

Verifiable conditions for the irreducibility and aperiodicity of Markov chains by analyzing underlying deterministic models

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We consider Markov chains that obey the following general non-linear state space model: $\Phi_{k+1} = F(\Phi_k, \alpha(\Phi_k, U_{k+1}))$ where the function F is C^1 while α is typically discontinuous and $\{U_k : k \in \mathbb{Z}_{>0}\}$ is an independent and identically distributed process. We assume that for all x , the random variable $\alpha(x, U_1)$ admits a density p_x such that $(x, w) \mapsto p_x(w)$ is lower semi-continuous.

We generalize and extend previous results that connect properties of the underlying deterministic control model to provide conditions for the chain to be φ -irreducible and aperiodic. By building on those results, we show that if a rank condition on the controllability matrix is satisfied for all x , there is equivalence between the existence of a globally attracting state for the control model and φ -irreducibility of the Markov chain. Additionally, under the same rank condition on the controllability matrix, we prove that there is equivalence between the existence of a steadily attracting state and the φ -irreducibility and aperiodicity of the chain. The notion of steadily attracting state is new. We additionally derive practical conditions by showing that the rank condition on the controllability matrix needs to be verified only at a globally attracting state (resp. steadily attracting state) for the chain to be a φ -irreducible T -chain (resp. φ -irreducible aperiodic T -chain).

Those results hold under considerably weaker assumptions on the model than previous ones that would require $(x, u) \mapsto F(x, \alpha(x, u))$ to be C^∞ (while it can be discontinuous here). Additionally the establishment of a *necessary and sufficient* condition on the control model for the φ -irreducibility and aperiodicity without a structural assumption on the control set is novel – even for Markov chains where $(x, u) \mapsto F(x, \alpha(x, u))$ is C^∞ .

We illustrate that the conditions are easy to verify on a non-trivial and non-artificial example of Markov chain arising in the context of adaptive stochastic search algorithms to optimize continuous functions in a black-box scenario.

Keywords: aperiodicity; controllability matrix; deterministic control model; Evolution Strategies; globally attracting state; irreducibility; Markov chains; T -chain

1. Introduction

Markov chain theory is widely applied for analyzing methods arising in different domains like machine learning, time series analysis, statistics or optimization. Prior to establishing stability properties like geometric ergodicity or using sample path theorems, one often needs to prove basic standard properties such as φ -irreducibility and aperiodicity (see, for instance, the Law of

Large Numbers in [11], Theorem 17.0.1, or the aperiodic and geometric ergodic theorems [11], Theorem 13.0.1 and 15.0.1). In addition, to prove the existence of an invariant probability distribution or geometric ergodicity, it is often practical to use *drift* conditions that roughly speaking state that outside a specific set, the conditional expected progress measured in terms of an appropriate non-negative potential function should be negative. The specific sets are typically so-called *small sets* and they need to be identified in order to prove a drift condition.

Establishing φ -irreducibility, aperiodicity and identifying small sets can turn out to be very challenging. In the domain of time series analysis, this observation was already done and several works developed tools to facilitate this task. This includes conditions on the model parameters of specific time series (e.g., bilinear models [14] or nonlinear autoregressive time series [5]), general conditions on the underlying deterministic control model [9], [11], Chapter 7, or on small or petite sets [2,4]. One notable work in the latter direction is presented in [4] where the equivalence between φ -irreducibility, aperiodicity and the T -chain property with conditions on reachable petite or small sets has been shown. Remarkably, these results hold under weak conditions (at most weak-Feller) for bounded positive kernels. However, in practice it can be difficult to show that a set is small or petite without having first shown that the Markov chain is a T -chain. Hence these conditions, albeit weak, can turn out to be difficult to verify. Small sets can be shown to exist using a rank condition on the controllability matrix as shown in [10], Proposition 2.1, or Proposition 4.1 in this paper, and then the results obtained in [4] are similar to the ones presented here.

For Markov chains following a non-linear state space model of the form

$$\Phi_{k+1} = G(\Phi_k, U_{k+1}), \quad k \in \mathbb{Z}_{\geq 0}, \quad (1)$$

where $G : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ is C^∞ with $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^p$ open sets and $\{U_k : k \in \mathbb{Z}_{>0}\}$ is an independent identically distributed (i.i.d.) process, independent of Φ_0 , some practical tools for proving φ -irreducibility, aperiodicity and identify that compact are small sets rely on investigating the underlying deterministic *control model* [11], Chapter 7, [9,10]. They connect structural and stability properties of the deterministic control model to structural and stability aspects of the associated Markov chains. In contrast to the works mentioned above [2,4], the particularly attracting feature is that the tools entail manipulating deterministic sequences of possible paths followed by the underlying deterministic algorithm and are thus relatively straightforward to verify, at the cost of being constrained to the model (1) where in particular G is C^∞ .

The assumption that G is C^∞ is quite restrictive for some nonlinear state space models. Particularly, many Markov chains arising in the context of adaptive randomized optimization algorithms are associated to functions G that are discontinuous. Yet, we show in this paper that most of the results presented in [11], Chapter 7, holding for chains following (1) generalize to a broader model presented below – that naturally arises in the context of adaptive comparison-based algorithms. More precisely, we consider Markov chains following the model

$$\Phi_{k+1} = F(\Phi_k, \alpha(\Phi_k, U_{k+1})), \quad k \in \mathbb{Z}_{\geq 0}, \quad (2)$$

where for all k , $\Phi_k \in \mathcal{X}$ with \mathcal{X} an open subset of \mathbb{R}^n , $\{U_k : k \in \mathbb{Z}_{>0}\}$ is an i.i.d. process valued in \mathcal{U} an open subset of \mathbb{R}^m , $F : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X}$ is a C^1 function with \mathcal{W} an open subset of \mathbb{R}^p , $\alpha : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{W}$ is a measurable function – that can typically be discontinuous – and

for all $x \in \mathcal{X}$, the distribution μ_x of the random variable $\alpha(x, U_1)$ admits a density $p_x(\cdot)$ such that the function $(x, w) \mapsto p_x(w)$ is lower semi-continuous with respect to both variables. In the models (1) and (2), n, m, p belong to $\mathbb{Z}_{>0}$. Note that (1) and (2) are linked via the relation $G(x, u) = F(x, \alpha(x, u))$, so discontinuities of α may render G discontinuous, hence not satisfying the smooth condition of [11], while F itself can be C^1 or smooth. Remark that model (2) is actually a strict generalization of (1): by setting $\alpha(x, u) = u$ in (2) we indeed recover (1) such that we will talk about a single model – model (2) – that can reduce to (1) if $\alpha(x, u) = u$.

Toy examples. To motivate the general model (2), consider first an additive random walk on \mathbb{R} defined by choosing for Φ_0 an arbitrary distribution on \mathbb{R} and for all $k \in \mathbb{Z}_{\geq 0}$ by

$$\Phi_{k+1} = \Phi_k + U_{k+1}, \quad (3)$$

where $\{U_k : k \in \mathbb{Z}_{>0}\}$ is an independent and identically distributed Gaussian process with each U_k distributed as a standard normal distribution, that is $U_k \sim \mathcal{N}(0, 1)$ for all k . This Markov chain follows the model (1) with $G(x, u) = x + u$ and hence model (2) with $F(x, u) = x + u$ and $\alpha(x, u) = u$. The random variable $\alpha(x, U_1)$ admits the density

$$p(w) = p_{\mathcal{N}}(w) := \frac{1}{\sqrt{2\pi}} \exp(-w^2/2). \quad (4)$$

We describe now a variation of this Markov chain where the update of Φ_k is also additive and more precisely $F(x, w) = x + w$. Consider indeed a simple (naive) iterative optimization algorithm on \mathbb{R} , aiming at *minimizing* an objective function $f : \mathbb{R} \rightarrow \mathbb{R}$ without using any derivatives of f . The algorithm delivers at each iteration k , an estimate of the optimum of the function encoded within the random variable Φ_k . The first estimate Φ_0 is sampled from an arbitrary distribution on \mathbb{R} . To obtain the estimate Φ_{k+1} from Φ_k , two candidate solutions are sampled around Φ_k

$$\tilde{\Phi}_{k+1}^i = \Phi_k + U_{k+1}^i, \quad \text{for } i = 1, 2,$$

where for all $k \geq 1$, $\{U_k^i : i = 1, 2\}$ are independent following each a normal distribution $\mathcal{N}(0, 1)$, and $\{U_k = (U_k^1, U_k^2) : k \geq 1\}$ is an independent and identically distributed process. Those candidate solutions are evaluated on the objective function f and ranked according to their f values. A permutation that contains the index of the ordered candidate solutions is extracted, that is, \mathcal{S} is a permutation of the set $\{1, 2\}$ such that¹

$$f(\Phi_k + U_{k+1}^{\mathcal{S}(1)}) \leq f(\Phi_k + U_{k+1}^{\mathcal{S}(2)}). \quad (5)$$

The new estimate of the solution corresponds to the best of the two sampled candidate solutions, that is

$$\Phi_{k+1} = \Phi_k + U_{k+1}^{\mathcal{S}(1)}. \quad (6)$$

¹The unicity of the permutation can be guaranteed by imposing that if $f(\Phi_k + U_{k+1}^{\mathcal{S}(1)}) = f(\Phi_k + U_{k+1}^{\mathcal{S}(2)})$ then $\mathcal{S}(1) = 1$.

This algorithm is a simplification of some randomized adaptive optimization algorithms and particularly of evolution strategies [7]. The update (6) loosely drives Φ_k towards better solutions. This algorithm is not meant to be a good optimization algorithm but serves as illustration for our general model. We will present more reasonable optimization algorithms in Section 5.

Remark that $U_{k+1}^{S(1)} = (U_{k+1}^1 - U_{k+1}^2)1_{\{f(\Phi_k + U_{k+1}^1) \leq f(\Phi_k + U_{k+1}^2)\}} + U_{k+1}^2$ and define the function $\alpha : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\alpha(x, (u^1, u^2)) = (u^1 - u^2)1_{\{f(x+u^1) \leq f(x+u^2)\}} + u^2 \quad (7)$$

such that $U_{k+1}^{S(1)} = \alpha(\Phi_k, (U_{k+1}^1, U_{k+1}^2))$. The update of Φ_k then satisfies

$$\Phi_{k+1} = \Phi_k + \alpha(\Phi_k, (U_{k+1}^1, U_{k+1}^2)) = F(\Phi_k, \alpha(\Phi_k, (U_{k+1}^1, U_{k+1}^2))), \quad (8)$$

where $F(x, w) = x + w$. Suppose that $f(x) = x^2$, then the function α is discontinuous. (Indeed, for u^1 and u^2 different such that $f(x + u^1) = f(x + u^2)$, a small (continuous) change in u^2 can lead to $\alpha(x, (u^1, u^2))$ jumping from u^1 to u^2 .) Hence, using the modeling via (1), the corresponding function $G(x, u) = x + \alpha(x, u)$ with $u = (u^1, u^2)$ is discontinuous and does not satisfy the basic assumptions of the model such that the results presented in [11], Chapter 7, cannot be directly applied.

For all x , the random variable $\alpha(x, U_1)$ admits a density equal to

$$p_x(w) = 2p_{\mathcal{N}}(w) \int 1_{\{(x+w)^2 < (x+u)^2\}} p_{\mathcal{N}}(u) du, \quad (9)$$

where $p_{\mathcal{N}}(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)$ is the density of a standard normal distribution. The function $(x, w) \mapsto p_x(w)$ is continuous as a consequence of the Lebesgue dominated convergence theorem and hence lower semi-continuous.

Those examples serve as illustration to the model underlying the paper. Yet φ -irreducibility, aperiodicity and identification of small sets for those Markov chains can be easily proven directly considering the transition kernel

$$P(x, A) = \int 1_A(F(x + w)) p_x(w) dw = \int 1_A(y) p_x(x - y) dy. \quad (10)$$

We will present in Section 5 a more complex example – where F is more complex and the density is lower semi-continuous but not continuous – where the tools developed in the paper are needed to easily prove φ -irreducibility, aperiodicity and identify small sets.

As sketched on this toy example, the function $\alpha(\cdot, \cdot)$ for defining a Markov chain following (2) naturally arises – in the context of randomized optimization – from the “selection” of the step to update the state of the algorithm: in this simple example, we select the best step via (5). This selection gives a discontinuous α function which leads to a discontinuous underlying G function.

While the Markov chains in (3) and (8) look sensibly different, we will see that both chains have the same underlying deterministic (control) model and consequently, the φ -irreducibility and aperiodicity of one chain implies the φ -irreducibility and aperiodicity of the other one.

Summary of main contributions. Our first contribution is to generalize the definition of the deterministic control model associated to a Markov chain following (1) [9–11] to a Markov chain following (2) by extending the definition of the control set to a set of open sets – indexed by the initial conditions and time steps – where the extended lower semi-continuous densities $p_x^k(\cdot)$ are strictly positive. Our first main result concerns the φ -irreducibility: We prove that under a proper controllability condition – formulated as a condition on the rank of the controllability matrix that needs to be satisfied for every x – the existence of a globally attracting state for the underlying deterministic control model is *equivalent* to the Markov chain being φ -irreducible. This result generalizes to our context Proposition 7.2.6 presented in [11]. It heavily relies on [10], Theorem 2.1(iii), which can easily be transposed to our setting. We then establish a similar result for the φ -irreducibility and aperiodicity of the chain. We introduce the notion of steadily attracting state (implying global attractivity) and we prove that under the same controllability condition, the existence of a steadily attracting state is *equivalent* to the φ -irreducibility and aperiodicity of the chain. This improves previous results derived for model (1) where the control set is assumed to be connected.

We additionally derive practical conditions by showing that the rank conditions on the controllability matrix needs to be satisfied at a globally attracting state (resp. steadily attracting state) *only* to imply the φ -irreducibility and T -chain property (resp. aperiodicity, φ -irreducibility and T -chain property) of the Markov chain under the existence of a globally attracting (resp. steadily attracting) state. For the aperiodicity, our assumption of existence of a steadily attracting state is significantly weaker than the one of asymptotic controllability of the deterministic model made in [9], Proposition 3.2. We illustrate how to use the practical conditions on a nontrivial example arising in stochastic optimization.

This paper is organized as follows. In Section 2, we introduce the Markov chain background necessary for the paper. In Section 3, we define the deterministic control model associated to the Markov chain and the different notions related to it. In Section 4, we present our main results related to φ -irreducibility and aperiodicity. In Section 5, we apply the results to the toy examples presented in this [Introduction](#) and to a stochastic optimization algorithm.

Notations. We denote \mathbb{Z} the set of integers, $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers $\{0, 1, 2, \dots\}$ and $\mathbb{Z}_{>0}$ the set of positive integers $\{1, 2, \dots\}$. We denote \mathbb{R}_{\geq} the set of nonnegative real numbers and $\mathbb{R}_{>}$ of positive real numbers. For $n \in \mathbb{Z}_{>0}$, we denote \mathbb{R}^n the n -dimensional set of real numbers and μ^{Leb} the n -dimensional Lebesgue measure. The Borel sigma-field of a topological space \mathcal{X} is denoted $\mathcal{B}(\mathcal{X})$. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $B(x, \varepsilon)$ denotes the open ball of center x and radius ε . For $A \in \mathcal{B}(\mathcal{X})$, we denote A^c the complement of A in \mathcal{X} . For u a real vector of \mathbb{R}^n , u^T denotes the transpose of u . For f and g real-valued functions, following the Bachmann-Landau notation, $g = o(f)$ denotes that g is a little-o of f .

2. Background on Markov chains

In this paper, we consider the state-space \mathcal{X} being an open subset of \mathbb{R}^n and equipped with its Borel sigma-field $\mathcal{B}(\mathcal{X})$. A *kernel* K is a function on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that $K(\cdot, A)$ is measurable for all A and for each $x \in \mathcal{X}$, $K(x, \cdot)$ is a signed measure.

A kernel K is *substochastic* if it is nonnegative and satisfies $K(x, \mathcal{X}) \leq 1$ for all $x \in \mathcal{X}$. It is a *transition kernel* if it satisfies $K(x, \mathcal{X}) = 1$ for all $x \in \mathcal{X}$. Given a Markov chain $\Phi = \{\Phi_k : k \in \mathbb{Z}_{\geq 0}\}$, we denote P^k , $k \in \mathbb{Z}_{>0}$, its *k-step transition kernel* defined by

$$P^k(x, A) = \Pr(\Phi_k \in A | \Phi_0 = x), \quad x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X}). \quad (11)$$

We shall be concerned with the question of φ -irreducibility, that is whether there exists a non-trivial measure φ on $\mathcal{B}(\mathcal{X})$ such that for all $A \in \mathcal{B}(\mathcal{X})$ with $\varphi(A) > 0$

$$\sum_{k \in \mathbb{Z}_{>0}} P^k(x, A) > 0, \quad \text{for all } x \in \mathcal{X}.$$

If such a φ exists, the chain is called φ -irreducible. A φ -irreducible Markov chain admits a *maximal irreducibility measure*, ψ , which dominates any other irreducibility measure, meaning for $A \in \mathcal{B}(\mathcal{X})$, $\psi(A) = 0$ implies $\varphi(A) = 0$ for any irreducibility measure φ (see [11], Theorem 4.0.1, for more).

We also need the notion of T -chain defined in the following way. Let b be a probability distribution on $\mathbb{Z}_{\geq 0}$, and let K_b denote the probability transition kernel defined by

$$K_b(x, A) := \sum_{k \in \mathbb{Z}_{\geq 0}} b(k) P^k(x, A), \quad x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X}). \quad (12)$$

Let T be a substochastic transition kernel which satisfies

$$K_b(x, A) \geq T(x, A), \quad \text{for all } x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X}),$$

and such that $T(\cdot, A)$ is a lower semi-continuous function for all $A \in \mathcal{B}(\mathcal{X})$. Then T is called a *continuous component* of K_b . If a Markov chain Φ admits a probability distribution b on $\mathbb{Z}_{\geq 0}$ such that K_b possesses a continuous component T satisfying $T(x, \mathcal{X}) > 0$ for all $x \in \mathcal{X}$, then Φ is called a *T-chain*.

A set $C \in \mathcal{B}(\mathcal{X})$ is called *petite* if there exists b a probability distribution on $\mathbb{Z}_{\geq 0}$ and ν_b a non-trivial measure on $\mathcal{B}(\mathcal{X})$ such that

$$K_b(x, A) \geq \nu_b(A), \quad \text{for all } x \in C, A \in \mathcal{B}(\mathcal{X}).$$

The set C is then called a ν_b -petite set.

Similarly, a set $C \in \mathcal{B}(\mathcal{X})$ is called *small* if there exists $k \in \mathbb{Z}_{>0}$ and ν_k a nontrivial measure on $\mathcal{B}(\mathcal{X})$ such that

$$P^k(x, A) \geq \nu_k(A), \quad \text{for all } x \in C, A \in \mathcal{B}(\mathcal{X}). \quad (13)$$

The set C is then called a ν_k -small set. Note that a ν_k -small set is ν_{δ_k} -petite, where δ_k is the Dirac measure at k .

Last, we will derive conditions for a Markov chain following (2) to be aperiodic. We therefore remind the definition of an aperiodic Markov chain. Suppose that Φ is a φ -irreducible Markov chain. For $d \in \mathbb{Z}_{>0}$, let $(D_i)_{i=1,\dots,d} \in \mathcal{B}(\mathcal{X})^d$ be a sequence of disjoint sets. We call $(D_i)_{i=1,\dots,d}$ a d -cycle if:

- (i) $P(x, D_{i+1}) = 1$ for all $x \in D_i$ and $i = 0, \dots, d-1 \pmod{d}$,
- (ii) $\varphi((\bigcup_{i=1}^d D_i)^c) = 0$ for all φ -irreducibility measure of Φ .

If Φ is φ -irreducible, there exists a d -cycle with $d \in \mathbb{Z}_{>0}$ [11], Theorem 5.4.4. The largest d for which there exists a d -cycle is called the *period* of Φ . If the period of Φ is 1, then Φ is called *aperiodic*.

For more on Markov chains theory, we refer to [11,12].

3. Deterministic control model: Definitions and first results

From now on, we consider a Markov chain defined via (2). We pose the following basic assumptions on the initial condition Φ_0 and the disturbance process $\mathbf{U} := \{U_k : k \in \mathbb{Z}_{>0}\}$.

- A1. (Φ_0, \mathbf{U}) are random variables on a probability space $(\Omega, \mathcal{F}, P_{\Phi_0})$;
- A2. Φ_0 is independent of \mathbf{U} ;
- A3. \mathbf{U} is an independent and identically distributed process.

We additionally assume that

- A4. For all $x \in \mathcal{X}$, the distribution μ_x of the random variable $\alpha(x, U_1)$ admits a density $p_x(\cdot)$, such that the function $(x, w) \mapsto p_x(w)$ is lower semi-continuous with respect to both variables;
- A5. The function $F : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X}$ is C^1 .

3.1. Deterministic control model (CM(F))

The attractive feature of the results presented in [11], Chapter 7, for Markov chains following model (1) is that they entail manipulating deterministic trajectories of an underlying deterministic control model. The Markov chains considered in the present paper strictly generalize the Markov chains following (1) by simply assuming that $\alpha(x, u) = u$. We here generalize the underlying deterministic control model introduced for Markov chains following (1) to Markov chains following (2).

We consider first the *extended transition map* function [10] $S_x^k : \mathcal{W}^k \rightarrow \mathcal{X}$ defined inductively for $k \in \mathbb{Z}_{\geq 0}$, $x \in \mathcal{X}$ and $\mathbf{w} = (w_1, \dots, w_k) \in \mathcal{W}^k$ by

$$\begin{aligned} S_x^k(\mathbf{w}) &:= F(S_x^{k-1}(w_1, \dots, w_{k-1}), w_k), & k \in \mathbb{Z}_{>0}, \\ S_x^0 &:= x. \end{aligned} \tag{14}$$

If the function F is C^k , then the function $(x, \mathbf{w}) \mapsto S_x^k(\mathbf{w})$ is also C^k with respect to both variables (Lemma A.2).

The extended probability density is the function p_x^k defined inductively for all $k \in \mathbb{Z}_{>0}$, $x \in \mathcal{X}$ and $\mathbf{w} = (w_1, \dots, w_k) \in \mathcal{W}^k$ by

$$\begin{aligned} p_x^k(\mathbf{w}) &:= p_x^{k-1}(w_1, \dots, w_{k-1}) p_{S_x^{k-1}(w_1, \dots, w_{k-1})}(w_k), \\ p_x^1(w_1) &:= p_x(w_1). \end{aligned} \quad (15)$$

Let $W_1 := \alpha(x, U_1)$ and $W_k := \alpha(S_x^{k-1}(W_1, \dots, W_{k-1}), U_k)$ for all $k \in \mathbb{Z}_{>0}$, the extended probability function p_x^k is a probability density function of (W_1, W_2, \dots, W_k) . In the case where $\alpha(x, u) = u$, denoting p the lower semi-continuous density of U_1 the extended probability density reduces to $p^k(\mathbf{w}) = p(w_1), \dots, p(w_k)$ (that is, in the case of a Markov chain that also follows (1)).

The function $(x, \mathbf{w}) \mapsto p_x^k(\mathbf{w})$ is lower semi-continuous as a consequence of the lower semi-continuity of the function $(x, w) \mapsto p_x(w)$ and of the continuity of the function $(x, w) \mapsto F(x, w)$ (Lemma A.3). This implies that the set defined for all $k \in \mathbb{Z}_{>0}$ and for all $x \in \mathcal{X}$ as

$$\mathcal{O}_x^k := \{\mathbf{w} \in \mathcal{W}^k \mid p_x^k(\mathbf{w}) > 0\} \quad (16)$$

is open. Remark that given that p_x^k is a density, \mathcal{O}_x^k is non-empty for all $k \in \mathbb{Z}_{>0}$.

In the case of a Markov chain following (1) modeled via $\alpha(x, u) = u$, the set \mathcal{O}_x^k is the k -fold product $\mathcal{O}_{\mathcal{X}}^k$ where $\mathcal{O}_{\mathcal{X}}$ is the open control set $\{x : p(x) > 0\}$.

The deterministic system

$$S_x^{k+1}(w_1, \dots, w_{k+1}) = F(S_x^k(w_1, \dots, w_k), w_{k+1}), \quad k \in \mathbb{Z}_{>0}$$

for $(w_1, \dots, w_k, w_{k+1})$ in the open set \mathcal{O}_x^{k+1} is the associated deterministic control model $\text{CM}(F)$ for Markov chains following (2). Given an initial condition $x \in \mathcal{X}$, the control model $\text{CM}(F)$ is characterized by F and the sets \mathcal{O}_x^k for $k \in \mathbb{Z}_{>0}$. Remark that the control sequence (w_1, \dots, w_k) lies in the set \mathcal{O}_x^k that depends on the initial condition x . In contrast when $\alpha(x, u) = u$, the control sequence (w_1, \dots, w_k) lies in $\mathcal{O}_{\mathcal{X}}^k$ which is independent of x . In analogy to this later case, the sets \mathcal{O}_x^k are termed control sets.

Example 1 (CM(F) for the toy examples). For the Markov chains defined via (3) and (8) and the densities (4) and (9), $\text{CM}(F)$ is defined by $F(x, w) = x + w$ and the control sets \mathcal{O}_x^k that equal \mathbb{R}^k for all $x \in \mathbb{R}$ and for all k .

Hence, since both examples share the same F and the same control sets \mathcal{O}_x^1 , we will see later on that proving the φ -irreducibility or aperiodicity for both Markov chains via the conditions we derive in the paper is the same.

The extended transition map function and the extended probability density can be used to express the transition kernel of Φ in the following way

$$P^k(x, A) = \int 1_A(S_x^k(\mathbf{w})) p_x^k(\mathbf{w}) d\mathbf{w}, \quad k \in \mathbb{Z}_{>0}, A \in \mathcal{B}(\mathcal{X}). \quad (17)$$

This expression will be useful in many proofs.

For a given initial point $x \in \mathcal{X}$, we will consider “deterministic paths” \mathbf{w} – where \mathbf{w} is in a control set \mathcal{O}_x^k – that bring to $A \in \mathcal{B}(\mathcal{X})$. We give more precisely the following definition of a k -steps path from x to A .

Definition 3.1 (k -steps path). For $x \in \mathcal{X}$, $A \in \mathcal{B}(\mathcal{X})$ and $k \in \mathbb{Z}_{>0}$, we say that $\mathbf{w} \in \mathcal{W}^k$ is a k -steps path from x to A if $\mathbf{w} \in \mathcal{O}_x^k$ and $S_x^k(\mathbf{w}) \in A$.

3.2. Accessibility, attracting and attainable states

Following [11], Chapter 7, for $x \in \mathcal{X}$ and $k \in \mathbb{Z}_{\geq 0}$ we define $A_+^k(x)$, the set of all states that can be reached from x in k steps by $\text{CM}(F)$, that is, $A_+^0(x) := \{x\}$ and

$$A_+^k(x) := \{S_x^k(\mathbf{w}) \mid \mathbf{w} \in \mathcal{O}_x^k\}.$$

The set of all points that can be reached from x is defined as

$$A_+(x) := \bigcup_{k \in \mathbb{Z}_{\geq 0}} A_+^k(x).$$

If for all $x \in \mathcal{X}$, $A_+(x)$ has nonempty interior, the deterministic control model $\text{CM}(F)$ is said to be *forward accessible* [8].

Example 2. For the Markov chains (3) and (8) with the common $\text{CM}(F)$ model defined in Example 1, $A_+^1(x) = \mathbb{R}$ and thus $A_+(x) = \mathbb{R}$. Therefore, $\text{CM}(F)$ is forward accessible.

We remind now the definition of a *globally attracting state* [11], Chapter 7. A point $x^* \in \mathcal{X}$ is called globally attracting if for all $y \in \mathcal{X}$,

$$x^* \in \Omega_+(y) := \bigcap_{N=1}^{+\infty} \overline{\bigcup_{k=N}^{+\infty} A_+^k(y)}. \quad (18)$$

As we show later on, the existence of a globally attracting state for $\text{CM}(F)$ is linked to the φ -irreducibility of the associated Markov chain. We establish here a basic yet useful proposition giving equivalent statements to the definition of a globally attracting state. This proposition will be heavily used in the different proofs.

Proposition 3.1 (Characterization of globally attracting states). *Suppose that Φ follows model (2) and that conditions A1–A4 hold. A point $x^* \in X$ is a globally attracting state if and only if one of the three following equivalent conditions holds:*

- (i) for all $y \in \mathcal{X}$, $x^* \in \overline{A_+(y)}$,
- (ii) for all $y \in \mathcal{X}$ and all open $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ containing x^* , there exists $k \in \mathbb{Z}_{>0}$ and a k -steps path from y to \mathcal{U} ,

(iii) for all $y \in \mathcal{X}$, there exists a sequence $\{y_k : k \in \mathbb{Z}_{>0}\}$ with $y_k \in A_+^k(y)$ from which a subsequence converging to x^* can be extracted.

We will show that in our context, globally attracting states for $CM(F)$ are equivalent to the notion of reachable states for the associated Markov chain. We remind that a point $x \in \mathcal{X}$ is called *reachable* [11], 6.1.2, for a Markov chain if for every open set $\mathcal{O} \in \mathcal{B}(\mathcal{X})$ containing x

$$\sum_{k=1}^{\infty} P^k(y, \mathcal{O}) > 0, \quad \forall y \in \mathcal{X}.^2$$

The equivalence between globally attracting states and reachable states relies on the following proposition.

Proposition 3.2. *Suppose that Φ follows model (2), that conditions A1–A4 hold and that the function F is continuous. Then for all $\mathcal{O} \in \mathcal{B}(\mathcal{X})$ open set, $x \in \mathcal{X}$ and $k \in \mathbb{Z}_{>0}$, the following statements are equivalent:*

- (i) *there exists a k -steps path from x to \mathcal{O} ,*
- (ii) *$P^k(x, \mathcal{O}) > 0$.*

We can now deduce the following corollary.

Corollary 3.1. *Suppose that Φ follows model (2), that conditions A1–A4 hold and that the function F is continuous. Then $x \in \mathcal{X}$ is globally attractive for $CM(F)$ if and only if it is reachable for the associated Markov chain.*

We introduce now two new definitions to characterize some specific states of the underlying deterministic control model. First of all, we have seen that a globally attracting state can be approached via the control model arbitrarily close from any other point in the state space. We introduce the notion of *attainable state* for states that can be visited in finite time from any other point. That is $x^* \in \mathcal{X}$ is an *attainable state* if

$$x^* \in A_+(y), \quad \text{for all } y \in \mathcal{X}. \tag{19}$$

This statement is actually equivalent to

$$x^* \in \bigcup_{k \in \mathbb{Z}_{>0}} A_+^k(y), \quad \text{for all } y \in \mathcal{X}, \tag{20}$$

²Another definition of reachability considers the sum from k equal 0 instead of 1. Both definitions are equivalent: $\sum_{k \in \mathbb{Z}_{>0}} P^k(y, \mathcal{O}) = \int_{\mathcal{X}} P(y, dz) \sum_{k \in \mathbb{Z}_{\geq 0}} P^k(z, \mathcal{O})$ with Tonelli's theorem. So if $\sum_{k \in \mathbb{Z}_{\geq 0}} P^k(z, \mathcal{O}) > 0$ for all $z \in \mathcal{X}$, $\sum_{k \in \mathbb{Z}_{>0}} P^k(y, \mathcal{O}) > 0$. The other implication is trivial, hence the equivalence between the two definitions.

which turns out to be often more practical to use.³ Comparing (19) and Proposition 3.1(i), we see that an attainable state is globally attracting. We will show in Proposition 3.6 that the existence of a globally attracting state under some conditions implies the existence of an attainable state.

Second, we introduce the notion of a *steadily attracting state*, whose existence as we will later on show, is linked to the φ -irreducibility and aperiodicity of the associated Markov chain. We say that a point x^* is a *steadily attracting state* if for any $y \in \mathcal{X}$ and any open $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ containing x^* , there exists $T \in \mathbb{Z}_{>0}$ such that for all $k \geq T$ there exists a k -steps path from y to \mathcal{U} . In the next proposition we state some quite immediate relations between globally attracting states and steadily attracting ones. Additionally, we provide a characterization of a steadily attracting state.

Proposition 3.3. *Suppose that Φ follows model (2) and that conditions A1–A4 hold. The following statements hold:*

- (i) *If $x^* \in \mathcal{X}$ is steadily attracting, then it is globally attracting.*
- (ii) *A state $x^* \in \mathcal{X}$ is steadily attracting if and only if for all $y \in \mathcal{X}$ there exists a sequence $\{y_k : k \in \mathbb{Z}_{>0}\}$ with $y_k \in A_+^k(y)$, which converges to x^* .*
- (iii) *Assume F is C^0 . If there exists a steadily attracting state, then all globally attracting states are steadily attracting.*

Proposition A.5 also establishes that under a controllability condition, a globally attracting state $x^* \in \mathcal{X}$ where we can come back in a and b steps with $\gcd(a, b) = 1$ is steadily attracting (that is $S_{x^*}^a(\mathbf{w}_a) = S_{x^*}^b(\mathbf{w}_b) = x^*$ for some $\mathbf{w}_a \in \mathcal{O}_{x^*}^a$ and $\mathbf{w}_b \in \mathcal{O}_{x^*}^b$); and that if a steadily attracting state exists, then such a globally attracting state necessarily exists.

An attainable state may also be steadily attracting under conditions stated in the following proposition, which will be key to showing that the existence of a steadily attracting state is necessary for the aperiodicity of the Markov chain.

Proposition 3.4. *Suppose that Φ follows model (2) and that conditions A1–A4 hold. Let $x^* \in \mathcal{X}$ be an attainable state, and consider the set*

$$E := \{a \in \mathbb{Z}_{>0} \mid \exists t_0 \in \mathbb{Z}_{\geq 0}, \forall t \geq t_0, x^* \in A_+^{\text{at}}(x^*)\}. \quad (21)$$

The following statements hold:

- (i) *E is not empty and for all $(a, b) \in E^2$, $\gcd(a, b) \in E$,*
- (ii) *if $\gcd(E) = 1$, then x^* is steadily attracting,*
- (iii) *if Φ is φ -irreducible then there exists a d -cycle, where $d = \gcd(E)$.*

Remark 1 (Control model and choice of density). For a Markov chain following model (2) under conditions A1–A5, the random variable $\alpha(x, U_1)$ admits different densities which differ on sets of null measure. Therefore, there is not a unique deterministic control model associated

³The equivalence can be easily seen. Indeed suppose that (19) holds. Let $y \in \mathcal{X}$ and $u \in \mathcal{O}_y^1$. Then $x^* \in A_+(S_y^1(u)) \subset \bigcup_{k \in \mathbb{Z}_{>0}} A_+^k(y)$. The other implication is immediate.

to a Markov chain following (2). The control sets $\{\mathcal{O}_x^k : k \in \mathbb{Z}_{>0}, x \in \mathcal{X}\}$ and the sets $\{A_+^k(x) : k \in \mathbb{Z}_{>0}, x \in \mathcal{X}\}$ may differ depending on the choice of the density for representing the Markov chain.

However, under conditions A1–A4 and that F is C^0 , the choice of a different lower semi-continuous density does not affect whether a point $x^* \in \mathcal{X}$ is globally or steadily attracting.⁴ Note that while a globally (resp. steadily) attracting state for a lower semi-continuous density is therefore globally (resp. steadily) attracting for any other density representing the same random variable (even non-lower semi-continuous densities), the converse does not hold in general.

Attainable states may depend on the choice of density. However, under conditions A1–A5 and if there exists $x^* \in \mathcal{X}$ a globally attracting state, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that the controllability matrix $C_{x^*}^k(\mathbf{w}^*)$ has rank n , then as a consequence of Proposition 3.6 the existence of an attainable state is independent of the choice of density.

Remark 2. In Proposition A.5 in the Appendix, we show that under a controllability condition, the existence of a steadily attracting state is equivalent to the existence of a globally attracting state x^* for which there exists paths of length a and b with $\gcd(a, b) = 1$ leading from x^* to x^* . This result is particularly useful to obtain practical conditions to prove that a globally attracting state is steadily attracting as stated in Lemma 5.1.

3.3. Controllability matrix and controllability condition

A central condition in many of our results is that the rank of the so-called controllability matrix is n . This condition is a generalization of the controllability condition for linear state-space models [11]. In this section, we give some background on controllability matrices and derive some first results related to the rank condition above. Our notations are borrowed from [10,11].

3.3.1. Controllability matrix: Definition and first properties

For an initial condition $y \in \mathcal{X}$ and a sequence $\{w_k \in \mathcal{W} : k \in \mathbb{Z}_{\geq 0}\}$, let $\{A_k, B_k : k \in \mathbb{Z}_{\geq 0}\}$ denote the matrices

$$A_k = A_k(y, w_1, \dots, w_{k+1}) := \left[\frac{\partial F}{\partial x} \right]_{(S_y^k, w_{k+1})}, \quad (22)$$

$$B_k = B_k(y, w_1, \dots, w_{k+1}) := \left[\frac{\partial F}{\partial w} \right]_{(S_y^k, w_{k+1})} \quad (23)$$

⁴Indeed, if x^* is globally attracting then for all $y \in \mathcal{X}$ and \mathcal{U}_{x^*} open neighborhood of x^* there exists \mathbf{w} a k -steps path from y to \mathcal{U}_{x^*} for some $k \in \mathbb{Z}_{>0}$; since p_y^k is lower semi-continuous, for any other density \tilde{p}_y^k equal almost everywhere to p_y^k , for any $\eta > 0$ there exists a $\mathbf{u} \in B(\mathbf{w}, \eta)$ such that $\tilde{p}_y^k(\mathbf{u}) > 0$. By continuity of S_y^k , we may then have a \mathbf{u} such that $S_y^k(\mathbf{u}) \in \mathcal{U}_{x^*}$ and $\tilde{p}_y^k(\mathbf{u}) > 0$, which implies that x^* is a globally attracting state for $(\tilde{p}_y)_{y \in \mathcal{X}}$. The same reasoning applies for steadily attracting states.

and let $C_y^k = C_y^k(w_1, \dots, w_k) \in \mathbb{R}^{n \times pk}$ denote the *generalized controllability matrix* (along the sequence (w_1, \dots, w_k))

$$C_y^k(w_1, \dots, w_k) := [A_{k-1} \cdots A_1 B_0 | \cdots | A_{k-1} B_{k-2} | B_{k-1}]. \quad (24)$$

Remark that from (22) and (23), it follows immediately that for $k \in \mathbb{Z}_{>0}$

$$\begin{aligned} B_k(y, w_1, \dots, w_{k+1}) &= B_{k-1}(S_y^1(w_1), w_2, \dots, w_{k+1}), \\ A_k(y, w_1, \dots, w_{k+1}) &= A_{k-1}(S_y^1(w_1), w_2, \dots, w_{k+1}), \end{aligned}$$

and therefore, the controllability matrix satisfies for $k \in \mathbb{Z}_{>0}$

$$C_y^k(w_1, \dots, w_k) = [A_{k-1} \cdots A_1 B_0 | C_x^{k-1}[S_y^1(w_1)](w_2, \dots, w_k)]. \quad (25)$$

Inductively, it follows that for $i = 1, \dots, k$, $C_{S_y^i(w_1, \dots, w_i)}^{k-i}(w_{i+1}, \dots, w_k)$ is a sub-matrix of $C_y^k(w_1, \dots, w_k)$. Additionally, the generalized controllability matrix C_y^k is the Jacobian matrix of the function $(w_1, \dots, w_k) \mapsto S_y^k(w_1, \dots, w_k)$, that is for $\mathbf{w}_0 \in \mathcal{W}^k$

$$C_y^k(\mathbf{w}_0) = \left[\frac{\partial S_y^k}{\partial w_1} \mid \cdots \mid \frac{\partial S_y^k}{\partial w_k} \right]_{\mathbf{w}_0}. \quad (26)$$

This formula is a consequence of the chain rule. We provide in the [Appendix](#) its derivation.

Remark 3. A central condition for our results will be that the rank of the controllability matrix $C_x^k(\mathbf{w})$ is n for some $x \in \mathcal{X}$, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$. This condition is equivalent to S_x^k being a submersion at \mathbf{w} , that is the differential of S_x^k in \mathbf{w} is surjective. This formulation was used in a previous version of this work [3]. It is arguably a more intuitive formulation, in particular when it is easier to manipulate directly the differential than the Jacobian matrix.

3.3.2. Accessibility and controllability condition

We show in the next proposition that under conditions A1–A5, if the rank condition on the controllability matrix is satisfied at a globally attracting state, it is satisfied at any x in \mathcal{X} .

Proposition 3.5. *Suppose that Φ follows model (2) and that conditions A1–A5 hold. Let $x^* \in \mathcal{X}$ be a globally attracting state. If there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$, then for all $x \in \mathcal{X}$, there exists $T \in \mathbb{Z}_{>0}$ and $\mathbf{u} \in \mathcal{O}_x^T$ for which $\text{rank}(C_x^T(\mathbf{u})) = n$.*

In the next proposition, we show that from a globally attracting state where the rank condition on the controllability matrix is satisfied, we can construct an attainable state. This proposition will later on allow us to use Proposition 3.4 to prove that if for all $x \in \mathcal{X}$ the rank condition is satisfied, then the existence of a steadily attracting state is a necessary condition for the aperiodicity of the Markov chain.

Proposition 3.6. *Suppose that Φ follows model (2) and that conditions A1–A5 hold. Let $x^* \in \mathcal{X}$ and suppose there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$.*

(i) *There exists \mathcal{U} a neighborhood of x^* such that for all $x \in \mathcal{U}$, there exists $\mathbf{w} \in \mathcal{O}_x^k$ for which $S_x^k(\mathbf{w}) = S_{x^*}^k(\mathbf{w}^*)$.*

(ii) *If x^* is globally attracting, then $S_{x^*}^k(\mathbf{w}^*)$ is attainable.*

When the Markov chain reduces to (1), that is $\alpha(x, u) = u$, and F is C^∞ , the rank condition holding for every x is equivalent to forward accessibility [8], Proposition 2.3. This results relies on the inverse function theorem and Sard’s theorem and can be easily generalized to the following proposition.

Proposition 3.7. *Suppose that Φ follows model (2) and that conditions A1–A5 hold. If for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ such that $\text{rank}(C_x^k(\mathbf{w})) = n$, then $\text{CM}(F)$ is forward accessible.*

Furthermore, if F is C^∞ , $\text{CM}(F)$ is forward accessible if and only if for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank}(C_x^k(\mathbf{w})) = n$.

Remark that the fact that the rank condition on the controllability matrix implies forward accessibility still holds for F a C^1 function was already noted in [8].

4. Main results

4.1. T -chain and irreducibility

In this section, we state our main results on the φ -irreducibility and T -chain property. On the one hand, we generalize the result holding for a Markov chain following (1) with F being C^∞ that if $\text{CM}(F)$ is forward accessible, the associated Markov chain is φ -irreducible if and only if $\text{CM}(F)$ admits a globally attracting state [11], Proposition 7.2.6. We prove more precisely that under the conditions A1–A5, if for all x , there exists k and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$ (we have seen that this condition implies forward accessibility), then the φ -irreducibility of a chain following (2) is equivalent to the existence of a globally attracting state.

We then derive a practical condition by showing that the existence of a globally attracting state where the rank condition is satisfied implies that the associated Markov chain is a φ -irreducible T -chain and thus that every compact set is petite. That is, we only need to find a globally attracting state and verify the rank condition at this state to prove the φ -irreducibility and T -chain property.

Those results rely on the generalization to our context of [10], Theorem 2.1(iii). In particular, we show that around a point $x \in \mathcal{X}$ where the rank of the controllability matrix $C_x^k(\mathbf{w})$ is n for some $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$, there exists an open small set containing x . More precisely we have the following result.

Proposition 4.1. *Suppose that Φ follows model (2) and that conditions A1–A5 are satisfied.*

(i) Let $x \in \mathcal{X}$, if $\text{rank } C_x^k(\mathbf{w}) = n$ for some $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$, then there exists $c > 0$, and open sets \mathcal{U}_x and $\mathcal{V}_x^{\mathbf{w}}$ containing x and $S_x^k(\mathbf{w})$ respectively, such that

$$P^k(y, A) \geq c\mu^{\text{Leb}}(A \cap \mathcal{V}_x^{\mathbf{w}}), \quad \text{for all } y \in \mathcal{U}_x, A \in \mathcal{B}(\mathcal{X}). \quad (27)$$

That is, \mathcal{U}_x is a v_k -small set where $v_k : A \mapsto c\mu^{\text{Leb}}(A \cap \mathcal{V}_x^{\mathbf{w}})$.

(ii) If furthermore F is C^∞ , and if for some $x \in \mathcal{X}$, there exists $k \in \mathbb{Z}_{>0}$, $c > 0$ and \mathcal{V} an open set such that

$$P^k(x, A) \geq c\mu^{\text{Leb}}(A \cap \mathcal{V}), \quad \text{for all } A \in \mathcal{B}(\mathcal{X}), \quad (28)$$

then $\text{rank } C_x^k(\mathbf{w}) = n$ for some $\mathbf{w} \in \mathcal{O}_x^k$, a k -steps path from x to \mathcal{V} .

The proof of this proposition is very similar to the proof of [10], Theorem 2.1. We present it in the [Appendix](#) where we additionally highlight the differences to the proof of [10], Theorem 2.1. Note that if F is C^∞ and (28) holds for some $x \in \mathcal{X}$, then by Proposition 4.1(i) and (ii), (28) holds for all y in an open neighborhood of x and \mathcal{V} a possibly smaller open set.

We deduce from Proposition 4.1 that if for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank}(C_x^k(\mathbf{w})) = n$, the state-space may be written as the union of open small sets and hence Φ is a T -chain (see [11], Proposition 6.2.3, Proposition 6.2.4). This result is formalized in the following corollary, which can be seen as a generalization to our model of [11], Proposition 7.1.5.

Corollary 4.1. *Suppose that Φ follows model (2) and that conditions A1–A5 are satisfied. Suppose that for all $x \in \mathcal{X}$, $\text{rank}(C_x^k(\mathbf{w})) = n$ for some $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$. Then \mathcal{X} can be written as the union of open small sets and thus Φ is a T -chain.*

Proof. Under the conditions of the corollary, the conditions of Proposition 4.1(i) are satisfied for all $x \in \mathcal{X}$. Hence, for all $x \in \mathcal{X}$, there exists \mathcal{U}_x an open v_k -small set containing x . A v_k -small set is a v_a -petite set (with a a Dirac distribution at k), hence, according to [11], Proposition 6.2.3, K_a possesses a continuous component T_x nontrivial on all of \mathcal{U}_x , and so in particular non trivial at x . Hence, according to [11], Proposition 6.2.4, Φ is a T -chain. \square

To prove the equivalence between φ -irreducibility and the existence of a globally attracting state, we first characterize the support of the maximal irreducibility measure in terms of globally attracting states. More precisely, the following holds.

Proposition 4.2. *Suppose that Φ is a ψ -irreducible Markov chain following model (2), with ψ the maximal irreducibility measure, that A1–A4 hold and that F is C^0 . Then*

$$\text{supp } \psi = \{x^* \in \mathcal{X} \mid x^* \text{ is globally attracting}\}. \quad (29)$$

Furthermore, let $x^* \in \mathcal{X}$ be globally attracting, then

$$\text{supp } \psi = \overline{A_+(x^*)}. \quad (30)$$

We are now ready to state our result generalizing [11], Proposition 7.2.6, to our model.

Theorem 4.1. *Suppose that Φ follows model (2) and that conditions A1–A5 are satisfied. Suppose that for all $x \in \mathcal{X}$, there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ such that $\text{rank } C_x^k(\mathbf{w}) = n$. Then Φ is φ -irreducible if and only if a globally attracting state exists.*

Proof. Suppose that for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ such that $\text{rank } C_x^k(\mathbf{w}) = n$. If Φ is a φ -irreducible chain, then by Proposition 4.2 any point of its support is globally attracting. Since φ is not trivial, its support is not empty and so there exists a globally attracting state.

Conversely, suppose that there exists x^* a globally attracting point, which is also reachable by Corollary 3.1. By hypothesis, according to Corollary 4.1, Φ is a T -chain and so by [11], Proposition 6.2.1, Φ is φ -irreducible, which concludes the proof. \square

As discussed above, in the particular case of $\alpha(x, u) = u$ and F is C^∞ , the similar result derived in [11], Proposition 7.2.6, states that if $\text{CM}(F)$ is forward accessible, the associated Markov chain is φ -irreducible if and only if $\text{CM}(F)$ admits a globally attracting state. In the more general context of bounded positive kernels, the existence of an open reachable⁵ petite set is equivalent to the Markov chain being a φ -irreducible T -chain for which compact sets are petite [4], Theorem 2.4.

From Proposition 3.5, we know that if the rank condition on the controllability matrix is satisfied at a globally attracting state, it is satisfied for all $x \in \mathcal{X}$. Hence, we can deduce the practical condition that if there exists a globally attracting state where the rank condition on the controllability matrix is satisfied, then the associated Markov chain is a φ -irreducible T -chain and thus every compact set is petite.

Theorem 4.2 (Practical Condition for φ -irreducibility). *Suppose that Φ follows model (2) and that conditions A1–A5 are satisfied. If there exists x^* a globally attracting state, and if $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$ for some $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$, then Φ is a φ -irreducible T -chain, and thus every compact set is petite.*

Proof. Suppose there exists x^* a globally attracting state, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$. By Proposition 3.5 for all $x \in \mathcal{X}$ there exists $t \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^t$ for which $\text{rank } C_x^t(\mathbf{w}) = n$. According to Corollary 4.1, Φ is a T -chain and according to Theorem 4.1, Φ is φ -irreducible. Hence, according to [11], Theorem 6.2.5, every compact set is petite. \square

4.2. Aperiodicity

We show now that the results of the previous section can be transposed to prove φ -irreducibility and aperiodicity of a Markov chain if we replace the condition of the existence of a globally attracting state by the existence of a steadily attracting state.

We first state the equivalence between the existence of a steadily attracting state and the φ -irreducibility and aperiodicity of the associated Markov chain.

⁵A set $A \in \mathcal{B}(\mathcal{X})$ is said reachable if for all $x \in \mathcal{X}$ there exists some $k \in \mathbb{Z}_{>0}$ for which $P^k(x, A) > 0$.

Theorem 4.3. *Consider a Markov chain Φ following the model (2) for which conditions A1–A5 are satisfied. If for all $x \in \mathcal{X}$, there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank}(C_x^k(\mathbf{w})) = n$, then Φ is a φ -irreducible aperiodic Markov chain if and only if there exists a steadily attracting state.*

A related result proving a necessary and sufficient condition for a chain to be aperiodic has been derived for Markov chains following (1) under the assumption that the control set $\mathcal{O}_{\mathcal{X}} = \{u \in \mathcal{U} | p(u) > 0\}$ is connected. More precisely, it has been shown that if there exists x^* a globally attracting state, then Φ is aperiodic if and only if $\overline{A_+(x^*)}$ is connected [11], Proposition 7.2.5, 7.3.4, Theorem 7.3.5. Note that the condition that $\mathcal{O}_{\mathcal{X}}$ is connected is critical: if a Markov chain has a non-connected set $\mathcal{O}_{\mathcal{X}}$, then the equivalence that $\overline{A_+(x^*)}$ is connected if and only if the Markov chain is aperiodic does not hold anymore. A trivial (albeit artificial) example is to take an i.i.d. sequence of random variables $\{U_k : k \in \mathbb{Z}_{>0}\}$ with non-connected support $(-2, -1) \cup (1, 2)$, and to consider it as our Markov chain of interest $\Phi_k = U_k$. Then $A_+^k(x) = (-2, -1) \cup (1, 2)$ for all $k \in \mathbb{Z}_{>0}$, and so $\overline{A_+(x)}$ is not connected for any $x \in \mathbb{R}$ but the Markov chain is aperiodic.

In the general context of a bounded positive weak-Feller transition kernel Q , if there exists B a ν_a -small set (i.e., $Q^a(x, A) \geq \nu_a(A)$ for all $x \in B$ and $A \in \mathcal{B}(\mathcal{X})$) and neighborhood of a reachable point x^* such that $Q^b(x^*, B) > 0$ for some $b \in \mathbb{Z}_{>0}$ for which $\text{gcd}(a, b) = 1$, then Q is a φ -irreducible aperiodic T -chain; and a slightly weaker form of the converse holds [4], Theorem 2.6. In our more limited context, this can be shown using Theorem 4.3 and Proposition A.5.

Similarly to Theorem 4.2, we now deduce the following practical condition to prove the aperiodicity of a Markov chain.

Theorem 4.4 (Practical condition for φ -irreducibility and aperiodicity). *Consider a Markov chain Φ following the model (2) for which conditions A1–A5 are satisfied. If there exists $x^* \in \mathcal{X}$ a steadily attracting state, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$, then Φ is an aperiodic φ -irreducible T -chain, and every compact set is small.*

We give here an outline of how the existence of steadily attracting state implies the aperiodicity of the Markov chain while the full proofs the theorems can be found in the [Appendix](#).

Proposition 4.1(i) allows to construct a set \mathcal{U} which is a neighborhood of x^* and a non-trivial measure $\mu_{\mathcal{V}}$ such that if a point $y \in \mathcal{X}$ can reach \mathcal{U} with a t -steps path, then the kernel $P^{t+k}(y, \cdot)$ dominates the measure $\mu_{\mathcal{V}}$. Since x^* is steadily attractive, \mathcal{U} can be reached from any point $y \in \mathcal{X}$ for all time $t \geq t_y$. Hence, $\sum_{t \in \mathbb{Z}_{>0}} P^{t+k}(y, \cdot)$ dominates $\mu_{\mathcal{V}}$ for all $y \in \mathcal{X}$, implying that $\mu_{\mathcal{V}}$ is an irreducibility measure.

Now consider $(D_i)_{i=1, \dots, d}$ a d -cycle. Since $\mu_{\mathcal{V}}$ is an irreducibility measure, there exists D_i such that $\mu_{\mathcal{V}}(D_i) > 0$, and with the fact that $P^{t+k}(y, \cdot)$ dominates $\mu_{\mathcal{V}}$ for all $t \geq t_y$, we deduce that D_i can be reached with positive probability from all $y \in \mathcal{X}$ in t steps with any $t \geq t_y$. By definition of the d -cycle, the Markov chain steps with probability one from a set of the cycle to the next, and so for any $m \in \mathbb{Z}_{\geq 0}$ the Markov chain goes in $md + 1$ steps from D_i to D_{i+1} with probability one. This contradicts that D_i can be reached from anywhere (including itself) in $md + 1$ steps for m large enough, unless $d = 1$ meaning the Markov chain is aperiodic.

Theorem 4.4 is a generalization of Proposition 3.2 in [9] where a Markov chain following (1) with G being C^∞ is φ -irreducible and aperiodic if the control model is forward accessible and asymptotically controllable, that is if there exists $x^* \in \mathcal{X}$ such that for all $y \in \mathcal{X}$ there exists

a sequence $\{w_k : k \in \mathbb{Z}_{>0}\}$ with $w_k \in \mathcal{O}_{\mathcal{X}}$ such that the sequence $\{S_x^k(w_1, \dots, w_k) : k \in \mathbb{Z}_{>0}\}$ converges to x^* . This latter condition of asymptotic controllability implies the existence of a steadily attracting state (as can easily be seen from Proposition 3.3), and is in fact quite stronger: for $x^* \in \mathcal{X}$ a steadily attracting state and any $y \in \mathcal{X}$ there exists a sequence $\{\mathbf{w}_k : k \in \mathbb{Z}_{>0}\}$ with $\mathbf{w}_k \in \mathcal{O}_{\mathcal{X}}^k$ such that $\{S_x^k(\mathbf{w}_k) : k \in \mathbb{Z}_{>0}\}$ converges to x^* , while in the context of asymptotic controllability \mathbf{w}_{k+1} would be restricted to (\mathbf{w}_k, u) with $u \in \mathcal{O}_{\mathcal{X}}$. Remark that hence asymptotic controllability forces for any $\varepsilon > 0$ the existence of $y \in \mathcal{X}$ and $u \in \mathcal{O}_{\mathcal{X}}$ such that $\|S_y^1(u) - y\| \leq \varepsilon$.⁶ This is a restriction on the possible models that can be considered which is not imposed if we consider the condition of the existence of a steadily attracting state. Indeed take an additive random walk on \mathbb{R} with increment distribution with a support of the type $(-\infty, a) \cup (a, +\infty)$ with $a > 0$. It is not difficult to prove that the associated control model is *not asymptotically controllable* while every $x \in \mathbb{R}$ is steadily attractive and the chain is φ -irreducible.

5. Applications

In this section, we illustrate how to use the different conditions we have derived to prove φ -irreducibility, aperiodicity and that compact are small sets for three examples of Markov chains. We consider first the toy examples presented in Section 1 and then turn to a more complex example where it would be very intricate – if not impossible – to prove φ -irreducibility and aperiodicity by hand. Before to tackle those examples, we summarize the methodology ensuing the results we have developed.

Methodology. The following steps need to be followed to apply Theorem 4.2 to prove that a Markov chain is a φ -irreducible T -chain (and thus that compact sets are petite).

- (i) Identify that the Markov chain follows model (2): exhibit the function F and either identify explicitly α or argue on its existence.
- (ii) Identify the density $p_x(\cdot)$ of $\alpha(x, U_1)$ for all $x \in \mathcal{X}$.
- (iii) Show that conditions A1 to A5 are satisfied. Particularly prove that F is C^1 and $(x, w) \mapsto p_x(w)$ is lower semi-continuous.
- (iv) Prove that there exists a globally attracting state x^* .
- (v) Show that there exists k and $\mathbf{w} \in \mathcal{O}_x^k$ such that $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$.

Similarly, to apply Theorem 4.4 to prove that the chain is a φ -irreducible aperiodic T -chain and that compact are small sets, we need to replace step (iv) above by proving that there exists a steadily attracting state. We highlight in Lemma 5.1 two practical conditions to facilitate the proof of existence of a steadily attracting state. Note also that to prove the existence of a globally (or steadily) attracting state, it is practical to have identified the control sets \mathcal{O}_x^k , yet it might be enough to only know \mathcal{O}_x^1 (see the proof of Proposition 5.3).

⁶Indeed for $t \in \mathbb{Z}_{>0}$ large enough, the convergence of $S_y^t(\mathbf{w}_k)$ to x^* forces $S_y^t(\mathbf{w}_k)$ and $S_y^{t+1}(\mathbf{w}_k, u)$ to be contained within the same ball of radius $\varepsilon/2$, and so $x := S_y^t(\mathbf{w}_k)$ is at distance at most ε from $S_x^1(u)$.

5.1. Toy examples

We consider the two examples introduced in Section 1. For those Markov chains, φ -irreducibility can be proven directly. Indeed, it follows from the expression of the transition kernel in (10) that for all A with strictly positive Lebesgue measure, for all x , $P(x, A) > 0$. It is also relatively straightforward to prove aperiodicity and that compact are small sets by minorizing (10) for all x in a compact C by $\int_A \min_{x \in C} p_x(x - y) dy$.

Yet in order to illustrate how to use the conditions derived in the paper, we show how Theorem 4.4 can be applied to show φ -irreducibility, aperiodicity and that every compact set is small. The function F associated to the chains defined in (3) and (8) equals $F(x, w) = x + w$ and it is thus C^∞ . Additionally, we have seen that both chains share the same control model $CM(F)$ which is additionally forward accessible (see Example 2). Thus according to Proposition 3.7, the rank condition on the controllability matrix is satisfied for all $x \in \mathbb{R}$. Hence in order to show the φ -irreducibility, aperiodicity, and that compact sets are small sets, according to Theorem 4.3 or Theorem 4.4, it remains to prove the existence of a steadily attracting state. We actually prove in the next proposition that *every* x in \mathbb{R} is steadily attracting.

Proposition 5.1. *Consider the control model $CM(F)$ defined in Example 1 associated to the Markov chains defined in (3) and (8). Then every $x \in \mathbb{R}$ is steadily attracting for $CM(F)$.*

Proof. Since $\mathcal{O}_x^k = \mathbb{R}^k$ and $S_y^k(\mathbf{w}) = y + w_1 + \dots + w_k$, for all x and all initial condition y , for all $k \geq 1$, the vector $\bar{\mathbf{w}} = (0, \dots, 0, y - x) \in \mathbb{R}^k = \mathcal{O}_y^k$ satisfies $S_y^k(\bar{\mathbf{w}}) = x$. This implies that x is a steadily attracting state. Remark that we have shown a stronger condition than steady attractivity because we have shown that we can exactly hit x in k steps for any $k \geq 1$. \square

Consequently, the assumptions of Theorem 4.3 are satisfied for the Markov chains defined in (3) and (8) and thus the chains are φ -irreducible, aperiodic T-chains and every compact set is small. According to Proposition 4.2, we also know that the support of the maximal irreducibility measure equals \mathbb{R} .

We have illustrated how the tools developed in the paper unify the study of φ -irreducibility, aperiodicity and the identification that compact are small sets for the two Markov chains (3) and (8) whose model defined via (1) is sensibly different with an associated G which is C^∞ for (3) and discontinuous for (8).

5.2. A step-size adaptive randomized algorithm optimizing scaling-invariant functions

We consider now a Markov chain stemming from an adaptive stochastic algorithm aiming at optimizing continuous optimization problems. Proving the stability of the chain is important because it implies the linear convergence (or divergence) of the underlying optimization algorithm which is generally difficult to prove for this type of algorithms. This example is not artificial: showing the φ -irreducibility, aperiodicity and that compact sets are small sets by hand without

the results of the current paper seems to be very arduous and actually motivated the development of the theory of this paper.

We consider a step-size adaptive stochastic search algorithm optimizing an objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The algorithm pertains to the class of so-called *Evolution Strategies* (ES) algorithms [16] that date back to the 70's. The algorithm is however related to information geometry: It was recently derived from taking the natural gradient of a joint objective function defined on the Riemannian manifold formed by the family of Gaussian distributions [6,13]. More precisely, let $X_0 \in \mathbb{R}^n$ and let $\{U_k : k \in \mathbb{Z}_{>0}\}$ be an i.i.d. sequence of random vectors where each U_k is composed of $\lambda \in \mathbb{Z}_{>0}$ components $U_k = (U_k^1, \dots, U_k^\lambda) \in \mathbb{R}^{n\lambda}$ with $\{U_k^i : i = 1, \dots, \lambda\}$ i.i.d. and following each a standard multivariate normal distribution $\mathcal{N}(0, I_n)$ where I_n denotes the identity matrix of size n . Given $(X_k, \sigma_k) \in \mathbb{R}^n \times \mathbb{R}_{>}$ the current state of the algorithm, λ candidate solutions centered on X_k are sampled using the vector U_{k+1} , i.e. for $i = 1, \dots, \lambda$

$$X_k + \sigma_k U_{k+1}^i, \quad (31)$$

where σ_k called the step-size of the algorithm corresponds to the overall standard deviation of $\sigma_k U_{k+1}^i$. Those solutions are ranked according to their f -values. More precisely, let \mathcal{S} be the permutation of λ elements such that

$$f(X_k + \sigma_k U_{k+1}^{\mathcal{S}(1)}) \leq f(X_k + \sigma_k U_{k+1}^{\mathcal{S}(2)}) \leq \dots \leq f(X_k + \sigma_k U_{k+1}^{\mathcal{S}(\lambda)}). \quad (32)$$

To break the possible ties and have an uniquely defined permutation \mathcal{S} , we can simply consider the natural order, that is, if for instance $\lambda = 2$ and $f(X_k + \sigma_k U_{k+1}^1) = f(X_k + \sigma_k U_{k+1}^2)$, then $\mathcal{S}(1) = 1$ and $\mathcal{S}(2) = 2$. The new estimate of the optimum X_{k+1} is formed by taking a weighted average of the $\mu (\geq 1)$ best directions (typically $\mu = \lambda/2$), that is

$$X_{k+1} = X_k + \sigma_k \kappa_m \sum_{i=1}^{\mu} \beta_i U_{k+1}^{\mathcal{S}(i)}, \quad (33)$$

where the sequence of weights $\{\beta_i : 1 \leq i \leq \mu\}$ sums to 1 (typically $\beta_1 \geq \dots \geq \beta_\mu$), and $\kappa_m > 0$ is called a learning rate. The step-size is adapted according to

$$\sigma_{k+1} = \sigma_k \exp\left(\frac{\kappa_\sigma}{2n} \left(\sum_{i=1}^{\mu} \beta_i (\|U_{k+1}^{\mathcal{S}(i)}\|^2 - n)\right)\right), \quad (34)$$

where $\kappa_\sigma > 0$ is a learning rate for the step-size. The equations (33) and (34) correspond to the xNES algorithm with covariance matrix restricted to $\sigma_k^2 I_n$ [6].

Consider a scaling-invariant function with respect to x^* , that is for all $\rho > 0$, $x, y \in \mathbb{R}^n$

$$f(x) \leq f(y) \Leftrightarrow f(x^* + \rho(x - x^*)) \leq f(x^* + \rho(y - x^*)). \quad (35)$$

Examples of scaling-invariant functions include $f(x) = \|x - x^*\|$ for any arbitrary norm on \mathbb{R}^n . It also includes non continuous functions, functions with non-convex sublevel sets. We assume w.l.g. that $x^* = 0$. On this class of functions, $\mathcal{Z} := \{Z_k = X_k/\sigma_k : k \in \mathbb{Z}_{\geq 0}\}$ is a homogeneous Markov chain that can be defined independently of the Markov chain (X_k, σ_k) in the following

manner [1], Proposition 4.1. Given $Z_k \in \mathbb{R}^n$, sample λ candidate solutions centered on Z_k using a vector U_{k+1} , i.e. for $1 \leq i \leq \lambda$

$$Z_k + U_{k+1}^i, \quad (36)$$

where similarly as for the chain (X_k, σ_k) , $\{U_k : k \in \mathbb{Z}_{\geq 0}\}$ are i.i.d. and each U_k is a vector of λ i.i.d. components following each a standard multivariate normal distribution. Those λ solutions are evaluated and ranked according to their f -values. Similarly to (32), the permutation \mathcal{S} containing the order of the solutions is extracted. This permutation can be uniquely defined if we break the ties as explained below (32). The update of Z_k then reads

$$Z_{k+1} = \frac{Z_k + \kappa_m \sum_{i=1}^{\mu} \beta_i U_{k+1}^{\mathcal{S}(i)}}{\exp(\frac{\kappa_m}{2n} (\sum_{i=1}^{\mu} \beta_i (\|U_{k+1}^{\mathcal{S}(i)}\|^2 - n)))}. \quad (37)$$

Let us now define the vector of selected steps as $W_{k+1} = (U_{k+1}^{\mathcal{S}(1)}, \dots, U_{k+1}^{\mathcal{S}(\mu)}) \in \mathbb{R}^{n\mu}$ and for $z \in \mathbb{R}^n$, $w \in \mathbb{R}^{n\mu}$ (with $w = (w^1, \dots, w^\mu)$)

$$F_{\text{xNES}}(z, w) = \frac{z + \kappa_m \sum_{i=1}^{\mu} \beta_i w^i}{\exp(\frac{\kappa_m}{2n} (\sum_{i=1}^{\mu} \beta_i (\|w^i\|^2 - n)))}, \quad (38)$$

such that

$$Z_{k+1} = F_{\text{xNES}}(Z_k, W_{k+1}).$$

The writing of the explicit function α such that $W_{k+1} = \alpha(Z_k, U_{k+1})$ is quite tedious in the general case. For the sake of simplicity, we only give it when $\mu = 1$ and $\lambda = 2$. In this case

$$W_{k+1} = (U_{k+1}^1 - U_{k+1}^2) 1_{\{f(Z_k + U_{k+1}^1) \leq f(Z_k + U_{k+1}^2)\}} + U_{k+1}^2 \quad (39)$$

that is for all $z \in \mathbb{R}^n$ and $u = (u^1, u^2) \in \mathbb{R}^{2n}$

$$\alpha(z, u) = (u^1 - u^2) 1_{\{f(z + u^1) \leq f(z + u^2)\}} + u^2.$$

The function α is typically discontinuous (similarly to the function α in (8)). Consider indeed a function f with level sets that are Lebesgue negligible, then if $f(z + u^1) = \dots = f(z + u^\lambda)$ while the $\{u^i : 1 \leq i \leq \lambda\}$ are all distincts, an arbitrarily small change in u^1 can lead to a different ranking and so to a non continuous change in $\alpha(z, u)$. In the next proposition, we derive $p_z(w)$ a density of W_{k+1} conditional to $Z_k = z$.

Proposition 5.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an objective function whose level sets are Lebesgue negligible. Let $\lambda \in \mathbb{Z}_{>0}$ and $\mu \in \mathbb{Z}_{>0}$ with $\mu \leq \lambda$. Let us define if $\mu = 1$*

$$p_z(w) = \lambda (1 - Q_z^f(w))^{\lambda-1} p_{\mathcal{N}}(w) \quad (40)$$

with $w \in \mathbb{R}^n$, $Q_z^f(w) = \Pr(f(z + \mathcal{N}) \leq f(z + w))$ with \mathcal{N} following a standard multivariate normal distribution in dimension n and $p_{\mathcal{N}}(u) = \frac{1}{(\sqrt{2\pi})^n} \exp(-u^T u/2)$ its density. If $\mu > 1$

$$p_z(w) = \frac{\lambda!}{(\lambda - \mu)!} \mathbf{1}_{\{f(z+w^1) < \dots < f(z+w^\mu)\}} (1 - Q_z^f(w^\mu))^{\lambda - \mu} p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\mu), \quad (41)$$

where $w = (w^1, \dots, w^\mu) \in \mathbb{R}^{n\mu}$. Then $p_z(w)$ is a density associated to $\alpha(z, U_1)$ (also a density of W_{k+1} conditionally that $Z_k = z$).

Assume the objective function f is continuous, it is not difficult to see that if $\mu = 1$, then $(z, w) \mapsto p_z(w)$ is continuous (and thus lower-semi continuous) and if $\mu > 1$ it is lower semi-continuous.

The stability of the homogeneous Markov chain \mathcal{Z} is one key to prove the linear convergence of the algorithm defined in (33) and (34) as stated in the next theorem.

Theorem 5.1 (Theorem 5.2 in [1]). *Let f be a scaling-invariant function with respect to 0. Assume that the Markov chain \mathcal{Z} defined in (36) and (37) is φ -irreducible, Harris-recurrent and positive with invariant probability measure π . Assume that $E_\pi[\lceil \ln \|z\| \rceil] < \infty$ and $E_\pi[f \mid \sum_{i=1}^\mu \beta_i (\|w^i\|^2 - n) \mid p_z(w) dw] < \infty$, then the xNES algorithm defined in (33) and (34) converges (or diverges) linearly almost surely, that is for all X_0, σ_0*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \ln \frac{\|X_k\|}{\|X_0\|} = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \frac{\sigma_k}{\sigma_0} = E_{z \sim \pi} \left[\int \sum_{i=1}^\mu \beta_i (\|w^i\|^2 - n) p_z(w) dw \right]. \quad (42)$$

Linear convergence happens if the convergence rate $E_{z \sim \pi}[f \sum_{i=1}^\mu \beta_i (\|w^i\|^2 - n) p_z(w) dw]$ is strictly negative. Given that this rate depends on the unknown invariant probability distribution π , we are often not able to prove the strict negativity. However, it is fairly easy to simulate precisely this convergence rate such that not knowing the sign is generally not problematic. The proof of this theorem relies on applying a Law of Large Numbers to the chain \mathcal{Z} . Hence, the stability properties that need to be shown correspond to the assumptions needed for \mathcal{Z} to satisfy a Law of Large Numbers. Positivity and Harris recurrence are typically proven by using Foster–Lyapunov drift conditions, that state the negativity of a drift function outside a small set. It is thus important to identify small sets for the chain. Irreducibility and aperiodicity are also needed because we typically establish a geometric drift and use the geometric ergodic theorem for φ -irreducible aperiodic chains [11], Theorem 15.0.1.

We will now explain how to use Theorem 4.4 to prove that \mathcal{Z} is a φ -irreducible aperiodic T -chain and compact sets are small sets for the chain. Remark first that assumption A1–A3 are satisfied following from the construction and definition of the algorithm, A4 is satisfied as it has been discussed above and the function F_{xNES} being C^1 , the assumption A5 is satisfied. The control sets \mathcal{O}_z^1 for $z \in \mathbb{R}^n$ are defined as $\{\mathbf{w} \in \mathbb{R}^{n\mu} \mid p_z(\mathbf{w}) > 0\}$ that is for $\mu = 1$, $\mathcal{O}_z^1 = \mathbb{R}^n$ and for $\mu > 1$, $\mathcal{O}_z^1 = \{\mathbf{w} \in \mathbb{R}^{n\mu} \mid f(z + w^1) < \dots < f(z + w^\mu)\}$. We prove in the next proposition that the null vector is a steadily attracting state for $\text{CM}(F_{\text{xNES}})$.

Proposition 5.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous with Lebesgue negligible level sets. Then 0 is steadily attracting for $\text{CM}(F_{\text{xNES}})$.*

Proof. We first assume that $\mu > 1$ and prove that for all $\varepsilon > 0$, for all $y \in \mathbb{R}^n$, there exists a 1-step path from y to $B(0, \varepsilon)$. (This latter property implies that 0 is globally attracting and is actually stronger as the time step to reach the neighborhood of 0 is independent of the initial point y .) Let $y \in \mathbb{R}^n$, since $\lim_{\|w\| \rightarrow +\infty} F_{\text{xNES}}(y, w) = 0$, there exists $r > 0$ such that if $\|w\| \geq r$ then $F_{\text{xNES}}(y, w) \in B(0, \varepsilon)$. Let us choose $u \in \mathbb{R}^{n\lambda}$ such that each u^i satisfies $\|u^i\| \geq r$ and additionally $f(y + u^i) \neq f(y + u^j)$ for all $i \neq j$ (we can find such an u because we have assumed that all the level sets of f are Lebesgue negligible). Let $\mathbf{w}_y = \alpha(y, u)$, then $\mathbf{w}_y \in \mathcal{O}_y^1$ and $\|\mathbf{w}_y\| \geq r$ such that $S_y^1(\mathbf{w}_y) \in B(0, \varepsilon)$. Hence, \mathbf{w}_y is a 1-step path from y to $B(0, \varepsilon)$.

Now, showing that a path from y to $B(0, \varepsilon)$ exists for all $t \geq 1$ is easy: take $\mathbf{w} \in \mathcal{O}_y^{t-1}$, and denote $\tilde{y} = S_y^{t-1}(\mathbf{w})$; by the previous reasoning, $\mathbf{w}_{\tilde{y}}$ is a 1-step path from \tilde{y} to $B(0, \varepsilon)$. Therefore, $(\mathbf{w}, \mathbf{w}_{\tilde{y}})$ is a t -steps path from y to $B(0, \varepsilon)$, and so 0 is steadily attracting.

In the case where $\mu = 1$, the proof is even simpler as we do not need to care for finding a step that derives from a vector $o \in \mathbb{R}^{n\lambda}$ that does not result in solutions on the same f -level sets. We omit the details than can be easily deduced from the previous case. \square

In the previous proof, we have shown that any neighborhood of 0 can be reached via a 1-step path from any starting point. This directly implies that 0 is steadily attracting. More generally, if we can reach any neighborhood of x^* in T steps from any initial point x – that is T is independent of the initial point x – then x^* is steadily attracting.

Another practical result to prove that a state x^* is steadily attracting holding under a controllability condition is first to prove that the point is globally attracting and then to show that there exists $\mathbf{w} \in \mathcal{O}_{x^*}^1$ that allows to stay in x^* , that is such that $S_{x^*}^1(\mathbf{w}) = x^*$. Those two practical conditions to prove that a state x^* is steadily attractive are formalized in the following lemma.

Lemma 5.1 (Practical Conditions for a Steadily Attracting State). *Suppose that conditions A1–A4 hold, and that F is continuous. Let $x^* \in \mathcal{X}$, the following holds:*

(i) *If for all \mathcal{U}_{x^*} neighborhood of x^* , there exists $T \in \mathbb{Z}_{>0}$ such that for any $y \in \mathcal{X}$ there exists a T -steps path from y to \mathcal{U}_{x^*} , then x^* is steadily attracting.*

(ii) *Suppose that assumption A5 holds and that for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank } C_x^k(\mathbf{w}) = n$. If x^* is globally attracting and if there exists $\mathbf{w}^* \in \mathcal{O}_{x^*}^1$ that allows to stay in x^* , that is such that $S_{x^*}^1(\mathbf{w}^*) = x^*$, then x^* is steadily attracting.*

The proof of the second point is not completely straightforward and is presented in the [Appendix](#) as a consequence of [Proposition A.5](#).

We have now seen that 0 is a steadily attracting state. We will prove that the rank condition is satisfied in 0. We prove more precisely the following proposition.

Proposition 5.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous with Lebesgue negligible level sets, then there exists $\mathbf{w}^* \in \mathcal{O}_0^1$ such that $\text{rank } C_0^1(\mathbf{w}^*)$ equals n , that is the rank condition is satisfied at the steadily attracting state 0.*

Proof. We prove that there exists $\mathbf{w}^* \in \mathcal{O}_0^1$ such that $C_0^1(\mathbf{w}^*)$ has rank n . Let $\mathbf{w}_0 = (0, \dots, 0) \in \mathbb{R}^{n\mu}$, we will first prove that $C_0^1(\mathbf{w}_0)$ has rank n , i.e. the differential of $w \rightarrow S_0^1(w) := F_{\text{xNES}}(0, w)$ at \mathbf{w}_0 is surjective. Let $h = (h_i)_{i=1, \dots, \mu} \in \mathbb{R}^{n\mu}$, then

$$\begin{aligned} S_0^1(\mathbf{w}_0 + h) &= \frac{\kappa_m \sum_{i=1}^{\mu} \beta_i h_i}{\exp\left(\frac{\kappa_m}{2n} (\sum_{i=1}^{\mu} \beta_i (\|h_i\|^2 - n))\right)} \\ &= S_0^1(\mathbf{w}_0) + \kappa_m \exp\left(\frac{\kappa_m}{2}\right) \left(\sum_{i=1}^{\mu} \beta_i h_i\right) (1 + o(\|h\|)). \end{aligned}$$

Hence, $DS_0^1(\mathbf{w}_0)(h) = \kappa_m \exp\left(\frac{\kappa_m}{2}\right) \sum_{i=1}^{\mu} \beta_i h_i$ which is a surjective linear map, which implies that the rank of $C_0^1(\mathbf{w}_0)$ is n . The point \mathbf{w}_0 is not in \mathcal{O}_0^1 , but since $w \rightarrow S_0^1(w)$ is C^1 , there exists $\mathcal{V}_{\mathbf{w}_0}$ an open neighborhood of \mathbf{w}_0 such that for all $\mathbf{v} \in \mathcal{V}_{\mathbf{w}_0}$, $C_0^1(\mathbf{v})$ has rank n . Finally since $\mathcal{V}_{\mathbf{w}_0} \cap \mathcal{O}_0^1$ is not empty (since f has Lebesgue negligible level sets, we can find μ distinct points arbitrarily close to zero with different f -values, the ranked vectors will belong to \mathcal{O}_0^1), there exists $\mathbf{w}^* \in \mathcal{O}_0^1$ such that $C_0^1(\mathbf{w}^*)$ has rank n . \square

The two previous lemmas prove that 0 is a steadily attracting state where the rank condition on the controllability matrix is satisfied. Hence, according to Theorem 4.3, the Markov chain \mathcal{Z} defined in (37) is an aperiodic φ -irreducible T -chain and every compact set is small.

Appendix

Regularity lemmas

We state two simple lemmas whose results are often used. The first one is given without any proof.

Lemma A.2. *Suppose that the function $(x, w) \mapsto F(x, w)$ is C^m for $m \in \mathbb{Z}_{\geq 0}$, then for all $k \in \mathbb{Z}_{> 0}$, the function $(x, \mathbf{w}) \mapsto S_x^k(\mathbf{w})$ defined in (14) is C^m .*

Lemma A.3. *Suppose that the function $p : (x, w) \mapsto p_x(w)$ is lower semi-continuous and the function $(x, w) \mapsto F(x, w)$ is continuous, then for all $k \in \mathbb{Z}_{> 0}$ the function $(x, \mathbf{w}) \mapsto p_x^k(\mathbf{w})$ defined in (15) is lower semi-continuous.*

Proof. According to Lemma A.2, the function $(x, \mathbf{w}) \mapsto S_x^k(\mathbf{w})$ is continuous, and by hypothesis, the function p is lower semi-continuous. Now suppose that the function $(x, \mathbf{w}) \mapsto p_x^k(\mathbf{w})$ is lower semi-continuous. The function $(x, \mathbf{w}, u) \mapsto p_{S_x^k(\mathbf{w})}(u)$ is lower semi-continuous as the composition of a continuous and a lower semi-continuous function. So by (15) the function $(x, \mathbf{w}, u) \mapsto p_x^{k+1}(\mathbf{w}, u)$ is lower semi-continuous as the product of two non-negative lower semi-continuous functions (see, for instance, [15], Proposition B.1). \square

Proof of Proposition 3.1

It is immediate that if $x^* \in \mathcal{X}$ is globally attractive then (i) holds. Indeed for all $y \in \mathcal{X}$, (18) implies that $x^* \in \overline{\bigcup_{k=1}^{+\infty} A_+^k(y)} \subset \overline{A_+(y)}$.

We now show that (i) implies (ii). Suppose that (i) holds, take $y \in \mathcal{X}$, \mathcal{U} an open set containing x^* , and $u \in \mathcal{O}_y^1$. Since from (i), $x^* \in \overline{A_+(S_y^1(u))}$, there exists $z \in A_+(S_y^1(u))$ such that $z \in \mathcal{U}$. Either $z \in A_+^0(S_y^1(u)) = \{S_y^1(u)\}$, then u is a 1-step path from y to \mathcal{U} or $z \in A_+^k(S_y^1(u))$ for $k > 0$ but then there exists $\mathbf{w} \in \mathcal{O}_{S_y^1(u)}^k$ such that $z = S_{S_y^1(u)}^k(\mathbf{w}) = S_y^{k+1}(u, \mathbf{w})$ and thus (u, \mathbf{w}) is a $k+1$ path from y to \mathcal{U} .

We now show that (ii) implies (iii). Suppose that (ii) holds, hence there exists $k_1 \in \mathbb{Z}_{>0}$ and \mathbf{w}_1 a k_1 -steps path from y to $B(x^*, 1)$. Let y_1 denote $S_{y_1}^{k_1}(\mathbf{w}_1)$. Inductively for $t \in \mathbb{Z}_{>0}$ there exists k_{t+1} and \mathbf{w}_{t+1} a k_{t+1} -steps path from y_t to $B(x^*, 1/(t+1))$, and we define y_{t+1} as $S_{y_t}^{k_{t+1}}(\mathbf{w}_{t+1})$. We then have $y_t \in A_+^{k_1+\dots+k_t}(y)$ with $k_i > 0$ for $i = 1, \dots, t$, so $\{y_t : t \in \mathbb{Z}_{>0}\}$ is a subsequence of a sequence of $\prod_{i \in \mathbb{Z}_{>0}} A_+^i(y)$. Finally, $y_t \in B(x^*, 1/t)$ so this subsequence converges to x^* .

Finally, we show that (iii) implies that x^* is a globally attracting state. Suppose that (iii) holds, that is for all $y \in \mathcal{X}$ there exists $\{y_k : k \in \mathbb{Z}_{>0}\}$ a sequence with $y_k \in A_+^k(y)$ from which we can extract a subsequence converging to x^* . Since $y_k \in A_+^k(y) \subset \bigcup_{i=k}^{\infty} A_+^i(y)$, and that for any $k \in \mathbb{Z}_{>0}$, the state x^* is the limit of a subsequence of $\{y_i : i \geq k\}$, we have $x^* \in \overline{\bigcup_{i \geq k} A_+^i(y)}$ for any $k \in \mathbb{Z}_{>0}$, and so (18) holds for all $y \in \mathcal{X}$.

Proof of Proposition 3.2

Let $x \in \mathcal{X}$, $\mathcal{O} \in \mathcal{B}(\mathcal{X})$ be an open set, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ be a k -steps path from x to \mathcal{O} . From the continuity of S_x^k (by Lemma A.2) there exists $\eta_1 > 0$ such that for all $\mathbf{u} \in B(\mathbf{w}, \eta_1)$, $S_x^k(\mathbf{u}) \in \mathcal{O}$. Since $\mathbf{w} \in \mathcal{O}_x^k$, $p_0 := p_x^k(\mathbf{w}) > 0$ and from the lower semi-continuity of p_x^k (by Lemma A.3), there exists $\eta_2 > 0$ such that for all $\mathbf{u} \in B(\mathbf{w}, \eta_2)$, $p_x^k(\mathbf{u}) > p_0/2$. Hence,

$$\begin{aligned} P^k(x, \mathcal{O}) &= \int_{\mathcal{O}_x^k} \mathbf{1}_{\mathcal{O}}(S_x^k(\mathbf{u})) p_x^k(\mathbf{u}) d\mathbf{u} \\ &\geq \int_{B(\mathbf{w}, \min(\eta_1, \eta_2))} \frac{p_0}{2} d\mathbf{u} > 0. \end{aligned}$$

Conversely, for any $x \in \mathcal{X}$, $k \in \mathbb{Z}_{>0}$ and $A \in \mathcal{B}(\mathcal{X})$

$$P^k(x, A) = \int_{\mathcal{O}_x^k} \mathbf{1}_A(S_x^k(\mathbf{u})) p_x^k(\mathbf{u}) d\mathbf{u}$$

and therefore if $P^k(x, A) > 0$, then there exists \mathbf{u} such that $p_x^k(\mathbf{u}) > 0$ and $S_x^k(\mathbf{u}) \in A$ that is, there exists a k -steps path from x to A .

Proof of Corollary 3.1

Let x^* be a reachable point. Then for all $y \in \mathcal{X}$ and $\mathcal{O} \in \mathcal{B}(\mathcal{X})$ open containing x^* , there exists $k \in \mathbb{Z}_{>0}$ such that $P^k(y, \mathcal{O}) > 0$. Hence from Proposition 3.2 there exists a k -steps from y to \mathcal{O} , and from Proposition 3.1, x^* is globally attracting.

Conversely, let x^* be globally attracting. From Proposition 3.1, for all $y \in \mathcal{X}$ and \mathcal{O} open neighborhood of x^* , there exists $k \in \mathbb{Z}_{>0}$ and a k -steps path from y to \mathcal{O} . From Proposition 3.2, we have $P^k(y, \mathcal{O}) > 0$, and so x^* is reachable.

Proof of Proposition 3.3

Let x^* be a steadily attracting state, then from Proposition 3.1(ii), we immediately find that x^* is globally attracting.

We now prove (ii). Suppose that x^* is steadily attracting, and let $y \in \mathcal{X}$. We will construct a sequence $(y_k)_{k \in \mathbb{Z}_{>0}}$ with $y_k \in A_+^k(y)$ converging to x^* . There exists $(T_i)_{i \in \mathbb{Z}_{>0}}$ a strictly increasing sequence of integers such that for all $t \geq T_i$, there exists a t -steps path from y to $B(x^*, 1/i)$. For any $k \in \mathbb{Z}_{>0}$, we construct the sequence y_k in the following way. If $k < T_1$, let y_k be any point of $A_+^k(y)$. Else for the largest $i \in \mathbb{Z}_{>0}$ such that $T_i \leq k$, there exists \mathbf{w}_k a k -steps path from y to $B(x^*, 1/i)$. Let $y_k = S_y^k(\mathbf{w}_k)$. Then $\{y_k : k \in \mathbb{Z}_{>0}\}$ with $y_k \in A_+^k(y)$ is a sequence converging to x^* .

Now suppose that for all $y \in \mathcal{X}$ there exists a sequence $\{y_k : k \in \mathbb{Z}_{>0}\}$ with $y_k \in A_+^k(y)$ which converges to x^* , and take \mathcal{U} a neighborhood of x^* . There exists T such that for all $k \geq T$, $y_k \in \mathcal{U}$. Since $y_k \in A_+^k(y)$, there exists $\mathbf{w}_k \in \mathcal{O}_y^k$ such that $S_y^k(\mathbf{w}_k) = y_k \in \mathcal{U}$. Hence, \mathbf{w}_k is a k -steps path from y to \mathcal{U} . Since such a \mathbf{w}_k exists for all $k \geq T$, this proves that x^* is steadily attracting.

We now prove (iii). Suppose that x^* is steadily attracting and that y^* is globally attracting. Then for \mathcal{U}_{y^*} an open neighborhood of y^* , there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_{x^*}^k$ such that $S_{x^*}^k(\mathbf{w}) \in \mathcal{U}_{y^*}$. By Lemma A.2 and A.3, the function $x \mapsto p_x^k$ is lower semi-continuous and the function $x \mapsto S_x^k(\mathbf{w})$ is continuous, so since \mathcal{U}_{y^*} is open there exists $\varepsilon > 0$ such that for all $x \in B(x^*, \varepsilon)$, $\mathbf{w} \in \mathcal{O}_x^k$ and $S_x^k(\mathbf{w}) \in \mathcal{U}_{y^*}$. And as x^* is steadily attracting, for all $z \in \mathcal{X}$ there exists $T \in \mathbb{Z}_{>0}$ such that for all $t \geq T$ there exists $\mathbf{u}_t \in \mathcal{O}_z^t$ for which $S_z^t(\mathbf{u}_t) \in B(x^*, \varepsilon)$. Therefore, $S_z^{t+k}(\mathbf{u}_t, \mathbf{w}) \in \mathcal{U}_{y^*}$ for all $t \geq T$, that is y^* is steadily attracting.

Proof of Proposition 3.4

We first prove (i). Since x^* is attainable, according to (20), there exists $a \in \mathbb{Z}_{>0}$ and $\mathbf{w}_a \in \mathcal{O}_{x^*}^a$ such that $x^* = S_{x^*}^a(\mathbf{w}_a)$. Hence for all $t \geq 0$, $x^* = S_{x^*}^{\text{at}}(\mathbf{w}_a, \dots, \mathbf{w}_a) \in A_+^{\text{at}}(x^*)$, meaning $a \in E$ and so E is not empty.

Take $(a, b) \in E^2$, and denote d the greatest common divisor of a and b . There exists $(t_a, t_b) \in \mathbb{Z}_{\geq 0}$ such that for all $t \geq t_a$, $x^* \in A_+^{\text{at}}(x^*)$ and for $t \geq t_b$, $x^* \in A_+^{bt}(x^*)$. To show that $d \in E$, we will show that there exists $k_0 \in \mathbb{Z}_{\geq 0}$ such that for all $t \geq 0$, there exists $(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2$ for which

$$(k_0 + t)d = ac_1 + bc_2. \quad (43)$$

Indeed if this holds as $x^* \in A_+^{a(c_1+t_a)}(x^*)$ and $x^* \in A_+^{b(c_2+t_b)}(x^*)$, $x^* \in A_+^{(k_0+t)d+at_a+bt_b}(x^*)$. Since d divides a and b , $x^* \in A_+^{(k_0+t+p)d}(x^*)$ for some $p \in \mathbb{Z}_{\geq 0}$. Hence, for all $t \geq k_0 + p$, $x^* \in A_+^{td}(x^*)$, i.e. $d \in E$.

It remains to show (43). According to Bézout's identity there exists $(u, v) \in \mathbb{Z}^2$ for which $au + bv = d$. If u or v is zero, then d equals b or a and so $d \in E$. Else, w.l.o.g. suppose that $v < 0$ and $u > 0$. Take $k \geq -va/d$ and $t \in \mathbb{Z}_{\geq 0}$. We will write $kb + td$ as a positive sum of a

and b . As a result of Euclidian division of t by a/d there exists $q \in \mathbb{Z}_{\geq 0}$ and $s \in \{0, \dots, a/d - 1\}$ such that $td = aq + sd$, and so $kb + td = kb + qa + sd$. Furthermore $sd = s(au + bv)$ so $kb + td = (k + sv)b + (q + su)a$, with the coefficients $k + sv$ and $q + su$ positive. Hence for all $t \geq 0$, $(kb/d + t)d = (k + sv)b + (q + su)a$ and hence we have proven (43) with $c_1 = q + su$, $c_2 = k + sv$.

We now prove (ii). If $\gcd(E) = 1$, then $1 \in E$, that is, there exists $t_0 \in \mathbb{Z}_{>0}$ such that for all $t \geq t_0$, $x^* \in A_+^t(x^*)$. Since x^* is attainable, for all $y \in X$ there exists $t_1 \in \mathbb{Z}_{>0}$ such that $x^* \in A_+^{t_1}(y)$, and so for all $t \geq t_0 + t_1$, $x^* \in A_+^t(y)$. Therefore for all \mathcal{U} neighborhood of x^* and for all $t \geq t_0 + t_1$ there exists $w \in \mathcal{O}_y^t$ such that $S_y^t(w) = x^* \in \mathcal{U}$, meaning x^* is steadily attracting.

We finally prove (iii). Let $d := \gcd(E)$ and for $i = 0, \dots, d - 1$ take $D_i := \bigcup_{r \in \mathbb{Z}_{\geq 0}} A_+^{rd+i}(x^*)$. Then:

- for all $y \in A_+^t(x^*)$, $P(y, A_+^{t+1}(x^*)) = 1$, so for $i = 0, \dots, d - 1$ and $y \in D_i$, $P(y, D_{i+1 \bmod d}) = 1$.
- take $y \in A_+^i(x^*)$. Since x^* is attainable there exists $t_0 \in \mathbb{Z}_{>0}$ such that $x^* \in A_+^{t_0}(y)$, so $x^* \in A_+^{t_0+i}(x^*)$ and $x^* \in A_+^{r(t_0+i)}(x^*)$ for all $r \geq 0$, i.e. $t_0 + i \in E$. As $d = \gcd(E)$, this implies that $i = k_1d - t_0$ for some $k_1 \in \mathbb{Z}_{>0}$. By the same reasoning if $y \in A_+^j(x^*)$ for some $j \in \mathbb{Z}_{\geq 0}$, then for some $k_2 \in \mathbb{Z}_{>0}$, $j = k_2d - t_0$ also. Therefore $i - j$ is a multiple of d , which implies that the sets $(D_i)_{i=0, \dots, d-1}$ are disjoint sets.
- $\bigcup_{i=0}^{d-1} D_i = A_+(x^*)$. Since for all $k \in \mathbb{Z}_{>0}$, $P^k(x^*, A_+(x^*)^c) = 0$, $\varphi((\bigcup_{i=0}^{d-1} D_i)^c) = 0$ for all φ irreducibility measure.

Hence, $(D_i)_{i=0, \dots, d-1}$ is a d -cycle.

Derivation of (26):

By chain rule, for $i = 1, \dots, k - 1$,

$$\begin{aligned} \left[\frac{\partial S_y^k}{\partial w_i} \right]_{(w_1, \dots, w_k)} &= \left[\frac{\partial F}{\partial x} \right]_{(S_y^{k-1}, w_k)} \left[\frac{\partial S_y^{k-1}}{\partial w_i} \right]_{(w_1, \dots, w_{k-1})} \\ &= A_{k-1} \left[\frac{\partial S_y^{k-1}}{\partial w_i} \right]_{(w_1, \dots, w_{k-1})} \\ &= A_{k-1} \cdots A_i \left[\frac{\partial S_y^i}{\partial w_i} \right]_{(w_1, \dots, w_i)}. \end{aligned}$$

Furthermore, for $i = 1, \dots, k$,

$$\left[\frac{\partial S_y^i}{\partial w_i} \right]_{(w_1, \dots, w_i)} = \left[\frac{\partial F}{\partial w} \right]_{(S_y^{i-1}, w_i)} = B_{i-1},$$

and so

$$\left[\frac{\partial S_y^k}{\partial w_i} \right]_{(w_1, \dots, w_k)} = A_{k-1} \cdots A_i B_{i-1}.$$

Proof of Proposition 3.5

The matrix $C_{x^*}^k(\mathbf{w}^*)$ has rank n and thus with the expression of the matrix given in (26), we can find i_1, \dots, i_n such that

$$\det \left[\frac{\partial S_{x^*}^k}{\partial w_{i_1}} \mid \cdots \mid \frac{\partial S_{x^*}^k}{\partial w_{i_n}} \right]_{\mathbf{w}^*} \neq 0.$$

Since the function $(x, \mathbf{w}) \mapsto S_x^k(\mathbf{w})$ is a C^1 function (by Lemma A.2), there exists $\mathcal{N}_{(x^*, \mathbf{w}^*)}$ a neighborhood of (x^*, \mathbf{w}^*) such that for all $(x, \mathbf{w}) \in \mathcal{N}_{(x^*, \mathbf{w}^*)}$

$$\det \left[\frac{\partial S_x^k}{\partial w_{i_1}} \mid \cdots \mid \frac{\partial S_x^k}{\partial w_{i_n}} \right]_{\mathbf{w}} \neq 0.$$

Let $r_1 > 0$ and $r_2 > 0$ such that $B(x^*, r_1) \times B(\mathbf{w}^*, r_2) \subset \mathcal{N}_{(x^*, \mathbf{w}^*)}$. Since x^* is a globally attracting state, for all $y \in \mathcal{X}$ according to Proposition 3.1 there exists $t_0 \in \mathbb{Z}_{>0}$ and \mathbf{u}_0 a t_0 -steps path from y to $B(x^*, r_1)$. Since $(S_y^{t_0}(\mathbf{u}_0), \mathbf{w}^*) \in \mathcal{N}_{(x^*, \mathbf{w}^*)}$, the matrix $C_{S_y^{t_0}(\mathbf{u}_0)}^k(\mathbf{w}^*)$ has rank n . Hence, taking $T = t_0 + k$ and $\mathbf{u} = (\mathbf{u}_0, \mathbf{w}^*)$, according to (25) the matrix $C_y^T(\mathbf{u})$ also has rank n .

Proof of Proposition 3.6

This result is a consequence of the implicit function theorem. Take $x^* \in \mathcal{X}$, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which the controllability matrix $C_{x^*}^k(\mathbf{w}^*)$ has rank n . We first prove that there exists \mathcal{U}_{x^*} a neighborhood of x^* such that for all $y \in \mathcal{U}_{x^*}$, there exists $\mathbf{u} \in \mathcal{O}_y^k$ such that $S_y^k(\mathbf{u}) = S_{x^*}^k(\mathbf{w}^*)$. Since for all $m \in \mathbb{Z}_{>0}$ the function $(x, \mathbf{w}) \mapsto p_x^m(\mathbf{w})$ is lower semi-continuous (as a consequence of Lemma A.3), there exists \mathcal{O} an open neighborhood of (x^*, \mathbf{w}^*) such that for all $(x, \mathbf{w}) \in \mathcal{O}$, $p_x^k(\mathbf{w}) > 0$. Since $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$, using the expression of the controllability matrix given in (26) (where w_i are coordinates rather than vectors), there exists integers (i_1, \dots, i_n) such that

$$\det \left[\frac{\partial S_{x^*}^k}{\partial w_{i_1}} \mid \cdots \mid \frac{\partial S_{x^*}^k}{\partial w_{i_n}} \right]_{\mathbf{w}^*} \neq 0.$$

Assume that $i_1 = kp - n + 1, \dots, i_n = kp$ (which can be imposed by considering a composition of S_x^k with a function permuting the variables). The partial differential of the C^1 function $(x, w_1, \dots, w_{kp}) \mapsto S_x^k(w_1, \dots, w_{kp})$ with respect to $(w_{kp-n+1}, \dots, w_{kp})$ is invertible at (x^*, \mathbf{w}^*) , so according to the implicit function theorem applied to this function restricted to \mathcal{O} , there exists \mathcal{U} and \mathcal{V} open neighborhoods of respectively $(x^*, w_1^*, \dots, w_{kp-n}^*)$ and $(w_{kp-n+1}^*, \dots, w_{kp}^*)$ such that $\mathcal{U} \times \mathcal{V} \subset \mathcal{O}$, and a C^1 function $g : \mathcal{U} \rightarrow \mathcal{V}$ such that $S_x^k(w_1, \dots, w_{kp-n}, g(x, w_1, \dots, w_{kp-n})) = S_{x^*}^k(\mathbf{w}^*)$ for all $(x, w_1, \dots, w_{kp-n}) \in \mathcal{U}$, which proves our first point by taking $\mathcal{U}_{x^*} = \{x \in \mathcal{X} \mid (x, w_1, \dots, w_{kp-n}) \in \mathcal{U}\}$.

Now suppose that x^* is globally attracting. Then according to Proposition 3.1(ii), for all $z \in \mathcal{X}$ there exists $t_0 \in \mathbb{Z}_{>0}$ and $\mathbf{v} \in \mathcal{O}_z^{t_0}$ such that $S_z^{t_0}(\mathbf{v}) \in \mathcal{U}_{x^*}$. And so there exists $\mathbf{u} \in \mathcal{O}_{S_z^{t_0}(\mathbf{v})}^k$ such that $S_z^{t_0+k}(\mathbf{v}, \mathbf{u}) = S_{x^*}^k(\mathbf{w}^*)$.

Proof of Proposition 3.7

Let $x^* \in \mathcal{X}$ such that there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* = (w_1^*, \dots, w_{kp}^*) \in \mathcal{O}_{x^*}^k$ such that $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$, and let us prove that $A_+^k(x^*)$ contains an open set (which would imply the forward accessibility of the control model, given that the condition holds for all $x^* \in \mathcal{X}$).

Since $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$, following (26) there exists integers $\{i_1, \dots, i_n\}$ such that

$$\det \left[\frac{\partial S_{x^*}^k}{\partial w_{i_1}} \mid \dots \mid \frac{\partial S_{x^*}^k}{\partial w_{i_n}} \right]_{\mathbf{w}^*} \neq 0.$$

Assume that $i_1 = kp - n + 1, \dots, i_n = kp$ (which can be imposed by considering the composition of $S_{x^*}^k$ with a function permuting the variables), let G denote the C^1 function $(u_1, \dots, u_n) \in \mathcal{O} \mapsto S_{x^*}^k(w_1^*, \dots, w_{kp-n}^*, u_1, \dots, u_n)$ with \mathcal{O} the set $\{(u_1, \dots, u_n) \in \mathbb{R}^n \mid (w_1^*, \dots, w_{kp-n}^*, u_1, \dots, u_n) \in \mathcal{O}_{x^*}^k\}$ (which is open since $\mathcal{O}_{x^*}^k$ is open). Then the Jacobian determinant of G is non-zero, and according to the inverse function theorem applied to G there exists $\mathcal{U} \subset \mathcal{O}$ and \mathcal{V} open neighborhoods of respectively $(w_{kp-n+1}^*, \dots, w_{kp}^*)$ and $S_{x^*}^k(\mathbf{w}^*)$, such that G is a bijection from \mathcal{U} to \mathcal{V} . Therefore $\mathcal{V} \subset A_+^k(x^*)$ which hence has nonempty interior.

Suppose now that F is C^∞ and that the control model is forward accessible. Then, for all $x \in \mathcal{X}$, $A_+(x)$ is not Lebesgue negligible. Since $\sum_{k \in \mathbb{Z}_{>0}} \mu^{\text{Leb}}(A_+^k(x)) \geq \mu^{\text{Leb}}(A_+(x)) > 0$, there exists $k \in \mathbb{Z}_{>0}$ such that $\mu^{\text{Leb}}(A_+^k(x)) > 0$ ($k \neq 0$ because $A_+^0(x) = \{x\}$ is Lebesgue negligible). Let N_x^k denote the set $\{\mathbf{w} \in \mathcal{O}_x^k \mid \mathbf{w} \text{ is a critical point of } S_x^k\}$ (i.e. $\mathbf{w} \in N_x^k$ if and only if $\text{rank}(C_x^k(\mathbf{w})) < n$). According to Lemma A.2 the function S_x^k is C^∞ , so we can apply Sard's theorem [17], Theorem II.3.1, to S_x^k which implies that the image of its critical points is Lebesgue negligible, i.e. $\mu^{\text{Leb}}(S_x^k(N_x^k)) = 0$. Since $\mu^{\text{Leb}}(S_x^k(N_x^k)) < \mu^{\text{Leb}}(A_+^k(x))$ and $A_+^k(x) = S_x^k(\mathcal{O}_x^k)$, N_x^k is a strict subset of \mathcal{O}_x^k meaning there exists $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank}(C_x^k(\mathbf{w})) = n$.

Proof of Proposition 4.1

In order to prove the proposition, we use the following lemma, which is identical to [10], Lemma 3.0, with the exception that the function G is here assumed to be C^1 instead of C^∞ , and that \mathbb{R}^n has in some places been replaced with \mathcal{X}_1 . These changes do not impact the proof of the lemma. Variable names have also been changed for consistency with the notations used in this article.

Lemma A.4 ([10], Lemma 3.0). *Let $\mathcal{X}_1 \subset \mathbb{R}^n$, $\widetilde{\mathcal{W}}_1 \subset \mathbb{R}^m$ and $\widehat{\mathcal{W}}_1 \subset \mathbb{R}^n$ be open sets and let $G : (x, \widetilde{w}, \widehat{w}) \in \mathcal{X}_1 \times \widetilde{\mathcal{W}}_1 \times \widehat{\mathcal{W}}_1 \mapsto z \in \mathcal{X}_1$ be a C^1 function such that the matrix $\partial G / \partial \widehat{w}$ has rank n at some $(x_0, \widetilde{w}_0, \widehat{w}_0) \in \mathcal{X}_1 \times \widetilde{\mathcal{W}}_1 \times \widehat{\mathcal{W}}_1$. Then*

(i) there exists an open set $\mathcal{X} \times \tilde{W} \times \hat{W} \subset \mathcal{X}_1 \times \tilde{W}_1 \times \hat{W}_1$ containing $(x_0, \tilde{w}_0, \hat{w}_0)$ such that for all $x \in \mathcal{X}$ the measure $\nu(x, \cdot)$ defined by

$$\nu(x, \cdot) : A \in \mathcal{B}(\mathcal{X}_1) \mapsto \int_{\tilde{W}} \int_{\hat{W}} \mathbf{1}_A(G(x, \tilde{w}, \hat{w})) d\tilde{w} d\hat{w}$$

is equivalent to Lebesgue measure on an open set \mathcal{R}_x .

(ii) there exists $c > 0$ and open sets \mathcal{U}_{x_0} and $\mathcal{V}_{x_0}^{(\tilde{w}_0, \hat{w}_0)}$ containing x_0 and $G(x_0, \tilde{w}_0, \hat{w}_0)$ respectively, such that for all $x \in \mathcal{X}$ and $A \in \mathcal{B}(\mathcal{X}_1)$

$$\nu(x, A) \geq c \mathbf{1}_{\mathcal{U}_{x_0}}(x) \mu^{\text{Leb}}(A \cap \mathcal{V}_{x_0}^{(\tilde{w}_0, \hat{w}_0)}).$$

The proof of Proposition 4.1 (i) differs from the proof in [10], Theorem 2.1, in (44) where the equation holds for $y \in B_\delta(x_0, \delta)$ (instead of for all $y \in \mathcal{X}$). This does not impact the rest of the proof. The detailed proof is given below. The proof of [10], Theorem 2.1, uses Sard's theorem to show that weak stochastic controllability implies that for all $x \in \mathcal{X}$, there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ such that the rank of $C_x^k(\mathbf{w})$ is n . We use here the same idea to show (ii), with a slight precision to show that \mathbf{w} may be taken in the open set \mathcal{V} of (28).

We are now ready to start the core of the proof of Proposition 4.1. Suppose that the controllability matrix $C_{x_0}^k(\mathbf{w}_0)$ has rank n for some $x_0 \in \mathcal{X}$, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}_0 \in \mathcal{O}_{x_0}^k$. Since the function $(x, \mathbf{w}) \mapsto p_x^k(\mathbf{w})$ is lower semi-continuous (as a consequence of Lemma A.3) and that $p_{x_0}^k(\mathbf{w}_0) > 0$, there exists $p_0 > 0$ and δ , such that $p_x^k(\mathbf{w}) > p_0$ for all $(x, \mathbf{w}) \in B(x_0, \delta) \times B(\mathbf{w}_0, \delta)$. Hence,

$$\begin{aligned} P^k(y, A) &= \int_{\mathcal{O}_y^k} \mathbf{1}_A(S_y^k(\mathbf{w})) p_y^k(\mathbf{w}) d\mathbf{w} \\ &\geq p_0 \int_{B(\mathbf{w}_0, \delta)} \mathbf{1}_A(S_y^k(\mathbf{w})) d\mathbf{w} \quad \text{for all } y \in B(x_0, \delta). \end{aligned} \tag{44}$$

Since $\text{rank}(C_{x_0}^k(\mathbf{w}_0)) = n$, we can extract a sequence of n integers (i_1, \dots, i_n) for which

$$\det \left[\frac{\partial S_{x_0}^k}{\partial w_{i_1}} \mid \dots \mid \frac{\partial S_{x_0}^k}{\partial w_{i_n}} \right]_{\mathbf{w}_0} \neq 0.$$

Hence, we can apply Lemma A.4 to the measure $\nu(y, A) = \int_{B(\mathbf{w}_0, \delta)} \mathbf{1}_A(S_y^k(\mathbf{w})) d\mathbf{w}$ by appropriately defining G in terms of S_x^k (separating $\mathbf{w} \in B(\mathbf{w}_0, \delta)$ into $\hat{\mathbf{w}} = (w_{i_1}, \dots, w_{i_n})$ and $\tilde{\mathbf{w}}$ equal to the other coordinates, and defining $G : (x, \tilde{\mathbf{w}}, \hat{\mathbf{w}}) \mapsto S_x^k(\mathbf{w})$ which ensures that $\partial G / \partial \hat{\mathbf{w}}(x_0, \tilde{\mathbf{w}}_0, \hat{\mathbf{w}}_0)$ has rank n). This shows the existence of open sets \mathcal{U}_{x_0} and $\mathcal{V}_{x_0}^{\mathbf{w}_0}$ containing x_0 and $S_{x_0}^k(\mathbf{w}_0)$ respectively, and a constant $c > 0$ such that

$$\int_{B(\mathbf{w}_0, \delta)} \mathbf{1}_A(S_x^k(\mathbf{w})) d\mathbf{w} \geq c \mathbf{1}_{\mathcal{U}_{x_0}}(y) \mu^{\text{Leb}}(A \cap \mathcal{V}_{x_0}^{\mathbf{w}_0}) \quad \text{for all } y \in \mathcal{X}. \tag{45}$$

Combining (44) and (45) shows that for all y in the open set $\mathcal{U}_{x_0} \cap B(x_0, \delta)$,

$$P^k(y, A) \geq cp_0 \mu^{\text{Leb}}(A \cap \mathcal{V}_{x_0}^{\mathbf{w}_0}),$$

which shows (i).

Now suppose that F is C^∞ , hence according to Lemma A.2 the function $(x, \mathbf{w}) \mapsto S_x^k(\mathbf{w})$ is also C^∞ for all $k \in \mathbb{Z}_{>0}$. Take $k \in \mathbb{Z}_{>0}$, $c > 0$ and $\mathcal{V} \in \mathcal{B}(\mathcal{X})$ an open set for which (28) holds. Let N denote the set of critical values of S_x^k , i.e.

$$N := \{S_x^k(\mathbf{w}) \in \mathcal{X} \mid \mathbf{w} \in \mathcal{O}_x^k \text{ and } \text{rank}(C_x^k(\mathbf{w})) < n\}.$$

By Sard's theorem, $\mu^{\text{Leb}}(N) = 0$. Hence, with (28) for $A = \mathcal{V} \setminus N$, $P^k(x, \mathcal{V} \setminus N) \geq c\mu^{\text{Leb}}(\mathcal{V} \setminus N \cap \mathcal{V}) = c\mu^{\text{Leb}}(\mathcal{V}) > 0$. Hence, there exists $\mathbf{w} \in \mathcal{O}_x^k$ such that $S_x^k(\mathbf{w}) \in \mathcal{V} \setminus N$ (otherwise we would have $P^k(x, \mathcal{V} \setminus N) = 0$). And since $S_x^k(\mathbf{w}) \notin N$, the controllability matrix $C_x^k(\mathbf{w})$ has rank n .

Proof of Proposition 4.2

We first prove that a point x^* of the support of ψ is globally attracting. By definition of the support, there exists N a closed set containing x^* with full ψ -measure (i.e., $\psi(N^c) = 0$). Let \mathcal{U} be an open neighborhood of x^* , then $\psi(\mathcal{U}) > 0$ (otherwise the closed set $N \setminus \mathcal{U}$ would have full ψ -measure without containing x^* , so x^* would not be in the support of ψ which is a contradiction). This imply that for all $y \in \mathcal{X}$, $\sum_{k \in \mathbb{Z}_{>0}} P^k(y, \mathcal{U}) > 0$, and so by Proposition 3.2 that there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ a k -steps path from y to \mathcal{U} . Therefore by Proposition 3.1, x^* is globally attracting.

Conversely, let $x^* \in \mathcal{X}$ be a globally attracting point. Suppose that $x^* \notin \text{supp } \psi$ then there exists N a closed set of full ψ -measure such that $x^* \notin N$. Then its complementary N^c is an open set containing x^* with null- ψ -measure. By Corollary 3.1 it is reachable, and so any open neighborhood of x^* has positive ψ measure, which is in contradiction with the set N^c previously defined. Therefore, $x^* \in \text{supp } \psi$.

We now prove that $\text{supp } \psi \subset \overline{A_+(x^*)}$ for $x^* \in \mathcal{X}$ a globally attracting state. Take $y^* \in \mathcal{X}$ a globally attracting state, by Proposition 3.1, $y^* \in \overline{A_+(x^*)}$ which implies that the set $\{z^* \in \mathcal{X} \mid z^* \text{ is globally attracting}\}$ is included in $\overline{A_+(x^*)}$ which itself as we have shown equals $\text{supp } \psi$.

Finally, we prove that $\overline{A_+(x^*)} \subset \text{supp } \psi$ by showing that any $y \in \overline{A_+(x^*)}$ is globally attracting. Indeed if this holds, then $\overline{A_+(x^*)} \subset \{z^* \in \mathcal{X} \mid z^* \text{ is globally attracting}\}$ which with (29) concludes the proof.

Take \mathcal{V} an open neighbourhood of y and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $S_{x^*}^k(\mathbf{w}^*) = y$. By continuity of F , $S_x^k(\mathbf{w}^*) \in \mathcal{V}$ for all x in an open neighborhood of x^* , \mathcal{U}_{x^*} . By lower semi-continuity of the function $(x, \mathbf{w}) \mapsto p_x^k(\mathbf{w})$, $\mathbf{w}^* \in \mathcal{O}_x^k$ for all $x \in \mathcal{U}_{x^*}$ (and taking a smaller \mathcal{U}_{x^*} if necessary). And x^* being globally attracting, for all $z \in \mathcal{X}$ there exists $T \in \mathbb{Z}_{>0}$ and $\mathbf{u}_z \in \mathcal{O}_z^T$ for which $S_z^T(\mathbf{u}_z) \in \mathcal{U}_{x^*}$, and so $S_z^{T+k}(\mathbf{u}_z, \mathbf{w}^*) \in \mathcal{V}$. Since this holds for all z and \mathcal{V} neighborhood of y , y is globally attracting.

Proof of Theorem 4.3

Assume that for all $x \in \mathcal{X}$, there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w} \in \mathcal{O}_x^k$ for which $\text{rank } C_x^k(\mathbf{w}) = n$, and assume that x^* is steadily attracting; we will prove that Φ is φ -irreducible and aperiodic. By hypothesis there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank}(C_{x^*}^k(\mathbf{w}^*)) = n$. According to Proposition 4.1(i), there exists $c > 0$ and open sets \mathcal{U}_{x^*} and $\mathcal{V}_{x^*}^{\mathbf{w}^*}$ containing respectively x^* and $S_{x^*}^k(\mathbf{w}^*)$

and such that

$$P^k(y, A) \geq c\mu^{\text{Leb}}(A \cap \mathcal{V}_{x^*}^{\mathbf{w}^*}) \quad \text{for all } y \in \mathcal{U}_{x^*}, A \in \mathcal{B}(\mathcal{X}). \quad (46)$$

For all $y \in \mathcal{X}$ and $t \in \mathbb{Z}_{>0}$, we may develop $P^{t+k}(y, A)$ as

$$\begin{aligned} P^{t+k}(y, A) &= \int_{\mathcal{X}} P^t(y, dz) P^k(z, A) \\ &\geq \int_{\mathcal{U}_{x^*}} P^t(y, dz) P^k(z, A) \\ &\geq \int_{\mathcal{U}_{x^*}} P^t(y, dz) c\mu^{\text{Leb}}(A \cap \mathcal{V}_{x^*}^{\mathbf{w}^*}) = P^t(y, \mathcal{U}_{x^*}) c\mu^{\text{Leb}}(A \cap \mathcal{V}_{x^*}^{\mathbf{w}^*}). \end{aligned} \quad (47)$$

As x^* is globally attractive, by Corollary 3.1 x^* is also reachable. Hence for all $y \in \mathcal{X}$, there exists a $t \in \mathbb{Z}_{>0}$ such that $P^t(y, \mathcal{U}_{x^*}) > 0$, and therefore by (47) the measure $\bar{\varphi} : A \mapsto \mu^{\text{Leb}}(A \cap \mathcal{V}_{x^*}^{\mathbf{w}^*})$ is an irreducibility measure.

Since the Markov chain Φ is φ -irreducible, according to [11], Theorem 5.4.4, there exists $d \in \mathbb{Z}_{>0}$ and disjoint sets $(D_i)_{i=0, \dots, d-1} \in \mathcal{B}(\mathcal{X})^d$ such that:

- (a) for $x \in D_i$, $P(x, D_{i+1 \pmod{d}}) = 1$, $i = 0, \dots, d-1 \pmod{d}$;
- (b) $\varphi((\bigcup_{i=0}^{d-1} D_i)^c) = 0$.

Note that (b) is usually stated with respect to the maximal irreducibility measure, which implies (b) for any irreducibility measure. Point (b) applied to $\bar{\varphi}$ implies the existence of $i \in \{0, \dots, d-1\}$ for which $\bar{\varphi}(D_i) > 0$, that is, $\mu^{\text{Leb}}(D_i \cap \mathcal{V}_{x^*}^{\mathbf{w}^*}) > 0$. Also since x^* is steadily attracting, for all $y \in \mathcal{X}$ there exists $t_0 \in \mathbb{Z}_{>0}$ such that for all $t \geq t_0$, there exists a t -steps path from y to \mathcal{U}_{x^*} and so by Proposition 3.2 $P^t(y, \mathcal{U}_{x^*}) > 0$. Together with (47), we obtain $P^{t+k}(y, D_i) \geq P^t(y, \mathcal{U}_{x^*}) c\mu^{\text{Leb}}(D_i \cap \mathcal{V}_{x^*}^{\mathbf{w}^*}) > 0$ for all $t \geq t_0$. Hence for $y \in D_i$ and for m large enough, $P^{md+1}(y, D_i) > 0$, which with (a) and the fact that the D_i are disjoint sets implies $d = 1$, that is, Φ is aperiodic.

Now suppose that no steadily attracting state exists, we will show that Φ is not irreducible and aperiodic. If no globally attracting state exists, then by Theorem 4.1 the Markov chain is not irreducible. Otherwise let x^* denote a globally attracting state, and take $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$. Then according to Proposition 3.6, the point $y^* := S_{x^*}^k(\mathbf{w}^*)$ is attainable, and according to Theorem 4.2, Φ is φ -irreducible. Since y^* is attainable yet not steadily attracting, by Proposition 3.4, the set

$$E := \{a \in \mathbb{Z}_{>0} \mid \exists t_0 \in \mathbb{Z}_{\geq 0}, \forall t \geq t_0, y^* \in A_+^{\text{at}}(y^*)\}$$

has a gcd larger than 1, and there exists a d -cycle with $d := \text{gcd}(E) > 1$. Hence, the period of the Markov chain is at least d , and so Φ is not aperiodic.

Proof of Theorem 4.4

Suppose that x^* is a steadily attracting point, and there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$. Then according to Proposition 3.5, for all $x \in \mathcal{X}$ there exists $t \in \mathbb{Z}_{>0}$ and

$\mathbf{w} \in \mathcal{O}_x^t$ for which $\text{rank } C_x^t(\mathbf{w}) = n$. According to Theorem 4.3, Φ is therefore a φ -irreducible and aperiodic Markov chain, and by Theorem 4.2, it is a T -chain for which compact sets are petite. Finally, according to [11], Theorem 5.5.7, since Φ is irreducible and aperiodic, every petite set is small.

Proof of Proposition 5.2

Let $U_1 = (U_1^1, \dots, U_1^\lambda) \in \mathbb{R}^{n\lambda}$ with each U_1^i following a standard multivariate normal distribution and $\{U_1^i : 1 \leq i \leq \lambda\}$ i.i.d. Let $W_1 = \alpha(z, U_1)$, let μ satisfying $1 < \mu < \lambda$, $w^\mu \in \mathbb{R}^n$ and $C(w^\mu)$ denote the event $\{f(z + w^\mu) \leq f(z + U_1^i), \text{ for all } i = \mu + 1, \dots, \lambda\}$. The following holds

$$\Pr(C(w^\mu)) = \left(\int \mathbf{1}_{f(z+w^\mu) \leq f(z+w)} p_{\mathcal{N}}(w) dw \right)^{\lambda-\mu} = (1 - Q_z^f(w^\mu))^{\lambda-\mu}. \quad (48)$$

Let \mathfrak{S}_λ for $\lambda \in \mathbb{Z}_{>0}$ denote the set of permutation with λ elements. For $\sigma \in \mathfrak{S}_{\lambda-\mu}$, let $C_\sigma(w^\mu)$ denote the event $C_\sigma(w^\mu) = \{f(z + w^\mu) < f(z + U_1^{\mu+\sigma(1)}) < \dots < f(z + U_1^{\mu+\sigma(\lambda-\mu)})\}$. Since the level sets of f are Lebesgue negligible,

$$\Pr(C(w^\mu)) = \Pr\left(\bigcup_{\sigma \in \mathfrak{S}_{\lambda-\mu}} C_\sigma(w^\mu)\right) = \sum_{\sigma \in \mathfrak{S}_{\lambda-\mu}} \Pr(C_\sigma(w^\mu)) = (\lambda - \mu)! \Pr(C_I(w^\mu)), \quad (49)$$

where I denotes the identity permutation. Let $x_1 \in \mathbb{R}^n, \dots, x_\mu \in \mathbb{R}^n$ and let denote for $x, y \in \mathbb{R}^n$, $x \leq y$ for $([x]_1 \leq [y]_1, \dots, [x]_n \leq [y]_n)$ where $[x]_i$ is the i th coordinate of the vector x .

$$\begin{aligned} & \Pr(W_1^1 \leq x_1, \dots, W_1^\mu \leq x_\mu) \\ &= \sum_{\sigma \in \mathfrak{S}_\lambda} \Pr(\{U_1^{\sigma(1)} \leq x_1, \dots, U_1^{\sigma(\mu)} \leq x_\mu\} \cap \{f(z + U_1^{\sigma(1)}) < \dots < f(z + U_1^{\sigma(\lambda)})\}) \\ &= \lambda! \Pr(\{U_1^1 \leq x_1, \dots, U_1^\mu \leq x_\mu\} \cap \{f(z + U_1^1) < \dots < f(z + U_1^\lambda)\}). \end{aligned}$$

By cutting the event $\{f(z + U_1^1) < \dots < f(z + U_1^\lambda)\}$ into $\{f(z + U_1^1) < \dots < f(z + U_1^\mu)\} \cap \{f(z + U_1^\mu) < \dots < f(z + U_1^\lambda)\}$ we find that

$$\begin{aligned} & \Pr(W_1^1 \leq x_1, \dots, W_1^\mu \leq x_\mu) \\ &= \lambda! \int \mathbf{1}_{w^1 \leq x_1} \cdots \mathbf{1}_{w^\mu \leq x_\mu} \mathbf{1}_{f(z+w^1) < \dots < f(z+w^\mu)} \\ & \quad \times \mathbf{1}_{f(z+w^\mu) < \dots < f(z+w^\lambda)} p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\lambda) dw^1 \cdots dw^\lambda \\ &= \lambda! \int \mathbf{1}_{w^1 \leq x_1} \cdots \mathbf{1}_{w^\mu \leq x_\mu} \mathbf{1}_{f(z+w^1) < \dots < f(z+w^\mu)} \\ & \quad \times \Pr(C_I(w^\mu)) p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\mu) dw^1 \cdots dw^\mu. \end{aligned}$$

Using (48) and (49), we find

$$\begin{aligned} & \Pr(W_1^1 \leq x_1, \dots, W_1^\mu \leq x_\mu) \\ &= \frac{\lambda!}{(\lambda - \mu)!} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_\mu} \mathbf{1}_{f(z+w^1) < \cdots < f(z+w^\mu)} \\ & \quad \times (1 - Q_z^f(w^\mu))^{\lambda - \mu} p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\mu) dw^1 \cdots dw^\mu, \end{aligned}$$

where in the previous equation, the integrals between $-\infty$ and the vector x_i stand for $\int_{-\infty}^{[x_i]_1} \cdots \int_{-\infty}^{[x_i]_n}$ where $[x_i]_j$ denotes the j th coordinate of the vector x_i . We directly deduce the expression of the density for $\mu > 1$ and $\mu < \lambda$ given in (41). When $\mu = \lambda$, then the density of $(W_1^1, \dots, W_1^\lambda)$ equals $p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\lambda)$ that can also be written as

$$\lambda! \mathbf{1}_{f(z+w^1) < \cdots < f(z+w^\mu)} p_{\mathcal{N}}(w^1) \cdots p_{\mathcal{N}}(w^\lambda)$$

which is the expression in (41) for $\mu = \lambda$ (by using that the levels sets of f are Lebesgue negligible, and so that with probability 1, one of the $\lambda!$ possible strict orderings of the $\{w_i : i = 1, \dots, \lambda\}$ according to $f(z + w^i)$ occurs).

In the case where $\mu = 1$, the indicator $\mathbf{1}_{f(z+w^1) < \cdots < f(z+w^\mu)}$ reduces to 1 and we deduce the expression in (40).

Proof of Lemma 5.1

The proof of Lemma 5.1(ii) is a direct consequence of the following proposition.

Proposition A.5. *Suppose that assumptions A1–A5 hold and that for all $x \in \mathcal{X}$ there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ such that $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$. Then there exists a steadily attracting state if and only if there exists a globally attracting state $x^* \in \mathcal{X}$, $(a, b) \in \mathbb{Z}_{>0}^2$ with $\text{gcd}(a, b) = 1$, $\mathbf{w}_a \in \mathcal{O}_{x^*}^a$ and $\mathbf{w}_b \in \mathcal{O}_{x^*}^b$ such that $S_{x^*}^a(\mathbf{w}_a) = S_{x^*}^b(\mathbf{w}_b) = x^*$.*

Proof. Suppose that x^* is a steadily attracting state. Under the conditions of Proposition A.5, there exists $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$, so by Proposition 3.6, $y^* := S_{x^*}^k(\mathbf{w}^*)$ is attainable and there exists \mathcal{V}_{x^*} a neighborhood of x^* such that for all $x \in \mathcal{V}_{x^*}$ there exists $\mathbf{w} \in \mathcal{O}_x^k$ for which $S_x^k(\mathbf{w}) = y^*$. Since x^* is steadily attracting, there exists $T \in \mathbb{Z}_{>0}$ such that for all $t \geq T$ there is a t -steps path from y^* to \mathcal{V}_{x^*} , which can be completed by a k -steps path to y^* . So for all $t \geq T$ there exists $\mathbf{w} \in \mathcal{O}_{y^*}^{t+k}$ for which $S_{y^*}^{t+k}(\mathbf{w}) = y^*$, which proves the first implication by choosing $t_1 \geq T$ and $t_2 \geq T$ such that $\text{gcd}(t_1 + k, t_2 + k) = 1$.

Now let x^* be a globally attracting state, $k \in \mathbb{Z}_{>0}$ and $\mathbf{w}^* \in \mathcal{O}_{x^*}^k$ for which $\text{rank } C_{x^*}^k(\mathbf{w}^*) = n$. Then by Proposition 3.6 $y^* := S_{x^*}^k(\mathbf{w}^*)$ is an attainable point, and there exists \mathcal{V}_{x^*} a neighborhood of x^* such that for all $x \in \mathcal{V}_{x^*}$ there exists $\mathbf{w} \in \mathcal{O}_x^k$ for which $S_x^k(\mathbf{w}) = y^*$. Suppose that there exists $(a, b) \in \mathbb{Z}_{>0}^2$, \mathbf{w}_a and \mathbf{w}_b such as stated in Proposition A.5. We prove that y^* is steadily attracting using Proposition 3.4, by showing that the set E in (21) has two elements \tilde{a} and \tilde{b} with $\text{gcd}(\tilde{a}, \tilde{b}) = 1$.

By Bézout’s identity, there exists $(c, d) \in \mathbb{Z}^2$ such that $ca + db = 1$. Suppose without loss of generality that $d < 0$ (and so $c > 0$). By lower semi-continuity of the function $x \mapsto$

$p_x^{ca}(\mathbf{w}_a, \dots, \mathbf{w}_a)$ and continuity of the function $x \mapsto S_x^{ca}(\mathbf{w}_a, \dots, \mathbf{w}_a)$ for all $\varepsilon > 0$ there exists $\varepsilon_a > 0$ such that if $x \in B(x^*, \varepsilon_a)$, then $(\mathbf{w}_a, \dots, \mathbf{w}_a) \in \mathcal{O}_x^{ca}$ and $S_x^{ca}(\mathbf{w}_a, \dots, \mathbf{w}_a) \in B(x^*, \varepsilon)$. We choose ε small enough to ensure that $B(x^*, \varepsilon) \subset \mathcal{V}_{x^*}$. Since x^* is globally attracting, there exists $t \in \mathbb{Z}_{>0}$ and $\mathbf{u} \in \mathcal{O}_{y^*}^t$ such that $S_{y^*}^t(\mathbf{u}) \in B(x^*, \varepsilon_a)$, and so $S_{y^*}^{t+ca}(\mathbf{u}, \mathbf{w}_a, \dots, \mathbf{w}_a) \in \mathcal{V}_{x^*}$. Hence, there exists $\mathbf{w} \in \mathcal{O}_{S_{y^*}^{t+ca}(\mathbf{u}, \mathbf{w}_a, \dots, \mathbf{w}_a)}^k$ such that $S_{y^*}^{t+ca+k}(\mathbf{u}, \mathbf{w}_a, \dots, \mathbf{w}_a, \mathbf{w}) = y^*$. Since the path $t+ca+k$ -steps path $(\mathbf{u}, \mathbf{w}_a, \dots, \mathbf{w}_a, \mathbf{w})$ can be retaken infinitely many times, $t+ca+k \in E$. The same reasoning holds for $-db$, and so $t-db+k \in E$ (the same t can be obtained by taking a t -steps path \mathbf{u} ensuring that $S_{y^*}^t(\mathbf{u}) \in B(x^*, \min(\varepsilon_a, \varepsilon_b))$). Finally, since $(t+ca+k) - (t-db+k) = 1$, $\gcd(E) = 1$ and so by Proposition 3.4 y^* is steadily attracting. \square

Acknowledgements

We are very grateful to the anonymous reviewers for their constructive comments that helped us to improve considerably the presentation of the results. A special thanks goes to Reviewer 3 who pointed out a much simpler approach allowing to present more straightforward proofs. His comments made us realize several important connections with previous results that we had missed in the first version of the paper and that are now presented.

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Received September 2015 and revised May 2017