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Asymptotics of random processes with immigration II: Convergence to stationarity

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Let $X_1, X_2, \dots$ be random elements of the Skorokhod space $D(\mathbb{R})$ and $\xi_1, \xi_2, \dots$ positive random variables such that the pairs $(X_1, \xi_1), (X_2, \xi_2), \dots$ are independent and identically distributed. We call the random process $(Y(t))_{t\in\mathbb{R}}$ defined by $Y(t) := \sum_{k\geq 0} X_{k+1}(t - \xi_1 - \dots - \xi_k)\mathbb{1}_{\{\xi_1+\dots+\xi_k \leq t\}}$, $t \in \mathbb{R}$ random process with immigration at the epochs of a renewal process. Assuming that X_k and ξ_k are independent and that the distribution of ξ_1 is nonlattice and has finite mean we investigate weak convergence of $(Y(t))_{t\in\mathbb{R}}$ as $t \to \infty$ in $D(\mathbb{R})$ endowed with the J_1-topology. The limits are stationary processes with immigration.

Keywords: random point process; renewal shot noise process; stationary renewal process; weak convergence in the Skorokhod space

1. Introduction

Denote by $D(\mathbb{R})$ the Skorokhod space of right-continuous real-valued functions which are defined on $\mathbb{R}$ and have finite limits from the left at each point of $\mathbb{R}$. Let $X := (X(t))_{t\in\mathbb{R}}$ be a random process with paths in $D(\mathbb{R})$ satisfying $X(t) = 0$ for all $t < 0$ and let ξ be a positive random variable. Further, let $(X_1, \xi_1), (X_2, \xi_2), \dots$ be i.i.d. copies of the pair (X, ξ) and denote by $(S_n)_{n\in\mathbb{N}_0}$ the zero-delayed random walk with increments ξ_k, that is,

$$S_0 := 0, \qquad S_n := \xi_1 + \dots + \xi_n, \qquad n \in \mathbb{N}.$$

Following [7], we call *random process with immigration* the process $Y := (Y(t))_{t\in\mathbb{R}}$ defined by

$$Y(t) := \sum_{k\geq 0} X_{k+1}(t - S_k), \qquad t \in \mathbb{R}. \tag{1}$$

The motivation for the term is discussed in [7] where the reader can also find a list of possible applications and some bibliographic comments.

Continuing the line of research initiated in [5–7], we are interested in weak convergence of random processes with immigration. We treat the situation when $\mathbb{E}[|X(t)|]$ is finite and tends to 0 quickly as $t \to \infty$ while $\mathbb{E}\xi < \infty$. Then Y is the superposition of a regular stream of freshly started processes with quickly fading contributions of the processes that started early. As t becomes large, these competing effects balance on a distributional level and Y approaches

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stationarity. Under these assumptions the joint distribution of (X, ξ) should affect the asymptotic behavior of Y. However, we refrain from investigating this by assuming that X and ξ are independent.

We are only aware of two papers which are concerned with weak convergence of processes Y to their stationary versions in the case when ξ has distribution other than exponential.[1] In [6], the authors prove weak convergence of the finite-dimensional distributions of $(\sum_{k\geq 0} h(ut - S_k)\mathbb{1}_{\{S_k\leq ut\}})_{u>0}$ as $t\to\infty$ for a deterministic function h. Notice that the time is assumed scaled (ut) in Theorem 2.1 of [6], whereas we translate the time $(u+t)$ in Theorem 2.2 of the present paper. Furthermore, the approach taken in [6], which is partly discussed in Remark 2.3, differs from that exploited here. Theorem 6.1 in [15] is a result about weak convergence of the one-dimensional distributions of $Y(t)$ as $t\to\infty$ for $X(t) = g(t,\eta)$, where $g : [0,\infty)\times\mathbb{R}\to[0,\infty)$ is a deterministic function satisfying certain conditions and η is a random variable independent of ξ. As has already been mentioned in [6], the cited theorem does not hold in the generality stated in [15].

2. Main results

Before we formulate our results, some preliminary work has to be done.

2.1. Stationary renewal processes and stationary random processes with immigration

Suppose that $\mu := \mathbb{E}\xi < \infty$, and that the distribution of ξ is nonlattice, that is, it is not concentrated on any lattice $d\mathbb{Z}$, $d > 0$. Further, we stipulate hereafter that the basic probability space on which $(X_k)_{k\in\mathbb{N}}$ and $(\xi_k)_{k\in\mathbb{N}}$ are defined is rich enough to accommodate

- an independent copy $(\xi_{-k})_{k\in\mathbb{N}}$ of $(\xi_k)_{k\in\mathbb{N}}$;
- a random variable ξ_0 which is independent of $(\xi_k)_{k\in\mathbb{Z}\setminus\{0\}}$ and has distribution

$$\mathbb{P}\{\xi_0\in \mathrm{d}x\} = \mu^{-1}\mathbb{E}[\xi\mathbb{1}_{\{\xi\in \mathrm{d}x\}}], \qquad x\geq 0;$$

- a random variable U which is independent of $(\xi_k)_{k\in\mathbb{Z}}$ and has the uniform distribution on $[0,1]$;
- a family $(X_k)_{k\in\mathbb{Z}}$ of i.i.d. random elements of $D(\mathbb{R})$ that is independent of $(\xi_k)_{k\in\mathbb{Z}}$ and U.

Set $\nu(t) := \inf\{k\in\mathbb{N}_0 : S_k > t\}$, $t\in\mathbb{R}$,

$$S_{-k} := -(\xi_{-1}+\cdots+\xi_{-k}), \qquad k\in\mathbb{N},$$

and

$$S_0^* := U\xi_0, \qquad S_{-1}^* := -(1-U)\xi_0, \qquad S_k^* = S_0^* + S_k, \qquad k\in\mathbb{N},$$
$$S_{-k}^* := S_{-1}^* + S_{-k+1}, \qquad k\in\mathbb{N}\setminus\{1\}.$$

[1] Various references related to a simpler situation when the distribution of ξ is exponential are given in [7].

Recall[2] that the distribution of both, S_0^* and $-S_{-1}^*$, coincides with the limiting distribution of the overshoot $S_{\nu(t)} - t$ and the undershoot $t - S_{\nu(t)-1}$ as $t \to \infty$:

$$\mathbb{P}\{S_0^* \in \mathrm{d}x\} = \mathbb{P}\{-S_{-1}^* \in \mathrm{d}x\} = \mu^{-1}\mathbb{P}\{\xi > x\}\mathbb{1}_{(0,\infty)}(x)\,\mathrm{d}x.$$

It is well known [19], Chapter 8, Theorem 4.1, that the point process $\sum_{k\in\mathbb{Z}} \delta_{S_k^*}$ is shift-invariant, that is, $\sum_{k\in\mathbb{Z}} \delta_{S_k^*}$ has the same distribution as $\sum_{k\in\mathbb{Z}} \delta_{S_k^*+t}$ for every $t \in \mathbb{R}$. In particular, the intensity measure of this process is a constant multiple of the Lebesgue measure where the constant can be identified as μ^{-1} by the elementary renewal theorem. In conclusion,

$$\mathbb{E}\left[\sum_{k\in\mathbb{Z}} \delta_{S_k^*}(\mathrm{d}x)\right] = \frac{\mathrm{d}x}{\mu}. \tag{2}$$

Fix any $u \in \mathbb{R}$. Since $\lim_{k\to-\infty} S_k^* = -\infty$ a.s., the sum

$$\sum_{k\le -1} X_{k+1}\big(u + S_k^*\big)\mathbb{1}_{\{S_k^* \ge -u\}}$$

is a.s. finite because the number of non-zero summands is a.s. finite. Define

$$Y^*(u) := \sum_{k\in\mathbb{Z}} X_{k+1}\big(u + S_k^*\big) = \sum_{k\in\mathbb{Z}} X_{k+1}\big(u + S_k^*\big)\mathbb{1}_{\{S_k^* \ge -u\}}$$

with the random variable $Y^*(u)$ being a.s. finite provided that the series $\sum_{k\ge 0} X_{k+1}(u + S_k^*)\mathbb{1}_{\{S_k \ge -u\}}$ converges in probability. It is natural to call $(Y^*(u))_{u\in\mathbb{R}}$ *the stationary random process with immigration.*

2.2. Convergence in $D(\mathbb{R})$

Consider the subset D_0 of the Skorokhod space $D(\mathbb{R})$ composed of those functions $f \in D(\mathbb{R})$ which have finite limits $f(-\infty) := \lim_{t\to-\infty} f(t)$ and $f(\infty) := \lim_{t\to+\infty} f(t)$. For $a, b \in \mathbb{R}$, $a < b$ let $d_0^{a,b}$ be the Skorokhod metric on $D[a, b]$, i.e.,

$$d_0^{a,b}(x, y) = \inf_{\lambda\in\Lambda_{a,b}} \left(\sup_{t\in[a,b]} \big|x\big(\lambda(t)\big) - y(t)\big| \vee \sup_{s\ne t} \left|\log\left(\frac{\lambda(t) - \lambda(s)}{t - s}\right)\right| \right),$$

where $\Lambda_{a,b} = \{\lambda : \lambda$ is a strictly increasing continuous function on $[a, b]$ with $\lambda(a) = a, \lambda(b) = b\}$. Following [12], Section 3, for $f, g \in D_0$, put

$$d_0(f, g) := d_0^{0,1}\big(\overline{\phi}(f), \overline{\phi}(g)\big),$$

where

$$\phi(t) := \log\big(t/(1-t)\big), \qquad t \in (0, 1), \qquad \phi(0) = -\infty, \qquad \phi(1) := +\infty$$

[2] See, for example, [17], Section 3.10.

and

$$\overline{\phi}: D_0 \to D[0,1], \qquad \overline{\phi}(x)(\cdot) := x\big(\phi(\cdot)\big), \qquad x \in D_0.$$

Then (D_0, d_0) is a complete separable metric space. Mimicking the argument given in Section 4 in [12] and using d_0 as a basis one can construct a metric d (its explicit form is of no importance here) on $D(\mathbb{R})$ such that $(D(\mathbb{R}), d)$ is a complete separable metric space. We shall need the following characterization of the convergence in $(D(\mathbb{R}), d)$, see Theorem 1(b) in [12] and Theorem 12.9.3(ii) in [21] for the convergence in $D[0,\infty)$.

Proposition 2.1. *Suppose $f_n, f \in D(\mathbb{R})$, $n \in \mathbb{N}$. The following conditions are equivalent*:

(i) *$f_n \to f$ in $(D(\mathbb{R}), d)$ as $n \to \infty$*;

(ii) *there exist*

$$\lambda_n \in \Lambda := \big\{\lambda : \lambda \textit{ is a strictly increasing continuous function on } \mathbb{R} \textit{ with } \lambda(\pm\infty) = \pm\infty\big\}$$

such that, for any finite a and b, $a < b$,

$$\lim_{n\to\infty} \max\Big\{\sup_{u\in[a,b]} \big|f_n\big(\lambda_n(u)\big) - f(u)\big|, \sup_{u\in[a,b]} \big|\lambda_n(u) - u\big|\Big\} = 0;$$

(iii) *for any finite a and b, $a < b$ which are continuity points of f it holds that $f_n|_{[a,b]} \to f|_{[a,b]}$ in $(D[a,b], d_0^{a,b})$ as $n \to \infty$, where $g|_{[a,b]}$ denotes the restriction of $g \in D(\mathbb{R})$ to $[a,b]$.*

2.3. Main result

Let $\mathcal{D}_X := \{t \geq 0 : \mathbb{P}\{X(t) \neq X(t-)\} > 0\}$ and $\Delta_X := \{a - b : a, b \in \mathcal{D}_X\}$. Note that $\mathcal{D}_X$, and hence Δ_X, may be empty. In the following, we write '$Z_t \Rightarrow Z$ as $t \to \infty$ on (S, d^*)' to denote weak convergence of processes on a complete separable metric space (S, d^*) and '$\stackrel{\mathrm{d}}{\to}$' to denote convergence in distribution of random variables or random vectors.

Our main result, Theorem 2.2, provides (a) sufficient conditions for weak convergence of the finite-dimensional distributions of $(Y(t+u))_{u\in\mathbb{R}}$ as $t \to \infty$ and (b) more restrictive sufficient conditions for weak convergence of the same processes in $(D(\mathbb{R}), d)$.

Theorem 2.2. *Suppose that*

- *X and ξ are independent*;
- *$\mu := \mathbb{E}\xi < \infty$*;
- *the distribution of ξ is nonlattice.*

(a) *If the function $G(t) := \mathbb{E}[|X(t)| \wedge 1]$ is directly Riemann integrable*[3] *(dRi) on $[0,\infty)$, then, for each $u \in \mathbb{R}$, the series $\sum_{k\geq 0} X_{k+1}(u + S_k^*)\mathbb{1}_{\{S_k^* \geq -u\}}$ is absolutely convergent with probability one, and, for any $n \in \mathbb{N}$ and any finite $u_1 < u_2 < \cdots < u_n$,*

$$\big(Y(t+u_1), \ldots, Y(t+u_n)\big) \stackrel{\mathrm{d}}{\to} \big(Y^*(u_1), \ldots, Y^*(u_n)\big), \qquad t \to \infty. \tag{3}$$

[3] See page 232 in [17] for the definition of direct Riemann integrability.

(b) *If, for some* $\varepsilon > 0$, *the function* $H_\varepsilon(t) := \mathbb{E}[\sup_{u\in[t,t+\varepsilon]} |X(u)| \wedge 1]$ *is dRi on* $[0,\infty)$, *and*

$$\mathbb{P}\{S_j \in \Delta_X\} = 0 \tag{4}$$

for each $j \in \mathbb{N}$, *then*

$$Y(t+u) \Rightarrow Y^*(u), \qquad t \to \infty \textit{ in } \big(D(\mathbb{R}), d\big). \tag{5}$$

Remark 2.3. Condition (4) needs to be checked only if the set $\mathcal{D}_X$ contains more than one element, and the distribution of ξ has a discrete component. Otherwise, it holds automatically.

Remark 2.4. Establishing weak convergence of finite-dimensional distributions followed by checking the tightness is the standard approach to proving weak convergence in the Skorokhod space. To prove Theorem 2.2, we take another route: the two statements of the theorem are treated independently, the main technical tool being the continuous mapping theorem applied to relation (19). It is known that, for any $\varepsilon > 0$, the sequences $(S_n)_{n\in\mathbb{N}_0}$ and $(S_k^*)_{k\in\mathbb{N}_0}$ can be coupled such that they become ε-close with probability one, see [13], pages 74–75. By exploiting this observation, it is proved in [6], Theorem 2.1, that $Y(t)$ converges in distribution to $Y^*(0)$ as $t \to \infty$ for deterministic X. Elaborating on the ideas of the proof of the cited theorem, we could have suggested another proof of (3) which would be intuitively more appealing than the proof given below. However, we have not been able to overcome the considerable technical obstacles arising when attempting to prove (5) in this way.

3. Discussion of the assumptions of Theorem 2.2

Suppose that X is as defined in the introduction, that is, with probability one, X takes values in $D(\mathbb{R})$ and $X(t) = 0$ for all $t < 0$. In this subsection, we first derive equivalent conditions for the direct Riemann integrability of the functions $G(t) = \mathbb{E}[|X(t)| \wedge 1]$ and $H_\varepsilon(t) = \mathbb{E}[\sup_{u\in[t,t+\varepsilon]} |X(u)| \wedge 1]$ that are more suitable for applications.

With probability one, X takes values in $D(\mathbb{R})$, and hence is continuous almost everywhere (a.e.). This carries over to $t \mapsto |X(t)| \wedge 1$. Now notice that if $t \mapsto |X(t)| \wedge 1$ is continuous at t_0 and $t_0 + \varepsilon$, then $t \mapsto \sup_{u\in[t,t+\varepsilon]}(|X(u)| \wedge 1)$ is continuous at t_0. Consequently, with probability one, the process $t \mapsto \sup_{u\in[t,t+\varepsilon]}(|X(u)| \wedge 1)$ is a.e. continuous. This implies that the set of t_0 such that $t \mapsto \sup_{u\in[t,t+\varepsilon]}(|X(u)| \wedge 1)$ is discontinuous at t_0 with positive probability has Lebesgue measure 0. From Lebesgue's dominated convergence theorem, we conclude that G and H_ε are a.e. continuous. Since G and H_ε are also bounded, they must be locally Riemann integrable. From this, we conclude that the direct Riemann integrability of G is equivalent to

$$\sum_{k\geq 0} \sup_{t\in[k,k+1)} \mathbb{E}\big[\big|X(t)\big| \wedge 1\big] < \infty, \tag{6}$$

while the direct Riemann integrability of H_ε is equivalent to

$$\sum_{k\geq 0} \sup_{t\in[k,k+1)} \mathbb{E}\Big[\sup_{u\in[t,t+\varepsilon]} \big(\big|X(u)\big| \wedge 1\big)\Big] < \infty. \tag{7}$$

Moreover, (7) is equivalent to

$$\sum_{k\geq 0}\mathbb{E}\Big[\sup_{u\in[k,k+1)}\big(\big|X(u)\big|\wedge 1\big)\Big]<\infty \tag{8}$$

which particularly implies that H_ε is dRi for every $\varepsilon>0$ whenever it is dRi for some $\varepsilon>0$. Indeed,

$$\begin{aligned}\sum_{k\geq 0}\mathbb{E}\Big[\sup_{u\in[k,k+1)}\big(\big|X(u)\big|\wedge 1\big)\Big] &\leq \sum_{k\geq 0}\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor\varepsilon^{-1}\rfloor}\sup_{u\in[k+j\varepsilon,k+(j+1)\varepsilon)}\big(\big|X(u)\big|\wedge 1\big)\Bigg]\\ &=\sum_{j=0}^{\lfloor\varepsilon^{-1}\rfloor}\sum_{k\geq 0}H_\varepsilon(k+j\varepsilon)\leq \sum_{j=0}^{\lfloor\varepsilon^{-1}\rfloor}\sum_{k\geq 0}\sup_{t\in[k,k+1)}H_\varepsilon(t).\end{aligned}$$

Thus (7) implies (8). To see that (8) implies (7) use (18) below with $a=0$ and $b=\varepsilon$.

Here are several cases in which (6) and (8) are equivalent:

(i) $X(t)\equiv h(t)$ a.s. for a deterministic function h;
(ii) there exists $t_0>0$ such that, with probability one, $|X(t)|$ is nonincreasing on $[t_0,\infty)$;
(iii) $\mathbb{P}\{|X(t)|\in(0,1)\}=0$ for all $t\geq 0$, $\tau:=\inf\{t\geq 0: X(t)=0\}<\infty$ a.s. and $X(t)=0$ for all $t\geq\tau$ a.s. in which case

$$(6)\Leftrightarrow(8)\Leftrightarrow\mathbb{E}\tau<\infty. \tag{9}$$

Indeed, in case (i), one can omit the expectations in (6) and (8) since X is deterministic. The resulting formulae coincide. In case (ii), for all $k\geq t_0$, $\sup_{t\in[k,k+1)}\mathbb{E}[|X(t)|\wedge 1]=\mathbb{E}[|X(k)|\wedge 1]$ and $\mathbb{E}[\sup_{u\in[k,k+1)}|X(u)|\wedge 1]=\mathbb{E}[|X(k)|\wedge 1]$. Hence, the infinite series in (6) and (8) coincide for all but finitely many terms. Finally, assume that X satisfies the assumptions of case (iii). We show that (9) holds.

"(6) $\Rightarrow\mathbb{E}\tau<\infty$": From the equality

$$\mathbb{E}\big[\big|X(t)\big|\wedge 1\big]=\mathbb{P}\big\{\big|X(t)\big|\geq 1\big\}=\mathbb{P}\{\tau>t\}$$

we deduce that

$$\infty>\sum_{k\geq 0}\sup_{t\in[k,k+1)}\mathbb{E}\big[\big|X(t)\big|\wedge 1\big]\geq\sum_{k\geq 0}\mathbb{P}\{\tau>k\}\geq\mathbb{E}\tau.$$

"$\mathbb{E}\tau<\infty\Rightarrow$ (8)": This implication follows from

$$\sum_{k\geq 0}\mathbb{E}\Big[\sup_{t\in[k,k+1)}\big(\big|X(t)\big|\wedge 1\big)\Big]=\mathbb{E}\Bigg[\sum_{k=0}^{\lfloor\tau\rfloor}\sup_{t\in[k,k+1)}\big(\big|X(t)\big|\wedge 1\big)\Bigg]\leq\mathbb{E}\big[\lfloor\tau\rfloor+1\big]<\infty.$$

Finally, the implication "(8) $\Rightarrow$ (6)" is obvious.

We now give an example in which X satisfies (6), yet does not satisfy (8).

Example 3.1. Let η be uniformly distributed on [0, 1] and set

$$X(t) := \sum_{k\geq 1} \mathbb{1}_{\{k+k^2\eta/(k^2+1)\leq t<k+\eta\}}, \qquad t \geq 0.$$

Then inequality (8) fails to hold, for $\sup_{t\in[k,k+1)}(|X(t)| \wedge 1) = \sup_{t\in[k,k+1)} X(t) = 1$ a.s. On the other hand, $\sup_{t\in[k,k+1)} \mathbb{E}[|X(t)| \wedge 1] = \sup_{t\in[k,k+1)} \mathbb{E}[X(t)] = (k^2 + 1)^{-1}$ for $k \in \mathbb{N}$, and inequality (6) holds true.

To make the distinction between (6) and (8) more transparent, we note that (8) entails the direct Riemann integrability of X with probability one and, as a consequence, $\lim_{t\to\infty} X(t) = 0$ a.s. On the other hand, the preceding example demonstrates that (6) only guarantees $\lim_{t\to\infty} X(t) = 0$ in probability.

We close this subsection with an example in which $Y(t)$ fails to converge in distribution, as $t \to \infty$.

Example 3.2. Let $X(t) = h(t) := (1 \wedge 1/t^2)\mathbb{1}_{\mathbb{Q}}(t)$, $t \geq 0$, where $\mathbb{Q}$ denotes the set of rationals. Observe that $G(t) = \mathbb{E}[|X(t)| \wedge 1] = h(t)$ is Lebesgue integrable but not Riemann integrable. Let the distribution of ξ be such that $\mathbb{P}\{\xi \in \mathbb{Q} \cap (0, 1]\} = 1$ and $\mathbb{P}\{\xi = r\} > 0$ for all $r \in \mathbb{Q} \cap (0, 1]$. Then the distribution of ξ is non-lattice. It is clear that $Y(t) = 0$ for $t \in \mathbb{R} \setminus \mathbb{Q}$. On the other hand, according to Example 2.6 in [6]

$$Y(t) \xrightarrow{\mathrm{d}} \sum_{k\geq 0} f\big(S_k^*\big) \qquad \text{as } t \to \infty, t \in \mathbb{Q},$$

where $f(t) = 1 \wedge 1/t^2$ for $t \geq 0$. Note that the latter random variable is positive a.s.

4. Applications

Suppose that the first three assumptions of Theorem 2.2 hold.

Example 4.1. Let $\mathbb{P}\{X(t) \geq 0\} = 1$ and $\mathbb{P}\{X(t) \in (0, 1)\} = 0$ for each $t \geq 0$. Suppose that, with probability one, X gets absorbed into the unique absorbing state 0. This means that the random variable $\tau := \inf\{t : X(t) = 0\}$ is a.s. finite, and $X(t) = 0$ for $t \geq \tau$. Then $\mathbb{E}\tau < \infty$ is necessary and sufficient for (3) to hold. Moreover, under the additional assumption (4), $\mathbb{E}\tau < \infty$ is equivalent to (5).

Indeed, if $\mathbb{E}\tau < \infty$, then (9) ensures that the functions $G(t) = \mathbb{E}[|X(t)| \wedge 1]$ and $H_\varepsilon(t) = \mathbb{E}[\sup_{u\in[t,t+\varepsilon]}(|X(u)| \wedge 1)]$ (for arbitrary $\varepsilon > 0$) are dRi on $[0, \infty)$. Therefore, (3) and, under the additional assumption (4), (5) follow from Theorem 2.2.

Suppose now that $\mathbb{E}\tau = \infty$. By the strong law of large numbers, for any $\rho \in (0, \mu)$, there exists an a.s. finite random variable M such that $S_k^* > (\mu - \rho)k$ for $k \geq M$. Therefore, for any $u \in \mathbb{R}$,

$$\begin{aligned}
&\sum_{k\geq 0} \mathbb{P}\big\{X_{k+1}\big(u + S_k^*\big)\mathbb{1}_{\{u+S_k^*\geq 0\}} \geq 1 | \big(S_j^*\big)_j\big\} \\
&\quad = \sum_{k\geq 0} \mathbb{1}_{\{u+S_k^*\geq 0\}} \mathbb{P}\big\{\tau_{k+1} > u + S_k^* | \big(S_j^*\big)_j\big\} = \sum_{k\geq \nu^*(-u)} \mathbb{P}\big\{\tau_{k+1} - u > S_k^* | \big(S_j^*\big)_j\big\} \\
&\quad \geq \sum_{k\geq M\vee \nu^*(-u)} \mathbb{P}\big\{\tau - u > (\mu - \rho)k | \big(S_j^*\big)_j\big\} = \infty \qquad \text{a.s.,}
\end{aligned}$$

where $\tau_k := \inf\{t : X_k(t) = 0\}$, and $\nu^*(-u) := \inf\{k \in \mathbb{N}_0 : S_k^* \geq -u\}$. Given $(S_j^*)_j$, the series $\sum_{k\geq 0} X_{k+1}(u+S_k^*)\mathbb{1}_{\{S_k^*\geq -u\}}$ does not converge a.s. by the three series theorem. Since the general term of the series is nonnegative, then, given $(S_j^*)_j$, this series diverges a.s. Hence, for each $u \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} X_{k+1}(u + S_k^*)\mathbb{1}_{\{S_k^*\geq -u\}} = \infty$ a.s., and (3) cannot hold.

(a) Let X be a subcritical or critical Bellman–Harris process (see Chapter IV in [2] for the definition and many properties) with a single ancestor, and let η and N be independent with η being distributed according to the life length distribution and N according to the offspring distribution of the process. Suppose that $\mathbb{P}\{\eta = 0\} = 0$, $\mathbb{P}\{N = 0\} < 1$ and $\mathbb{P}\{N = 1\} < 1$. Then Y is a subcritical or critical Bellman–Harris process with (single) immigration at the epochs of a renewal process. Y satisfies (3) if and only if $\mathbb{E}\tau < \infty$. In [16], Theorem 1, the same criterion is derived for the convergence of the one-dimensional distributions via an analytic argument. Under the condition $\mathbb{E}\eta < \infty$, which entails $\mathbb{E}\tau < \infty$, weak convergence of the one-dimensional distributions of a subcritical process with immigration was proved in Theorem 3 in [8]. Notice that in the two cited papers multiple immigration is allowed. Finally we note that relation (5) is equivalent to $\mathbb{E}\tau < \infty$ under the additional condition (4), which holds, for instance, if the distribution of η is continuous.

(b) Suppose that $X(t) = \mathbb{1}_{\{\eta>t\}}$ for a nonnegative random variable η. Observe that X is a (degenerate) Bellman–Harris process with $\mathbb{P}\{N = 0\} = 1$. Because of its simplicity and its numerous applications the corresponding process Y has received considerable attention in the literature. We only mention the following interpretations:

– $Y(t)$ is the number of busy servers at time t in the $GI/G/\infty$ queue [9];
– $Y(t)$ is the number of active downloads at time t in a computer network [11,14];
– $Y(t)$ is the difference between the number of visits to the segment $[0,t]$ of the standard random walk $(S_n)_{n\in\mathbb{N}_0}$ and the perturbed random walk $(S_n + \eta_{n+1})_{n\in\mathbb{N}_0}$, where (η_n) are independent copies of η [1].

In this case $\tau = \eta$. Hence the corresponding Y satisfies (3) and, under the additional assumption (4), (5) if and only if $\mathbb{E}\eta < \infty$.

(c) Let X be a birth and death process with $X(0) = i \in \mathbb{N}$ a.s. Suppose that X is eventually absorbed at 0 with probability one. Since (4) holds we conclude that the corresponding Y satisfies (5) if and only if $\mathbb{E}\tau < \infty$. A criterion for the finiteness of $\mathbb{E}\tau$ expressed in terms of infinitesimal intensities is given in [10], Theorem 7.1 on page 149.

Example 4.2. Let $X(t) = \eta f(t)$, where η is a random variable independent of ξ, and $f : \mathbb{R} \to \mathbb{R}$, with $f(t) = 0$ for $t < 0$, belongs to $D(\mathbb{R})$.

(a) Suppose that $\mathbb{P}\{\eta = b\} = 1$, and that the function $t \to |f(t)| \wedge 1$ is dRi. Then relation (3) and, under the additional assumption (4), relation (5) hold. Weak convergence of one dimensional distributions was proved in Theorem 2.1 in [6] under the assumption that the function $t \mapsto |f(t)|$ is dRi, not assuming, however, that $f \in D(\mathbb{R})$. [4]

(b) Suppose $f(t) = e^{-at}$, $a > 0$. If $\mathbb{E}[\log^+ |\eta|] < \infty$, then the nonincreasing function

$$H_\varepsilon(t) = \mathbb{E}\Big[\sup_{u \in [t, t+\varepsilon]} \big(|\eta| e^{-au} \wedge 1\big)\Big] = \mathbb{E}\big[|\eta| e^{-at} \wedge 1\big]$$

is integrable, hence dRi. Further, (4) holds since X is a.s. continuous. Thus Theorem 2.2 implies (5). If $\mathbb{E}[\log^+ |\eta|] = \infty$, then, by [4], Theorem 2.1,

$$\lim_{n \to \infty} \left| \sum_{k=0}^{n} \eta_{k+1} \exp\big(-a S_k^*\big) \right| = \infty$$

in probability where $\eta_1, \eta_2, \ldots$ are i.i.d. copies of η. The latter implies that (3) and (5) cannot hold.

5. Proof of Theorem 2.2

Let $M_p(\mathbb{R})$ be the set of Radon point measures on $\mathbb{R}$ with the topology of vague convergence $\stackrel{v}{\to}$, and let δ_{x_0} denote the probability measure concentrated at point $x_0 \in \mathbb{R}$. Recall that, for $m_n, m \in M_p(\mathbb{R})$,

$$m_n \stackrel{v}{\to} m, \qquad n \to \infty$$

if and only if

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x) m_n(\mathrm{d}x) = \int_{\mathbb{R}} f(x) m(\mathrm{d}x)$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}^+$ with compact support. According to Proposition 3.17 in [18], there is a metric ρ on $M_p(\mathbb{R})$ which makes $(M_p(\mathbb{R}), \rho)$ a complete separable metric space, while convergence in this metric is equivalent to vague convergence. Further, for later use, recall that any $m \in M_p(\mathbb{R})$ has a representation of the form $m = \sum_{k \in \mathbb{Z}} \delta_{t_k}$ for $t_k \in \mathbb{R}$. Moreover, this representation is unique subject to the constraints $t_k \le t_{k+1}$ for all $k \in \mathbb{Z}$ and $t_{-1} < 0 \le t_0$. The t_k are given by

$$t_k = \begin{cases} \inf\big\{t \ge 0 : m\big([0, t]\big) \ge k + 1\big\}, & \text{if } k \ge 0; \\ -\inf\big\{t \ge 0 : m\big([-t, 0)\big) \ge -k\big\}, & \text{if } k < 0. \end{cases} \tag{10}$$

Before we prove Theorem 2.2, we give three auxiliary lemmas. Lemma 5.1 given next and the continuous mapping theorem are the key technical tools in the proof of Theorem 2.2.

[4]If $f \in D(\mathbb{R})$, then f is bounded on compact intervals, and the function $t \mapsto |f(t)| \wedge 1$ is dRi if and only if so is $t \mapsto |f(t)|$.

Lemma 5.1. *Assume that* $\mathbb{E}\xi < \infty$ *and that the distribution of* ξ *is nonlattice. Then*

$$\sum_{k\geq 0} \delta_{t-S_k} \Rightarrow \sum_{j\in\mathbb{Z}} \delta_{S_j^*}, \qquad t\to\infty$$

on $(M_p(\mathbb{R}),\rho)$.

Proof. Let $h:\mathbb{R}\to\mathbb{R}^+$ be a continuous function with a compact support. According to Proposition 3.19 on page 153 in [18], it suffices to prove

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\Big(-\sum_{k\geq 0} h(t-S_k)\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\sum_{j\in\mathbb{Z}} h\big(S_j^*\big)\Big)\Big].$$

Let $A := \inf\{t : h(t)\neq 0\} > -\infty$ and $g(t) := h(t+A)$, $t\in\mathbb{R}$. Then $g(t)=0$ for $t<0$ and g is dRi on $\mathbb{R}^+$ as a continuous function with compact support. Hence, Theorem 2.1 in [6] applies and yields

$$\sum_{k\geq 0} g(t-S_k) \xrightarrow{\mathrm{d}} \sum_{k\geq 0} g\big(S_k^*\big), \qquad t\to\infty.$$

This implies convergence of the associated Laplace transforms, hence,

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(-\sum_{k\geq 0} h(t-S_k)\Big)\Big] &= \mathbb{E}\Big[\exp\Big(-\sum_{k\geq 0} g(t-A-S_k)\Big)\Big] \\
&\to \mathbb{E}\Big[\exp\Big(-\sum_{k\geq 0} g\big(S_k^*\big)\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\sum_{k\geq 0} h\big(S_k^*+A\big)\Big)\Big] \\
&= \mathbb{E}\Big[\exp\Big(-\sum_{k\in\mathbb{Z}} h\big(S_k^*+A\big)\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\sum_{k\in\mathbb{Z}} h\big(S_k^*\big)\Big)\Big]
\end{aligned}$$

as $t\to\infty$ where the next-to-last equality is due to the fact that $h(S_k^*+A)=0$ for $k<0$ while the last equality follows from the distributional shift invariance of $\sum_{k\in\mathbb{Z}}\delta_{S_k^*}$ [19], Chapter 8, Theorem 4.1. □

Lemma 5.2. *Suppose that* $t_n\to t$ *on* $\mathbb{R}$ *and* $f_n\to f$ *in* $(D(\mathbb{R}),d)$, *as* $n\to\infty$. *Then*

$$f_n(t_n+\cdot)\to f(t+\cdot), \qquad n\to\infty$$

in $(D(\mathbb{R}),d)$.

Proof. Without loss of generality, we assume that $t=0$. It suffices to prove that there exist $\lambda_n\in\Lambda$, $n\in\mathbb{N}$ such that, for any $-\infty<a<b<\infty$,

$$\lim_{n\to\infty} \max\Big\{\sup_{u\in[a,b]}\big|\lambda_n(u)-u\big|, \sup_{u\in[a,b]}\big|f_n\big(t_n+\lambda_n(u)\big)-f(u)\big|\Big\} = 0. \tag{11}$$

By assumption, $f_n \to f$ in $(D(\mathbb{R}), d)$. Hence, there are $\mu_n \in \Lambda$, $n \in \mathbb{N}$ such that, for any $-\infty < a < b < \infty$,

$$\lim_{n\to\infty} \max\Big\{ \sup_{u\in[a,b]} \big|\mu_n(u) - u\big|, \sup_{u\in[a,b]} \big|f_n\big(\mu_n(u)\big) - f(u)\big|\Big\} = 0. \tag{12}$$

Put $\lambda_n(u) := \mu_n(u) - t_n$ and note that $\lambda_n \in \Lambda$. Then (12) can be rewritten as

$$\lim_{n\to\infty} \max\Big\{ \sup_{u\in[a,b]} \big|\lambda_n(u) - u + t_n\big|, \sup_{u\in[a,b]} \big|f_n\big(t_n + \lambda_n(u)\big) - f(u)\big|\Big\} = 0$$

which is equivalent to (11), for $\lim_{n\to\infty} t_n = 0$. □

Remark 5.3. As was kindly communicated to us by one of the referees the counterpart of Lemma 5.2 with $(D(\mathbb{R}), d)$ replaced by $(D[0,\infty), d_1)$, where d_1 is the standard J_1-metric, may fail to hold. Take, for instance, $f_n(t) := f(t) := \mathbb{1}_{[1,\infty)}(t)$ and $t_n := 1 - n^{-1}$. Then $f_n(t_n) = 0$ does not converge to $f(1) = 1$ as $n \to \infty$ which implies that $f_n(t_n + \cdot)$ do not converge to $f(1 + \cdot)$ in $(D[0,\infty), d_1)$.

Denote by $D(\mathbb{R})^{\mathbb{Z}}$ the Cartesian product of countably many copies of $D(\mathbb{R})$ endowed with the topology of componentwise convergence via the metric

$$d^{\mathbb{Z}}\big((f_k)_{k\in\mathbb{Z}}, (g_k)_{k\in\mathbb{Z}}\big) := \sum_{k\in\mathbb{Z}} 2^{-|k|}\big(d(f_k, g_k) \wedge 1\big).$$

Note that $(D(\mathbb{R})^{\mathbb{Z}}, d^{\mathbb{Z}})$ is a complete and separable metric space. Now consider the metric space $(M_p(\mathbb{R}) \times D(\mathbb{R})^{\mathbb{Z}}, \rho^*)$ where $\rho^*(\cdot,\cdot) := d^{\mathbb{Z}}(\cdot,\cdot) + \rho(\cdot,\cdot)$ (i.e., convergence is defined componentwise). As the Cartesian product of complete and separable spaces, $(M_p(\mathbb{R}) \times D(\mathbb{R})^{\mathbb{Z}}, \rho^*)$ is complete and separable.

For fixed $c > 0$, $l \in \mathbb{N}$ and $(u_1, \dots, u_l) \in \mathbb{R}^l$, define the mapping $\phi_c^{(l)} : M_p(\mathbb{R}) \times D(\mathbb{R})^{\mathbb{Z}} \to \mathbb{R}^l$ by

$$\phi_c^{(l)}\big(m, \big(f_k(\cdot)\big)_{k\in\mathbb{Z}}\big) := \bigg(\sum_k f_k(t_k + u_j)\mathbb{1}_{\{|t_k|\le c\}}\bigg)_{j=1,\dots,l}$$

and the mapping $\phi_c : M_p(\mathbb{R}) \times D(\mathbb{R})^{\mathbb{Z}} \to D(\mathbb{R})$ by

$$\phi_c\big(m, \big(f_k(\cdot)\big)_{k\in\mathbb{Z}}\big) := \sum_k f_k(t_k + \cdot)\mathbb{1}_{\{|t_k|\le c\}},$$

where in the definition of $\phi_c^{(l)}$ and ϕ_c, the t_k are given by (10). It can be checked that $\phi_c^{(l)}$ and ϕ_c are measurable mappings. For $f \in D(\mathbb{R})$, denote by $\mathrm{Disc}(f)$ the set of discontinuity points of f on $\mathbb{R}$.

Lemma 5.4. *The mapping $\phi_c^{(l)}$ is continuous at all points $(m, (f_k)_{k\in\mathbb{Z}})$ for which $m(\{-c, 0, c\}) = 0$ and for which $u_1, \dots, u_l$ are continuity points of $f_k(t_k + \cdot)$ for all $k \in \mathbb{Z}$. ϕ_c is continuous* *at all points* $(m, (f_k)_{k\in\mathbb{Z}})$ *satisfying* $m(\{-c,0,c\}) = 0$ *and* $\mathrm{Disc}(f_k(t_k + \cdot)) \cap \mathrm{Disc}(f_j(t_j + \cdot)) = \varnothing$ *for* $k \neq j$.

Proof. Let $c > 0$ and suppose that

$$\big(m_n, \big(f_k^{(n)}\big)_{k\in\mathbb{Z}}\big) \to \big(m, (f_k)_{k\in\mathbb{Z}}\big), \qquad n \to \infty \tag{13}$$

on $(M_p(\mathbb{R}) \times D(\mathbb{R})^{\mathbb{Z}}, \rho^*)$ where $m(\{-c,0,c\}) = 0$. Then, in particular, $m_n \stackrel{v}{\to} m$ as $n \to \infty$. Since $m(\{-c,0,c\}) = 0$, we can apply Theorem 3.13 in [18], which says that $m_n([-c,0]) = m([-c,0]) =: r_-$ and $m_n([0,c]) = m([0,c]) =: r_+$ for all sufficiently large n. For these n, with the definition of $t_k^{(n)}$ and t_k according to (10), we have

$$m_n\big(\cdot \cap [-c,0]\big) = \sum_{k=1}^{r_-} \delta_{t_{-k}^{(n)}}, \qquad m_n\big(\cdot \cap [0,c]\big) = \sum_{k=0}^{r_+-1} \delta_{t_k^{(n)}},$$

$$m\big(\cdot \cap [-c,0]\big) = \sum_{k=1}^{r_-} \delta_{t_{-k}} \quad \text{and} \quad m_n\big(\cdot \cap [0,c]\big) = \sum_{k=0}^{r_+-1} \delta_{t_k},$$

where, of course, the empty sum is understood to be 0. Theorem 3.13 in [18] further implies that there is convergence of the points of m_n in $[-c,0]$ to the points of m in $[-c,0]$ and analogously with $[-c,0]$ replaced by $[0,c]$. Since m has no point at 0, this implies that $t_k^{(n)} \to t_k$ as $n \to \infty$ for $k = -r_-, \dots, r_+ - 1$. On the other hand, (13) entails $\lim_{n\to\infty} f_k^{(n)} = f_k$ in $(D(\mathbb{R}), d)$ for $k = -r_-, \dots, r_+ - 1$. Therefore, Lemma 5.2 ensures that

$$f_k^{(n)}\big(t_k^{(n)} + \cdot\big) \to f_k(t_k + \cdot), \qquad n \to \infty \tag{14}$$

in $(D(\mathbb{R}), d)$ for $k = -r_-, \dots, r_+ - 1$.

Now assume that $u_1, \dots, u_l$ are continuity points of $f_k(t_k + \cdot)$ for all $k \in \mathbb{Z}$. We show that then

$$\phi_c^{(l)}\big(m_n, \big(f_k^{(n)}\big)_{k\in\mathbb{Z}}\big) \to \phi_c^{(l)}\big(m, (f_k)_{k\in\mathbb{Z}}\big), \qquad n \to \infty. \tag{15}$$

Indeed, in the given situation, (14) implies that

$$\big(f_k^{(n)}\big(t_k^{(n)} + u_1\big), \dots, f_k^{(n)}\big(t_k^{(n)} + u_l\big)\big) \to \big(f_k(t_k + u_1), \dots, f_k(t_k + u_l)\big), \qquad n \to \infty$$

for $k = -r_-, \dots, r_+ - 1$. Summation of these relations over $k = -r_-, \dots, r_+ - 1$ yields (15).

Theorem 4.1 in [20] tells us that addition on $D(\mathbb{R}) \times D(\mathbb{R})$ is continuous at those (x,y) for which $\mathrm{Disc}(x) \cap \mathrm{Disc}(y) = \varnothing$. Since this immediately extends to any finite number of summands, we conclude that relations (14) entail

$$\phi_c\big(m_n, \big(f_k^{(n)}\big)_{k\in\mathbb{Z}}\big) = \sum_{k=-r_-}^{r_+-1} f_k^{(n)}\big(t_k^{(n)} + \cdot\big) \to \sum_{k=-r_-}^{r_+-1} f_k(t_k + \cdot) = \phi_c\big(m, (f_k)_{k\in\mathbb{Z}}\big), \qquad n \to \infty$$

in $(D(\mathbb{R}), d)$ provided that $\mathrm{Disc}(f_k(t_k + \cdot)) \cap \mathrm{Disc}(f_j(t_j + \cdot)) = \varnothing$ for $k \neq j$. $\square$

Proof of Theorem 2.2. We start by showing that the Lebesgue integrability of $G(t) = \mathbb{E}[|X(t)| \wedge 1]$ ensures $|Y^*(u)| < \infty$ a.s. for each $u \in \mathbb{R}$. To this end, fix $u \in \mathbb{R}$ and set $\mathcal{Z}_k := X_{k+1}(u + S_k^*)\mathbb{1}_{\{S_k^* \geq -u\}}$, $k \in \mathbb{Z}$. We infer

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \mathbb{E}\big[|\mathcal{Z}_k| \wedge 1\big] &= \sum_{k\in\mathbb{Z}} \mathbb{E}\big[\big(\big|X_{k+1}\big(u + S_k^*\big)\big| \wedge 1\big)\mathbb{1}_{\{S_k^* \geq -u\}}\big] \\ &= \sum_{k\in\mathbb{Z}} \mathbb{E}\big[G\big(u + S_k^*\big)\mathbb{1}_{\{S_k^* \geq -u\}}\big] = \frac{1}{\mu}\int_0^\infty G(x)\,\mathrm{d}x < \infty\end{aligned} \tag{16}$$

having utilized (2) for the last equality. Therefore, $\sum_{k\geq 0} |\mathcal{Z}_k| < \infty$ a.s. by the two-series theorem which implies $|Y^*(u)| < \infty$ a.s.

Next, we prove that direct Riemann integrability of $H_\varepsilon(t) = \mathbb{E}[\sup_{u\in[t,t+\varepsilon]} |X(u)| \wedge 1]$ for some $\varepsilon > 0$ implies that Y^* takes values in $D(\mathbb{R})$ a.s. Since locally uniform limits of elements from $D(\mathbb{R})$ are again in $D(\mathbb{R})$, it suffices to check that $Y^*(u) := \sum_{k\in\mathbb{Z}} X_{k+1}(u + S_k^*)$ converges uniformly on every compact interval a.s. To this end, fix $a, b \in \mathbb{R}$, $a < b$. It suffices to consider the case when $a, b \in \mathbb{Z}$ (otherwise, replace a by $\lfloor a\rfloor$ and b by $\lceil b\rceil$). Now notice that

$$\sup_{u\in[a,b]} \sum_{k\in\mathbb{Z}}\big(\big|X_{k+1}\big(u + S_k^*\big)\big| \wedge 1\big) \leq \sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]}\big(\big|X_{k+1}\big(u + S_k^*\big)\big| \wedge 1\big)$$

and that the series on the right-hand side of this equation is measurable since the X_{k+1}, $k \in \mathbb{Z}$ take values in $D(\mathbb{R})$. Further, we observe that

$$\begin{aligned}\mathbb{E}\Big[\sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]}\big(\big|X_{k+1}\big(u + S_k^*\big)\big| \wedge 1\big)\Big] &= \mathbb{E}\Big[\sum_{k\in\mathbb{Z}} \mathbb{E}\Big[\sup_{u\in[a,b]}\big(\big|X\big(u + S_k^*\big)\big| \wedge 1\big)\Big|\big(S_j^*\big)_{j\in\mathbb{Z}}\Big]\Big] \\ &= \frac{1}{\mu}\int_{\mathbb{R}} \mathbb{E}\Big[\sup_{u\in[a,b]}\big(\big|X(u+t)\big| \wedge 1\big)\Big]\,\mathrm{d}t \\ &\leq \frac{1}{\mu}\sum_{k\in\mathbb{Z}} \sup_{t\in[k,k+1)} \mathbb{E}\Big[\sup_{u\in[a+t,b+t]}\big|X(u)\big| \wedge 1\Big],\end{aligned} \tag{17}$$

where the last equality is a consequence of (2). To check that the last series converges, put $r := b - a$ and notice that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sup_{t\in[k,k+1)} \mathbb{E}\Big[\sup_{u\in[a+t,b+t]}\big|X(u)\big| \wedge 1\Big] &\leq \sum_{k\in\mathbb{Z}} \mathbb{E}\Big[\sup_{u\in[a+k,b+k+1)}\big|X(u)\big| \wedge 1\Big] \\ &\leq \sum_{j=0}^{r}\sum_{k\in\mathbb{Z}} \mathbb{E}\Big[\sup_{u\in[a+k+j,a+k+j+1)}\big|X(u)\big| \wedge 1\Big] \\ &= \sum_{j=0}^{r}\sum_{k\in\mathbb{Z}} \mathbb{E}\Big[\sup_{u\in[k,k+1)}\big|X(u)\big| \wedge 1\Big] \\ &= (r+1)\sum_{k\geq 0} \mathbb{E}\Big[\sup_{u\in[k,k+1)}\big|X(u)\big| \wedge 1\Big] < \infty,\end{aligned} \tag{18}$$

where the last equality follows from the fact that $X(u)=0$ for $u<0$, while the finiteness of the last series is secured by (8). Thus, $\sum_{k\in\mathbb{Z}}(|X_{k+1}(u+S_k^*)|\wedge 1)$ converges uniformly on $[a,b]$ a.s. Since the set of $a,b\in\mathbb{Z}$ with $a<b$ is countable, Y^* is indeed $D(\mathbb{R})$-valued a.s.

Using Lemma 5.1 and recalling that the space $M_p(\mathbb{R})\times D(\mathbb{R})^{\mathbb{Z}}$ is separable we infer

$$\Big(\sum_{k\geq 0}\delta_{t-S_k},(X_{k+1})_{k\in\mathbb{Z}}\Big)\Rightarrow\Big(\sum_{k\in\mathbb{Z}}\delta_{S_k^*},(X_{k+1})_{k\in\mathbb{Z}}\Big),\qquad t\to\infty \tag{19}$$

on $M_p(\mathbb{R})\times D(\mathbb{R})^{\mathbb{Z}}$ by Theorem 3.2 in [3].

Proof of (5). We shall use Lemma 5.4. To this end, observe that each S_j^* has an absolutely continuous distribution. In particular, $\sum_{j\in\mathbb{Z}}\delta_{S_j^*}(\{-c,0,c\})=0$ a.s. for every $c>0$. Further, for $i<j$, let $D_{i+1}:=\mathrm{Disc}(X_{i+1})$, $D_{j+1}:=\mathrm{Disc}(X_{j+1})$. For a set $A\subset\mathbb{R}$ and $b\in\mathbb{R}$, we write $A-b$ for the set $\{a-b:a\in A\}$. With this notation, we have

$$\begin{aligned}
&\mathbb{P}\big\{\mathrm{Disc}\big(X_{i+1}\big(S_i^*+\cdot\big)\big)\cap\mathrm{Disc}\big(X_{j+1}\big(S_j^*+\cdot\big)\big)\neq\varnothing\big\}\\
&\quad=\mathbb{P}\big\{\big(D_{i+1}-S_i^*\big)\cap\big(D_{j+1}-S_j^*\big)\neq\varnothing\big\}\\
&\quad\leq\mathbb{P}\big\{\big(\big(D_{i+1}-S_i^*\big)\setminus\big(\mathcal{D}_X-S_i^*\big)\big)\cap\big(D_{j+1}-S_j^*\big)\neq\varnothing\big\}\\
&\qquad+\mathbb{P}\big\{\big(\mathcal{D}_X-S_i^*\big)\cap\big(\big(D_{j+1}-S_j^*\big)\setminus\big(\mathcal{D}_X-S_j^*\big)\big)\neq\varnothing\big\}\\
&\qquad+\mathbb{P}\big\{\big(\mathcal{D}_X-S_i^*\big)\cap\big(\mathcal{D}_X-S_j^*\big)\neq\varnothing\big\}.
\end{aligned}$$

We now argue that the last three summands vanish. As to the first, using the independence of (X_{i+1},X_{j+1}) and (S_i^*,S_j^*) and conditioning with respect to (S_i^*,S_j^*) we conclude that it suffices to show that $\mathbb{P}\{((D_{i+1}-s_i^*)\setminus(\mathcal{D}_X-s_i^*))\cap(D_{j+1}-s_j^*)\neq\varnothing\}=0$ for fixed $s_i^*,s_j^*\in\mathbb{R}$, $s_i^*<s_j^*$. Since $X_{i+1}(s_i^*+\cdot)$ and $X_{j+1}(s_j^*+\cdot)$ are independent, we can argue as in the proof of Lemma 4.3 in [20]:

$$\begin{aligned}
&\mathbb{P}\big\{\big(\big(D_{i+1}-s_i^*\big)\setminus\big(\mathcal{D}_X-s_i^*\big)\big)\cap\big(D_{j+1}-s_j^*\big)\neq\varnothing\big\}\\
&\quad=\int\mathbb{P}\big\{X_{i+1}\big(s_i^*+\cdot\big)\in A(y)\big\}\mathbb{P}\big\{X\big(s_j^*+\cdot\big)\in\mathrm{d}y\big\},
\end{aligned}$$

where $A(y)=\{x\in D(\mathbb{R}):(\mathrm{Disc}(x)\setminus(\mathcal{D}_X-s_i^*))\cap\mathrm{Disc}(y)\neq\varnothing\}$. For any $y\in D(\mathbb{R})$,

$$\mathbb{P}\big\{X_{i+1}\big(s_i^*+\cdot\big)\in A(y)\big\}=\sum_{t\in\mathrm{Disc}(y)\setminus(\mathcal{D}_X-s_i^*)}\mathbb{P}\big\{X_{i+1}\big(s_i^*+t\big)\neq X_{i+1}\big(\big(s_i^*+t\big)-\big)\big\}=0.$$

The second term can be treated similarly. As to the third term, $\mathbb{P}\{(\mathcal{D}_X-S_i^*)\cap(\mathcal{D}_X-S_j^*)\neq\varnothing\}\leq\mathbb{P}\{S_j^*-S_i^*\in\Delta_X\}$. If $i\geq 0$ or $j<0$, then $S_j^*-S_i^*\overset{\mathrm{d}}{=}S_{j-i}$. If $i<0\leq j$, then $S_i^*-S_j^*\overset{\mathrm{d}}{=}\xi_0+S_{i-j-1}$. Since the sets of atoms of distributions of ξ_0 and ξ are the same, we conclude that $\mathbb{P}\{S_j^* - S_i^* \in \Delta_X\} = 0$ by (4). Consequently, $\mathbb{P}\{\mathrm{Disc}(X_{j+1}(S_j^* + \cdot)) \cap \mathrm{Disc}(X_{i+1}(S_i^* + \cdot)) \neq \varnothing\} = 0$ for $i \neq j$. This justifies using Lemma 5.4, according to which ϕ_c is a.s. continuous at $(\sum_{k\in\mathbb{Z}} \delta_{S_k^*}, (X_{k+1})_{k\in\mathbb{Z}})$. Applying now the continuous mapping theorem to (19) yields

$$\begin{aligned} Y_c(t,u) &:= \sum_{k\geq 0} X_{k+1}(u+t-S_k)\mathbb{1}_{\{|t-S_k|\leq c\}} \\ &\overset{\mathrm{d}}{=} \phi_c\Bigg(\sum_{k\geq 0} \delta_{t-S_k}, (X_{k+1})_{k\in\mathbb{Z}}\Bigg) \Rightarrow \phi_c\Bigg(\sum_{k\in\mathbb{Z}} \delta_{S_k^*}, (X_{k+1})_{k\in\mathbb{Z}}\Bigg) \\ &= \sum_{k\in\mathbb{Z}} X_{k+1}\big(u+S_k^*\big)\mathbb{1}_{\{|S_k^*|\leq c\}} =: Y_c^*(u), \qquad t\to\infty \end{aligned} \tag{20}$$

in $(D(\mathbb{R}), d)$. Here, it should be noticed that one equality in the relation above is distributional rather than pathwise. However, since $(X_{k+1})_{k\in\mathbb{Z}}$ is an i.i.d. sequence, its distribution is invariant under permutations. And, what is more, due to the independence between the sequences $(\xi_k)_{k\in\mathbb{N}}$ and $(X_k)_{k\in\mathbb{Z}}$, the distribution of $(X_{k+1})_{k\in\mathbb{Z}}$ is even invariant under $(\xi_k)_{k\in\mathbb{N}}$-measurable permutations.

Using Proposition 2.1 and following the reasoning in the proof of Proposition 4.18 in [18], we conclude that in order to prove (5) it suffices to check that

$$Y(t+u) \Rightarrow Y^*(u), \qquad t\to\infty \tag{21}$$

in $(D[a,b], d_0^{a,b})$ for any a and b, $a<b$ which are not fixed discontinuities of Y^*. To this end, first observe that (20) implies

$$Y_c(t,\cdot) \Rightarrow Y_c^*(\cdot), \qquad t\to\infty \tag{22}$$

in $(D[a,b], d_0^{a,b})$ for any a and b, $a<b$ which are not fixed discontinuities of Y_c^*. It can be checked that for each fixed $c>0$ the set of fixed discontinuities of Y_c^* is a subset of the set of fixed discontinuities of Y^*. Since Y^* has paths in $D(\mathbb{R})$ and is stationary, the set of fixed discontinuities of Y^* is empty. Hence, (22) holds for any a and b, $a<b$. Now (21) follows from Theorem 4.2 in [3] if we can prove that

$$Y_c^* \Rightarrow Y^*, \qquad c\to\infty \tag{23}$$

in $(D[a,b], d_0^{a,b})$ and that

$$\lim_{c\to\infty}\limsup_{t\to\infty} \mathbb{P}\big\{d_0^{a,b}\big(Y_c(t,\cdot), Y(t+\cdot)\big) > \varepsilon\big\} = 0 \tag{24}$$

for all $\varepsilon>0$ and any $a,b\in\mathbb{R}$, $a<b$.

Proof of (24). Since $d_0^{a,b}$ is dominated by the uniform metric on $[a,b]$ it suffices to prove that

$$\lim_{c\to\infty}\limsup_{t\to\infty} \mathbb{P}\Bigg\{\sup_{u\in[a,b]}\Bigg|\sum_{k\geq 0} X_{k+1}(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\Bigg| > \varepsilon\Bigg\} = 0 \tag{25}$$

for all $\varepsilon > 0$ and any $a, b \in \mathbb{R}$, $a < b$.

To prove (25), set $M_k(t) := \sup_{u\in[a,b]} |X_k(u+t)|$, $k \in \mathbb{Z}$ and $K(t) := \mathbb{E}[M_1(t) \wedge 1]$ and write

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{u\in[a,b]}\Big|\sum_{k\geq 0} X_{k+1}(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\Big| > \varepsilon\Big\} \\
&\quad\leq \mathbb{P}\Big\{\sum_{k\geq 0} M_{k+1}(t-S_k)\mathbb{1}_{\{|t-S_k|>c\}} > \varepsilon\Big\} \\
&\quad\leq \mathbb{P}\Big\{\sum_{k\geq 0} M_{k+1}(t-S_k)\mathbb{1}_{\{|t-S_k|>c, M_{k+1}(t-S_k)\leq 1\}} > \varepsilon/2\Big\} \\
&\qquad + \mathbb{P}\Big\{\sum_{k\geq 0} M_{k+1}(t-S_k)\mathbb{1}_{\{|t-S_k|>c, M_{k+1}(t-S_k)> 1\}} > \varepsilon/2\Big\} \\
&\quad\leq \frac{2}{\varepsilon}\mathbb{E}\Big[\sum_{k\geq 0} K(t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\Big] + \sum_{k\geq 0}\mathbb{P}\big\{|t-S_k|>c,\, M_{k+1}(t-S_k)>1\big\},
\end{aligned}$$

where the last inequality follows from Markov's inequality. (18) implies

$$\sum_{k\in\mathbb{Z}} \sup_{t\in[k,k+1)} K(t) < \infty.$$

This together with the local Riemann integrability of K imply that K is dRi on $\mathbb{R}$. Consequently,

$$\lim_{t\to\infty} \mathbb{E}\Big[\sum_{k\geq 0} K(t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\Big] = \mu^{-1}\int_{\{|t|>c\}} K(t)\,\mathrm{d}t$$

by the key renewal theorem. Observe also that the last expression tends to zero as $c \to \infty$. Further, the function $t \mapsto \mathbb{P}\{M_1(t) > 1\}$ is dRi on $\mathbb{R}$, for

$$\mathbb{P}\big\{M_1(t) > 1\big\} \leq \mathbb{E}\big[M_1(t) \wedge 1\big] = K(t).$$

Again from the key renewal theorem, we conclude that

$$\lim_{t\to\infty}\sum_{k\geq 0}\mathbb{P}\big\{|t-S_k|>c,\, M_{k+1}(t-S_k)>1\big\} = \frac{1}{\mu}\int_{\{|x|>c\}}\mathbb{P}\big\{M_1(x)>1\big\}\,\mathrm{d}x.$$

The last expression tends to 0 as $c \to \infty$, which establishes (25).

Proof of (23). In fact, we claim that even the stronger statement $Y_c^* \to Y^*$ as $c \to \infty$ in $(D(\mathbb{R}), d)$ a.s. holds. To prove this, we fix arbitrary $a, b \in \mathbb{Z}$, $a < b$ and observe that it is sufficient to check that the right-hand side of

$$\begin{aligned}
\sup_{u\in[a,b]}\big|Y_c^*(u) - Y^*(u)\big| &= \sup_{u\in[a,b]}\Big|\sum_{k\in\mathbb{Z}} X_{k+1}\big(u+S_k^*\big)\mathbb{1}_{\{|S_k^*|>c\}}\Big| \\
&\leq \sum_{k\in\mathbb{Z}}\sup_{u\in[a,b]}\big|X_{k+1}\big(u+S_k^*\big)\big|\mathbb{1}_{\{|S_k^*|>c\}}
\end{aligned}$$

tends to zero as $c \to \infty$ a.s. To this end, notice that

$$\sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]} \big(\big|X_{k+1}\big(u+S_k^*\big)\big| \wedge 1\big)$$

has finite expectation by (17) and (18) and, hence, $\sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]} |X_{k+1}(u+S_k^*)| < \infty$ a.s. Therefore,

$$\sup_{u\in[a,b]} \big|Y_c^*(u) - Y^*(u)\big| \le \sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]} \big|X_{k+1}\big(u+S_k^*\big)\big| \mathbb{1}_{\{|S_k^*|>c\}} \to 0$$

as $c \to \infty$ a.s. by the monotone (or dominated) convergence theorem. The asserted a.s. convergence of Y_c^* to Y^* as $c \to \infty$ in $(D(\mathbb{R}), d)$ follows.

Proof of (3). Fix $l \in \mathbb{N}$ and real numbers $\alpha_1, \dots, \alpha_l$ and $u_1, \dots, u_l$. For $k \in \mathbb{Z}$, the number of jumps of X_{k+1} is at most countable a.s. Since the distribution of S_k^* is absolutely continuous, and S_k^* and X_{k+1} are independent we infer

$$\mathbb{P}\big\{S_k^* + u \in \mathrm{Disc}(X_{k+1})\big\} = 0$$

for any $u \in \mathbb{R}$. According to Lemma 5.4, for every $c > 0$, the mapping $\phi_c^{(l)}$ is a.s. continuous at $(\sum_{k\in\mathbb{Z}} \delta_{S_k^*}, (X_{k+1})_{k\in\mathbb{Z}})$. Now apply the continuous mapping theorem to (19) twice (first using the map $\phi_c^{(l)}$ and then the map $(x_1, \dots, x_l) \mapsto \alpha_1 x_1 + \dots + \alpha_l x_l$) to obtain that

$$\sum_{i=1}^{l} \alpha_i Y_c(t, u_i) \xrightarrow{\mathrm{d}} \sum_{i=1}^{l} \alpha_i Y_c^*(u_i), \qquad t \to \infty.$$

The proof of (3) is complete if we verify

$$\sum_{i=1}^{l} \alpha_i Y_c^*(u_i) \xrightarrow{\mathrm{d}} \sum_{i=1}^{l} \alpha_i Y^*(u_i), \qquad c \to \infty \tag{26}$$

and

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\left\{\left|\sum_{i=1}^{l} \alpha_i \sum_{k\ge 0} X_{k+1}(u_i + t - S_k) \mathbb{1}_{\{|t-S_k|>c\}}\right| > \varepsilon\right\} = 0 \tag{27}$$

for all $\varepsilon > 0$. As to (26), we claim that the stronger statement $Y_c^*(u) \to Y^*(u)$ as $c \to \infty$ a.s. for all $u \in \mathbb{R}$ holds. Indeed, as we have shown in (16),

$$\mathbb{E}\bigg[\sum_{k\in\mathbb{Z}} \big|X_{k+1}\big(u+S_k^*\big)\big| \wedge 1\bigg] < \infty,$$

in particular, $\sum_{k\in\mathbb{Z}}|X_{k+1}(u+S_k^*)|<\infty$ a.s. Hence, by the monotone (or dominated) convergence theorem,

$$\big|Y_c^*(u)-Y^*(u)\big|\leq\sum_{k\in\mathbb{Z}}\big|X_{k+1}\big(u+S_k^*\big)\big|\mathbb{1}_{\{|S_k^*|>c\}}\to 0$$

as $c\to\infty$ a.s. Further, (27) is a consequence of

$$\lim_{c\to\infty}\limsup_{t\to\infty}\mathbb{P}\left\{\left|\sum_{k\geq 0}X_{k+1}(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\right|>\varepsilon\right\}=0$$

for every $u\in\mathbb{R}$. Write

$$\begin{aligned}
&\mathbb{P}\left\{\left|\sum_{k\geq 0}X_{k+1}(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\right|>\varepsilon\right\}\\
&\quad\leq\mathbb{P}\left\{\sum_{k\geq 0}\big|X_{k+1}(u+t-S_k)\big|\mathbb{1}_{\{|t-S_k|>c\}}>\varepsilon\right\}\\
&\quad\leq\mathbb{P}\left\{\sum_{k\geq 0}\big|X_{k+1}(u+t-S_k)\big|\mathbb{1}_{\{|t-S_k|>c,|X_{k+1}(u+t-S_k)|\leq 1\}}>\varepsilon/2\right\}\\
&\qquad+\mathbb{P}\left\{\sum_{k\geq 0}\big|X_{k+1}(u+t-S_k)\big|\mathbb{1}_{\{|t-S_k|>c,|X_{k+1}(u+t-S_k)|>1\}}>\varepsilon/2\right\}\\
&\quad\leq\frac{2}{\varepsilon}\mathbb{E}\left[\sum_{k\geq 0}G(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\right]+\sum_{k\geq 0}\mathbb{P}\Big\{|t-S_k|>c,\,\big|X_{k+1}(u+t-S_k)\big|>1\Big\}
\end{aligned}$$

and observe that since G is dRi on $[0,\infty)$, so is $t\mapsto G(u+t)$ on $\mathbb{R}$ whence

$$\lim_{c\to\infty}\lim_{t\to\infty}\mathbb{E}\left[\sum_{k\geq 0}G(u+t-S_k)\mathbb{1}_{\{|t-S_k|>c\}}\right]=\frac{1}{\mu}\lim_{c\to\infty}\int_{\{|x|>c\}}G(u+x)\,\mathrm{d}x=0.$$

In view of

$$\mathbb{P}\Big\{\big|X(u+t)\big|>1\Big\}\leq G(u+t),$$

the function $t\mapsto\mathbb{P}\{|X(u+t)|>1\}$ is dRi on $\mathbb{R}$, which gives

$$\begin{aligned}
&\lim_{c\to\infty}\lim_{t\to\infty}\sum_{k\geq 0}\mathbb{P}\Big\{|t-S_k|>c,\,\big|X_{k+1}(u+t-S_k)\big|>1\Big\}\\
&\quad=\frac{1}{\mu}\lim_{c\to\infty}\int_{\{|x|>c\}}\mathbb{P}\Big\{\big|X(u+x)\big|>1\Big\}\,\mathrm{d}x=0.
\end{aligned}$$

This finishes the proof of (27). Equation (26) can be checked along the same lines. We omit the details. $\square$

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