Bernoulli **1(4)**, 1995, 371–380

On Brownian disconnection exponents

WENDELIN WERNER

CNRS and University of Cambridge, Statistical Laboratory, DPMMS, 16 Mill Lane, Cambridge CB2 1SB, UK

We derive an upper bound for the disconnection exponent γ of two-dimensional Brownian motion. More precisely, we show that $\gamma \leq \frac{1}{2} - (\log 2)^2/(2\pi^2) < 0.475\,66$. This implies in particular that γ is not equal to its trivial upper bound (i.e. $\frac{1}{2}$). We also derive similar estimates of disconnection exponents for several planar Brownian motions and intersection exponents.

Keywords: Brownian motion; disconnection exponent; intersection exponent

1. Introduction

The initial purpose of this short paper is to prove that the disconnection exponent for two-dimensional Brownian motion is not equal to $\frac{1}{2}$ (which is its trival upper bound). Let us recall the exact definition of this exponent and some known facts. Let $(B_t, t \geq 0)$ denote a planar Brownian motion started from $B_0 = 1$, and put, for all $R > 1$,

$$T_R = \inf\{t \geq 0, |B_t| = R\}.$$

We say that $B_{[0,t]}$ disconnects 0 from ∞ if there is a subpath of $\{B_s; s \leq t\}$ which makes a closed loop around 0. We then define:

$$p_R = P(B_{[0,T_R]} \text{ does not disconnect 0 from } \infty).$$

We are interested in the asymptotic behaviour of p_R as $R \to \infty$. A subadditivity argument (see, for example, Lawler 1991, Proposition 5.5.1) shows that

$$\lim_{R \to \infty} \frac{-\log p_R}{\log R} = \gamma$$

for some finite strictly positive constant γ called the disconnection exponent for two-dimensional Brownian motion. It is known that

$$\frac{1}{2\pi} \leq \gamma \leq \frac{1}{2}.$$

The left-hand inequality has been derived by Burdzy and Lawler (1990b); the right-hand inequality corresponds to the fact that the probability that B does not hit the ray $(-\infty, 0]$ before T_R decays asymptotically as $(2/\pi)R^{-1/2}$, when $R \to \infty$. It has been conjectured by physicists that $\gamma = \frac{1}{4}$ (see Duplantier and Kwon 1988, Duplantier *et al.* 1993).

Lawler and Puckette (1994) have shown that the corresponding disconnection exponent for the simple two-dimensional random walk exists and is also equal to γ. We are going to prove the

1350-4864

following estimate:

Proposition 1

$$\gamma \leq \frac{1}{2} - \frac{1}{2}\left(\frac{\log 2}{\pi}\right)^2 < 0.475\,66.$$

We now briefly explain the outline of our proof. Throughout this paper, we will identify $\mathbb{C}$ with $\mathbb{R}^2$. It is straightforward that

$$p_R \geq P(\exists g : (0, R] \to \mathbb{R}, \text{ continuous}, \forall t \in [0, T_R], Z_t \neq |Z_t| \exp\{ig(|Z_t|)\}).$$

The image under a conformal mapping of a planar Brownian motion is a time-changed planar Brownian motion. Therefore, using the analyticity of the exponential mapping – in other words, using the skew-product representation of B (see, for example, Itô and McKean 1965, p. 265) – if $Z = (X + \mathrm{i}Y)$ is a complex Brownian motion started from 0, and if

$$\hat{T}_r = \inf\{t > 0, X_t = r\}$$

where $r = \log R (> 0)$ (we will use this notation throughout this paper), then

$$p_R \geq P(\exists f : (-\infty, r] \to \mathbb{R}, \text{ continuous}, \forall t \in [0, \hat{T}_r), |Y_t - f(X_t)| < \pi).$$

For all functions $f : (-\infty, r] \to \mathbb{R}$, we put:

$$A_f^r = \{\forall t \in [0, \hat{T}_r), |Y_t - f(X_t)| < \pi\}.$$

Beurling's (1933) projection theorem on harmonic measure in a disc – see Ahlfors (1973, Theorem 3.6), Oksendal (1983) for a short probabilistic proof, or Werner (1994) for an alternative approach – shows readily that for continuous functions f, $P(A_f^r) < P(A_0^r)$ as soon as $f \not\equiv 0$. Nevertheless, we are going to show (using conformal invariance arguments and extremal distance) that if the growth rate of f is absolutely bounded by $M > 0$ (that is, if for all $(x, x') \in (-\infty, r]^2$, $|f(x) - f(x')| \leq M|x - x'|$) and if f is odd (that is, for all $x \in [0, r]$, $f(x) = -f(-x)$), then

$$P(A_f^r) \geq \frac{1}{\pi}\exp\left(-\frac{r}{2}(1 + M^2)\right).$$

Then, it suffices to observe that (for some fixed c and for all M) it is possible to find at least $m > \exp(crM)$ such functions $(f_i)_{1 \leq i \leq m}$ (we will choose piecewise linear functions) for which all the sets $A_{f_i}^r$ are disjoint; then:

$$p_R \geq \sum_{i=1}^{m} P(A_{f_i}^r) \geq \frac{1}{\pi}\exp(cMr)\exp\left(-\frac{r}{2}(1 + M^2)\right) \geq \frac{1}{\pi}\exp\left\{-r\left(\frac{1}{2} - \epsilon\right)\right\}$$

for some $\epsilon > 0$ and for a sufficiently small M, which shows that $\gamma \leq \frac{1}{2} - \epsilon < \frac{1}{2}$.

We will also adapt the same arguments to prove an analogue of Proposition 1 for several planar Brownian motions. More precisely, in Section 5, we define a disconnection exponent γ_n corresponding to the event that the union of n independent planar Brownian paths stopped at their respective hitting times of a big circle, does not disconnect 0 from ∞, and we show the following:

Proposition 2 For all $n \geq 2$,

$$\gamma_n \leq \frac{n}{2} - \frac{(\log 2)^2}{2\pi^2 n}.$$

γ_n is not equal to its trivial upper bound $n/2$. In particular, $\gamma_2 < 0.988 < 1$. This is still far from the conjectured value (it has been conjectured, using a relationship with 'self-avoiding planar Brownian motion' for which it is believed that its Hausdorff dimension is $\frac{4}{3}$ (Mandelbrot 1982) that $\gamma_2 = 2 - \frac{4}{3} = \frac{2}{3}$; see Burdzy and Lawler (1990b), but it might be the first step in proving that the fractal dimension of 'self-avoiding planar Brownian motion' is strictly bigger than 1 (since $2 - \gamma_2 > 1$).

Let us mention that disconnection exponents and techniques involved have a lot of similarities with intersection exponents, which have given rise to much interest in recent years (see, for example, Burdzy and Lawler 1990a, 1990b, Burdzy *et al.* 1989, Lawler 1989, 1993, and, in particular, Lawler 1991, Chapter 5, in which all known information on intersection exponents can be found). We will also point out (in the last section) that arguments used in the proof of Proposition 1 can also be adapted to provide analogous upper bounds for some intersection exponents of two-dimensional Brownian motion.

2. Preliminaries

We now recall some well-known facts on hitting times by linear Brownian motion which we will use in the following sections. Let $W = (W_t, t \geq 0)$ and $W' = (W'_t, t \geq 0)$ be two independent one-dimensional Brownian motions started from 0. For all $a > 0$, we put:

$$\tilde{T}_a = \inf\{t > 0, |W_t| = a\},$$

$$\tilde{T}'_a = \inf\{t > 0, |W'_t| = a\}.$$

Then, for all $a > 0$, $b > 0$, $t > 0$ and $\lambda \geq 0$,

$$E\{\exp(-\lambda^2 \tilde{T}_a/2)\} = \frac{1}{\cosh(\lambda a)}, \tag{1}$$

$$\frac{2}{\pi}\exp\left(\frac{-\pi^2 t}{8a^2}\right) \leq P(\tilde{T}_a > t) \leq \frac{4}{\pi}\exp\left(\frac{-\pi^2 t}{8a^2}\right), \tag{2}$$

$$\frac{2}{\pi\cosh\{\pi a/(2b)\}} \leq P(\tilde{T}_a < \tilde{T}'_b) \leq \frac{4}{\pi\cosh\{\pi a/(2b)\}}. \tag{3}$$

Equation (1) can be found in Revuz and Yor (1991, Chapter II, Proposition 3.7); (2) is an easy consequence of the explicit law of $\tilde{T}_1$ (see, for example, Port and Stone 1978, p. 52); and (3) is a straightforward consequence of (1) and (2).

Let us recall that if $\mathscr{T}_a = \inf\{t > 0, W_t = a\}$, one has (see, for example Revuz and Yor 1991, Chapter II, Proposition 3.7)

$$E\{\exp(-\lambda^2 \mathscr{T}_a/2)\} = \exp(-a\lambda),$$

so that (2) yields:

$$\frac{2}{\pi}R^{-1/2} \leq P(\mathcal{T}_{\log R} < \tilde{T}'_\pi) \leq \frac{4}{\pi}R^{-1/2} \tag{4}$$

which provides immediately (via the skew-product representation),

$$p_R \geq P(A_0^r) \geq \frac{2}{\pi}R^{-1/2}$$

and the upper bound $\gamma \leq \frac{1}{2}$.

3. Extremal distance estimates

We are going to estimate extremal distances for particular sets. We refer to Ahlfors (1973, Chapter 4) for properties of extremal length and extremal distance, but recall some facts in this paper.

Let us fix $r > 0$ and an odd continuous function $f : [-r, r] \to \mathbb{R}$ such that its growth rate is absolutely bounded by $\tan\theta$, for some fixed $\theta \in (0, \pi/2)$. We then define:

$$\Omega = \{x + \mathrm{i}y\,;\, x \in (-r, r), f(x) - \pi < y < f(x) + \pi\}$$

and $A = -r + \mathrm{i}(f(-r) - \pi)$, $B = -r + \mathrm{i}(f(-r) + \pi)$, $C = r + \mathrm{i}(f(r) + \pi)$ and $D = r - \mathrm{i}(f(r) - \pi)$.

We also put:

$$I = \{x + \mathrm{i}(f(x) + \pi);\ x \in [-r, r]\}$$

$$J = \{x + \mathrm{i}(f(x) - \pi);\ x \in [-r, r]\}.$$

Then, $\partial\Omega = [AB] \cup I \cup [CD] \cup J$, where $[AB]$ denotes the segment linking A and B in the plane. Note that, as f is odd, Ω is symmetric with respect to 0.

Let $(Z_t, t \geq 0)$ be a complex Brownian motion started from 0 and let us denote (for all compact sets K),

$$T(K) = \inf\{t > 0;\ Z_t \in K\}.$$

Our aim is to give a lower bound for the probability that Z hits $\partial\Omega$ on $[AB] \cup [CD]$. More precisely, we are going to prove that

$$P(T(I \cup J) > T([AB] \cup [CD])) \geq \frac{2}{\pi}\exp\left(\frac{-r}{2\cos^2\theta}\right). \tag{5}$$

Proof of (5)

There exists a unique $\ell > 0$, for which there exists a (unique) conformal one-to-one mapping $\Phi : \Omega \to (-\ell, \ell) \times (-1, 1)$, such that the continuous extension $\bar{\Phi} : \bar{\Omega} \to [-\ell, \ell] \times [-1, 1]$ of Φ satisfies $\bar{\Phi}(A) = -\ell - i$, $\bar{\Phi}(B) = -\ell + i$, $\bar{\Phi}(C) = \ell + i$ and $\bar{\Phi}(D) = \ell - i$ (see, for example, Ahlfors 1973, p. 52 on the conformal mapping of a quadrilateral in a rectangle or Ahlfors 1973, para. 4.10 on configurations with a single modulus). By unicity of this conformal mapping (see, for example, Ahlfors 1966, 1973) and using a symmetry argument (recall that Ω is symmetric with respect to 0), it is straightforward to see that $\Phi(0) = 0$. We now put $\Omega' = \Phi(\Omega) = (-\ell, \ell) \times (-1, 1)$, $A' = -\ell - i$, $B' = -\ell + i$, $C' = \ell + i$ and $D' = \ell - i$.

The extremal distance in a rectangle of area ab between two non-adjacent sides of length a of this rectangle is equal to b/a (see, for example, Ahlfors 1973, p. 53), so that, if $d_F(G,H)$ denotes the extremal distance between G and H in F,

$$\ell^{-1} = \mathrm{d}_{\Omega'}([B'C'],[D'A']) = \mathrm{d}_{\Omega}(I,J). \tag{6}$$

As the growth rate of f is absolutely bounded by $\tan\theta$, it is very easy to see that, with respect to the Euclidean metric in Ω, the length of a path joining I to J in Ω is greater than or equal to $2\pi\cos\theta$. Since the Euclidean area of Ω is $4\pi r$, it follows (from the definition of extremal distance in Ahlfors 1973, Definition 4.1) that

$$\mathrm{d}_{\Omega}(I,J) \geq \frac{\pi\cos^2\theta}{r}. \tag{7}$$

Now, (6) and (7) imply that

$$\ell \leq \frac{r}{\pi\cos^2\theta}.$$

But by conformal invariance, and using inequality (3), in the notation of Section 2,

$$\begin{aligned}
P(T(I\cup J) > T([AB]\cup[CD])) &= P(T([B'C']\cup[D'A']) > T([A'B']\cup[C'D'])) \\
&= P(\tilde{T}_1 > \tilde{T}'_\ell) \\
&\geq \frac{2}{\pi\cosh(\pi\ell/2)} \\
&\geq \frac{2}{\pi}\exp(-\pi\ell/2) \\
&\geq \frac{2}{\pi}\exp\left(\frac{-r}{2\cos^2\theta}\right)
\end{aligned}$$

which proves (5). □

4. Conclusion of the proof of Proposition 1

We use the same notation as in the introduction and put

$$\bar{T}_r = \inf\{t > 0; |X_t| = r\}.$$

Recall that $\hat{T}_r = \inf\{t > 0; X_t = r\}$. We now fix $\theta \in (0,\pi/2)$, define N as being the integer part of $(r\tan\theta)/\pi$ and put $\Lambda_N = \{-1,1\}^N$. For all $U = (u_1,\dots,u_N) \in \Lambda_N$, we define the function $f_U : [-r,r] \to \mathbb{R}$ as follows:

- $f_U(rn/N) = -f_U(-rn/N) = \sum_{1\leq i\leq n} \pi u_i$ for all $n \in \{1,\dots,N\}$.
- $f_U(0) = 0$.
- For all $n \in \{-N,\dots,N-1\}$, f_U is continuous and linear on $[nr/N,(n+1)r/N]$.

This definition implies immediately that the growth rate of f_U is absolutely bounded by $\pi N/r \leq \tan\theta$. We put:

$$D'_U = \{\forall t < \bar{T}_r, |Y_t - f_U(X_t)| < \pi\}$$

$$D_U = \{\forall t < \bar{T}_r, |Y_t - f_U(X_t)| < \pi \text{ and } \hat{T}_r = \bar{T}_r\}.$$

If D_U is satisfied, then for all $n \in \{1, \dots, N\}$,

$$\pi\left(-1 + \sum_{1 \le i \le n} u_i\right) < Y_{\hat{T}_{(nr/N)}} < \pi\left(1 + \sum_{1 \le i \le n} u_i\right).$$

Consequently, the events $(D_U)_{U \in \Lambda_N}$ are disjoint.

Moreover, for any $U \in \Lambda_N$, there exists a continuous function f (any function $f : (-\infty, r] \to \mathbb{R}$, such that $f = f_U$ on $[-r, r]$ would suffice), such that $D_U \subset A_f$; hence,

$$p_R \ge P(\cup_{U \in \Lambda_N} D_U) \ge \sum_{U \in \Lambda_N} P(D_U).$$

By symmetry, $P(D_U) = P(D'_U)/2$ for all $U \in \Lambda_N$. Hence, (5) yields that for all $U \in \Lambda_N$,

$$P(D_U) \ge \frac{1}{\pi} \exp\left(\frac{-r}{2\cos^2\theta}\right).$$

Consequently,

$$\begin{aligned} p_R &\ge \frac{2^N}{\pi} \exp\left(\frac{-r}{2\cos^2\theta}\right) \\ &\ge \frac{1}{2\pi} \exp\left(\frac{(\log 2)\tan\theta}{\pi} r - \frac{r}{(2\cos^2\theta)}\right) \\ &\ge \frac{1}{2\pi} \exp\left\{\left(-\frac{1}{2} + \frac{\log 2}{\pi}\tan\theta - \frac{\tan^2\theta}{2}\right)\log R\right\}. \end{aligned}$$

If we choose θ such that $\tan\theta = (\log 2)/\pi$, then: $p_R \ge (2\pi)^{-1} R^\alpha$ with

$$\alpha = -\frac{1}{2} + \frac{1}{2}\left(\frac{\log 2}{\pi}\right)^2,$$

which completes the proof of Proposition 1.

5. Disconnection exponents for several Brownian motions

Definition

We fix $n \ge 2$. Let us consider n independent planar Brownian motions $B^1, \dots, B^n$ started from $B^1_0 = x_1, \dots, B^n_0 = x_n$ under the probability measure $P_{(x_1,\dots,x_n)}$. For all $j \in \{1, \dots, n\}$ and $R > 1$, we put

$$T^j_R = \inf\{t > 0, |B^j_t| = R\}.$$

We then define for all $x = (x_1, \dots, x_n) \in C^n$ (where $C = C(0,1) = \{z; |z| = 1\}$ denotes the unit circle),

$$p_x(R) = P_x(B^1_{[0,T^1_R]} \cup \dots \cup B^n_{[0,T^n_R]} \text{ does not disconnect } 0 \text{ from } \infty)$$

and

$$p^*_n(R) = \sup_{x \in C^n} p_x(R).$$

It is easy to see that the classical subadditivity argument can be adapted (see, for example, Lawler 1991, p. 152) to show that, for all $R > 1$, $R' > 1$,

$$p^*_n(RR') \le p^*_n(R) p^*_n(R')$$

and consequently (see, for example, Lawler 1991, Lemma 5.2.1),

$$\lim_{R \to \infty} \frac{-\log p^*_n(R)}{\log R} = \gamma_n > 0$$

for some deterministic constant γ_n.

Proof of Proposition 2

Let us first notice that (4) shows readily that $\gamma_n \le n/2$ (which corresponds to the estimate of the probability that none of the n Brownian motions hits the ray $(-\infty, 0]$ before hitting a large circle). We are now going to prove Proposition 2. We use the same notation as in Sections 1 and 4. We consider n independent planar Brownian motions $Z^1 = X^1 + \mathrm{i}Y^1, \dots, Z^n = X^n + \mathrm{i}Y^n$ started from 0, and put, for all $r > 0$, for all $j \in \{1, \dots, n\}$,

$$\hat{T}^j_r = \inf\{t > 0, X^j_t = r\}.$$

We have

$$\begin{aligned}
p^*_n(R) &\ge p_{(1,\dots,1)}(R) \\
&\ge P(\exists f : (-\infty, r] \to \mathbb{R}, \text{ continuous}, \forall j \in \{1, \dots, n\}, \forall t \in [0, \hat{T}^j_r], |Y^j_t - f(X^j_t)| < \pi) \\
&\ge \sum_{U \in \Lambda_N} P(D_U)^n \\
&\ge \frac{2^N}{\pi} \exp\left(\frac{-nr}{2\cos^2\theta}\right) \\
&\ge \frac{1}{2\pi} \exp\left\{ \left(-\frac{n}{2} + \frac{\log 2}{\pi} \tan\theta - \frac{n}{2} \tan^2\theta \right) \log R \right\}.
\end{aligned}$$

Proposition 2 follows, taking $\tan\theta = (\log 2)/(n\pi)$.

Remarks

It is straightforward to show that for all $n \ge 1$, $m \ge 1$, (we put $\gamma_1 = \gamma$),

$$\gamma_{m+n} \geq \gamma_m + \gamma_n.$$

Proposition 5.2.1. in Lawler (1991) then yields

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = \eta,$$

where $\eta = \sup_{n>0}(\gamma_n/n)$. As $\gamma_n \leq n/2$ for all $n \geq 1$, $\gamma \leq \frac{1}{2}$.

Using the exact value (i.e. $\xi(2,1) = 2$ with the notation of Section 6 below) of the intersection exponent of two paths versus one path (see Lawler 1989, or 1991, Chapter 5, Eq. (5.9)), and the analyticity of the mapping $z \to z^2$, it is possible to show that $\gamma_3 \geq 1$, which implies that $\eta \geq \frac{1}{3}$. Needless to say this last estimate is very crude (one expects γ_n/n to be strictly increasing). In Burdzy and Werner (1994), we show that $\gamma_6 > 2$, which then implies that $\eta > \frac{1}{3}$. It would, of course, be of interest to identify η; $\eta = \frac{1}{2}$ looks like a reasonable conjecture.

6. Intersection exponents

We now remark that the same arguments can be adapted to show that intersection exponents for two-dimensional Brownian motion (or two-dimensional random walks, since they are equal, as shown in Burdzy and Lawler 1990a) are not equal to their 'trivial' upper bounds.

Let us briefly recall (see Burdzy and Lawler 1990a, and Lawler 1991 for more details) the definition of these exponents: If $n \geq 1$, $m \geq 1$, if $B^1, \ldots, \ldots, B^{n+m}$ denote independent two-dimensional Brownian motions started from $B_0^1 = \ldots = B_n^0 = 1$ and $B_0^{n+1} = \ldots = B_0^{n+m} = -1$, and if for all $j \in \{1, \ldots, n+m\}$, and $R > 1$,

$$T_R^j = \inf\{t > 0, |B_t^j| = R\},$$

then the exponent $\xi(n,m)$ is defined as follows:

$$\xi(n,m) = \lim_{R\to\infty} \frac{-\log P((\cup_{1\leq j\leq n} B^j_{[0,T_R^j]}) \cap (\cup_{n+1\leq j\leq n+m} B^j_{[0,T_R^j]}) = \emptyset)}{\log R}.$$

The exact value of $\xi(n,m)$ is not known except for $\xi(2,1) = 2$ (see Lawler 1989); see also Burdzy and Lawler (1990b) and Lawler (1993) for some estimates.

The 'trivial' upper bound of $\xi(n,m)$ corresponds to the fact that if $B^1, \ldots, B^n$ stay in a wedge W and if $B^{n+1}, \ldots, B^{n+m}$ stay in W^c, then they do not intersect. More precisely, if $W = \{re^{i\theta}, |\theta| < \pi\lambda\}$ with $\lambda < 1$, then (4) shows immediately that

$$P(B^1_{[0,T_R^1]} \subset W) \geq \frac{2}{\pi} R^{-1/(2\lambda)},$$

and consequently:

$$\xi(n,m) \leq \frac{n}{2\lambda} + \frac{m}{2(1-\lambda)}.$$

Hence, for $\lambda = \sqrt{n}/(\sqrt{n} + \sqrt{m})$:

$$\xi(n,m) \leq \frac{(\sqrt{n}+\sqrt{m})^2}{2}. \tag{8}$$

The arguments of the proof of Proposition 1 can be easily adapted to obtain (slightly) improved upper bounds, which imply that inequality (8) is strict. If $Z^1 = X^1 + \mathrm{i}Y^1, \ldots, Z^{n+m} = X^{n+m} + \mathrm{i}Y^{n+m}$ denote $n+m$ planar Brownian motions started from $Z_0^1 = \ldots = Z_0^n = 0$, $Z_0^{n+1} = \ldots = Z_0^{n+m} = \mathrm{i}\pi$, and if, for all $j \in \{1, \ldots, n+m\}$ and all $r > 0$, we put

$$\hat{T}_r^j = \inf\{t > 0,\ X_t^j = r\},$$

it is easy to see that for all $\lambda \in (0,1)$

$$\begin{aligned} &P((\cup_{1\le j\le n} B^j_{[0,T^j_R]}) \cap (\cup_{n+1\le j\le n+m} B^j_{[0,T^j_R]}) = \emptyset) \\ &\quad \ge P(\exists f : (-\infty, r),\ \text{continuous},\ \forall i \in \{1,\ldots,n\},\ \forall t \in [0, \hat{T}_r^i],\ |Y_t^i - f(X_t^i)| < \lambda\pi, \\ &\qquad \text{and}\ \forall j \in \{n+1,\ldots,n+m\},\ \forall t \in [0, \hat{T}_r^j],\ |Y_t^j - f(X_t^j) - \pi| < (1-\lambda)\pi). \end{aligned}$$

The same argument as in Section 5.2 (we can define N as the integer part of $(r\tan\theta)/(\lambda\pi)$) then shows easily that for all $\theta \in (0, \pi/2)$ and $\lambda \in (0,1)$,

$$\xi(n,m) \le (1 + \tan^2\theta)\left(\frac{n}{2\lambda} + \frac{m}{2(1-\lambda)}\right) - \frac{\log 2}{\lambda\pi}\tan\theta.$$

For $\lambda = \sqrt{n}/(\sqrt{n} + \sqrt{m})$ and $\tan\theta = (\log 2)/\{\pi\sqrt{n}(\sqrt{n} + \sqrt{m})\}$, the previous inequality yields:

$$\xi(n,m) \le \frac{(\sqrt{n} + \sqrt{m})^2}{2} - \frac{(\log 2)^2}{2n\pi^2}. \tag{9}$$

Note that this choice of (λ, θ) is not optimal if $m > n$, so that in this case, inequality (9) is strict.

These estimates give new information on $\xi(n,m)$ only for $m \ge 3$; Burdzy and Lawler (1990b) have used the explicit value $\xi(2,1) = 2$ to show that $\xi(1,1) < \frac{3}{2}$, which is a much better upper bound than (9) for $n = m = 1$. Similarly, Lawler (1993) proved that $\xi(2,2) \le 3$, which is also a much better estimate than (9). However, their methods do not seem to give any upper bounds for $\xi(n,m)$ for $m \ge 3$.

Note that Beurling's theorem yields readily that for all $n \ge 1$, $m \ge 1$, $\xi(n,m) \ge \gamma + (n+m-1)/2$. Hence, for all $n \ge 1$, $m \ge 1$,

$$\frac{1}{2} \le \frac{\xi(n,m)}{n} < \frac{1}{2}(1 + \sqrt{m/n})^2 \tag{10}$$

and consequently,

$$\lim_{n\to\infty} \frac{\xi(n,k)}{n} = \frac{1}{2} \tag{11}$$

for every fixed $k \ge 1$. Again, it would be interesting to identify the limits $\alpha(p,q)$ of $\xi(pn, qn)/n$ as $n \to \infty$ for fixed $p \ge 1$, $q \ge 1$. The subadditivity argument shows that $\alpha(p,q)$ exists and the previous estimates show that $\alpha(p,q) \in [(p+q)/2, (\sqrt{p} + \sqrt{q})^2/2]$.

We take this opportunity to make the following remark: (11) shows that for all $k \ge 1$, $\lim_{n\to\infty}(\xi(3n-k, k)/n) = \frac{3}{2}$. On the other hand, $\xi(2n, n) \ge n\xi(2,1) = 2n$, so that for all $k \ge 1$ and for all large enough n, $\xi(3n-k, k) < \xi(2n, n)$. This suggests the following conjecture: for all $N \ge 4$, $1 \le n < n' \le N/2$,

$$\xi(N-n, n) < \xi(N-n', n'). \tag{12}$$

Acknowledgements

This paper would not exist without very helpful and enlightening discussions with Chris Burdzy, to whom I am very grateful. I also thank Greg Lawler, who initially suggested this problem.

References

Ahlfors, L.V. (1966) *Complex Analysis.* New York: McGraw-Hill.

Ahlfors, L.V. (1973) *Conformal Invariants,* Topics in Geometric Function Theory. New York: McGraw-Hill.

Beurling, A. (1933) Etudes sur un problème de majoration. Thèse, Uppsala.

Burdzy, K. and Lawler, G.L. (1990a) Non-intersection exponents for random walk and Brownian motion. Part I: Existence and an invariance principle. *Probab. Theory Related Fields,* **84**, 393–410.

Burdzy, K. and Lawler, G.L. (1990b) Non-intersection exponents for random walk and Brownian motion. Part II: Estimates and applications to a random fractal. *Ann. Probab.*, **18**, 981–1009.

Burdzy, K., Lawler, G.L. and Polaski, T. (1989) On the critical exponent for random walk intersections, *J. Statist. Phys.*, **56**, 1–12.

Burdzy, K. and Werner, W. (1994) No triple point of planar Brownian motion is accessible. *Ann. Probab.* (to appear).

Duplantier, B. and Kwon, K.-H. (1988) Conformal invariance and intersection of random walks. *Phys. Rev. Lett.*, **61**, 2514–2517.

Duplantier, B., Lawler, G.F., Le Gall, J.F. and Lyons, T.J. (1993) The geometry of the Brownian curve. In *Probabilités et Analyse Stochastique, Tables Rondes de St-Chéron Janvier 1992. Bull. Sci. Math.* (2), **117**.

Itô, K. and McKean, H.P. (1965) *Diffusion Processes and Their Sampling Paths.* New York: Springer-Verlag.

Lawler, G.L. (1989) Intersection of random walks with random sets. *Israel J. Math.*, **65**, 113–132.

Lawler, G.L. (1991) *Intersection of random walks.* Boston: Birkhäuser.

Lawler, G.L. (1993) A discrete analogue of a theorem of Makarov. *Combin. Probab. Comput.*, **2**, 181–200.

Lawler, G.L. and Puckette, E.E. (1994) The disconnecting exponent for simple random walk. Preprint.

Mandelbrot, B.B. (1982) *The Fractal Geometry of Nature.* New York: Freeman.

Oksendal, B. (1983) Projection estimates for harmonic measure. *Ark. Mat.*, **21**, 191–203.

Port, S.C. and Stone, C.J. (1978) *Brownian Motion and Classical Potential Theory.* New York: Academic Press.

Revuz, D. and Yor, M. (1991) *Continuous Martingales and Brownian Motion.* Berlin: Springer-Verlag.

Werner, W. (1994) Beurling's projection theorem via linear Brownian notion. *Math. Proc. Cambridge Phil. Soc.* (to appear).

Received July 1994 and revised February 1995