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The third term in lens surgery polynomials

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Abstract. It is well-known that the second highest coefficient of the Alexander polynomial of any lens space knot in S^3 is -1. We show that if the third highest coefficient of the Alexander polynomial $\Delta_K(t)$ of a lens space knot K in S^3 is non-zero, then $\Delta_K(t)$ coincides with the Alexander polynomial of the $(2, 2g+1)$-torus knot, where g is the Seifert genus of K.

1. Introduction

1.1. Lens space knots. If a knot K in a homology sphere Y yields a lens space by an integral Dehn surgery, then we call K a *lens space knot* in Y. The result obtained by the Dehn surgery of a knot K in Y with slope p is written by $Y_p(K)$. Hence, the lens space surgery is presented as $Y_p(K) = L(p, q)$ for integers p and q. The homology class represented by the dual knot of the surgery is identified with an element k in $(\mathbb{Z}/p\mathbb{Z})^\times$. Precisely, it will be explained in Section 2. The pair (p, k) is called a *lens surgery parameter*.

We call a polynomial $\Delta(t)$ a *lens surgery polynomial (in Y)* if there exists a lens space knot K in Y such that $\Delta(t) = \Delta_K(t)$, the Alexander polynomial of K. Lens surgery polynomials have the following interesting properties.

In [6], Ozsváth and Szabó proved that any lens surgery polynomials in S^3 are flat and alternating. Here, a polynomial is said to be *flat*, if the absolute values of all coefficients of the polynomial are smaller than or equal to 1. Also, a polynomial is said to be *alternating*, if the non-zero coefficients of the polynomial are alternating sign in order. We say that a polynomial $\Delta(t)$ is *trivial*, if $\Delta(t) = 1$.

Any lens space knot with trivial Alexander polynomial in S^3 is isotopic to the unknot due to [5]. Generally, if K is a lens space knot, then the degree of the Alexander polynomial coincides with the Seifert genus $g(K)$.

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In this paper we use the following notation for coefficients of a lens surgery polynomial:

$$\Delta(t) = t^{-g} \sum_{i=0}^{2g} \alpha_i t^i = \sum_{i=-g}^{g} a_i t^i.$$

Note that $\alpha_i = a_{i-g}$. By the symmetry of the Alexander polynomial we obtain $a_i = a_{-i}$ and $\alpha_i = \alpha_{2g-i}$.

We consider non-trivial lens surgery polynomials from now. Then, due to the author [8] and Hedden and Watson [4], any lens surgery polynomial in S^3 has the following form around the top term t^g:

$$\Delta(t) = t^g - t^{g-1} + \cdots.$$

In [4], it is shown that the Alexander polynomial of any L-space knot has the same form. Namely, the second highest coefficient from the top is -1.

In [8], it is proven that the second highest coefficient of any lens space knot in any L-space homology sphere is -1. From now, we use the word 'the second term' or 'the third term' as the meanings of 'the second highest term' or 'the third highest term' respectively.

1.2. The third term in lens surgery polynomial. Since the second term of the Alexander polynomial of a lens space knot is always non-zero, that term cannot be used to investigate some sort of characteristics of lens space knots. However, Teragaito asked if the third terms might capture some characteristics of lens space knots.

1.2.1. *Main question and theorem.*

QUESTION 1 (Teragaito). *If a non-trivial lens surgery polynomial in S^3 has the following form:*

$$\Delta(t) = t^g - t^{g-1} + t^{g-2} + \cdots,$$

then does $\Delta(t)$ coincide with $\Delta_{T(2,2g+1)}(t)$?

In other words, if a lens surgery polynomial is not the Alexander polynomial of the $(2, 2g+1)$-torus knot for any integer g, then does $\alpha_2 = 0$ hold?

Here $T(p,q)$ is the right-handed (p,q)-torus knot. Our main aim in this paper is to answer this question affirmatively.

THEOREM 1. *Question 1 is true.*

This theorem holds even if the lens space knot K lies in an L-space homology sphere and satisfies $2g(K) \leq p$ because the condition is exactly the same as the case of S^3. Very recently, in [1], J. Caudell gave an alternative proof for Theorem 1 by classifying some changemaker lattices.

In [8, Theorem 1.15] gave a criterion for a lens space knot K to satisfy $\Delta_K(t) = \Delta_{T(2,2g+1)}(t)$ for some positive integer g. We can also say that Theorem 1 gives a new criterion for a lens space knot to have the same Alexander polynomial as that of $T(2, 2g+1)$.

1.2.2. *Realization of lens surgery.* We fix the following terminology.

Definition 1. Let p, k be relatively prime positive integers. If a lens surgery $Y_p(K) = L(p, q)$ on a homology sphere Y has the lens surgery parameter (p, k), then we say that the parameter (p, k) *is realized by a lens space knot K in Y*.

Corollary 1. *Let K be a lens space knot in S^3 with the surgery parameter (p, k). The Alexander polynomial $\Delta_K(t)$ has the following form around the top term t^g:*

$$\Delta_K(t) = t^g - t^{g-1} + t^{g-2} + \cdots,$$

if and only if (p, k) is realized by $T(2, 2g+1)$.

The condition in this corollary is also equivalent to the condition of $k = 2$. We ask the next question:

Question 2. *Let K be a non-trivial lens space knot with $\alpha_2 = 0$. If $\alpha_3 \neq 0$, then what kind of knots are the lens space surgery realized by?*

The typical cases are $T(3, 3n \pm 1)$ and the pretzel knot $Pr(-2, 3, 7)$:

$$\Delta_{T(3,3n\pm1)}(t) = t^g(1 - t^{-1} + t^{-3} + \cdots), \qquad (g = 3n - 1 \pm 1)$$
$$\Delta_{Pr(-2,3,7)}(t) = t^5(1 - t^{-1} + t^{-3} - t^{-4} + t^{-5} - t^{-6} + t^{-7} - t^{-9} + t^{-10}).$$

1.3. The cases of lens space knots $K_{p,k}$ in $Y_{p,k}$. Consider a simple $(1,1)$-knot in a lens space yielding a homology sphere by some integral Dehn surgery. The 'simple' is defined in [7] and [10]. If such a simple $(1,1)$-knot generates the 1st homology group of the lens space, we can always find such a slope. Hence any simple $(1,1)$-knot is parameterized by relatively prime integers (p, k). The dual knot is a lens space knot in the homology sphere. The dual knot is denoted by $K_{p,k}$ and the homology sphere by $Y_{p,k}$. The readers should probably understand these facts by reading [7] and [10]. The main result in [3] gave a formula of the Alexander polynomial of $K_{p,k}$ by using p, k. Here we give the following conjecture:

CONJECTURE 1. *If the third term of the symmetrized Alexander polynomial* $\Delta_{K_{p,k}}(t)$ *is non-zero, then* $\Delta_{K_{p,k}}(t)$ *coincides with* $\Delta_{T(2,2g+1)}(t)$ *for some integer* g, *in other words,* $k = 2$ *holds.*

This conjecture can be easily checked by a computer program based on the formula in [3]. The author checked the conjecture up to $p \leq 600$ with a computer aid. Conjecture 1 is true under a little strong condition that $Y_{p,k}$ is homeomorphic to S^3, because of Theorem 1. The essential part is whether $Y_{p,k}$ is homeomorphic to S^3, if the third term of $\Delta_{K_{p,k}}$ is non-zero. Notice that in [8] the author proved that $k = 2$ holds if and only if $Y_{p,k}$ is homeomorphic to S^3 and $K_{p,k}$ is isotopic to $T(2, 2g+1)$ for some integer g. This condition is also equivalent to the equality $\Delta_{K_{p,k}}(t) = \Delta_{T(2,2g+1)}(t)$.

2. Preliminaries and Proofs

2.1. Brief preliminaries. Here we define the lens surgery parameter (p, k).

DEFINITION 2. Let K be a knot in a homology sphere Y. Suppose that $Y_p(K) = L(p, q)$ and the dual knot $\tilde{K}$ has $[\tilde{K}] = k[C] \in H_1(L(p, q), \mathbb{Z})$ for some orientation of $\tilde{K}$. Here the dual knot is the core knot in the solid torus attached by the Dehn surgery. Furthermore, C is either of the core circles of a genus one Heegaard decomposition of $L(p, q)$. Then we call (p, k) a *lens surgery parameter* for $Y_p(K) = L(p, q)$. The integer k is called a *dual class.*

If $L(p, q)$ is obtained by a Dehn surgery of a homology sphere with slope p, the surgery parameter (p, k) is relatively prime and $q = \pm k^2 \bmod p$. These facts are due to [8]. Note that we adopt the orientation of $L(p, q)$ as the one of the p/q-surgery of the unknot in S^3.

The ambiguity of the orientation of $\tilde{K}$ and the choice of the core circles of a genus one Heegaard decomposition give (at most) four possibilities of the dual class k_0, $-k_0$, k_0^{-1}, $-k_0^{-1}$ (in $\mathbb{Z}/p\mathbb{Z}$), for some integer k_0. We always take the minimal integer k as a representative satisfying $0 < k < p/2$.

For any integer i we define the integer $[i]_p$ to be the integer with $i \equiv [i]_p \bmod p$ and $-\frac{p}{2} < [i]_p \leq \frac{p}{2}$. Let k_2 be the absolute value of the integer $[k']_p$, where k' is an integer satisfying $kk' \equiv 1 \bmod p$. We call k_2 *the second dual class* of the surgery parameter. We set $kk_2 \equiv e \bmod p$, $e = \pm 1$, $m = \frac{kk_2 - e}{p}$, $q = [k^2]_p$, $q_2 = [(k_2)^2]_p$, $c = \frac{(k-1)(k+1-p)}{2}$ and for any non-zero integer ℓ

$$I_\ell := \begin{cases} \{1, 2, \dots, \ell\} & \ell > 0 \\ \{\ell + 1, \dots, -1, 0\} & \ell < 0. \end{cases}$$

From these data, we can compute the i-th coefficient a_i of the Alexander polynomial of K due to [9]. Here we use a modified form introduced in [8].

Proposition 1 (Proposition 2.3 in [8]). *Let K be a lens space knot in S^3. For any integer i with $|i| \leq p/2$, the i-th coefficient of the Alexander polynomial is computed as follows*

$$a_i = -e(m - \#\{j \in I_k \mid [q_2(j + ki + c)]_p \in I_{ek_2}\}).$$

To prove Theorem 1, we use the *non-zero curve* defined in [8]. Here let us define a non-zero curve for a lens space knot K in S^3. First, we will consider coefficients $\bar{a}_i$ obtained by extending a_i to any integer i. For any integer i with $-\frac{p}{2} < i \leq \frac{p}{2}$

$$\bar{a}_i = \begin{cases} a_i & |i| \leq g(K) \\ 0 & \text{otherwise.} \end{cases}$$

Here $g(K)$ is the Seifert genus of K. For a general integer i, we define $\bar{a}_i$ to be $\bar{a}_i = \bar{a}_{[i]_p}$. In other words, the coefficients $\bar{a}_i$ are obtained by extending coefficients a_i with period p by considering the estimates in [2] and [5].

We define an *A-matrix* and a *dA-matrix* as follows:

$$A_{i,j} := \bar{a}_{k_2(i+jek-c)}, \qquad dA_{i,j} := A_{i,j} - A_{i-1,j},$$

where $c := \frac{(k-1)(k+1-p)}{2}$. Due to the formula (9) in [8, Lemma 2.6], we have

$$dA_{i,j} = E_{ek_2}(q_2 i + k_2(j+e)) - E_{ek_2}(q_2 i + k_2 j) = \begin{cases} 1 & [q_2 i + k_2 j]_p \in I_{-k_2} \\ -1 & [q_2 i + k_2 j]_p \in I_{k_2} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

$$E_y(x) := \begin{cases} \operatorname{sgn}(y) & [x]_p \in I_y \\ 0 & \text{otherwise.} \end{cases}$$

We put $A_{i,j}$ on each lattice point (i,j) in $\mathbb{Z}^2 \subset \mathbb{R}^2$. For a non-zero coefficient $A_{i,j}$ we draw a horizontal positive or negative arrow on (i,j) according to $A_{i,j} = 1$ or -1 respectively, where a positive (resp. negative) arrow means a horizontal arrow with positive (resp. negative) in the i-direction. After that, we connect the horizontally adjacent arrows with the same orientation and compatibly connect arrows around the non-zero $dA_{i,j}$ as in [8]. Then we can obtain an infinite family of simple curves on $\mathbb{R}^2$ with no finite ends (i.e., they are properly embedded curves in $\mathbb{R}^2$). The arrows are non-increasing with respect to the j-coordinate. We call the curves *non-zero curves*.

Proposition 2 ([8]). *Any non-zero curve for any lens space knot in S^3 is included in a non-zero region $\mathcal{N}$. In each non-zero region there is a single component non-zero curve.*

Here a non-zero region $\mathcal{N}$ (introduced in [8]) is defined as follows. First, we consider the union of $2g+1$ box-shaped neighborhoods of vertical sequent lattice points corresponding to $a_g, a_{g-1}, \dots, a_{-g}$. Two adjacent box neighborhoods are overlapped with a horizontal unit segment. Next, we take the infinite parallel copies moved by $n \cdot \mathbf{v}$ where $\mathbf{v}$ is the vector $(1, -k_2)$ and n is any integer. We denote the union of the infinite parallel copies by $\mathcal{N}$ and call it a *non-zero region*. Moving a non-zero region $\mathcal{N}$ by $n \cdot (0, p)$ for any integer n, we obtain infinite non-zero regions on $\mathbb{R}^2$.

The following lemma is essential to prove the main theorem. This is also the case of $m = 0$ in [8, Lemma 4.4].

Lemma 1. *If there exist integers i_0, j_0 such that $dA_{i_0, j_0} = -dA_{i_0, j_0+1} = -1$, then for any integer i, there are no two adjacent zeros in the sequence $\{dA_{i,s} \mid s \in \mathbb{Z}\}$.*

Proof. We assume the existence of integers i_0, j_0 satisfying $dA_{i_0, j_0} = -dA_{i_0, j_0+1} = -1$. Let x_j be $i_0 + (j_0 + j)ek$. Using the formula (1), we have

$$[q_2 x_{-1}]_p \in I_{-k_2},$$
$$[q_2 x_0] = [q_2 x_{-1} + k_2]_p \in I_{k_2},$$

and

$$[q_2 x_1] = [q_2 x_{-1} + 2k_2]_p \in I_{-k_2}.$$

Here we use $q_2 ek = k_2 \bmod p$. Hence, the sequence $[q_2 x_{s-1}]_p$ $(s = 0, 1, 2, \dots)$ starts at $[q_2 x_{-1}]_p$ and first returns in I_{-k_2} at $s = 2$. Therefore, we have $p - k_2 < (q_2 x + 2k_2) - q_2 x < p + k_2$ and this means $p < 3k_2$.

We suppose $dA_{n,j} = dA_{n,j+1} = 0$ for some integers n and j. Then $[q_2(n + jek)]_p, [q_2(n + jek) + k_2]_p \notin I_{-k_2} \cup I_{k_2}$. This implies $p - k_2 - k_2 \geq k_2$. This contradicts the inequality above. Hence, if for an integer n, the sequence $\{dA_{n,s} \mid s \in \mathbb{Z}\}$ has no adjacent zeros, for any integer i the same thing holds because the $\{dA_{i,s} \mid s \in \mathbb{Z}\}$ is a parallel copies of $\{dA_{n,s} \mid s \in \mathbb{Z}\}$. Hence, the desired conclusion is satisfied. □

Note that this lemma holds for any relatively prime positive integers (p, k). Actually, to prove this lemma we do not require that the matrices A and dA come from a lens space knot surgery in S^3. In particular, if any $K_{p,k}$ in $Y_{p,k}$ (defined in Section 1.3) has a non-zero third term in the Alexander polynomial, then $p < 3k_2$ holds. To prove Conjecture 1, first we should probably classify $(Y_{p,k}, K_{p,k})$ in the case of $p < 3k_2$.

2.2. Proof of Theorem 1. We now porve Theorem 1.

Proof. Let K be a lens space knot with lens surgery parameter (p, k) and with $g = g(K)$. Suppose that $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_2 = 1$. Now we assume that $2g - k_2 \geq 3$. Since any non-zero curve has no finite ends, $\alpha_3 = -1$ holds naturally. Hence, we can assume that

$$\begin{cases} \alpha_0 = 1 \\ \alpha_1 = -1 \\ \alpha_2 = 1 \\ \alpha_3 = -1. \end{cases} \tag{$*$}$$

Let i, j be fixed integers with $k_2(i + jek - c) = -g \bmod p$. Then $A_{i,j} = \alpha_0 = 1$. $A_{i-1,j} = A_{i-1,j+1} = A_{i-1,j+2} = A_{i-1,j+3} = 0$, because any non-zero curve is included in a non-zero region $\mathcal{N}$ due to Proposition 2.

We notice that the assumption of Lemma 1 is satisfied. Thus, we have $dA_{i,j} = 1$, $dA_{i,j+1} = -1$, $dA_{i,j+2} = 1$, and $dA_{i,j+3} = -1$. The local values for matrices A and dA are drawn in the top pictures in Figure 1.

Our situation falls into the following two cases:

(I) $A_{i+1,j+1} = -1$, and $A_{i+1,j+2} = 1$;

(II) $A_{i+1,j+1} = 0$, and $A_{i+1,j+2} = 0$.

(I) and (II) correspond to the bottom two pictures of Figure 1. In the case of (I), we obtain $dA_{i+1,j+1} = dA_{i+1,j+2} = 0$. This contradicts Lemma 1.

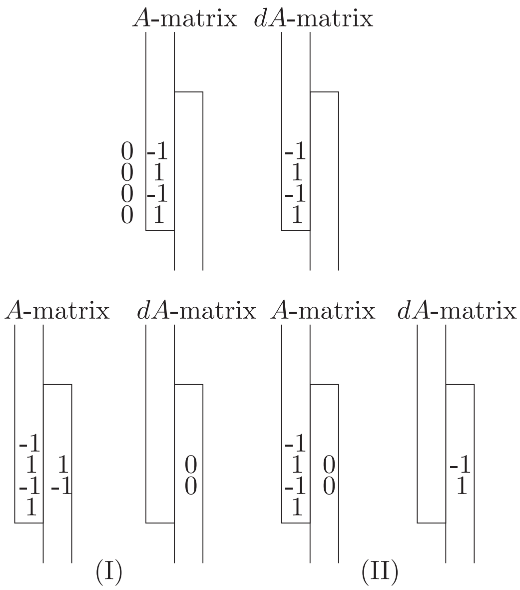

Fig. 1. The top picture: parts of A and dA matrices satisfying $(*)$; the bottom pictures: parts of A and dA matrices of cases (I) and (II).

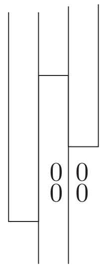

Fig. 2. An A-matrix of (II).

Next, consider the case of (II). We claim $A_{i+2,j+1} = A_{i+2,j+2} = 0$. The picture of the A-matrix is as in Figure 2. If $A_{i+2,j+1}$ or $A_{i+2,j+2}$ is non-zero, then the non-zero term is included in the non-zero region right next to $\mathscr{N}$, because there is only one non-zero curve in any non-zero region (Proposition 2). This implies that by seeing the j-coordinate in $\mathbb{R}^2$, we have

$$p - 2k_2 \leq 2.$$

Since $2k_2 < p$, we have $p = 2k_2 + 1$ or $2k_2 + 2$. The equality $p = 2k_2 + 1$ means it gives an integral lens space surgery on $T(2, 2g + 1)$. We consider the case of $p = 2k_2 + 2$. Since p, k_2 are relatively prime, k_2 is an odd number. The equality $p = 2k_2 + 2$ can give $k_2^2 - 1 = \frac{k_2 - 1}{2} p \equiv 0 \bmod p$. It is a lens space surgery yielding $L(p, 1)$. Due to [5], this case is the one of $k_2 = 1$. Thus, the claim above is true.

According to [8, Proposition 4.2], there are no lens space surgeries satisfying $2 \leq -2g + k_2$, hence, the remaining cases satisfy $-2 \leq -2g + k_2 \leq 1$. By using [8, Theorems 1.15 and 4.20], each of these cases is realized by an integral lens space surgery on $T(2, 2g + 1)$, $T(3, 4)$ or $Pr(-2, 3, 7)$. The knots $T(3, 4)$ and $Pr(-2, 3, 7)$ do not satisfy $(*)$. Thus, the remained cases are the ones of $\Delta_K(t) = \Delta_{T(2,2g+1)}(t)$. □

2.3. Proof of Corollary 1. We prove Corollary 1.

Proof. Let K be a lens space knot in S^3 with parameter (p, k). If $\Delta_K(t)$ has $\alpha_0 = -\alpha_1 = \alpha_2 = 1$, then $\Delta_K(t) = \Delta_{T(2,2g+1)}(t)$ holds by Theorem 1 for some positive integer g. Using [8, Theorem 1.15], this parameter is realized by the $(2, 2g + 1)$-torus knot. □

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References

[1] J. Caudell, A note on changemaker lattices and Alexander polynomials of lens space knots, arXiv:2007.11039.

[2] J. Greene, The lens space realization problem, Ann. of Math. (2) 177 (2013), no. 2, 449–511.

[3] K. Ichihara, T. Saito and M. Teragaito, Alexander polynomials of doubly primitive knots, Proc. Amer. Math. Soc. 135 (2007), 605–615.

[4] M. Hedden and T. Watson, On the geography and botany of knot Floer homology, Selecta Mathematica April 2018, Volume 24, Issue 2, pp. 997–1037.

[5] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó, Monopoles and lens space surgeries, Ann. of Math. (2) 165 (2007), no. 2, 457–546.

[6] P. Ozsváth and Z. Szabó, On knot Floer homology and lens space surgeries, Topology 44 (2005), no. 6, 1281–1300.

[7] T. Saito, Knots in lens spaces with the 3-sphere surgery, Algebr. Geom. Topol. 8 (2008), no. 1, 53–79.

[8] M. Tange, On the Alexander polynomial of lens space knots, Topology Appl. 275 (2020), 107124, 37 pp.

[9] M. Tange, Ozsváth-Szabó's correction term and lens surgery, Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 1, 119–134.

[10] M. Tange, Homology spheres yielding lens spaces, Proceedings of the Gökova Geometry-Topology Conference 2017, 73–121.

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