname{diag}(\mu, \dots, \mu), \qquad \mu e^{-\mathbf{i}\mathbb{A}} = e^{-\mathbf{i}\mathbb{A}}\mu, \qquad \mu \in \mathrm{Sp}(1).$$

Proof. It follows from

$$\langle \mathbf{p}_{i-1}\mu_{i-1}, \mathbf{p}_i\mu_i \rangle = 1, \qquad i = 2, \dots, m$$

that all those μ_i with i odd are equal, and so do for all those μ_i with i even. The fact $\langle \mathbf{p}_1\mu_1, \mathbf{p}_3\mu_3 \rangle = -e^{-\mathbf{i}\mathbb{A}}$ implies $\mu_1 = \mu_3$. Hence $\mu_1 = \mu_2 = \cdots = \mu_m :=$ μ and $\mu e^{-\mathbf{i}\mathbb{A}} = e^{-\mathbf{i}\mathbb{A}}\mu$. □

Set $t = \frac{(m-1)(m-2)}{2}$. We can represent a semi-normalized Gram matrix by a t-vector:

$$v_G = (g_{13}, g_{14}, g_{24}, \dots, g_{1m}, \dots, g_{m-2,m}). \tag{16}$$

Also we represent

$$G = G(v_G). \tag{17}$$

Recall that two Hermitian matrices H and $\tilde{H}$ are equivalent if there exists a diagonal matrix D such that $\tilde{H} = D^*HD$ (see [8, 12]). By Lemma 3.1, we obtain the following result.

LEMMA 3.2. *Let G and $\tilde{G}$ be two semi-normalized Gram matrices represented by v_G and $V_{\tilde{G}}$. Then $\tilde{G}$ and G are equivalent if and only if*

$$\mathbf{O}_{v_G} = \mathbf{O}_{v_{\tilde{G}}}. \tag{18}$$

From this, Proposition 3.2 can be reformulated as follows.

PROPOSITION 3.3. *Let $v = (v_1, \dots, v_t)$ with $v_1 = -e^{-\mathbf{i}\mathbb{A}}$, $\mathbb{A} \in [0, \pi/2]$. Let $i(G(v)) = (n_+, n_-, n_0)$. Then $G(v)$ is a semi-normalized Gram matrix associated with some ordered m-tuple $\mathfrak{p} = (p_1, \dots, p_m)$ of distinct isotropic points in $\partial\mathbf{H}^n_{\mathbb{H}}$ if and only if*

$$n_+ \leq n, \qquad n_- = 1, \qquad n_+ + n_- + n_0 = m. \tag{19}$$

DEFINITION 3.3.

$$V(n,m) = \{v = (v_1, \dots, v_t) : i(G(v)) = (n_+, n_-, n_0) \text{ with } n_+ \leq n,\ n_- = 1\}.$$

By Lemma 3.2, there is an equivalent relation in $V(n,m)$ defined by (18). Therefore the configuration space $\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}})$ can be thought of as the quotient of $V(n,m)$ under this equivalent relation. That is

$$\mathscr{M}(n,m;\partial\mathbf{H}^n_{\mathbb{H}}) = V(n,m)/\simeq.$$

Based on this observation, we are ready to construct the parameter space $\mathbb{M}(n,m)$ for $V(n,m)/\simeq$ with the rotation-normalized algorithm. We mainly rely on Lemma 2.1 to execute the rotation-normalized algorithm.

This procedure can be described conceptually as follows:

In case $\mathbb{A} = 0$, or equivalently, $-e^{-\mathbf{i}\mathbb{A}} = -1$, we basically need to find two entries v_i and v_j in $v \in V(n,m)$ with $\Im(v_i)$ and $\Im(v_j)$ being linearly independent to specific the parameters for its representing equivalent class, whilst only a quaternion in $\mathbb{H} - \mathbb{C}$ in the case of $\mathbb{A} \neq 0$.

The above conceptual description is a motivation of the definition of the following sets.

Let

$$\mathbb{R}^{2+} = \{v \in \mathbb{H} : v = x_0 + x_1\mathbf{i} + x_2\mathbf{j}, x_2 > 0\},$$

$$\mathbb{R}^{1+} = \{v \in \mathbb{C} : v = x_0 + x_1\mathbf{i}, x_1 > 0\}.$$

Definition 3.4. *We define the following sets.*

$$P(\mathbb{C}) = \{v \in V(n,m) : v_1 \notin \mathbb{R}, v_i \in \mathbb{C}, \ \textit{for} \ \ i = 2, \dots, t\};$$

$$P(j) = \{v \in V(n,m) : v_1 \notin \mathbb{R}, v_i \in \mathbb{C}, \ \textit{for} \ \ i < j, v_j \in \mathbb{R}^{2+}\}, \qquad j = 2, \dots, t;$$

$$Z(\mathbb{R}) = \{v \in V(n,m) : v_i \in \mathbb{R} \ \ \textit{for} \ \ i = 1, \dots, t\};$$

$$Z(\mathbb{C}, i) = \{v \in V(n,m) : v_t \in \mathbb{R}, \ \textit{for} \ \ t < i, v_i \in \mathbb{R}^{1+}\}, \qquad i = 2, \dots, t;$$

$$Z(i,j) = \{v \in V(n,m) : v_t \in \mathbb{R}, t < i, v_i \in \mathbb{R}^{1+}; \ v_t \in \mathbb{C}, t < j, v_j \in \mathbb{R}^{2+}\}$$

for $j = 2, \dots, t, \ 2 \le i < j.$

We remark that the sets defined above is roughly divided by two cases: $\mathbb{A} \neq 0$ and $\mathbb{A} = 0$. Each case is refined according to the positions in which Lemma 2.1 acts. Roughly speaking, such a $Z(i,j)$ looks like

$$Z(i,j) = (\underbrace{-1, \mathbb{R}^*, \dots, \mathbb{R}^*}_{i-1}, \mathbb{R}^{1+}, \underbrace{\mathbb{C}^*, \dots, \mathbb{C}^*}_{j-i-1}, \mathbb{R}^{2+}, \mathbb{H}^*, \dots, \mathbb{H}^*).$$

Let

$$P(n,m) = P(\mathbb{C}) \cup P(j), \qquad Z(n,m) = Z(\mathbb{R}) \cup Z(\mathbb{C}, j) \cup Z(i,j)$$

and

$$\mathbb{M}(n,m) = P(n,m) \cup Z(n,m).$$

Theorem 3.1. $\mathbb{M}(n,m)$ *is a parameter space of* $V(n,m)/\simeq$.

Proof. Let $v = (v_1, \dots, v_t) \in V(n,m)$, where $v_1 = -e^{-\mathbf{i}\mathbb{A}}$. We define a map

$$\psi : \mathbf{O}_v \in V(n,m)/\simeq \ \to \mathbb{M}(n,m) \tag{20}$$

by the following steps:

The equivalent class $\mathbf{O}_v$ with $\mathbb{A} \neq 0$ will be mapped to an element in $P(n,m)$. It is obvious that $\bar{\mu}v\mu \in V(n,m)$ if and only if $\mu \in U(1)$. If all entries of v are complex numbers, then $\mathbf{O}_v$ is represented by v itself. Equivalently, the parameter of $\mathbf{O}_v$ assigned by ψ in $\mathbb{M}(n,m)$ is v which belongs to $\in P(\mathbb{C})$. Otherwise, let j be the smallest index among entries of v such that $v_j \in \mathbb{H} - \mathbb{C}$. Let $\mu = \mu(\Im(v_1), \Im(v_j))$ given by (7). Therefore $\mathbf{O}_v$ is assigned to the parameter $\bar{\mu}v\mu$, which belongs to $P(j)$.

The equivalent class $\mathbf{O}_v$ with $\mathbb{A} = 0$ belongs to $Z(n,m)$. More precisely, if all entries of v are reals, then $\mathbf{O}_v$ is represented by v itself belonging to $Z(\mathbb{R})$. We divide the remainder into two cases. If all entries of v are complex numbers with i being the smallest index such that $v_i \in \mathbb{C} - \mathbb{R}$. Let $\mu = \nu(\Im(v_i))$ be given by (6). Then we assign $\mathbf{O}_v$ to $\bar{\mu} v \mu$, which belongs to $Z(\mathbb{C}, j)$. For the latter case, let i be the smallest index such that $v_i \in \mathbb{C} - \mathbb{R}$ and j the smallest index such that $v_j \in \mathbb{H} - \mathbb{C}$. Let $\mu = \mu(\Im(v_i), \Im(v_j))$. Then we assign $\mathbf{O}_v$ to $\bar{\mu} v \mu$, which belongs to $Z(i, j)$.

By Lemma 2.1 and the constructions of $P(n,m)$ and $Z(n,m)$ above, the map ψ is bijection. Therefore $\mathbb{M}(n,m)$ is a parameter space of $V(n,m)/\simeq$. $\square$

THEOREM 3.2. *The configuration space $\mathscr{M}(n,m;\partial\mathbf{H}_{\mathbb{H}}^n)$ is homeomorphic to $\mathbb{M}(n,m)$.*

PROOF. Let $m(\mathfrak{p}) \in \mathscr{M}(n,m;\partial\mathbf{H}_{\mathbb{H}}^n)$ be the point represented by $\mathfrak{p} = (p_1, \dots, p_m)$. We can get a semi-normalized Gram matrix G with arbitrary lift of $\mathfrak{p}$. Proposition 3.3 and Theorem 3.1 imply that we can define a map

$$\tau : m(\mathfrak{p}) \in \mathscr{M}(n,m;\partial\mathbf{H}_{\mathbb{H}}^n) \to \psi(v_G) \in \mathbb{M}(n,m).$$

This map is a bijection. Such a map is a homeomorphism because $\mathbb{M}(n,m)$ has the topology structure induced from $\mathbb{H}^t$. $\square$

We conclude this section by some remarks. Firstly, if we allow $m = 3$ in our process then we get the parameter of quaternionic Cartan's angular invariant $\mathbb{A}$ (in fact a complex number $-e^{-\mathbf{i}\mathbb{A}}$); while the case of $m = 4$ is exactly the result in [8]. Secondly it seems that the parameters of m-tuples in $Z(\mathbb{R})$, $Z(\mathbb{R}) \cup Z(\mathbb{C}, i) \cup P(\mathbb{C})$ can be thought of as m-tuples living in a copy of $\partial\mathbf{H}_{\mathbb{R}}^n$ and $\partial\mathbf{H}_{\mathbb{C}}^n$, respectively.

4. The structure of Gram matrices of points on $\mathbb{P}(V_+)$

The main purpose of this section is to refine the structures of Gram matrices. These refined structures are crucial in introducing new invariants in non regular case and the block-normalized algorithm in regular case.

4.1. 1-normalized Gram matrices.

PROPOSITION 4.1. *Let $\mathfrak{p} = (p_1, \dots, p_m)$ be an m-tuple of pairwise distinct points in $\mathbb{P}(V_+)$. Then the equivalence class of Gram matrices associated to $\mathfrak{p}$ contains a matrix $G = (g_{ij})$ with*

$$g_{ii} = 1, \quad i = 1, \dots, m, \qquad g_{1j} \geq 0, \quad j = 2, \dots, m.$$

PROOF. Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ be an arbitrary lift of $\mathfrak{p}$. We want to obtain a diagonal matrix D_1 such that $G(\mathbf{p}D_1)$ is the desired Gram matrix.

We may assume that $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 1$ by noticing that

$$\langle \mathbf{p}_i \lambda_i, \mathbf{p}_i \lambda_i \rangle = 1, \qquad \text{for } \lambda_i = \sqrt{\frac{1}{\langle \mathbf{p}_i, \mathbf{p}_i \rangle}}.$$

For $i = 2, \ldots, m$, let

$$\lambda_i = \begin{cases} \frac{\langle \mathbf{p}_1, \mathbf{p}_i \rangle}{|\langle \mathbf{p}_1, \mathbf{p}_i \rangle|}, & \text{provided } \langle \mathbf{p}_1, \mathbf{p}_i \rangle \neq 0; \\ 1, & \text{otherwise.} \end{cases} \tag{21}$$

Then there exists a $\lambda_1 \in \mathrm{Sp}(1)$ such that $\overline{\lambda_1}\,\overline{\lambda_3} \langle \mathbf{p}_2, \mathbf{p}_3 \rangle \lambda_2 \lambda_1$ is a complex number with non-negative imaginary part if $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle \neq 0$. Then

$$G(\mathbf{p}_1 \lambda_1, \mathbf{p}_2 \lambda_2 \lambda_1, \ldots, \mathbf{p}_m \lambda_m \lambda_1)$$

is the desired Gram matrix. In other words, $G(\mathbf{p} D_1)$ is the desired Gram matrix with

$$D_1 = \mathrm{diag}\left(\sqrt{\frac{1}{\langle \mathbf{p}_1, \mathbf{p}_1 \rangle}} \lambda_1, \sqrt{\frac{1}{\langle \mathbf{p}_2, \mathbf{p}_2 \rangle}} \lambda_2 \lambda_1, \ldots, \sqrt{\frac{1}{\langle \mathbf{p}_m, \mathbf{p}_m \rangle}} \lambda_m \lambda_1 \right). \tag{22}$$

□

DEFINITION 4.1. *The Gram matrix G as in Proposition 4.1 of the form*

$$G = (g_{ij}) = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} & \cdots & g_{1m} \\ g_{12} & 1 & g_{23} & g_{24} & \cdots & g_{2m} \\ g_{13} & \overline{g_{23}} & 1 & g_{34} & \cdots & g_{3m} \\ g_{14} & \overline{g_{24}} & \overline{g_{34}} & 1 & \cdots & g_{4m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{1m} & \overline{g_{2m}} & \overline{g_{3m}} & \overline{g_{4m}} & \cdots & 1 \end{pmatrix} \tag{23}$$

is called the 1*-normalized Gram matrix.*

The following result can be shown similarly as Proposition 3.2.

THEOREM 4.1 ([13, Proposition 3.2]). *Let $G = (g_{ij})$ be a Hermitian $m \times m$-matrix, $m > 2$ with*

$$g_{ii} = 1, \quad i = 1, \ldots, m, \qquad g_{1j} \geq 0, \quad j = 2, \ldots, m.$$

Let $i(G) = (n_+, n_-, n_0)$. Then G is a 1*-normalized Gram matrix associated with an m-tuple of pairwise distinct points in $\mathbb{P}(V_+)$ if and only if*

$$1 \leq n_+ + n_- \leq n + 1, \quad n_+ \leq n, \quad n_- \leq 1, \qquad n_+ + n_- + n_0 = m. \tag{24}$$

4.2. The parameter space of $\mathscr{M}(n,2;\mathbb{P}(V_+))$. In this subsection, we will construct the parameter space of $\mathscr{M}(n,2;\mathbb{P}(V_+))$. We need the following lemma, which is easy to be verified. We refer to [2, 6] for more details of $\mathrm{Sp}(1,1)$.

LEMMA 4.1. *Let $g \in \mathrm{Sp}(2,1)$ and $\mathbf{e}_2 = (0,1,0)^T \in \mathbb{H}^{2,1}$ such that $g\mathbf{e}_2 = \mathbf{e}_2\mu$. Then g is of the form*

$$g = \begin{pmatrix} a & 0 & b \\ 0 & \mu & 0 \\ c & 0 & d \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(1,1) \qquad \text{and} \qquad \mu \in \mathrm{Sp}(1).$$

THEOREM 4.2. *The configuration space $\mathscr{M}(n,2;\mathbb{P}(V_+))$ is homeomorphic to $\{t \in \mathbb{R} : t \geq 0\}$.*

PROOF. It is obvious that we can work in $\mathbb{H}^{2,1}$ in this situation. By Proposition 4.1, we only need to show that there exists a $g \in \mathrm{Sp}(2,1)$ such that $g\mathbf{p}_1 = \mathbf{q}_1\lambda_1$ and $g\mathbf{p}_2 = \mathbf{q}_2\lambda_2$ when $G((\mathbf{p}_1,\mathbf{p}_2)) = G((\mathbf{q}_1,\mathbf{q}_2)) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ with $t \geq 0$. By noting Proposition 2.2, we only need to consider the case $t \neq 0$. Observe that $t \neq 0$ implies $\lambda_1 = \lambda_2$. Since $\mathrm{Sp}(2,1)$ acts transitively on $\mathbb{P}(V_+)$, we may further assume that

$$\mathbf{p}_1 = \mathbf{q}_1 = (0,1,0)^T, \qquad \mathbf{p}_2 = (x_1,t,x_3), \qquad \mathbf{q}_2 = (y_1,t,y_3)^T,$$

where $|x_3|^2 - |x_1|^2 = |y_3|^2 - |y_1|^2 = t^2 - 1$. By Lemma 4.1, we need to find an element $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(1,1)$ mapping $(x_1,x_3)^T$ to $(y_1,y_3)^T\mu$. The fact that $\mathrm{Sp}(1,1)$ acts doubly transitively on $\partial\mathbf{H}^1_{\mathbb{H}}$, transitively on $\mathbf{H}^1_{\mathbb{H}}$, and on $\mathbb{P}(V_+)$ respectively, concludes the proof. $\square$

4.3. The structure of Gram matrices of points on $\mathbb{P}(V_+)$. In what follows, we assume that $G(\mathbf{p})$ is already a 1-normalized Gram matrix.

PROPOSITION 4.2. *Let $\mathbf{p} = (\mathbf{p}_1,\dots,\mathbf{p}_t)$ be a t-tuple of pairwise distinct points in $\mathbb{P}(V_+)$ satisfying*

$$\langle \mathbf{p}_i,\mathbf{p}_j \rangle = 1, \qquad i,j = 1,\dots,t$$

and $V = \mathrm{span}\{\mathbf{p}_1,\dots,\mathbf{p}_l\}$. Then there exists a unique fibre $\mathbf{z}\mathbb{H} \in V_0$ such that

$$V \subset \mathbf{z}^{\perp}, \qquad V \cap V_0 = \mathbf{z}\mathbb{H}, \qquad V \cap V_- = \varnothing.$$

In fact

$$\mathbf{z} = \mathbf{p}_2 - \mathbf{p}_1, \qquad V = \mathrm{span}\{\mathbf{p}_1, \mathbf{p}_2\} = \mathrm{span}\{\mathbf{z}, \mathbf{p}_1\}.$$

PROOF. Let $\mathbf{u} = \mathbf{p}_1\lambda_1 + \mathbf{p}_2\lambda_2 \in V$. Then

$$\langle \mathbf{u}, \mathbf{u}\rangle = |\lambda_1|^2 + |\lambda_2|^2 + 2\Re(\bar{\lambda}_2\lambda_1) \geq 0. \tag{25}$$

Note that $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ if and only if $\lambda_1 = -\lambda_2$. Hence $(\mathbf{p}_2 - \mathbf{p}_1)\mathbb{H}$ is the unique fibre of the intersection $\mathrm{span}\{\mathbf{p}_1, \mathbf{p}_2\} \cap V_0$ and $\mathrm{span}\{\mathbf{p}_1, \mathbf{p}_2\} \cap V_- = \emptyset$. By noting that $\langle \mathbf{p}_i - \mathbf{p}_j, \mathbf{p}_i - \mathbf{p}_j\rangle = 0$ and $\langle \mathbf{p}_i - \mathbf{p}_j, \mathbf{p}_2 - \mathbf{p}_1)\rangle = 0$ for $i \neq j$, by Proposition 2.4, we have $(\mathbf{p}_2 - \mathbf{p}_1)\mathbb{H} = (\mathbf{p}_i - \mathbf{p}_j)\mathbb{H}$. Since $\langle \mathbf{p}_i, (\mathbf{p}_1 - \mathbf{p}_2)\rangle = 0$, $i = 1, \dots, t$, we have $V \subset (\mathbf{p}_2 - \mathbf{p}_1)^{\perp}$. It follows from $\mathbf{p}_i - \mathbf{p}_1 \in (\mathbf{p}_2 - \mathbf{p}_1)\mathbb{H}$ that there exist λ_i such that

$$\mathbf{p}_i = \mathbf{p}_1 + (\mathbf{p}_2 - \mathbf{p}_1)\lambda_i = \mathbf{p}_2\lambda_i + \mathbf{p}_1(1 - \lambda_i), \qquad i = 1, \dots, t.$$

This implies that

$$V = \mathrm{span}\{\mathbf{p}_1, \mathbf{p}_2\} = \mathrm{span}\{z, \mathbf{p}_1\}$$

and therefore $V \cap V_0 = \mathbf{z}\mathbb{H}$, $V \cap V_- = \emptyset$. □

The information of λ_i disappears in the sub Gram matrix $G((\mathbf{p}_1, \dots, \mathbf{p}_t))$. Moreover, such information can not be rebuilt through the relationships with other points in some situations. This implies that the Gram matrix loses the configuration information of such a t-tuple. We provide the following explicit example in ball model to illustrate this phenomenon.

EXAMPLE 4.1. *Let* $\mathbf{z} = (1, 0, 1)^T \in V_0$ *and* $\mathbf{p}_1 = (0, 1, 0) \in V+$. *Let* $\mathbf{p}_i = \mathbf{p}_1 + i\mathbf{z}$, $i = 2, 3$. *Then*

$$G((\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)) = G((\mathbf{p}_3, \mathbf{p}_2, \mathbf{p}_1)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We claim that $(\mathbb{P}(\mathbf{p}_1), \mathbb{P}(\mathbf{p}_2), \mathbb{P}(\mathbf{p}_3))$ *and* $(\mathbb{P}(\mathbf{p}_3), \mathbb{P}(\mathbf{p}_2), \mathbb{P}(\mathbf{p}_1))$ *are not* $\mathrm{PSp}(2,1)$-*congruent.*

PROOF (Proof of the Claim). Suppose that the two triples above are $\mathrm{PSp}(2,1)$-congruent. Then there exists a $g \in \mathrm{Sp}(2,1)$ such that

$$g\mathbf{p}_1 = \mathbf{p}_3\lambda_1, \qquad g\mathbf{p}_2 = \mathbf{p}_2\lambda_2, \qquad g\mathbf{p}_3 = \mathbf{p}_1\lambda_3.$$

It follows from

$$\langle g\mathbf{p}_i, g\mathbf{p}_i\rangle = \langle g\mathbf{p}_i, g\mathbf{p}_j\rangle = \langle \mathbf{p}_i, \mathbf{p}_i\rangle = 1$$

that $\lambda_i \in \mathrm{Sp}(1)$ and $\overline{\lambda_j}\lambda_i = 1$, and therefore $\lambda_1 = \lambda_2 = \lambda_3 := \lambda$. Hence

$$g2\mathbf{z} = g(\mathbf{p}_2 - \mathbf{p}_1) = (\mathbf{p}_2 - \mathbf{p}_3)\lambda = -\mathbf{z}\lambda,$$

which contradicts

$$g\mathbf{z} = g(\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{p}_1 - \mathbf{p}_2)\lambda = -\mathbf{z}2\lambda. \qquad \square$$

If V is parabolic, by Proposition 4.2, we can refine Theorem 2.2 as follows.

Proposition 4.3. *Let* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m) \in \mathbb{H}_{n+1,m}$, $V = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ *and*

$$\dim_q V = k+1, \qquad i(G(\mathbf{p})) = i(\mathbf{p}^* J \mathbf{p}) = (k, 0, m-k).$$

Then $S(m) = \{1, \dots, m\}$ *has a partition:*

$$S_i = \{s_{i1}, \dots, s_{it_i}\}, \qquad s_{i1} < \cdots < s_{it_i}, \qquad i = 1, \dots, k \tag{26}$$

with the properties

$$S(m) = \bigcup_{i=1}^{k} S_i; \qquad \langle \mathbf{p}_{s_{il}}, \mathbf{p}_{s_{id}} \rangle = 1, \quad 1 \le l, d \le t_i; \qquad \langle \mathbf{p}_{s_{il}}, \mathbf{p}_{s_{jd}} \rangle = 0, \quad i \ne j \tag{27}$$

and in each

$$\mathbf{p}_{S_i} := (\mathbf{p}_{s_{i1}}, \dots, \mathbf{p}_{s_{it_i}}),$$

we can not partition likewise as in (27).

There exists a common $\mathbf{z}_0 \in V_0$ *such that* $\mathbf{p} \in \mathbf{z}_0^{\perp}$ *and*

$$\mathbf{p}_{s_{il}} = \mathbf{p}_{s_{i1}} + \mathbf{z}_0 \lambda_{il}, \qquad 1 < l \le \mathrm{Card}(S_i), \qquad i = 1, \dots, k, \tag{28}$$

where $\mathrm{Card}(S_i)$ *is the cardinality of* S_i. *We define*

$$V_i = \mathrm{span}\{\mathbf{p}_{s_{i1}}, \dots, \mathbf{p}_{s_{it_i}}\} = \mathrm{span}\{\mathbf{p}_{s_{i1}}, \mathbf{z}_0\}, \qquad i = 1, \dots, k. \tag{29}$$

If V is not parabolic, we can refine Theorem 2.2 as follows.

Proposition 4.4. *Let* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m) \in \mathbb{H}_{n+1,m}$, $V = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ *and*

$$\dim_q V = k+1, \qquad i(G(\mathbf{p})) = i(\mathbf{p}^* J \mathbf{p}) = (k, 1, m-k-1) \qquad \text{or}$$
$$(k+1, 0, m-k-1).$$

Then $S(m) = \{1, \dots, m\}$ *has a partition:*

$$S_i = \{s_{i1}, \dots, s_{it_i}\}, \qquad s_{i1} < \cdots < s_{it_i}, \qquad i = 1, \dots, s \tag{30}$$

with the properties

$$S(m) = \bigcup_{i=1}^{s} S_i; \qquad \langle \mathbf{p}_{s_{il}}, \mathbf{p}_{s_{jd}} \rangle = 0, \quad i \neq j \tag{31}$$

and in each $\mathbf{p}_{S_i} := (\mathbf{p}_{s_{i1}}, \dots, \mathbf{p}_{s_{it_i}})$*, we can not partition likewise as above.*

It is helpful to keep in mind that there are no relationships among the blocked-entries corresponding to each components $\mathbf{p}_{S_i}$ in the diagonal matrix D in (4). This is the motivation of refinement of Theorem 2.2. Furthermore, when V is not parabolic, we still need to partition the components S_i in some situations.

5. The moduli problem on $\mathbb{P}(V_+)$ of non regular case

We will work on the Siegel domain in this section. We will construct invariants which describe the $\mathrm{PSp}(n,1)$-congruence classes when V is parabolic.

We first recall the following fact of isometries in $\mathrm{Sp}(n,1)$ fixing ∞.

LEMMA 5.1 (c.f. [10, Lemma 3.3.1]). *Let* $\mathbf{z}_\infty = (1, 0, \dots, 0, 0)^T$, $\mathbb{P}(\mathbf{z}_\infty) = \infty$ *and*

$$G_\infty = \{g \in \mathrm{Sp}(n,1) : g(\infty) = \infty\}.$$

Then $g \in G_\infty$ *is of the form*

$$g = \begin{pmatrix} \lambda & \gamma^* & s \\ 0 & U & \beta \\ 0 & 0 & \mu \end{pmatrix}, \tag{32}$$

where $\lambda, \mu, s \in \mathbb{H}, \beta, \gamma \in \mathbb{H}^{n-1}$, $U \in \mathrm{Sp}(n-1)$, $|\bar{\mu}\lambda| = 1$, $\Re(\bar{\mu}s) = -\frac{1}{2}|\beta|^2$, $\beta = -U\gamma\mu$.

Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$ be two ordered m-tuples of pairwise distinct points in $\mathbb{P}(V_+)$ such that $V(\mathbf{p})$ and $V(\mathbf{q})$ are parabolic. Observe that if $\mathbf{p}$ and $\mathbf{q}$ are $\mathrm{PSp}(n,1)$-congruent, then they have the same structure given by Proposition 4.3. Since $\mathrm{Sp}(n,1)$ acts doubly transitively on $\partial\mathbf{H}^n_{\mathbb{H}}$, we can further assume that $\mathbf{p}, \mathbf{q} \in \mathbf{z}_\infty^\perp$. As showed by Example 4.1, besides the information of structure, other conditions are needed for $\mathbf{p}$, $\mathbf{q}$ being $\mathrm{PSp}(n,1)$-congruent.

In what follows, we assume that $m \geq 3$, $V(\mathbf{p}) = \mathrm{span}\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ is parabolic and $V(\mathbf{p}) \subset \mathbf{z}_\infty^\perp$. It is obvious that

$$\mathbf{z}_\infty^\perp = (z_1, \dots, z_n, 0)^T := (z_1, \alpha^T, 0)^T.$$

Therefore the action of $g \in G_\infty$ on $\mathbf{z}_\infty^\perp$ can be expressed by

$$g : \begin{pmatrix} z_1 \\ \alpha \\ 0 \end{pmatrix} \to \begin{pmatrix} \lambda z_1 + \gamma^* \alpha \\ U\alpha \\ 0 \end{pmatrix}.$$

The restriction of the Hermitian form $\langle , \rangle$ on $\mathbf{z}_\infty^\perp$ is the usual inner product on $\mathbb{H}^{n-1}$, i.e.,

$$\langle (k_1, \alpha_1, 0)^T, (k_2, \alpha_2, 0)^T \rangle = \alpha_2^* \alpha_1.$$

For g with the form (32), we define the map

$$\Pi : g \in G_\infty \to \tilde{g} = \begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix} \in \tilde{G}_\infty. \tag{33}$$

Then Π is a homomorphism with

$$\ker(\Pi) = \left\{ \begin{pmatrix} 1 & 0 & s \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \Re(s) = 0 \right\}$$

and its homomorphic image $\tilde{G}_\infty = \Pi(G_\infty)$ is a subgroup of $\mathrm{GL}(n, \mathbb{H})$. The action of G_∞ on $\mathbf{z}_\infty^\perp$ can be expressed by the projection action of $\tilde{G}_\infty$ on $\mathbb{HP}^{n-1} = (z_1, \alpha^T)^T$.

By noting Proposition 4.3 and $G(\mathbf{p})$ being a 1-normalized Gram matrix, we have

$$\tilde{\mathbf{p}}_{s_{il}} = (k_{il}, \alpha_i^T)^T, \qquad \text{for } s_{il} \in S_i \tag{34}$$

and

$$\alpha_i^* \alpha_i = 1, \qquad 1 \le i \le k.$$

Therefore there exists a $U \in \mathrm{Sp}(n-1)$ such that $g = \mathrm{diag}(1, U, 1) \in \mathrm{Sp}(n, 1)$ satisfying

$$U(\alpha_1, \ldots, \alpha_k) = (\mathbf{e}_1, \ldots, \mathbf{e}_k), \tag{35}$$

where $\mathbf{e}_i$, $1 \le i \le k$ are k vectors in the standard basis of $\mathbb{H}^{n-1}$. Therefore we may further reformulate (34) as

$$\tilde{\mathbf{p}}_{s_{il}} = (k_{il}, e_i^T)^T, \qquad \text{for } s_{il} \in S_i. \tag{36}$$

In order to parameterize the moduli space, we introduce the following map ϕ to give the corresponding coordinates in $\mathbb{H} \cup \infty$ for vectors in $V_i = \mathrm{span}\{\mathbf{p}_{s_{i1}}, \mathbf{z}_\infty\}$:

$$\phi(\mathbf{z}_\infty) = \infty, \qquad \phi(\tilde{\mathbf{p}}_{s_{il}}) = k_{il}, \qquad 1 \le l \le t_i;\ 1 \le i \le k. \tag{37}$$

Let $h = \begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix}$ with $U(\mathbf{e}_1, \dots, \mathbf{e}_k) = (\mathbf{e}_1, \dots, \mathbf{e}_k)$ and $\gamma = (c_1, \dots, c_{n-1})^T$. Note that

$$\begin{pmatrix} \lambda & \gamma^* \\ 0 & U \end{pmatrix} \begin{pmatrix} k_{il} \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \lambda k_{il} + \gamma^* \mathbf{e}_i \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \lambda k_{i1} + c_i \\ \mathbf{e}_i \end{pmatrix}. \tag{38}$$

This means that the restriction of h in V_i is

$$h_i : k_{il} \to \lambda k_{i1} + c_i, \qquad 1 \le i \le k. \tag{39}$$

The above treatment can be thought of as introducing the inhomogeneous coordinates in each V_i. From this point of view, the restriction of an element $\tilde{g}$ of form (33) to V_i is a quaternionic Möbius transformation in Γ_∞, the isotropy group at ∞ in $\mathrm{PS}_\triangle L(2, \mathbb{H})$ [7].

Summarizing the above descriptions, we have so far defined a map

$$\Pi_i : g \in G_\infty \to h_i = \begin{pmatrix} \lambda & c_i \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \tag{40}$$

and the action of g on $\mathbf{z}_\infty^\perp$ is inherited by the actions of h_i on V_i, which is identified with $\overline{\mathbb{H}}$.

Observe that the coordinates defined by (37) contain the information of λ_{il} in (28). To distinguish between $\mathrm{PSp}(n,1)$-congruence classes of m-tuples in degenerate case is the same as distinguishing the h_i-congruence classes in V_i for all i. For this purpose, we need to introduce new geometric invariants which are invariant under the action of h_i.

Definition 5.1 ([2, Definition 4.2]). *The quaternionic cross-ratio of four points $z_1, z_2, z_3, z_4 \in \mathbb{H} \cup \infty$ is defined as*

$$[z_1, z_2, z_3, z_4] = (z_1 - z_3)(z_1 - z_4)^{-1}(z_2 - z_4)(z_2 - z_3)^{-1}.$$

Lemma 5.2 ([2, Proposition 4.1]). *Given three distinct $z_1, z_2, z_3 \in \mathbb{H}$, the element $f \in \mathrm{PS}_\triangle L(2, \mathbb{H})$ defined by*

$$f(z) = (z_3 - z_2)(z_3 - z_1)^{-1}(z - z_1)(z - z_2)^{-1} \tag{41}$$

maps z_1 to 0, z_2 to ∞ and z_3 to 1. Moreover, all elements $f \in \mathrm{PS}_\triangle L(2, \mathbb{H})$ with the same property are of the form:

$$\lambda I_2 \circ f(z) = \lambda f(z) \lambda^{-1}$$

with $\lambda \in \mathbb{H} - \{0\}$.

Note that $f \in \mathrm{PS}_{\triangle}L(2, \mathbb{H})$ fixing 0, 1, ∞ is of the form $f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I_2$. Based on this observation and [2, Proposition 4.4], the cross-ratios enjoy the following properties.

Lemma 5.3. (1) *For any* $z \in \mathbb{H}$ *with* $z \neq 0$ *and* $z \neq 1$, *we have*

$$[z, 1, 0, \infty] = z.$$

(2) *Given distinct points* $z_1, z_2, z_3, z \in \overline{\mathbb{H}}$,

$$[f(z), f(z_3), f(z_2), f(z_1)] = \lambda_f [z, z_3, z_2, z_1] \lambda_f^{-1},$$

where λ_f *is a quaternion solely depending on* $f \in \mathrm{PS}_{\triangle}L(2, \mathbb{H})$. *In particular, for* h_i *given by* (39), *we have*

$$[h_i(z), h_i(z_3), h_i(z_2), h_i(z_1)] = \lambda [z, z_3, z_2, z_1] \lambda^{-1}.$$

Definition 5.2. *We introduce the following geometric invariants in component* $\mathbf{p}_{S_i} = (\mathbf{p}_{s_{i1}}, \dots, \mathbf{p}_{s_{it_i}})$ *when* $\mathrm{Card}(S_i) \geq 3$:

$$\chi(\tilde{\mathbf{p}}_{s_{il}}, \tilde{\mathbf{p}}_{s_{ij}}, \tilde{\mathbf{p}}_{s_{it}}) = (k_{il} - k_{it})(k_{ij} - k_{it})^{-1} = [k_{il}, k_{ij}, k_{it}, \infty]. \tag{42}$$

We mention that since the points $\tilde{\mathbf{p}}_{s_{i1}}$, $\tilde{\mathbf{p}}_{s_{ij}}$, $\tilde{\mathbf{p}}_{s_{it}}$ are all distinct, χ is finite and $\chi \neq 0, 1$. Therefore

$$\chi \in \mathbb{H} - \{0, 1\}.$$

To sort out the conditions for $\mathbf{p}$ and $\mathbf{q}$ being $\mathrm{PSp}(n, 1)$-congruent, without loss of generality, we may assume that $\mathbf{p}, \mathbf{q} \in \mathbf{z}_{\infty}^{\perp}$ have the same structure given by Proposition 4.3. We denote the corresponding coordinates of $\tilde{\mathbf{q}}_{S_i}$ by

$$\tilde{\mathbf{q}}_{s_{il}} = (w_{il}, e_i^T)^T, \qquad \text{for } s_{il} \in S_i \tag{43}$$

and compute the corresponding invariants of $\mathbf{q}$ in the same manner as these of $\mathbf{p}$.

We first obtain the necessary and sufficient condition of two triples being $\mathrm{PSp}(n, 1)$-congruent directly.

Proposition 5.1. $\tilde{\mathbf{p}}_i = (\tilde{\mathbf{p}}_{s_{i1}}, \tilde{\mathbf{p}}_{s_{i2}}, \tilde{\mathbf{p}}_{s_{i3}})$ *and* $\tilde{\mathbf{q}}_i = (\tilde{\mathbf{q}}_{s_{i1}}, \tilde{\mathbf{q}}_{s_{i2}}, \tilde{\mathbf{q}}_{s_{i3}})$ *are* $\mathrm{PSp}(n, 1)$-*congruent if and only if there exists a* $\lambda \in \mathbb{H} - \{0\}$ *such that*

$$\chi(\tilde{\mathbf{p}}_{s_{i1}}, \tilde{\mathbf{p}}_{s_{i2}}, \tilde{\mathbf{p}}_{s_{i3}}) = \lambda \chi(\tilde{\mathbf{q}}_{s_{i1}}, \tilde{\mathbf{q}}_{s_{i2}}, \tilde{\mathbf{q}}_{s_{i3}}) \lambda^{-1}.$$

Proof. Assume that $\tilde{\mathbf{p}}_i$ and $\tilde{\mathbf{q}}_i$ are $\mathrm{PSp}(n, 1)$-congruent. Let $\mathbf{p}_i$ and $\mathbf{q}_i$ be the corresponding triples in $\mathbb{H}^{n,1}$. Then there exists a $g \in G_{\infty}$ such that $g(\mathbf{p}_{s_{i1}}, \mathbf{p}_{s_{i2}}, \mathbf{p}_{s_{i3}}) = (\mathbf{q}_{s_{i1}} v_1, \mathbf{q}_{s_{i2}} v_2, \mathbf{q}_{s_{i3}} v_3)$. As before, we know that $v_1 = v_2 = v_3 :=$

v and $|v| = 1$. This implies that $\tilde{g}\tilde{\mathbf{p}}_i = \tilde{\mathbf{q}}_i v$, i.e.,

$$\lambda k_{il} + \gamma^* \mathbf{e}_i = w_{il} v, \qquad U\mathbf{e}_i = \mathbf{e}_i v, \qquad l = 1, 2, 3.$$

Therefore

$$\lambda(k_{i2} - k_{i3}) = (w_{i2} - w_{i3})v, \qquad \lambda(k_{i1} - k_{i3}) = (w_{i1} - w_{i3})v.$$

Hence

$$\lambda(k_{i1} - k_{i3})(k_{i2} - k_{i3})^{-1}\lambda^{-1} = (w_{i1} - w_{i3})(w_{i2} - w_{i3})^{-1}.$$

Conversely, suppose that $\chi(\tilde{\mathbf{p}}_{s_{i1}}, \tilde{\mathbf{p}}_{s_{i2}}, \tilde{\mathbf{p}}_{s_{i3}}) \sim \chi(\tilde{\mathbf{q}}_{s_{i1}}, \tilde{\mathbf{q}}_{s_{i2}}, \tilde{\mathbf{q}}_{s_{i3}})$. Then there exists a λ such that

$$\lambda(k_{i1} - k_{i3})(k_{i2} - k_{i3})^{-1}\lambda^{-1} = (w_{i1} - w_{i3})(w_{i2} - w_{i3})^{-1}.$$

We may further require that $|\lambda| = \frac{|k_{i1} - k_{i3}|}{|w_{i1} - w_{i3}|}$. Let $v = (w_{i1} - w_{i3})^{-1}\lambda(k_{i1} - k_{i3})$. Then

$$\lambda(k_{i1} - k_{i3}) = (w_{i1} - w_{i3})v, \qquad \lambda(k_{i2} - k_{i3}) = (w_{i2} - w_{i3})v.$$

From the above two equalities, we have $\lambda(k_{i1} - k_{i2}) = (w_{i1} - w_{i2})v$. We can find a $\gamma \in \mathbb{H}^{n-1}$ and a $U \in \mathrm{Sp}(n-1)$ satisfying

$$\lambda k_{i1} + \gamma^* \mathbf{e}_i = w_{i1} v, \qquad U\mathbf{e}_i = \mathbf{e}_i v.$$

The above equalities also imply

$$\lambda k_{i2} + \gamma^* \mathbf{e}_i = w_{i2} v, \qquad \lambda k_{i3} + \gamma^* \mathbf{e}_i = w_{i3} v.$$

With λ, γ, U above, we can construct a $g \in G_\infty$ of the form (32) satisfying

$$g(\mathbf{p}_{s_{i1}}, \mathbf{p}_{s_{i2}}, \mathbf{p}_{s_{i3}}) = (\mathbf{q}_{s_{i1}} v_1, \mathbf{q}_{s_{i2}} v_2, \mathbf{q}_{s_{i3}} v_3). \qquad \square$$

Translating Example 4.1 from ball model to Siegel domain model, one has an instance of positive points:

$$\mathbf{p}_1 = (0, 1, 0)^T, \qquad \mathbf{p}_2 = (2\sqrt{2}, 1, 0)^T, \qquad \text{and} \qquad \mathbf{p}_3 = (3\sqrt{2}, 1, 0)^T.$$

Observe that $\chi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 3$ and $\chi(\mathbf{p}_3, \mathbf{p}_2, \mathbf{p}_1) = \frac{3}{2}$. Thus $(\mathbb{P}(\mathbf{p}_1), \mathbb{P}(\mathbf{p}_2), \mathbb{P}(\mathbf{p}_3))$ and $(\mathbb{P}(\mathbf{p}_3), \mathbb{P}(\mathbf{p}_2), \mathbb{P}(\mathbf{p}_1))$ are not $\mathrm{PSp}(2,1)$-congruent. For the case of more than three points, it is convenient to use the quaternionic cross-ratios.

PROPOSITION 5.2. *Let $\mathbf{p} = (z_1, \ldots, z_m)$ and $\mathbf{q} = (w_1, \ldots, w_m)$ be two ordered m-tuples of pairwise distinct points in $\overline{\mathbb{H}}$, $m \geq 4$. Then $\mathbf{p}$ and $\mathbf{q}$ are congruent with respect to the diagonal action of $\mathrm{PS}_\triangle L(2, \mathbb{H})$ if and only if there exists a $\lambda \in \mathbb{H} - \{0\}$ such that*

$$[z_j, z_3, z_2, z_1] = \lambda[w_j, w_3, w_2, w_1]\lambda^{-1}, \qquad 4 \leq \forall j \leq m. \tag{44}$$

PROOF. If there is an $f \in \mathrm{PS}_{\triangle}L(2, \mathbb{H})$ such that $f(z_j) = w_j$, $j = 1, \ldots, m$. Then by Lemma 5.3, the conditions of (44) hold.

Conversely, assume that

$$[z_j, z_3, z_2, z_1] = \lambda[w_j, w_3, w_2, w_1]\lambda^{-1}, \qquad 4 \le \forall j \le m.$$

By Lemma 5.2 we can find $f, g \in \mathrm{PS}_{\triangle}L(2, \mathbb{H})$ such that $f(z_3) = 1$, $f(z_2) = 0$, $f(z_1) = \infty$, and $g(w_3) = 1$, $g(w_2) = 0$, $g(w_1) = \infty$. It follows from Lemma 5.3 that

$$f(z_j) = [f(z_j), 1, 0, \infty] = [f(z_j), f(z_3), f(z_2), f(z_1)] = \lambda_f[z, z_3, z_2, z_1]\lambda_f^{-1}$$

and

$$g(w_j) = [g(w_j), 1, 0, \infty] = [g(w_j), g(w_3), g(w_2), g(w_1)] = \lambda_g[w, w_3, w_2, w_1]\lambda_g^{-1}.$$

Therefore our assumption implies that

$$h(z_j) = g^{-1} \circ \lambda_g(\lambda_f\lambda)^{-1}I_2 \circ f(z_i) = w_i, \qquad i = 1, \ldots, m.$$

Hence $\mathbf{p}$ and $\mathbf{q}$ are $\mathrm{PS}_{\triangle}L(2, \mathbb{H})$-congruent. □

DEFINITION 5.3. *For $S_i = \{s_{i1}, \ldots, s_{it_i}\}$, $i = 1, \ldots, k$ with $\mathrm{Card}(S_i) > 3$, we associate with S_i the following geometric invariants:*

$$\chi_{i0} = \chi(\mathbf{p}_{s_{il}}, \mathbf{p}_{s_{i2}}, \mathbf{p}_{s_{i3}}), \qquad \chi_{i1} = \chi(\mathbf{p}_{s_{il}}, \mathbf{p}_{s_{i2}}, \mathbf{p}_{s_{i4}}), \ldots, \chi_{i(t_i-3)} = \chi(\mathbf{p}_{s_{il}}, \mathbf{p}_{s_{i2}}, \mathbf{p}_{s_{it_i}})$$

and

$$X_i(\mathbf{p}) = (\chi_{i0}, \ldots, \chi_{i(t_i-3)}).$$

Let $X(\mathbf{p})$ be the vector whose components consisting of $X_i(\mathbf{p})$ above.

Taking $z_1 = w_1 = \infty$ in Proposition 5.2, we get the following proposition.

PROPOSITION 5.3. *Let $\mathbf{p}_{S_i}$ and $\mathbf{q}_{S_i}$ belong to $\mathbf{z}_\infty^\perp$ with the same Gram matrix whose entries are all equal to* 1. *Then $\mathbf{p}_{S_i}$ and $\mathbf{q}_{S_i}$ are $\mathrm{PSp}(n, 1)$-congruent if and only if there exists a $\lambda \in \mathbb{H} - \{0\}$ such that*

$$X_i(\mathbf{p}) = \lambda X_i(\mathbf{q})\lambda^{-1}.$$

We still need to generalize the above result to the case of $G(\mathbf{p})$ and $G(\mathbf{q})$ having stratum structures.

PROPOSITION 5.4. *Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)$ and $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m)$ be two m-tuples of pairwise distinct positive points of non regular case. We also assume that $\mathbf{p}$ and $\mathbf{q}$ have the same structure given by Proposition* 4.3 *with the property*

$\mathrm{Card}(S_i) \geq 3$ *for some i. Then* $\mathbf{p}$ *and* $\mathbf{q}$ *are* $\mathrm{PSp}(n,1)$*-congruent if and only if there exists a* $\lambda \in \mathbb{H} - \{0\}$ *such that*

$$X(\mathbf{p}) = \lambda X(\mathbf{q})\lambda^{-1}.$$

PROOF. Without loss of generality, we assume that $\mathbf{p}, \mathbf{q} \in \mathbf{z}_\infty^\perp$. If there is an $f \in \mathrm{PSp}(n,1)$ such that $f(\mathbf{p}_i) = \mathbf{q}_i$, $j = 1, \dots, m$. Then $f \in G_\infty$ is of the form

$$f = \begin{pmatrix} \lambda & \gamma^* & \star \\ 0 & U & \star \\ 0 & 0 & \star \end{pmatrix} \tag{45}$$

and $\mathbf{p}$, $\mathbf{q}$ must have the same structure given by Proposition 4.3. Here and in what follows, $\star$ stands for an arbitrary entry satisfying constraint that the corresponding matrix f belongs to $\mathrm{Sp}(n,1)$. By our normalization, we have

$$U(\mathbf{e}_1, \dots, \mathbf{e}_k) = (\mathbf{e}_1, \dots, \mathbf{e}_k) \tag{46}$$

and in each block of index S_i, we also have

$$\lambda k_{il} + \gamma^* \mathbf{e}_i = w_{il}, \qquad 1 \leq l \leq \mathrm{Card}(S_i). \tag{47}$$

Therefore we have $X(\mathbf{p}) = \lambda X(\mathbf{q})\lambda^{-1}$.

Conversely, suppose that $X(\mathbf{p}) = \lambda X(\mathbf{q})\lambda^{-1}$. By Proposition 5.3, for two specific blocks $\mathbf{p}_{S_i}$ and $\mathbf{q}_{S_i}$, we can construct an element $f_i \in \mathrm{Sp}(n,1)$ of the form

$$f_i = \begin{pmatrix} \lambda_i & \gamma_i^* & \star \\ 0 & U_i & \star \\ 0 & 0 & \star \end{pmatrix}$$

such that

$$U_i e_i = e_i, \qquad \lambda_i k_{il} + \gamma_i^* \mathbf{e}_i = w_{il}, \qquad 1 \leq l \leq \mathrm{Card}(S_i). \tag{48}$$

It is a pleasant surprise that we can adjust f_i to a suitable transformation which works for $\mathbf{p}$ wholly as follows. First, it follows from Lemma 5.3 that $\lambda_i = \lambda$. Let $U \in \mathrm{Sp}(n-1)$ having the property $U(\mathbf{e}_1, \dots, \mathbf{e}_k) = (\mathbf{e}_1, \dots, \mathbf{e}_k)$. It is obvious that

$$h_i = \begin{pmatrix} \lambda & \gamma_i^* & \star \\ 0 & U & \star \\ 0 & 0 & \star \end{pmatrix} \tag{49}$$

also maps $\mathbf{p}_{S_i}$ to $\mathbf{q}_{S_i}$. Note that $k \leq n-1$. Let

$$\gamma = (\gamma_1^* \mathbf{e}_1, \dots, \gamma_k^* \mathbf{e}_k, \star, \dots, \star)^T$$

and

$$h = \begin{pmatrix} \lambda & \gamma^* & \star \\ 0 & U & \star \\ 0 & 0 & \star \end{pmatrix}. \tag{50}$$

Then one has the equations (47), and therefore $\mathbf{p}$ and $\mathbf{q}$ are congruent up to h. $\square$

By the above proof and Subsection 4.2, we have the following result which means that structures of Gram matrices determine their congruent classes when $\mathrm{Card}(S_i) \leq 2$ for all i.

Proposition 5.5. *Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$ be two m-tuples of pairwise distinct positive points of non regular case with the same structure given by Proposition* 4.3 *and* $\mathrm{Card}(S_i) \leq 2$, $i = 1, \dots, k$. *Then $\mathbf{p}$ and $\mathbf{q}$ are* $\mathrm{PSp}(n,1)$*-congruent.*

In order to describe the parameter space, we need the following result.

Proposition 5.6. *The coordinates of $\mathbf{O}_{X(\mathbf{p})}$ given by the rotation-normalized algorithm is well defined.*

Proof. If both $h_1, h_2 \in \mathrm{PSp}(n,1)$ map V to a subspace of $\mathbf{z}_\infty^\perp$, then the coordinates in (37) may be different from each other, which implies that $X(\mathbf{p})$ in Definition 5.3 is dependent on the map ϕ in (37). However, since $h_1^{-1}h_2 \in G_\infty$, Lemma 5.3 and Proposition 5.4 imply that the coordinates of $\mathbf{O}_{X(\mathbf{p})}$ given by the rotation-normalized algorithm is well defined. $\square$

Summarizing the previous results, we obtain the main result of this section.

Theorem 5.1. *Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ be an m-tuple of pairwise distinct positive points given by Proposition* 4.3. *Then the* $\mathrm{PSp}(n,1)$*-congruence class of $\mathbf{p}$ is determined uniquely by the partition structure of $S(m) = \bigcup_{i=1}^k S_i$ and the coordinates of $\mathbf{O}_{X(\mathbf{p})}$ given by the rotation-normalized algorithm.*

Therefore the moduli space can be described as follows.

Theorem 5.2. *The moduli space of $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ given by Proposition* 4.3 *can be identified with the set $\mathbb{M}_1 \times \cdots \times \mathbb{M}_k$, where $\mathbb{M}_1 \times \cdots \times \mathbb{M}_k$ are the coordinates of $\mathbf{O}_{X(\mathbf{p})}$ given by the rotation-normalized algorithm.*

6. The moduli problem on $\mathbb{P}(V_+)$ of regular case

In this section, we describe the moduli space of configurations of quaternionic $(n-1)$-dimensional submanifolds when V is not parabolic in conceptual style. The basic idea is to find a partition of $S(m) = \{1, \dots, m\}$ to perform the rotation-normalized algorithm in each block.

We begin with 1-normalized matrix of $\mathbf{p}$. Proposition 4.4 roughly shows that we can treat the mutually orthogonal blocks $\mathbf{p}_{S_i} = (\mathbf{p}_{s_{i1}}, \dots, \mathbf{p}_{s_{it_i}})$, $i = 1, \dots, s$ separately. Equivalently, we can perform the rotation-normalized algorithm separately. This is the structure of Gram matrix at top level. For each block $\mathbf{p}_{S_i}$, there may still exist 0s in $G(\mathbf{p}_{S_i})$. We may need to partition S_i into more small blocks to perform the rotation-normalized algorithm. We call such a partition process, together with similar 1-normalized process in each small blocks, *the block-normalized algorithm*. The output of the block-normalized algorithm is a special kind of Gram matrix, which is still not unique and can be viewed as an equivalent class. We still need to apply the rotation-normalized algorithm to get the parameters.

We describe the block-normalized algorithm conceptually as follows.

The block-normalized algorithm:

Step 1: Let O_{il} be the number of entries being zero in ilth row of $G(\mathbf{p}_{S_i})$ and record the set of columns of these entries being nonzero as P_{il}. Let $n_i = \min\{O_{i1}, \dots, O_{it_i}\}$ and K_i the set of indices il such that $O_{il} = n_i$. Let c_{i1} be the smallest integer in K_i and denote the corresponding P_{il} of c_{i1} as S_{i1}. In other words, c_{i1} is the smallest index in $S_i = \{s_{i1}, \dots, s_{it_i}\}$ such that the cardinality of nonzero entries in the c_{i1}th row of $G(S_i)$ is the largest among those of the others; the set of columns of nonzero entries is recorded as S_{i1}. It is obvious that $c_{i1} \in S_{i1}$.

Step 2: Repeating the process in *Step* 1 for the remainder of $S_i - S_{i1}$, we obtain c_{i2} and S_{i2}. It is obvious that we can continue this process only finite steps. We denote by τ_i the number of steps and record the corresponding numbers in each step as c_{ij} and S_{ij} for $1 \le j \le \tau_i$. Then we have

$$S_i = \bigcup_{j=1}^{\tau_i} S_{ij}.$$

Step 3: In each subindex set S_{ij}, we perform the c_{ij}-normalized process to $G(\mathbf{p}_{S_{ij}})$. We denote such result of the sub Gram matrix as $G_b(\mathbf{p}_{S_{ij}})$. In other words, the entries of $G_b(\mathbf{p}_{S_{ij}})$ have the following properties:

$$g_{tt} = 1, \qquad g_{c_{ij}t} \ge 0, \qquad g_{tc_{ij}} \ge 0, \qquad t \in S_{ij}.$$

As in (22) of Section 4, we record the corresponding normalized sub-diagonal matrix as D_{ij}. That is

$$G_b(\mathbf{p}_{S_{ij}}) = G(\mathbf{p}_{S_{ij}} D_{ij}).$$

Step 4: Let

$$D_i = \mathrm{diag}(D_{i1}, \dots, D_{i\tau_i}), \qquad D_b = \mathrm{diag}(D_1, \dots, D_s). \tag{51}$$

We define

$$G_b(\mathbf{p}_{S_i}) = G(\mathbf{p}_{S_i} D_i) \tag{52}$$

and

$$G_b(\mathbf{p}) = G(\mathbf{p} D_b). \tag{53}$$

Definition 6.1. *The Gram matrix $G_b(\mathbf{p})$ obtained by the above block-normalized algorithm is called the block-normalized matrix of $G(\mathbf{p})$.*

We mention that our strategy in the block-normalized algorithm is from parts to entirety. We deal with the diagonal blocks separately. In this scale $\mathbf{p}_{S_i}$ and $\mathbf{p}_{S_j}$ are totally independent. In each block $\mathbf{p}_{S_i}$, all processes are explicitly recorded by the corresponding sub-diagonal matrices $D_i = \mathrm{diag}(D_{i1}, \dots, D_{i\tau_i})$. In this way the entries $\langle \mathbf{p}_{s_{il}}, \mathbf{p}_{s_{id}} \rangle$ in off-diagonal blocks of $\mathbf{p}_{S_i}$ are all determined definitely by D_i. We describe the structure of $G_b(\mathbf{p})$ in the following proposition in more details.

Proposition 6.1. *The block-normalized Gram matrix $G_b(\mathbf{p})$ has the following characteristics.*

(1) *If we view the block-normalized Gram matrix $G_b(\mathbf{p})$ in its permuted position with index S_i, then $G_b(\mathbf{p})$ consists of blocks submatrix $G_b(\mathbf{p}_{S_i})$, and the entries of the corresponding off-diagonal blocks matrices are zero (see* (30) *in Proposition* 4.4*).*

(2) *In the c_{i1}th row (and column) of submatrix $G_b(\mathbf{p}_{S_i})$, the first* $\mathrm{Card}(S_{i1})$ *entries are nonzero real numbers, and the others are zeros (see Step* 2 *of the block-normalized algorithm).*

(3) *In the c_{i2}th row (and column) of submatrix $G_b(\mathbf{p}_{S_i})$, the entries with index between* $\mathrm{Card}(S_{i1}) + 1$ *and* $\mathrm{Card}(S_{i1}) + \mathrm{Card}(S_{i2})$ *are nonzero real numbers, the entries with index bigger than* $\mathrm{Card}(S_{i1}) + \mathrm{Card}(S_{i2})$ *are zeros; the entries in the c_{ij}th row (and column) of submatrix $G_b(\mathbf{p}_{S_i})$ can be described similarly when $j = 3, \dots, \tau_i$.*

(4) *$G_b(\mathbf{p}_{S_i})$ can not be block diagonal according to our partition in Proposition* 4.4.

Similarly to Lemma 3.1, we have the following result.

Lemma 6.1. *Suppose that $G(\mathbf{q})$ is a block-normalized Gram matrix $G_b(\mathbf{p})$ for $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$. Then $G(\mathbf{q} D_r)$ is still a block-normalized Gram matrix with*

$$D_r = \mathrm{diag}(\mu_1, \dots, \mu_m)$$

if only if every μ_t *with* $t \in S_{ij}$ *is the same quaternion of modulus* 1*, i.e.,*

$$\mu_t = \mu_{ij}, \qquad \forall t \in S_{ij}, \text{ where } \mu_{ij} \in \mathrm{Sp}(1).$$

Summarizing the previous treatments, we have the following procedure.

THEOREM 6.1. *Let* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ *be an m-tuple of pairwise distinct positive point given by Proposition* 4.4. *We can assign the* $\mathrm{PSp}(n,1)$*-congruence class of* $\mathbf{p}$ *a coordinate as follows.*

(1) *Obtain a block-normalized matrix* $G(\mathbf{p}D_b)$ *by performing the block-normalized algorithm, where* D_b *is given by* (51).
(2) *Perform the rotation-normalized algorithm to each block* S_{ij} *(as the case of m-tuple of* $\mathbb{P}(V_0)$ *in Section* 3*). This is equivalent to choosing a specific* $\mu_{ij} \in \mathrm{Sp}(1)$. *Combine them to the corresponding whole rotation normalized diagonal matrix* D_r.
(3) *The independent entries of*

$$G(\mathbf{p}D_bD_r),$$

that is, all the entries above the diagonal entries, are the desired coordinate of the $\mathrm{PSp}(n,1)$*-congruent class of* $\mathbf{p}$.

We now are ready to give a conceptual description of the parameter space $\mathbb{M}(n,m)$ in regular case. We mimic conceptually the method used in Section 3.2 as follows.

The procedure of constructing parameter space:

For a partition $\mathscr{S}$ of $S(m) = \{1, \dots, m\}$ as

$$S_i = \{s_{i1}, \dots, s_{it_i}\}, \qquad s_{i1} < \cdots < s_{it_i}, \qquad i = 1, \dots, s$$

with sub partitions

$$S_i = \bigcup_{j=1}^{\tau_i} S_{ij}.$$

Let $\mathrm{Card}(S_{ij}) = \sigma_{ij}$. As in Section 3.2, we construct the parameter space $\mathbb{M}(n,\sigma_{ij})$ of S_{ij}. Let

$$\mathbb{M}(n,i) = \mathbb{M}(n,\sigma_{i1}) \times \cdots \mathbb{M}(n,\sigma_{i\tau_i}) \times \mathbb{C}_i,$$

where the set of $\mathbb{C}_i$ is the corresponding space of the off-diagonal sub-blocks. Let

$$\mathbb{M}(n,m,\mathscr{S}) = \mathbb{M}(n,i) \times \cdots \mathbb{M}(n,s).$$

The Hermitian matrix constructed from the entries of the parameter space $\mathbb{M}(n,m,\mathscr{S})$ should subject to analogous constraints as those of Theorem 4.1. Then the union of the parameter spaces determined by all possible partitions

$$\mathbb{M}(n,m) = \bigcup_{\mathscr{S}} \mathbb{M}(n,m,\mathscr{S}) \tag{54}$$

is a parameter space of the configuration space $\mathscr{M}(n,m;\mathbb{P}(V_+))$ when V is not parabolic.

Therefore, the moduli space can be described as follows.

THEOREM 6.2. *The moduli space of* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)$ *given by Proposition* 4.4 *can be identified with the set*

$$\mathbb{M}(n,m) = \bigcup_{\mathscr{S}} \mathbb{M}(n,m,\mathscr{S}). \tag{55}$$

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