Hiroshima Math. J., **52** (2022), 139–152
doi:10.32917/h2016091

# Remarks on the signatures of invariant pseudo-Kähler metrics on generalized flag manifolds

Takumi Yamada

(Received November 18, 2016)
(Revised December 3, 2021)

**Abstract.** A pseudo-Kähler manifold is a natural generalization of a Kähler manifold. It is well-known that any generalized flag manifold has pseudo-Kähler metrics. Moreover, there exists a $T$-root system corresponding to a generalized flag manifold. In this paper, we investigate the signatures of invariant pseudo-Kähler metrics on a generalized flag manifold of which the $T$-root system becomes one of the irreducible reduced root systems (in general, a $T$-root system is not an irreducible reduced root system).

## 1. Introduction

A pseudo-Riemannian manifold $(M, g)$ with a complex structure $J$ is called a *pseudo-Kähler manifold* if $g$ is a pseudo-Hermitian metric and the fundamental 2-form $\omega_g$ is closed. Thus, a pseudo-Kähler manifold is a natural generalization of a Kähler manifold. We call such a pseudo-Riemannian metirc $g$ a *pseudo-Kähler metric*, and the fundamental 2-form $\omega_g$ the *pseudo-Kähler structure*.

Dorfmeister–Guan [6] proved that a compact homogeneous pseudo-Kähler manifold is biholomorphic to the product of a generalized flag manifold and a complex torus as in the Kähler case ([4, 8]). Alekseevsky–Perelomov [1] proved that any generalized flag manifold admits an invariant Kähler-Einstein metric by using $T$-roots (see [1, 2] for the details of $T$-roots).

In this paper, we showed signatures of invariant pseudo-Kähler metrics on generalized flag manifolds from a viewpoint of $T$-root systems. Let $G^{\mathbb{C}}/U$ be a generalized flag manifold. In the previous paper [9], we showed that if the $T$-root system corresponding to $G^{\mathbb{C}}/U$ is of type $A_l$, $B_2$, or $G_2$, then the signatures of the invariant pseudo-Kähler metrics on $G^{\mathbb{C}}/U$ can be indeed computed.

For the case of $G^{\mathbb{C}}/U \cong SU(6)/S(U(1) \times U(2) \times U(3))$, we also showed a relation between the patterns of the signatures of invariant pseudo-Kähler

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The author is supported by JSPS KAKENHI Grant number JP16K05131.
2020 *Mathematics Subject Classification*. Primary 53C30; Secondary 53C50.
*Key words and phrases*. pseudo-Kähler metric, flag manifolds, T-root system.

metrics and the moduli space of the invariant complex structures on $G^{\mathbb{C}}/U$ (see [9, Example 3.3, Remark 3.4] for the details). In this paper, we are interested in the general cases where the $T$-root systems associated to generalized flag manifolds satisfy the root systems axioms. The main result is the following:

Main Theorem. *Let $G^{\mathbb{C}}/U$ be a generalized flag manifold. Suppose that the $T$-root system associated to $G^{\mathbb{C}}/U$ satisfies the root system axioms. Then we give an algorithm to compute the signatures of invariant pseudo-Kähler metrics on $G^{\mathbb{C}}/U$ in terms of the root reflections (see Section 3).*

## 2. Preliminaries

Let $G$ be a connected compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of $G$, and $\mathfrak{h}$ a maximal abelian subalgebra. We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ the complexifiactions of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. We identify an element of the root system $R$ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ with the corresponding element of $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ via the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of the root system $R$, and $\{\Lambda_1, \dots, \Lambda_l\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\Pi$. We denote by $R^+$ the set of all positive roots relative to $\Pi$. Let $\Pi_0$ be a subset of $\Pi$ and we set $\Pi - \Pi_0 = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, where $1 \le i_1 < \cdots < i_r \le l$. We set $[\Pi_0] = R \cap \{\Pi_0\}_{\mathbb{Z}}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the subspace of $\mathfrak{h}_0$ generated by $\Pi_0$. We set $\mathfrak{t} = \{H \in \mathfrak{h}_0 \mid (H, \Pi_0) = 0\}$. Then $\{\Lambda_{i_1}, \dots, \Lambda_{i_r}\}$ is a basis of $\mathfrak{t}$. We consider the restriction map

$$\kappa : \mathfrak{h}_0 \to \mathfrak{t}^* \qquad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

and set $R_T = \kappa(R)$. Each element of $R_T$ is called a *$T$-root*. The collection of hyperplanes $\{\kappa(\alpha) = 0\}$ corresponding to $T$-roots decomposes space $\mathfrak{t}$ into a finite number of cones, which are called *$T$-chambers*. We denote by $B(C)$ a basis of $\mathfrak{t}^*$ corresponding to a $T$-chamber $C$. We also denote by $R_T^+(C)$ the set of the positive $T$-roots corresponding to a $T$-chamber $C$.

Let $G^{\mathbb{C}}$ be a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and $U$ a parabolic subgroup of $G^{\mathbb{C}}$. Then the homogeneous complex manifold $G^{\mathbb{C}}/U$ is compact simply connected and $G$ acts transitively on $G^{\mathbb{C}}/U$. Note also that $K = G \cap U$ is a connected closed subgroup of $G$ and $G^{\mathbb{C}}/U$ can be identified with $G/K$ as a homogeneous manifold of $G$.

We set

$$Z_{\mathfrak{t}} = \left\{ \Lambda \in \mathfrak{t} \;\middle|\; \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for each } \alpha \in R \right\}.$$

Then $Z_{\mathfrak{t}}$ is a lattice in $\mathfrak{t}$ generated by $\{\Lambda_{i_1}, \dots, \Lambda_{i_r}\}$. For each $\Lambda \in Z_{\mathfrak{t}}$, there exists a unique holomorphic character $X_\Lambda$ of $U$ such that $X_\Lambda(\exp H) = \exp \Lambda(H)$ for each $H \in \mathfrak{h}^{\mathbf{C}}$.

Let $F_\Lambda$ denote the holomorphic line bundle of $G^{\mathbf{C}}/U$ associated to the principal bundle $U \to G^{\mathbf{C}} \to G^{\mathbf{C}}/U$ by the holomorphic character $X_\Lambda$. Then correspondence $\Lambda \mapsto F_\Lambda$ gives a homomorphism from $Z_{\mathfrak{t}}$ to $H^1(G^{\mathbf{C}}/U, \mathcal{O}^*)$. Then it is well known that the homomorphism

$$Z_{\mathfrak{t}} \xrightarrow{F} H^1(G^{\mathbf{C}}/U, \mathcal{O}^*) \xrightarrow{c_1} H^2(G^{\mathbf{C}}/U, \mathbf{Z})$$

is an isomorphism. In particular,

$$b_2(M) = \dim \mathfrak{t}.$$

There exists a one-to-one correspondence between the $T$-roots $\xi$ and the irreducible submodules $\mathfrak{m}_\xi$ of the $Ad_G(K)$-module $\mathfrak{m}^{\mathbf{C}}$ given by

$$R_T \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha^{\mathbf{C}}.$$

Then we have a decomposition of the $Ad_G(K)$-module $\mathfrak{m}^{\mathbf{C}}$:

$$\mathfrak{m}^{\mathbf{C}} = \sum_{\xi \in R_T} \mathfrak{m}_\xi.$$

Let us denote by $R_T^+$ the set of all positive $T$-roots, that is, the restriction of the system $R^+$ on $\mathfrak{t}$, and by $\tau$ the complex-conjugation of $\mathfrak{g}^{\mathbf{C}}$ which is with respect to the real form $\mathfrak{g}$. Then we have a decomposition of $Ad_G(K)$-module $\mathfrak{m}$ into irreducible submodules:

$$\mathfrak{m} = \sum_{\xi \in R_T^+} (\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau.$$

We denote by $\omega_\alpha$ $(\alpha \in R)$ the complex linear form on $\mathfrak{g}^{\mathbf{C}}$ which is dual to the basis vectors $E_\alpha$, where we take a Weyl basis $E_\alpha \in \mathfrak{g}_\alpha^{\mathbf{C}}$:

$$\omega_\alpha(E_\beta) = \delta_{\alpha\beta}, \qquad \omega(\mathfrak{h}^{\mathbf{C}}) = \{0\}.$$

For integers $j_1, \dots, j_r$ with $(j_1, \dots, j_r) \neq (0, \dots, 0)$, we set

$$R(j_1, \dots, j_r) = \left\{ \sum_{j=1}^{l} m_j \alpha_j \in R^+ \,\middle|\, m_{i_1} = j_1, \dots, m_{i_r} = j_r \right\}.$$

We set $R_{\mathfrak{m}}^+ = R^+ - [\Pi_0]$. Note that

$$R^+ - [\Pi_0] = \bigcup_{j_1, \dots, j_r} R(j_1, \dots, j_r).$$

By definition, for $\xi = j_1\kappa(\alpha_{i_1}) + \cdots + j_r\kappa(\alpha_{i_r})$, we have

$$(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau = \sum_{\alpha \in R(j_1,\dots,j_r)} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

For $R(j_1,\dots,j_r) \neq \emptyset$, we define an $Ad_G(K)$-invariant subspace $\mathfrak{m}(j_1,\dots,j_r)$ of $\mathfrak{g}$ by

$$\mathfrak{m}(j_1,\dots,j_r) = \sum_{\alpha \in R(j_1,\dots,j_r)} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

Then we have a decomposition of $\mathfrak{m}$ into a mutually non-equivalent irreducible $Ad_G(K)$-modules $\mathfrak{m}(j_1,\dots,j_r)$:

$$\mathfrak{m} = \sum_{j_1,\dots,j_r} \mathfrak{m}(j_1,\dots,j_r).$$

We set $m(j_1,\dots,j_r) = \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{m}(j_1,\dots,j_r)$ and

$$m(\xi) = \frac{1}{2}\dim_{\mathbb{R}} \sum_{\kappa(\alpha)=\xi} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}$$

for each $T$-root $\xi$.

There exists a natural one-to-one correspondence between $R_T$ and the set $\{R(j_1,\dots,j_r) \neq \emptyset\}$ given by

$$R_T \ni \xi = j_1\kappa(\alpha_{i_1}) + \cdots + j_r\kappa(\alpha_{i_r}) \mapsto R(j_1,\dots,j_r).$$

We have a decomposition of $\mathfrak{m}$ into a mutually non-equivalent irreducible $Ad_G(K)$-modules:

$$\mathfrak{m} = \sum_{\xi \in R_T^+} (\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau = \sum_{j_1,\dots,j_r} \mathfrak{m}(j_1,\dots,j_r).$$

The following formula gives an isomorphism between $\mathfrak{t}^*$ and $H^2(G/K,\mathbb{R})$:

$$\mathfrak{t}^* \ni \lambda \mapsto \eta(\lambda) = -\frac{1}{2\pi\sqrt{-1}}\,d\lambda = -\frac{\sqrt{-1}}{2\pi}\sum_{\alpha \in R_{\mathfrak{m}}^+} (\lambda,\alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha} \in H^2(G/K,\mathbb{R}),$$

where we consider $\lambda$ as a complex linear form on $\mathfrak{g}$ by extending.

THEOREM 1 ([9], cf. [7]). *Let $\omega \in H^2(G^{\mathbb{C}}/U,\mathbb{Z})$ be a pseudo-Kähler structure on a generalized flag manifold $G^{\mathbb{C}}/U$. Then there exists an invariant pseudo-Kähler structure $\omega_0 = \eta(\lambda)$ such that $\omega$ and $\omega_0$ have the same signature.*

By the above theorem, on a generalized flag manifold $G^{\mathbb{C}}/U$, in order to study the signatures of pseudo-Kähler metrics of which fundamental classes are contained in $H^2(G^{\mathbb{C}}/U, \mathbb{Z})$, we can suppose that the pseudo-Kähler metrics are invariant.

## 3. Signatures of pseudo-Kähler metrics

In this section, we consider the signatures of invariant pseudo-Kähler metrics on generalized flag manifolds from a viewpoint of $T$-root systems. We fix an invariant complex structure on each generalized flag manifold.

From now on, we assume that a $T$-root system $R_T$ corresponding to $G^{\mathbb{C}}/U$ becomes one of the irreducible reduced root systems. We denote the Weyl group of $R_T$ by $W_T$. We denote by $C_0$ the chamber which contains $\frac{1}{2}\sum_{\alpha \in R_{\mathfrak{m}}^+} \alpha$.

Let $V$ be a vector space over $\mathbb{R}$, and $R$ a root system in $V$. Let $C$ be a chamber of $R$. We denote by $S_\beta$ ($\beta \in B(C)$) the reflection in a hyperplane $\beta = 0$. Let $B(C) = \{\xi_1, \dots, \xi_l\}$. Then a hyperplane $L_{\xi_i} : \xi_i = 0$ is called a *wall* of $C$, and $L_{\xi_i} \cap \bar{C}$ is called a *facet* of $C$, where $\bar{C}$ is the closure of $C$. A facet $F$ is said to be a *panel* of $C$ if $F$ is of codimension 1. Two chambers $C$ and $C'$ are said to be *adjoining* if they have a common panel $F$: then either $C = C'$ or $F = \bar{C} \cap \bar{C}'$. Assume that $C$ and $C'$ are adjoining and $C \neq C'$. Let $F \subset L_\xi$, where $\xi \in B(C)$, be a common panel. Then we have $C' = S_\xi(C)$, because $F$ is a subset of a wall of $S_\xi(C)$ and $C \neq S_\xi(C)$. Then the following theorem on the Weyl chambers is known ([5]; Chapter VI, Theorem 2).

THEOREM 2. (i) *The Weyl group $W$ of a root system $R$ acts simply-transitively on the set of chambers.*
(ii) *Let $C$ be a chamber. Then $\bar{C}$ is a fundamental domain for $W$.*
(iii) *$C$ is an open simplicial cone.*
(iv) *Let $L_1, L_2, \dots, L_l$ be walls of $C$. For all $i$, there exists a unique indivisible root $\alpha_i$ such that $L_i = L_{\alpha_i}$, and such that $\alpha_i$ lies on the same side of $L_i$ as $C$.*
(v) *The set $\{\alpha_1, \dots, \alpha_l\}$ is a basis of $R$.*

Note that $B(S_\xi(C)) = \{S_\xi(\xi_1), \dots, S_\xi(\xi_l)\}$. Thus, we have the following:

LEMMA 1. *Let $C$ be a chamber. Then there exists an element $w = S_{\beta_t} \circ \cdots \circ S_{\beta_1} \in W$ such that $\beta_k \in B(S_{\beta_{k-1}} \circ \cdots \circ S_{\beta_1}(C_0))$ for each $k$, and $w(C_0) = C$.*

PROOF. By the definition of chambers, note that there exists a sequence $(C_0, C_1, \dots, C_t)$ of chambers such that $C_{k-1}$ and $C_k$ are adjoining for $1 \leq k \leq$

$n$, $C_{k-1} \neq C_k$, and $C_t = C$. By the above argument, since $C_k = S_{\beta_k}(C_{k-1})$ for some $\beta_k \in B(C_{k-1})$, we have our lemma. $\square$

By using the fact that $S_\beta(R^+(C) - \{\beta\}) = R^+(C) - \{\beta\}$ and $S_\beta(\beta) = -\beta$ for $\beta \in B(C)$, we see the following:

LEMMA 2 ([9]). *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold. Suppose that the $T$-root system $R_T$ corresponding to $G^{\mathbf{C}}/U$ becomes one of the irreducible reduced root systems. Let $C$ be a $T$-chamber. Let $\lambda \in C$, $\beta \in B(C)$. Then*

$$\sum_{\substack{\alpha \in R\backslash[\Pi_0] \\ \kappa(\alpha) \in R_T^+(C)}} (S_\beta(\lambda), \kappa(\alpha))\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha}$$

$$= \sum_{\substack{\alpha \in R\backslash[\Pi_0] \\ \kappa(\alpha) \in R_T^+(C) \\ \kappa(\alpha) \neq \beta}} (\lambda, S_\beta(\alpha))\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha} + \sum_{\substack{\alpha \in R\backslash[\Pi_0] \\ \kappa(\alpha) \in R_T^+(C) \\ \kappa(\alpha) = \beta}} (\lambda, -\beta)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha}.$$

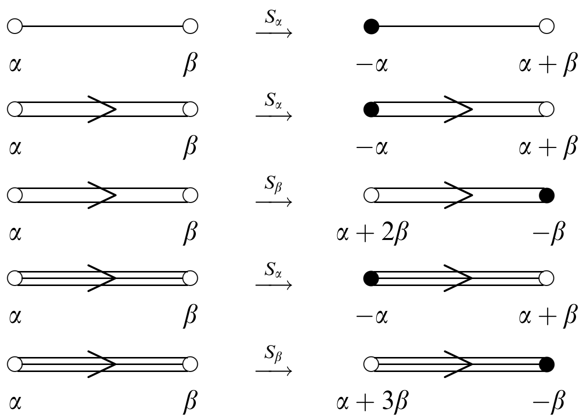
Hence, we have following proposition:

PROPOSITION 1. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold. Suppose that the $T$-root system $R_T$ corresponding to $G^{\mathbf{C}}/U$ becomes one of the irreducible reduced root systems. Let $C$ be a $T$-chamber, and $\eta(\lambda) = -\frac{\sqrt{-1}}{2\pi}\sum_{\alpha \in R_{\mathfrak{m}}^+}(\lambda, \alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha}$ a pseudo-Kähler metric of type $2(p, q)$ corresponding to $\lambda \in C$, and $\beta \in B(C)$. By the reflection in a hyperplane $\beta = 0$, if $(\beta, \lambda) > 0$ (resp. $(\beta, \lambda) < 0$), then $\eta(S_\beta(\lambda))$ is of type $2(p - m(\beta), q + m(\beta))$ (resp. $2(p + m(\beta), q - m(\beta))$).*

Let us fix a basis of the irreducible reduced root system $R_T$, we denote by $\{\alpha_1, \dots, \alpha_l\}$ the fixed basis in a usual manners ([5]), for example, if $R_T$ is $A_l$, then $\sum_{k=i}^{j} \alpha_k$ $(1 \leq i < j \leq l)$ is a positive root.

THEOREM 3. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold. Suppose that the $T$-root system $R_T$ corresponding to $G^{\mathbf{C}}/U$ becomes one of the irreducible reduced root system. Let $C$ be a $T$-chamber, and $\eta(\lambda)$ a pseudo-Kähler metric of type $2(p, q)$ corresponding to $\lambda \in C$. Then there exists an element $w = S_{\beta_t} \circ \cdots \circ S_{\beta_1}$ $\in W_T$ such that $\beta_k \in B(S_{\beta_{k-1}} \circ \cdots \circ S_{\beta_1}(C_0))$ for each $k$, and $w(C_0) = C$. Considering the Dynkin diagrams corresponding to a basis $\{S_{\beta_k} \circ \cdots \circ S_{\beta_1}(\alpha_1), \dots,$ $S_{\beta_k} \circ \cdots \circ S_{\beta_1}(\alpha_l)\}$, and reflections in a hyperplane $\beta_k = 0$ for $k = 1, \dots, t$, the signature of the invariant pseudo-Kähler metric $\eta(\lambda)$ on $G^{\mathbf{C}}/U$ can be written as $2(n - \sum_{i=1}^{t} m(\beta_i), \sum_{i=1}^{t} m(\beta_i))$.*

PROOF. By Lemma 1, there exists an element $w = S_{\beta_t} \circ \cdots \circ S_{\beta_1} \in W_T$ such that $\beta_k \in B(S_{\beta_{k-1}} \circ \cdots \circ S_{\beta_1}(C_0))$ for each $k$, and $w(C_0) = C$. By using the properties of reflections, that is,

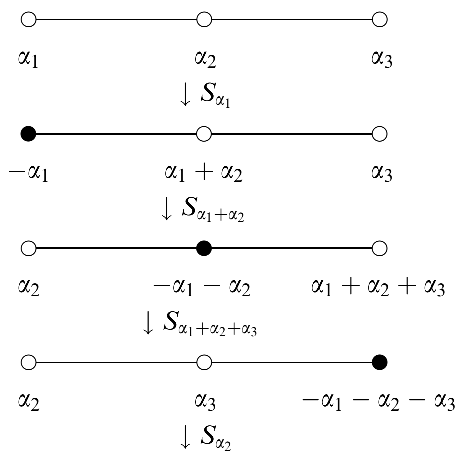


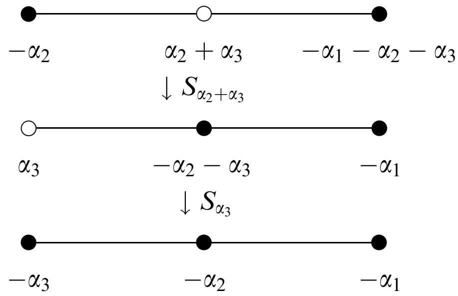

we have a series of the Dynkin diagram which is obtained by a reflection of some roots such that their bases are different each other (see examples below). Then by Proposition 1, we see the signatures of pseudo-Kähler metrics of generalized flag manifolds. □

Remark 1. *In* [9], *we proved that if* $2(n - m(j_1, \dots, j_r), m(j_1, \dots, j_r))$ *is a signature of some invariant pseudo-Kähler metric, then* $\sum_{k=1}^{r} j_k \alpha_{i_k} \in B(C_0)$. *Hence, it is not true that any* $2(n - \sum_{i=1}^{t} m(\beta_i), \sum_{i=1}^{t} m(\beta_i))$ *becomes the signature of an invariant pseudo-Kähler metrics on* $G^{\mathbb{C}}/U$ *without conditions on* $\{\beta_1, \dots, \beta_t\}$, *where* $\beta_i$ *is a* $T$*-root for each* $i$.

In the rest of this paper, we denote $n - p$ of a signature $2(n-p, p)$ by $**$ when $p$ is complicated, where $n = \dim G^{\mathbb{C}}/U$ for short.

For example, consider the following sequence of labeled Dynkin diagrams of type $A_3$ obtaining stepwisely by the root reflection corresponding to a certain vertex of each diagram. (black points indicate the negative roots):






By Proposition 1, we see that for the generalized flag manifold $SU(l_1+l_2+l_3+l_4)/S(U(l_1)\times U(l_2)\times U(l_3)\times U(l_4))$, there exist pseudo-Kähler metrics of type

$$2(**,m(1,0,0)),\ 2(**,m(1,0,0)+m(1,1,0)),$$
$$2(**,m(1,0,0)+m(1,1,0)+m(1,1,1)),$$
$$2(**,m_{\alpha_1-\mathrm{seq.}}+m(0,1,0)),\ 2(**,m_{\alpha_1-\mathrm{seq.}}+m_{\alpha_2-\mathrm{seq.}}),$$

where

$$m_{\alpha_1-\mathrm{seq.}}=m(1,0,0)+m(1,1,0)+m(1,1,1),\qquad m_{\alpha_2-\mathrm{seq.}}=m(0,1,0)+m(0,1,1).$$

When we assign the type of a root system $R_T$, we have the following:

Theorem 4. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold such that the $T$-root system corresponding to $G^{\mathbf{C}}/U$ is a root system of type $A_l$. For a positive root $\sum_{k=i}^{j}\alpha_k$, there are pseudo-Kähler metrics of type $2(n-\sum_{k=i}^{h}m(\beta_k),\sum_{k=i}^{h}m(\beta_k))$, where $\beta_k=\sum_{s=i}^{k}\alpha_s$, and $h=i,\dots,j$.*

Proof. Let $\lambda_0\in C_0$. Then the signature of $\eta(\lambda_0)$ is $2(n,0)$. Consider the elements $S_{\beta_t}\circ\cdots\circ S_{\beta_i}\in W_T$ $(t=i,\dots,j)$. Let $2(p(t-1),q(t-1))$ be the signature of $\eta(S_{\beta_{t-1}}\circ\cdots\circ S_{\beta_i}(\lambda_0))$. Since $S_{\beta_s}(\alpha_t)=\alpha_t$ for $s=i,\dots,t-2$ and $S_{\beta_{t-1}}(\alpha_t)=\beta_t$, we see $\beta_t\in B(S_{\beta_{t-1}}\circ\cdots\circ S_{\beta_i}(C_0))$. Since $(\beta_t,\lambda_0)>0$, the signature of $\eta(S_{\beta_t}\circ\cdots\circ S_{\beta_i}(\lambda_0))$ is $2(p(t-1)-m(\beta_t),q(t-1)+m(\beta_t))$ by Lemma 2 and Proposition 1. $\square$

Theorem 5. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold such that the $T$-root system corresponding to $G^{\mathbf{C}}/U$ is a root system of type $B_l$. For a positive root $\sum_{k=i}^{j-1}\alpha_k+2\sum_{k=j}^{l}\alpha_k$, there exist pseudo-Kähler metrics of type*

(1) $2(n-\sum_{k=i}^{h}m(\beta_k),\sum_{k=i}^{h}m(\beta_k))$ $(i\le h\le l)$,

(2) $2(n-\sum_{k=i}^{l}m(\beta_k)-\sum_{s=0}^{h}m(\gamma_{l-s}),\sum_{k=i}^{l}m(\beta_k)+\sum_{s=0}^{h}m(\gamma_{l-s}))$ $(0\le h\le l-j-1)$,

*where $\beta_k=\sum_{s=i}^{k}\alpha_s$, and $\gamma_t=\sum_{k=i}^{t-1}\alpha_k+2\sum_{k=t}^{l}\alpha_k$.*

Proof. Let $\lambda_0 \in C_0$. The case 1 is obtained from the elements $S_{\beta_t} \circ \cdots \circ S_{\beta_i} \in W_T$ $(t = i, \dots, l)$ by the same arguments as in the proof of Theorem 4. For the case 2, consider the elements $S_{\gamma_{l-s}} \circ \cdots \circ S_{\gamma_l} \circ S_{\beta_l} \circ \cdots \circ S_{\beta_i} \in W_T$ $(s = 0, \dots, j+1)$. Since

$$S_{\beta_{l-1}}(\beta_{l-1}) = -\beta_{l-1}, \qquad S_{\beta_l}(-\beta_{l-1}) = \gamma_l, \qquad S_{\beta_t}(-\beta_{t-1}) = \alpha_t, \qquad S_{\beta_u}(\alpha_t) = \alpha_t$$

for $t = i+1, \dots, l-1$ and $u = t+1, \dots, l-1$, we have $\gamma_l, \alpha_{i+1}, \dots, \alpha_{l-1} \in B(S_{\beta_l} \circ \cdots \circ S_{\beta_i}(C_0))$. Since $S_{\gamma_{l-s+1}}(\alpha_{l-s}) = \gamma_{l-s}$, $S_{\gamma_{l-s+u}}(\alpha_{l-s}) = \alpha_{l-s}$ for $u \geq 2$, we see $\gamma_{l-s} \in B(S_{\gamma_{l-s+1}} \circ \cdots \circ S_{\gamma_l} \circ S_{\beta_l} \circ \cdots \circ S_{\beta_i}(C_0))$. Let $2(p(s-1), q(s-1))$ be the signature of $\eta(S_{\gamma_{l-s+1}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$. Then the signature of $\eta(S_{\gamma_{l-s}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ is $2(p(s-1) - m(\gamma_{l-s}), q(s-1) + m(\gamma_{l-s}))$ by Lemma 2 and Proposition 1. □

Theorem 6. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold such that the $T$-root system corresponding to $G^{\mathbf{C}}/U$ is a root system of type $C_l$. For a positive root $\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{l-1} \alpha_k + \alpha_l$, there exist pseudo-Kähler metrics of type*

(1) $2(n - \sum_{k=i}^{h} m(\beta_k), \sum_{k=i}^{h} m(\beta_k))$ $(i \leq h \leq l-1)$,

(2) $2(n - \sum_{k=i}^{l-1} m(\beta_k) - m(\gamma_i), \sum_{k=i}^{l-1} m(\beta_k) + m(\gamma_i))$,

(3) $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\gamma_i), \sum_{k=i}^{l} m(\beta_k) + m(\gamma_i))$,

(4) $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\gamma_i) - \sum_{s=0}^{h} m(\delta_{l-s}), \sum_{k=i}^{l} m(\beta_k) + m(\gamma_i) + \sum_{s=0}^{h} m(\delta_{l-s}))$ $(0 \leq h \leq l-j+1)$,

*where $\beta_k = \sum_{s=i}^{k} \alpha_s$, $\gamma_i = 2\sum_{k=i}^{l-1} \alpha_k + \alpha_l$, and $\delta_t = \sum_{k=i}^{t-1} \alpha_k + 2\sum_{k=t}^{l-1} \alpha_k + \alpha_l$.*

Proof. Let $\lambda_0 \in C_0$. The case 1 is obtained from the elements $S_{\beta_t} \circ \cdots \circ S_{\beta_i} \in W_T$ $(t = i, \dots, l-1)$ by the same arguments as in the proof of Theorem 4. For cases 2 and 3, consider the elements

$$S_{\gamma_i} \circ S_{\beta_{l-1}} \circ \cdots \circ S_{\beta_i}, \qquad S_{\beta_l} \circ S_{\gamma_i} \circ S_{\beta_{l-1}} \circ \cdots \circ S_{\beta_i}.$$

Since $S_{\beta_{l-1}}(\alpha_l) = \gamma_i$ and $S_{\gamma_i}(-\beta_{l-1}) = \beta_l$, the signatures of $\eta(S_{\gamma_i} \circ S_{\beta_{l-1}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ and $\eta(S_{\beta_l} \circ S_{\gamma_i} \circ S_{\beta_{l-1}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ are $2(n - \sum_{k=i}^{l-1} m(\beta_k) - m(\gamma_i), \sum_{k=i}^{l-1} m(\beta_k) + m(\gamma_i))$, $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\gamma_i), \sum_{k=i}^{l} m(\beta_k) + m(\gamma_i))$, respectively. For the case 4, consider the elements

$$S_{\delta_{l-s}} \circ \cdots \circ S_{\delta_{l-1}} \circ S_{\beta_l} \circ S_{\gamma_i} \circ S_{\beta_{l-1}} \circ \cdots \circ S_{\beta_i} \qquad (s = 0, \dots, j-1).$$

Note that

$$\alpha_{i+1}, \dots, \alpha_{l-2} \in B(S_{\beta_l} \circ S_{\gamma_i} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i}(C_0))$$

because $S_{\beta_t}(-\beta_{t-1}) = \alpha_t$ for $t = i+1, \dots, l-2$. Since

$$S_{\beta_l}(\alpha_{l-1}) = \delta_{l-1}, \qquad S_{\delta_{l-s}}(\alpha_{l-s-1}) = \delta_{l-s-1},$$

if $2(p(s-1), q(s-1))$ is the signature of $\eta(S_{\delta_{l-s+1}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$. Then the signature of $\eta(S_{\delta_{l-s}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ is $2(p(s-1) - m(\delta_{l-s}), q(s-1) + m(\delta_{l-s}))$ by Lemma 2 and Proposition 1. □

Theorem 7. *Let $G^{\mathbf{C}}/U$ be a generalized flag manifold such that the $T$-root system corresponding to $G^{\mathbf{C}}/U$ is a root system of type $D_l$. For a positive root $\sum_{k=i}^{j-1}\alpha_k + 2\sum_{k=j}^{l-2}\alpha_k + \alpha_{l-1} + \alpha_l$, there exist pseudo-Kähler metrics of type*

(1) $2(n - \sum_{k=i}^{h} m(\beta_k), \sum_{k=i}^{h} m(\beta_k))$ $(i \leq h \leq l-1)$,

(2) $2(n - \sum_{k=i}^{l-1} m(\beta_k) - m(\beta_{l-2}+\alpha_l), \sum_{k=i}^{l-1} m(\beta_k) + m(\beta_{l-2}+\alpha_l))$,

(3) $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\beta_{l-2}+\alpha_l), \sum_{k=i}^{l} m(\beta_k) + m(\beta_{l-2}+\alpha_l))$,

(4) $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\beta_{l-2}+\alpha_l) - \sum_{s=2}^{h} m(\gamma_{l-s}), \sum_{k=i}^{l} m(\beta_k) + m(\beta_{l-2}+\alpha_l) + \sum_{s=2}^{h} m(\gamma_{l-s}))$ $(2 \leq h \leq l-j+1)$,

*where* $\beta_k = \sum_{s=i}^{k}\alpha_s$, *and* $\gamma_t = \sum_{k=i}^{t-1}\alpha_k + 2\sum_{k=t}^{l-2}\alpha_k + \alpha_{l-1} + \alpha_l$.

Proof. Let $\lambda_0 \in C_0$. The case 1 is obtained from the elements $S_{\beta_t} \circ \cdots \circ S_{\beta_i} \in W_T$ $(t = i, \dots, l-1)$ by the same arguments as in the proof of Theorem 4. For cases 2 and 3, consider

$$S_{\beta_{l-2}+\alpha_l} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i}, \qquad S_{\beta_l} \circ S_{\beta_{l-2}+\alpha_l} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i}.$$

Since

$$S_{\beta_{l-2}}(\alpha_l) = \beta_{l-2} + \alpha_l, \qquad S_{\beta_{l-2}+\alpha_l}(\alpha_{l-1}) = \beta_l,$$

the signatures of $\eta(S_{\beta_{l-2}+\alpha_l} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ and $\eta(S_{\beta_l} \circ S_{\beta_{l-2}+\alpha_l} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ are $2(n - \sum_{k=i}^{l-1} m(\beta_k) - m(\beta_{l-2}+\alpha_l), \sum_{k=i}^{l-1} m(\beta_k) + m(\beta_{l-2}+\alpha_l))$ and $2(n - \sum_{k=i}^{l} m(\beta_k) - m(\beta_{l-2}+\alpha_l), \sum_{k=i}^{l} m(\beta_k) + m(\beta_{l-2}+\alpha_l))$, respectively. For the case 4, consider the elements

$$S_{\gamma_{l-s}} \circ \cdots \circ S_{\gamma_{l-2}} \circ S_{\beta_l} \circ S_{\beta_{l-2}+\alpha_l} \circ S_{\beta_{l-2}} \circ \cdots \circ S_{\beta_i} \qquad (s = 2, \dots, j-1).$$

Note that

$$\alpha_{i+1}, \dots, \alpha_{l-2} \in B(S_{\beta_l} \circ S_{\gamma_i} \circ S_{\beta_{l-2}+\alpha_l} \circ \cdots \circ S_{\beta_i}(C_0))$$

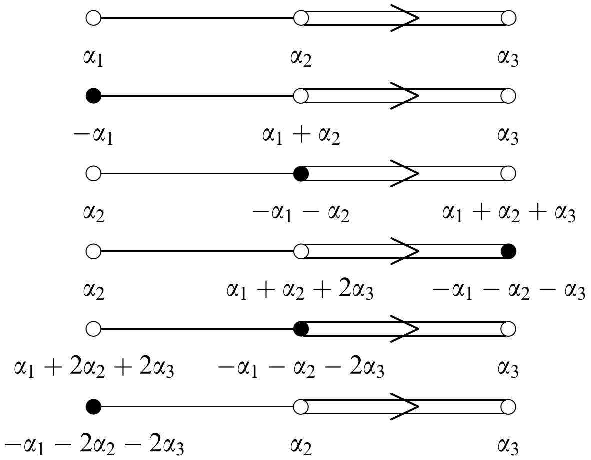
because $S_{\beta_t}(-\beta_{t-1}) = \alpha_t$ for $t = i+1, \dots, l-2$. Since

$$S_{\beta_l}(\alpha_{l-2}) = \gamma_{l-2}, \qquad S_{\gamma_{l-s}}(\alpha_{l-s-1}) = \gamma_{l-s-1},$$

if $2(p(s-1), q(s-1))$ is the signature of $\eta(S_{\delta_{l-s+1}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$. Then the signature of $\eta(S_{\delta_{l-s}} \circ \cdots \circ S_{\beta_i}(\lambda_0))$ is $2(p(s-1) - m(\gamma_{l-s}), q(s-1) + m(\gamma_{l-s}))$ by Lemma 2 and Proposition 1. □

From now on, we omit the notations $\downarrow S_{\xi}$ on a sequence of labeled Dynkin diagrams obtaining stepwisely by the root reflection corresponding to a certain vertex of each diagram.

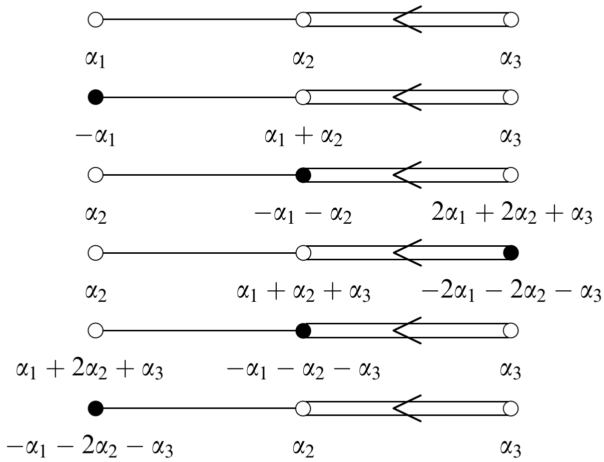
Example 1. *Consider the following sequence of labeled Dynkin diagrams of type $B_3$ obtaining stepwisely by the root reflection corresponding to a certain vertex of each diagram:*




*Then we see that for a generalized flag manifold such that the T-root system is of type $B_3$, there exist pseudo-Kähler metrics of type*

$$2(**, m(1,0,0)),\ 2(**, m(1,0,0)+m(1,1,0)),$$
$$2(**, m(1,0,0)+m(1,1,0)+m(1,1,1)),$$
$$2(**, m(1,0,0)+m(1,1,0)+m(1,1,1)+m(1,1,2)),$$
$$2(**, m(1,0,0)+m(1,1,0)+m(1,1,1)+m(1,1,2)+m(1,2,2)).$$

Example 2. *Consider the following sequence of labeled Dynkin diagrams of type $C_3$ obtaining stepwisely by the root reflection corresponding to a certain vertex of each diagram:*



*Then we see that for a generalized flag manifold such that the T-root system is of type $C_3$, there exist pseudo-Kähler metrics of type*

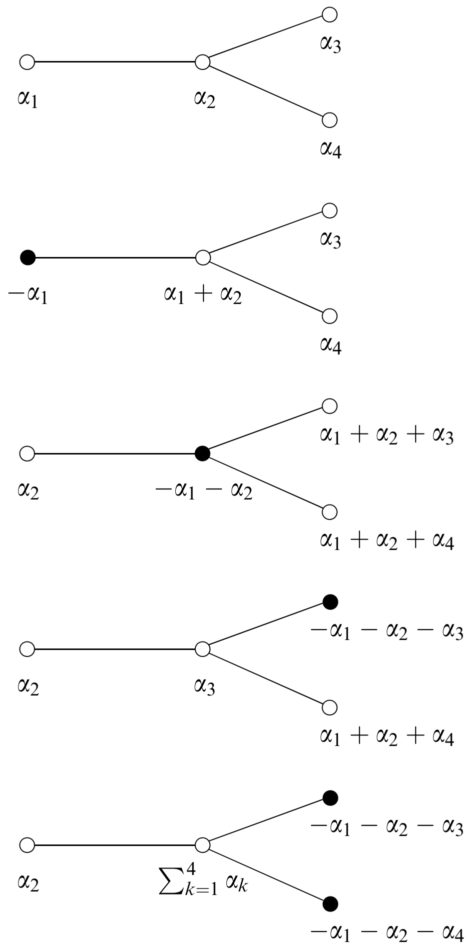
$$2(**, m(1,0,0)),\ 2(**, m(1,0,0) + m(1,1,0)),$$

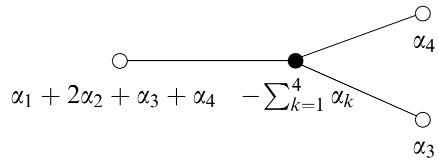
$$2(**, m(1,0,0) + m(1,1,0) + m(2,2,1)),$$

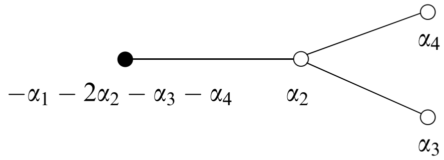
$$2(**, m(1,0,0) + m(1,1,0) + m(2,2,1) + m(1,1,1)),$$

$$2(**, m(1,0,0) + m(1,1,0) + m(2,2,1) + m(1,1,1) + m(1,1,2)).$$

Example 3. *Consider the following sequence of labeled Dynkin diagrams of type $D_4$ obtaining stepwisely by the root reflection corresponding to a certain vertex of each diagram:*






*Then we see that for a generalized flag manifold such that the T-root system is of $D_4$, there are pseudo-Kähler metrics of type*

$$2(**, m(1,0,0,0)),\ 2(**, m(1,0,0,0) + m(1,1,0,0)),$$
$$2(**, m(1,0,0,0) + m(1,1,0,0) + m(1,1,1,0)),$$
$$2(**, m(1,0,0,0) + m(1,1,0,0) + m(1,1,1,0) + m(1,1,0,1)),$$
$$2(**, m(1,0,0,0) + m(1,1,0,0) + m(1,1,1,0) + m(1,1,0,1) + m(1,1,1,1)),$$
$$2(**, m(1,0,0,0) + m(1,1,0,0) + m(1,1,1,0) + m(1,1,0,1) + m(1,1,1,1) + m(1,2,1,1)).$$

## Acknowledgement

The author would like to express his deep appreciation to Professor Yusuke Sakane for valuable advices and encouragements.

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*Takumi Yamada*
*Department of Mathematics*
*Interdisciplinary Faculty of Science and Engineering*
*Shimane University*
*Nishikawatsu-cho* 1060
*Matsue* 690-8504 *Japan*
*E-mail: t_yamada@riko.shimane-u.ac.jp*