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Simulation of volatility modulated Volterra processes using hyperbolic stochastic partial differential equations

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We propose a finite difference scheme to simulate solutions to a certain type of hyperbolic stochastic partial differential equation (HSPDE). These solutions can in turn estimate so called volatility modulated Volterra (VMV) processes and Lévy semistationary (LSS) processes, which is a class of processes that have been employed to model turbulence, tumor growth and electricity forward and spot prices. We will see that our finite difference scheme converges to the solution of the HSPDE as we take finer and finer partitions for our finite difference scheme in both time and space. Finally, we demonstrate our method with an example from the energy finance literature.

Keywords: finite difference scheme; hyperbolic stochastic partial differential equations; Lévy semistationary processes; volatility modulated Volterra processes

1. Introduction

This paper is concerned with developing a finite difference scheme to simulate the so-called mild solution to a particular hyperbolic stochastic partial differential equation. Our motivation for considering this scheme is to explore alternative methods to simulate so-called volatility modulated Volterra (VMV) processes (for definition, see (2.1)). Volatility modulated Volterra processes can be simulated by means of numerical integration, but due to the integrands depending on the time parameter, numerical integration is cumbersome since at each time step one needs to perform a complete re-integration. Thus, we propose an alternative method to simulate these volatility modulated Volterra processes, as the boundary solution of a hyperbolic stochastic partial differential equation.

We note that a special type of Volatility modulated Volterra processes are so-called Lévy semistationary (LSS) processes, which are processes that are stationary under a stationarity assumption on the volatility process. These processes have recently been proposed in the framework of modelling electricity and commodity prices, see Barndorff-Nielsen, Benth and Veraart [1–3], although they were initially employed as modelling tools for turbulence and tumor growth. It has been pointed out that the class of Lévy semistationary processes can indeed catch many of the stylised features, such as spikes and mean-reversion, that have been observed in electricity and commodity markets. The mean reversion of Lévy semistationary processes is in probability,

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and the high spikes are facilitated by the volatility process and jumps in the driving Lévy process. Thus, it is highly relevant in the energy setting to have an effective simulating algorithm for derivative pricing purposes.

Employing the finite difference scheme to simulate volatility modulated Volterra processes as opposed to numerical integration has the following advantages. As we have already noted we obtain the volatility modulated Volterra process as the boundary solution of the stochastic partial differential equation, but in order to obtain a full trajectory of the boundary with the finite difference scheme we need to numerically solve the stochastic partial differential equation on a triangular grid. Therefore when simulating our trajectory, we get the solution of the stochastic partial differential equation for free on the triangular grid. In order to simulate a value in a particular point $(t + \Delta t, x)$ in the grid, we need to know the values at the previous time step (t, x) as well as at the spatial step above $(t, x + \Delta x)$. Given initial and boundary conditions we may even solve the stochastic partial differential equation recursively on a rectangular grid. We shall show that under certain conditions, the finite difference scheme converges to the corresponding mild solution. Moreover, given a stochastic partial differential equation and a discretization we give a recipe for quantifying the error of the finite difference scheme in $L^2(\mathbb{P})$.

The rest of the paper is structured as follows. In Section 2, we start by introducing volatility modulated Volterra processes and discussing some preliminary results on them which we shall refer to later in the paper. While in Section 3, we proceed to introduce our hyperbolic stochastic partial differential equation, its mild solution and how we can obtain the volatility modulated Volterra process as the boundary of the mild solution, under rather general conditions. Subsequently in Section 4, we introduce the main contribution of this paper, namely the finite difference scheme for simulating the hyperbolic stochastic partial differential equation. Furthermore in that section we discuss convergence results for the finite difference scheme. Finally in Section 5, we present some numerical examples from the energy literature using our finite difference scheme, before reaching our concluding remarks in Section 6.

2. Volatility modulated Volterra processes

Throughout this paper, we shall assume that we are working on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ which satisfies the usual conditions, that is, the probability space is complete, the σ-algebras $\mathcal{F}_t$ include all the sets in $\mathcal{F}$ of zero probability and the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is right-continuous. Note that the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is indexed by $\mathbb{R}$. Following Barndorff-Nielsen *et al.* [1–3] we define the volatility modulated Volterra process (VMV process henceforth) to be a process of the type

$$X(t) = \mu + \int_{-\infty}^{t} p(t,s)a\left(s^-\right)\mathrm{d}s + \int_{-\infty}^{t} g(t,s)\sigma\left(s^-\right)\mathrm{d}L(s), \tag{2.1}$$

for $t \in \mathbb{R}$, where μ is a constant, $\{L(t)\}_{t\in\mathbb{R}}$ is a (two-sided) Lévy process which is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, g and p are real-valued deterministic kernel functions and $\{\sigma(t)\}_{t\in\mathbb{R}}$ and $\{a(t)\}_{t\in\mathbb{R}}$ are càdlàg processes which are adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$. Here the stochastic integral can be taken to be defined in the manner developed by Basse-O'Connor *et al.* [4]. However although VMV processes can be defined for a rather big class of Lévy processes we shall in fact only be concerned with VMV processes that are driven by square integrable Lévy processes. Hence, due to the left limits in the integrand processes, which imply predictability, the stochastic integration can also be defined in the sense of Protter [14]. In addition to the assumption that $\{L(t)\}_{t\in\mathbb{R}}$ is square integrable, we shall also assume it to have a zero mean and thus a martingale. Since if L has a drift we may observe that

$$L(t) = mt + \big(L(t) - mt\big),$$

where $m = \mathbb{E}[L(1)]$ and $\{L(t) - mt\}_{t\in\mathbb{R}}$ is a square integrable martingale. Thus, we may rewrite (2.1) as

$$X(t) = \mu + \int_{-\infty}^{t} \big(p(t,s)a\big(s^-\big) + mg(t,s)\sigma\big(s^-\big)\big)\,\mathrm{d}s + \int_{-\infty}^{t} g(t,s)\sigma\big(s^-\big)\,\mathrm{d}M(s),$$

where $M(t) = L(t) - mt$ for all $t \in \mathbb{R}$. Moreover, we shall assume that

$$\mathbb{E}\big[a(t)^2\big] \vee \mathbb{E}\big[\sigma(t)^2\big] < C \tag{2.2}$$

for some constant $C \geq 1$ and all $t \in \mathbb{R}$. Using this assumption, we may conclude by Minkowski's integral inequality and Itô isometry that

$$\begin{aligned}\mathbb{E}\big[X^2(t)\big] &\leq 3\left(\mu^2 + \mathbb{E}\left[\left(\int_{-\infty}^{t} p(t,s)a\big(s^-\big)\,\mathrm{d}s\right)^2\right] + \mathbb{E}\left[\left(\int_{-\infty}^{t} g(t,s)\sigma\big(s^-\big)\,\mathrm{d}L(s)\right)^2\right]\right)\\ &\leq 3C\left(\mu^2 + \left(\int_{-\infty}^{t} \big|p(t,s)\big|\,\mathrm{d}s\right)^2 + \int_{-\infty}^{t} g^2(t,s)\,\mathrm{d}s\right).\end{aligned}$$

Thus the VMV process (2.1) is well defined as an element in $L^2(\mathbb{P})$ if in addition to fulfilling (2.2) the deterministic kernel functions furthermore fulfill

$$p(t,\cdot) \in L^1((-\infty,t)) \quad \text{and} \quad g(t,\cdot) \in L^2((-\infty,t)) \tag{2.3}$$

for all $t \in \mathbb{R}$. In the sequel, we shall always assume that conditions (2.2) and (2.3) are fulfilled, which in turn imply that $X(t) \in L^2(\mathbb{P})$ for all $t \in \mathbb{R}$.

Of particular interest in many applications is the case when $p(t,s) = p(t-s)$ and $g(t,s) = g(t-s)$, that is, when

$$X(t) = \mu + \int_{-\infty}^{t} p(t-s)a\big(s^-\big)\,\mathrm{d}s + \int_{-\infty}^{t} g(t-s)\sigma\big(s^-\big)\,\mathrm{d}L(s). \tag{2.4}$$

Under the additional conditions that the processes $\{a(t)\}_{t\geq 0}$ and $\{\sigma(t)\}_{t\geq 0}$ are stationary, the process (2.4) is stationary. In particular, we remark that condition (2.2) holds when a and σ are stationary. Hence, like Barndorff-Nielsen *et al.* [1–3], we shall refer to processes of the type (2.4) as Lévy semistationary processes (or LSS processes). It is furthermore worth noting that in the case when $p(t,s) = p(t-s)$ and $g(t,s) = g(t-s)$ condition (2.3) is equivalent to $p \in L^1(\mathbb{R}_+)$ and $g \in L^2(\mathbb{R}_+)$.

Now under the assumption that the stochastic processes $\{a(t)\}_{t\in\mathbb{R}}$ and $\{\sigma(t)\}_{t\in\mathbb{R}}$ are independent to each other and the driving Lévy process $\{L(t)\}_{t\in\mathbb{R}}$, we have the following result for LSS processes of the type (2.1), which is based on a result in [5].

Proposition 2.1. *Assume that $\{a(t)\}_{t\in\mathbb{R}}$ and $\{\sigma(t)\}_{t\in\mathbb{R}}$ are independent to each other and the driving Lévy process $\{L(t)\}_{t\in\mathbb{R}}$. Then it holds for processes of the type (2.1) that*

$$\mathbb{E}\big[X(t)\big] = \int_{-\infty}^{t} p(t,s)\mathbb{E}\big[a\big(s^-\big)\big]\,\mathrm{d}s \tag{2.5}$$

and

$$\mathbb{E}\big[X^2(t)\big] = \mathbb{E}\bigg[\bigg(\mu + \int_{-\infty}^{t} p(t,s)a\big(s^-\big)\,\mathrm{d}s\bigg)^2\bigg] + \mathbb{E}\big[L^2(1)\big]\int_{-\infty}^{t} g^2(t,s)\mathbb{E}\big[\sigma\big(s^-\big)^2\big]\,\mathrm{d}s. \tag{2.6}$$

In particular if $\{\sigma(t)\}_{t\in\mathbb{R}}$ is stationary then it holds that

$$\mathbb{E}\big[X^2(t)\big] = \mathbb{E}\bigg[\bigg(\mu + \int_{-\infty}^{t} p(t,s)a\big(s^-\big)\,\mathrm{d}s\bigg)^2\bigg] + \mathbb{E}\big[L^2(1)\big]\mathbb{E}\big[\sigma^2(0)\big]\int_{-\infty}^{t} g^2(t,s)\,\mathrm{d}s.$$

Proof. The characteristic function of the stochastic integral

$$\widetilde{X}(t) = \int_{-\infty}^{t} g(t,s)\sigma\big(s^-\big)\,\mathrm{d}L(s)$$

may be computed by conditioning on the process $\{\sigma(t)\}_{t\in\mathbb{R}}$:

$$\varphi_{\widetilde{X}(t)}(\theta) = \mathbb{E}\big[\exp\big(\mathrm{i}\theta\widetilde{X}(t)\big)\big] = \mathbb{E}\bigg[\exp\bigg(\int_{-\infty}^{t} \psi\big(\theta g(t,s)\sigma\big(s^-\big)\big)\,\mathrm{d}s\bigg)\bigg],$$

where ψ is the cumulant of $L(1)$, that is, the log-characteristic function of $L(1)$ (see Proposition 2.6 in [15]). We observe that

$$\mathbb{E}\big[\widetilde{X}(t)\big] = -\mathrm{i}\psi'(0)\int_{-\infty}^{t} g(t,s)\mathbb{E}\big[\sigma\big(s^-\big)\big]\,\mathrm{d}s = 0$$

since $\mathbb{E}[L(1)] = \psi'(0) = 0$ by assumption. Hence, (2.5) follows. Furthermore, we find

$$\mathbb{E}\big[\widetilde{X}^2(t)\big] = -\psi''(0)\int_{-\infty}^{t} g^2(t,s)\mathbb{E}\big[\sigma\big(s^-\big)^2\big]\,\mathrm{d}s.$$

So (2.6) follows by independence of the processes a, σ and L. □

In other words, we know everything there is to know about the second order structure of VMV processes under the assumption that a and σ are independent to each other and the driving Lévy process.

In Section 3, we will describe how one can view VMV processes (2.1) by processes that solve a particular stochastic partial differential equation, with given initial and boundary conditions. The solution to the stochastic partial differential equation can in turn be estimated numerically by a finite difference method that we will introduce, which will have the same initial and boundary conditions. For simulating purposes, the initial condition must be finite and therefore the following lemma will prove useful.

Lemma 2.2. *For given VMV processes,*

$$X_1(t) = \int_{-\infty}^{t} p(t,s)a\bigl(s^-\bigr)\,\mathrm{d}s + \int_{-\infty}^{t} g(t,s)\sigma\bigl(s^-\bigr)\,\mathrm{d}L(s)$$

and

$$X_2(t) = \int_{-\infty}^{t} q(t,s)a\bigl(s^-\bigr)\,\mathrm{d}s + \int_{-\infty}^{t} h(t,s)\sigma\bigl(s^-\bigr)\,\mathrm{d}L(s),$$

where $\{a(t)\}_{t\in\mathbb{R}}$ and $\{\sigma(t)\}_{t\in\mathbb{R}}$ satisfy condition (2.2) and the deterministic kernel functions p, q, g and h are square integrable in the sense of (2.3) it holds that

$$\mathbb{E}\bigl[\bigl|X_1(t) - X_2(t)\bigr|^2\bigr] = C\bigl(\bigl\|p(t,\cdot) - q(t,\cdot)\bigr\|^2_{L^1((-\infty,t))} + \bigl\|g(t,\cdot) - h(t,\cdot)\bigr\|^2_{L^2((-\infty,t))}\bigr),$$

for a constant $C > 0$ and all $t \in \mathbb{R}$. In particular when $p(t,s) = p(t-s)$, $g(t,s) = g(t-s)$, $q(t,s) = q(t-s)$ and $h(t,s) = h(t-s)$ it holds that

$$\mathbb{E}\bigl[\bigl|X_1(t) - X_2(t)\bigr|^2\bigr] = C\bigl(\|p - q\|^2_{L^1(\mathbb{R}_+)} + \|g - h\|^2_{L^2(\mathbb{R}_+)}\bigr),$$

for a constant $C > 0$.

Proof. We may apply Proposition 2.1 to obtain

$$\begin{aligned}\mathbb{E}\bigl[\bigl|X_1(t) - X_2(t)\bigr|^2\bigr] &= \mathbb{E}\biggl[\biggl(\int_{-\infty}^{t} \bigl(p(t,s) - q(t,s)\bigr)a\bigl(s^-\bigr)\,\mathrm{d}s\biggr)^2\biggr] \\ &\quad + \mathbb{E}\bigl[L^2(1)\bigr]\int_{-\infty}^{t} \bigl(g(t,s) - h(t,s)\bigr)^2\mathbb{E}\bigl[\sigma\bigl(s^-\bigr)^2\bigr]\,\mathrm{d}s.\end{aligned}$$

Moreover it holds by Minkowski's integral inequality that

$$\mathbb{E}\biggl[\biggl(\int_{-\infty}^{t} \bigl(p(t,s) - q(t,s)\bigr)a\bigl(s^-\bigr)\,\mathrm{d}s\biggr)^2\biggr] \le \mathbb{E}\bigl[a\bigl(s^-\bigr)^2\bigr]\biggl(\int_{-\infty}^{t} \bigl|\bigl(p(t,s) - q(t,s)\bigr)\bigr|\,\mathrm{d}s\biggr)^2.$$

Now the result follows by (2.2). $\square$

Now for a given VMV process (2.1) satisfying conditions (2.2) and (2.3) we may employ Lemma 2.2 to approximate it with proper stochastic integrals, that is, integrals over compact intervals. That is, for a given $t \in \mathbb{R}$, let $r < t$ be a constant and consider the truncated kernel functions $\widetilde{p}(t,s) = 1_{\{s \geq r\}} p(t,s)$ and $\widetilde{g}(t,s) = 1_{\{s \geq r\}} g(t,s)$. Then due to (2.3) it holds that

$$\begin{aligned} &\|p(t,\cdot) - \widetilde{p}(t,\cdot)\|^2_{L^1((-\infty,t))} + \|g(t,\cdot) - \widetilde{g}(t,\cdot)\|^2_{L^2((-\infty,t))} \\ &= \int_{-\infty}^{r} \big(|p(t,s)| + g^2(t,s)\big)\,\mathrm{d}s \downarrow 0 \end{aligned} \tag{2.7}$$

as $r \downarrow -\infty$. So by Lemma 2.2 we may approximate the VMV process (2.1) in a fixed point $t \in \mathbb{R}$ arbitrarily well by a process

$$X(t) = \mu + \int_r^t p(t,s) a\big(s^-\big)\,\mathrm{d}s + \int_r^t g(t,s)\sigma\big(s^-\big)\,\mathrm{d}L(s), \tag{2.8}$$

where $r < t$.

3. Modelling VMV processes as boundary solutions to HSPDEs

In this section, we will "raise" the dimension of our VMV process (2.1) to obtain a stochastic process that can be viewed as a mild solution of a particular hyperbolic stochastic partial differential equation (HSPDE henceforth). To this end, we need to define the HSPDE and the concept of a mild solution. For references on stochastic partial differential equations and mild solutions, we refer to [8,13].

For a given $t_0 \in \mathbb{R}$ let us assume that $\{M_t\}_{t \geq t_0}$ is a square integrable càdlàg martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ that satisfies the usual conditions. Furthermore, let $\mathcal{P}$ denote the σ-algebra of predictable sets on $[t_0, \infty) \times \Omega$, that is, the smallest σ-algebra of subsets of $[t_0, \infty) \times \Omega$ containing all sets of the form $(s,t] \times B$, where $s, t \geq t_0$ and $B \in \mathcal{F}_s$. Suppose we have a given Hilbert space of univariate real-valued functions on $\mathbb{R}_+$, denoted by F, and predictable (i.e., $\mathcal{P}$-measurable) and adapted mappings $\alpha : [t_0, \infty) \times \Omega \to F$ and $\beta : [t_0, \infty) \times \Omega \to F$. Let us consider the stochastic partial differential equation

$$\mathrm{d}Y(t) = \big(AY(t) + \alpha(t)\big)\,\mathrm{d}t + \beta(t)\,\mathrm{d}M(t), \tag{3.1}$$

with the initial condition $Y(t_0) = Y_0$, where Y_0 is a square integrable $\mathcal{F}_{t_0}$-measurable random variable with values in F. Here we assume that A is a (potentially unbounded) infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ of bounded operators on the Hilbert space F. Where by a strongly continuous semigroup on F we mean that the family of bounded linear operators $\{S(t)\}_{t \geq 0}$ satisfies the following three conditions:

1. $S(0) = I$, where I is the identity operator on F,
2. $S(s) \circ S(t) = S(t+s)$ for all $s, t \geq 0$,
3. $\lim_{t \downarrow 0} \|S(t)f - f\|_F = 0$ for all $f \in F$.

Note that a family of bounded linear operators $\{S(t)\colon\ t \geq 0\}$ on a Banach space that satisfies the above conditions is called a C_0-*semigroup*. The domain of A,

$$\mathcal{D}(A) = \left\{ f \in F\colon\ \lim_{t \downarrow 0} \frac{S(t)f - f}{t} \text{ exists} \right\}$$

will in general be a proper subset of F, but it is always a dense subset in F, and its action on $\mathcal{D}(A)$ is given by

$$Af = \lim_{t \downarrow 0} \frac{S(t)f - f}{t}, \tag{3.2}$$

for $f \in \mathcal{D}(A)$. For references on operator semigroups see, for example, [10]. Since A is in general not a bounded operator, the notion of a strong solution in the sense of

$$Y(t) = Y_0 + \int_{t_0}^{t} \big(AY(s) + \alpha(s)\big)\,\mathrm{d}s + \int_{t_0}^{t} \beta(s)\,\mathrm{d}M(s)$$

may not always make sense, since $Y(s)$ might not be in the domain of A. Therefore, the notion of a mild solution to (3.1) has been introduced in the literature. A *mild solution* to the equation (3.1) is a recast of the differential equation (3.1):

$$Y(t) = S(t - t_0)Y_0 + \int_{t_0}^{t} S(t-s)\alpha(s)\,\mathrm{d}s + \int_{t_0}^{t} S(t-s)\beta(s)\,\mathrm{d}M(s). \tag{3.3}$$

In order for the mild solution to be well defined, we need to impose some conditions on the coefficient functions of (3.1). The first integral $\int_{t_0}^{t} S(t-s)\alpha(s)\,\mathrm{d}s$ is taken to be defined as a Bochner integral and is thus well defined if the integrand $s \mapsto S(t-s)\alpha(s)$ is measurable and

$$\int_{t_0}^{t} \big\| S(t-s)\alpha(s) \big\|_F \,\mathrm{d}s < \infty. \tag{3.4}$$

Note that the measurability of the integrand from $([t_0, \infty), \mathcal{B}([t_0, \infty)))$ to $(F, \mathcal{B}(F))$ follows from the strong continuity of the operator semigroup. As for the stochastic integral $\int_{t_0}^{t} S(t-s)\beta(s)\,\mathrm{d}M(s)$, recall that by the Doob–Meyer decomposition, for each càdlàg square integrable martingale $\{M(t)\}_{t \geq t_0}$ there exists a unique increasing predictable process, called the *angle bracket* of M, denoted by $\{\langle M\rangle(t)\}_{t \geq t_0}$ such that $\langle M\rangle(t_0) = 0$ and $\{M^2(t) - \langle M\rangle(t)\}_{t \geq t_0}$ is a martingale. For predictable integrands β the following Itô isometry holds:

$$\mathbb{E}\left[\left\| \int_{t_0}^{t} \beta(s)\,\mathrm{d}M(s) \right\|_F^2\right] = \mathbb{E}\left[\int_{t_0}^{t} \|\beta(s)\|_F^2 \,\mathrm{d}\langle M\rangle(s)\right],$$

see, for example, [13]. Now for the stochastic integral to be well defined we need to ensure that the integrand $s \mapsto S(t-s)\beta(s)$ is predictable, which follows from the strong continuity of the semigroup, and that it is an element of the space of integrands, that is, that

$$\mathbb{E}\left[\left\| \int_{t_0}^{t} S(t-s)\beta(s)\,\mathrm{d}M(s) \right\|_F^2\right] = \mathbb{E}\left[\int_{t_0}^{t} \big\|S(t-s)\beta(s)\big\|_F^2 \,\mathrm{d}\langle M\rangle(s)\right] < \infty, \tag{3.5}$$

for all $t \geq t_0$.

In what follows, when we work with solutions to (3.1) we will always mean mild solutions of the above type (3.3). Now let us reconsider the VMV Volterra model (2.1). Notice that this equation bears a resemblance to the mild solution (3.3). However, there are some differences which need to be addressed.

First of all, we need to make an assumption on the operator semigroup $\{S(t)\}_{t\geq 0}$ which is present in (3.3) and the function space it operates on. Our assumption will be that $\{S(t)\}_{t\geq 0}$ is the strongly continuous semigroup of (left) translation operators on F, defined by

$$\big(S(t)f\big)(x) = f(t+x) \tag{3.6}$$

for all $f \in F$ and $x \geq 0$. In this case, it follows from (3.2) that $A = \partial/\partial x$ is a differential operator on F. Clearly this operator semigroup fulfils the first two algebraic conditions regardless of the selection of the Hilbert space F. Whereas the third condition by contrast is a topological one, and thus dependent upon the norm of the Hilbert space. In our setting we propose to use as state space a Hilbert space proposed by Filipović [11] in the setting of HJM [12] dynamics. For a positive increasing function $w : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\int_0^\infty w(x)^{-1}\,\mathrm{d}x < \infty$ it is defined as the space of absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_0^\infty f'(x)^2 w(x)\,\mathrm{d}x < \infty,$$

endowed with the inner product

$$\langle f, g\rangle_w = f(0)g(0) + \int_0^\infty f'(x)g'(x)w(x)\,\mathrm{d}x.$$

It is easy to see that the norm induced by this inner product satisfies the strong continuity condition. Moreover for a given $x \geq 0$, it holds that the evaluation functional $\delta_x : F \to \mathbb{R}$ defined by $\delta_x(f) = f(x)$ is uniformly bounded, see [8]. This will in turn allow us to evaluate the mild solution (3.3) in any point $x \geq 0$, provided (3.4) and (3.5) hold.

The second issue is that the VMV process is defined on $\mathbb{R}$, whereas a mild solution to a HSPDE is only defined on a half line $[t_0, \infty)$. For our purposes, we simply cut the domain of the VMV process in the following way. For a given $t \in \mathbb{R}$, we assume that there exists a $t_0 < t$ such that we can approximate $X(t)$ in (2.1) by the process

$$X_{t_0}(t) = \mu + \int_{t_0}^t p(t,s)a\big(s^-\big)\,\mathrm{d}s + \int_{t_0}^t g(t,s)\sigma\big(s^-\big)\,\mathrm{d}L(s) \tag{3.7}$$

in $L^2(\mathbb{P})$. Lemma 2.2 confirms that this is possible. Now having truncated the integration domain, we would like to think of (3.7) as the boundary of a mild solution (3.3). By adding a spatial component, x, to the above equation we get something which we may interpret as a mild solution of a HSPDE under assumption (3.6) on the operator semigroup. For $x \geq 0$, we raise the dimension of the truncated VMV Volterra model by considering the field

$$Y(t,x) = \mu + \int_{t_0}^t p(t+x,s)a\big(s^-\big)\,\mathrm{d}s + \int_{t_0}^t g(t+x,s)\sigma\big(s^-\big)\,\mathrm{d}L(s). \tag{3.8}$$

Now notice that the process $\{Y(t,\cdot)\}_{t\geq t_0}$ can be viewed as a mild solution to the HSPDE (3.1), where $\alpha(t) = p(t+\cdot,t)a(t-)$, $\beta(t) = g(t+\cdot,t)\sigma(t-)$, $A = \partial/\partial x$, $M = L$, and $Y_0 = 0$. Indeed by considering these coefficient functions for the HSPDE (3.1) under assumption (3.6) on the operator semigroup one obtains that the mild solution (3.3) of the HSPDE (3.1) and the process defined in (3.8) coincide. Thus, the VMV Volterra process (3.7) is the boundary solution to the HSPDE, which in turn approximates the general VMV process (2.1).

Given the proposed function space selection let us recall the integrability conditions (3.4) and (3.5) and inspect what they translate into in the case of VMV processes. Let us for simplicity focus on the stochastic integral. In the case, when $M = L$ is a Lévy process it holds that $\langle L\rangle(t) = C_1 t$, where $C_1 = \mathrm{Var}[L(1)] > 0$ is a constant. If we furthermore recall the square integrability condition (2.2) on the volatility, condition (3.5) reduces to

$$\int_{t_0}^{t} \left\| g(t+\cdot, s)\right\|_w^2 \,\mathrm{d}s = \int_{t_0}^{t} g^2(t,s)\,\mathrm{d}s + \int_{t_0}^{t}\int_0^{\infty} \left(g'(t+x,s)\right)^2 w(x)\,\mathrm{d}x\,\mathrm{d}s < \infty.$$

In particular, this implies that $\|g(t,\cdot)\|_{L^2((t_0,t))} < \infty$ holds. Further strengthening the condition by letting $t_0 \downarrow -\infty$ and assuming that

$$\int_{-\infty}^{t} \left\| g(t+\cdot, s)\right\|_w^2 \,\mathrm{d}s < \infty \tag{3.9}$$

implies that the condition (2.3) is satisfied by g. Thus we observe that assuming that (3.9) holds for g and p, is sufficient for our purposes with VMV processes and for the solution of the HSPDE to be well defined.

In what follows, for a given discretization $t_1 < t_2 < \cdots < t_N$ in the time domain, to simulate a trajectory $\{X(t_n)\}_{n=0}^N$ of the VMV Volterra process (2.1), we propose the following two step procedure:

1. Truncate the integration domain of (2.1) from $\mathbb{R}$ to $[t_0, \infty)$. Where $t_0 \le t_1$ is such that $\|X(t) - X_{t_0}(t)\|_{L^2(\mathbb{P})}$ is close to zero for $t \ge t_1$.
2. Raise the dimension of the truncated VMV Volterra model by considering the field (3.8). Now simulate the HSPDE (3.1) with $\alpha(t) = p(t+\cdot, t)a(t-)$, $\beta(t) = g(t+\cdot,t)\sigma(t-)$, $M = L$ and $Y_0 = 0$ under assumption (3.6) on the operator semigroup using the finite difference scheme that will be introduced in Section 4. The trajectory $\{X(t_n)\}_{n=0}^N = \{Y(t_n, 0)\}_{n=0}^N$ is obtained as the boundary solution of the HSPDE.

In many cases, one may even be interested in more than just the boundary, as the following example shows.

Example 3.1. In Section 5 of [2] Barndorff-Nielsen *et al.*, derive a model for pricing electricity forward contracts based on general Lévy driven Volterra electricity spot prices. Thus deseasonalized electricity spot prices $\{X(t)\}_{t\in\mathbb{R}}$ are generally modelled by VMV processes of the type

$$X(t) = \int_{-\infty}^{t} g(t,s)\sigma\left(s^-\right)\mathrm{d}L(s),$$

where the components of the integral fulfil all the necessary conditions listed in Section 2. Examples of kernel functions considered by Barndorff-Nielsen *et al.* [2] include $g(t,s) =$

$\exp(-\alpha(t-s))$ for a constant $\alpha > 0$, and $g(t,s) = \sigma/(t-s+b)$ for constants $\sigma, b > 0$. Under certain integrability conditions, forward price dynamics $F_t(T)$ may be derived as an expression involving the volatility modulated Volterra process

$$\int_{-\infty}^{t} g(T,s)\sigma\left(s^-\right) \mathrm{d}L(s).$$

Here T is time of delivery. Letting $x = T - t$ we may write

$$\int_{-\infty}^{t} g(T,s)\sigma\left(s^-\right) \mathrm{d}L(s) = \int_{-\infty}^{t} g(t+x,s)\sigma\left(s^-\right) \mathrm{d}L(s),$$

and we are back to our mild solution. Hence, in a practical context, we are interested in simulating the joint spot-forward price dynamics. This can be done by simulating the mild solution of the corresponding HSPDE. Hence, in an energy market context, the finite difference scheme approach gives a joint simulation of spot and forward prices for all maturities directly without re-integration at each maturity.

We shall return to this example in Section 5, after we have discussed our finite difference scheme.

4. The finite difference scheme

This section presents the main contribution of this paper, namely a finite difference scheme for simulating solution fields for the HSPDE (3.1), under the assumption $A = \partial/\partial x$.

Now let us introduce the following notation for the finite difference method. Let $\Delta x > 0$ and $\Delta t > 0$ denote the discrete steps in space and time respectively, and denote by

$$y_j^n \approx Y(t_0 + n\Delta t)(j\Delta x)$$

the approximation of the solution of (3.1) at the point $(t_0 + n\Delta t, j\Delta x)$, where $n = 0, \dots, N$ and $j = 0, \dots, J$ for some $J, N \in \mathbb{N}$. From our HSPDE (3.1) with $A = \partial/\partial x$, using forward finite difference, that is, by using the approximations $\mathrm{d}Y(t) \approx Y(t+\Delta t) - Y(t)$, $\mathrm{d}t \approx \Delta t$, $\mathrm{d}M(t) \approx M(t+\Delta t) - M(t)$ and $AY(t) \approx (Y(t)(\cdot + \Delta x) - Y(t))/\Delta x$, we derive the finite difference scheme

$$y_j^{n+1} = \lambda y_{j+1}^n + (1-\lambda) y_j^n + \alpha_j^n \Delta t + \beta_j^n \Delta M^n, \tag{4.1}$$

where $\lambda = \Delta t/\Delta x$, $x_j = j\Delta x$, $t_n = t_0 + n\Delta t$, $\alpha_j^n = \alpha(t_n)(x_j)$, $\beta_j^n = \beta(t_n)(x_j)$ and $\Delta M^n = M(t_{n+1}) - M(t_n)$. Clearly, one should adjust the initial value so that it fits with the initial value of the HSPDE one is interested in simulating, that is, by setting $y_j^0 = Y_0(x_j)$ for all $j = 0, \dots, J$. For instance in our VMV applications (recall (3.8)) this means letting $y_j^0 = \mu$, for all $j = 0, \dots, J$. We furthermore note that information about the initial values are sufficient. Since in order to obtain a value at a given point (t_{n+1}, x_j) the scheme requires information about the values at the previous time steps (t_n, x_j) and (t_n, x_{j+1}). Thus for a fixed j' in order to calculate the trajectory

$\{y_{j'}^n\}_{n=1}^N$ we only need information about the previous values on a triangular grid, that is, we need to know the values of

$$\begin{aligned}
&y_{j'+N}^0;\\
&y_{j'+N-1}^0, y_{j'+N-1}^1;\\
&\vdots\\
&y_{j'+1}^0, y_{j'+1}^1, \dots, y_{j'+1}^{N-1},
\end{aligned} \tag{4.2}$$

all of which may be obtained from the initial values. Hence, to simulate the random field $\{Y(t_n)(x_j)\}_{j=0,n=0}^{J,N}$ which is the solution of the HSPDE (3.1) on a rectangular grid, for a given initial value, without knowing the values at the boundary (x_J), using the finite difference scheme (4.1), we propose the following.

1. Simulate ΔM^n, for $n = 0, \dots, N-1$.
2. Compute the values of the triangular grid (4.2) where $j' = J$.
3. Compute the values of the rectangular grid, using values from the triangular grid where necessary.

We remark that in some cases it may however be natural to impose a boundary condition on the spatial dimension. In the case of LSS processes with $p(t,s) = p(t-s)$ and $g(t,s) = g(t-s)$ one could use Lemma 2.2 to assume $y_J^n = 0$, for all $n = 0, \dots, N$, if x_J is big enough, since (2.3) implies that $p \in L^1(\mathbb{R}_+)$ and $g \in L^2(\mathbb{R}_+)$, so they vanish at infinity.

As in the case of a finite difference scheme for the standard advection partial differential equation, one needs some constraints on the discrete steps, that is, $(\Delta x, \Delta t)$, to guarantee its stability. The stability condition of Courant, Friedrichs, and Lewy (the CFL condition, see [9]) is needed to ensure the stability of our finite difference scheme (4.1). In our case this translates into the necessary constraint

$$\Delta t \le \Delta x, \tag{4.3}$$

which we assume to hold.

For the rest of this section, we will study the convergence properties of the finite difference scheme. Given our function space F of real-valued functions equipped with a supremum norm it will be convenient for our analysis to define the following family of bounded linear operators on F. Given positive $\Delta x > 0$ and $\Delta t > 0$ corresponding to the steps of the finite difference scheme in space and time respectively let us consider the family $\{T_{\Delta x,\Delta t}\}_{\Delta x>0,\Delta t>0}$ which is defined by

$$T_{\Delta x,\Delta t} = I + \Delta t \frac{S(\Delta x) - I}{\Delta x}, \tag{4.4}$$

for all $\Delta x > 0$, $\Delta t > 0$, where I denotes the identity operator on F and $S(\Delta x)$ is the left shift operator whose action on F is given by (3.6). The following lemma will be useful for proving convergence of the finite difference scheme.

Lemma 4.1. *For given steps $\Delta x > 0$ in space and $\Delta t > 0$ in time, the finite difference scheme (4.1) admits the representation*

$$y_j^n = T^n y_j^0 + \sum_{i=0}^{n-1} T^{n-1-i} \alpha_j^i \Delta t + \sum_{i=0}^{n-1} T^{n-1-i} \beta_j^i \Delta M^i \tag{4.5}$$

for all $n = 0, \dots, N$ and $j = 0, \dots, J$, where $T = T_{\Delta x, \Delta t}$ is defined by (4.4) and where $T^n = T^{\circ n}$ denotes the composition of the operator T with itself n times and $T^0 = I$.

Proof. We proceed by means of induction on n. The identity (4.5) clearly holds for $n = 0$ and all $j = 0, \dots, J$. Supposing that the identity (4.5) is satisfied by some $n \geq 0$ and all $j = 0, \dots, J$, we obtain the following.

$$\begin{aligned}
y_j^{n+1} &= \lambda y_{j+1}^n + (1-\lambda) y_j^n + \alpha_j^n \Delta t + \beta_j^n \Delta M^n \\
&= T^n y_j^0 + \lambda \big(T^n y_{j+1}^0 - T^n y_j^0 \big) \\
&\quad + \sum_{i=0}^{n-1} \big(T^{n-1-i} \alpha_j^i + \lambda \big(T^{n-1-i} \alpha_{j+1}^i - T^{n-1-i} \alpha_j^i \big) \big) \Delta t + \alpha_j^n \Delta t \\
&\quad + \sum_{i=0}^{n-1} \big(T^{n-1-i} \beta_j^i + \lambda \big(T^{n-1-i} \beta_{j+1}^i - T^{n-1-i} \beta_j^i \big) \big) \Delta M^i + \beta_j^n \Delta M^n \\
&= T T^n y_j^0 + \sum_{i=0}^{n-1} \big(T T^{n-1-i} \alpha_j^i \big) \Delta t + \alpha_j^n \Delta t + \sum_{i=0}^{n-1} \big(T T^{n-1-i} \beta_j^i \big) \Delta M^i + \beta_j^n \Delta M^n \\
&= T^{n+1} y_j^0 + \sum_{i=0}^{n} T^{n-i} \alpha_j^i \Delta t + \sum_{i=0}^{n} T^{n-i} \beta_j^i \Delta M^i .
\end{aligned}$$

This completes the proof. □

The above lemma characterizes the finite difference scheme (4.1) for a given discretization as the sum of three entities which, under appropriate conditions, will converge to their corresponding parts in the mild solution (3.3) as we consider finer and finer partitions in time and space. More precisely, we will employ the fact that the composed operator T^n, where $T = T_{\Delta x, \Delta t}$ is defined by (4.4), converges to the left shift operator $S(t_n - t_0)$ as we consider finer and finer partitions in first time and then space.

Let us take a closer look on the family (4.4) of operators. The following lemma will be employed later for proving a convergence result on the finite difference scheme.

Lemma 4.2. *Suppose $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a function that satisfies the Lipschitz condition*

$$\mathbb{E}\big[\big| \beta(x_1) - \beta(x_2) \big|^2 \big] \leq L |x_1 - x_2|^2$$

for all $x_1, x_2 \geq 0$ *where* $L > 0$ *is a constant. Then*

$$\mathbb{E}\big[\big|T^m\beta(x) - S(t)\beta(x)\big|^2\big] \leq Lt(\Delta x - \Delta t),$$

where $\{S(t)\}_{t\geq 0}$ *denotes the left shift semigroup* (3.6) *and* T *is defined in* (4.4) *with* $\Delta t = t/m$ *and* $\Delta t \leq \Delta x$, *for all* $x \geq 0$, $t > 0$ *and* $m \geq 1$.

Proof. Let $\lambda = \Delta t/\Delta x$ and suppose first that $\lambda = 1$, then clearly $T = S(\Delta x)$ and $T^m = S(t)$. Now suppose that $\lambda < 1$, and observe that by the binomial theorem it holds that

$$\begin{aligned} T^m\beta(x) &= (1-\lambda)^m\left(I + \frac{\lambda}{1-\lambda}S(\Delta x)\right)^m\beta(x) \\ &= \sum_{k=0}^{m}\binom{m}{k}\lambda^k(1-\lambda)^{m-k}\beta(x+k\Delta x) = \mathbb{E}'\big[\beta(x+\Delta x Z)\big], \end{aligned}$$

where Z denotes a binomial random variable with parameters m and λ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, with expectation operator denoted by $\mathbb{E}'$. Now recall that a binomial random variable has expected value $m\lambda$ and variance $m\lambda(1-\lambda)$, from which it is easy to deduce that the random variable $\Delta x Z$ has expected value t and variance $t(\Delta x - \Delta t)$. Thus by the Cauchy–Schwarz inequality, the Fubini theorem (or the linearity of the expected value) and the Lipschitz condition

$$\begin{aligned} \mathbb{E}\big[\big|T^m\beta(x) - \beta(x+t)\big|^2\big] &\leq \mathbb{E}\big[\mathbb{E}'\big[\big|\beta(x+\Delta x Z) - \beta(x+t)\big|^2\big]\big] \\ &\leq L\mathbb{E}'\big[|\Delta x Z - t|^2\big] \\ &= Lt(\Delta x - \Delta t). \end{aligned}$$

This concludes the proof. □

In the context of the finite difference scheme (4.1), $t - t_0$ corresponds to the length of the time interval $[t_0, t]$ and taking $\Delta t = t_{n+1} - t_n$ for all $n = 0, \ldots, N-1$. In light of Lemma 4.2, it is interesting to comment on the difference between employing our finite difference scheme as opposed to numerical integration. Supposing that we are mainly interested in the boundary solution ($x = 0$) of a particular HSPDE on a given time grid $t_0 < \cdots < t_N = t$, then one could estimate its mild solution at a particular time step t by means of numerical integration in the following way:

$$\widetilde{Y}(t)(0) = S(t-t_0)Y_0(0) + \sum_{i=0}^{N-1}S(t-t_{i+1})\alpha(t_i)(0)\Delta t + \sum_{i=0}^{N-1}S(t-t_{i+1})\beta(t_i)(0)\Delta M^i. \quad (4.6)$$

By comparison to (4.5), one sees that the above equation is quite similar. Moreover, Lemma 4.2 provides us with some evidence that the equation (4.5) for $j = 0$ and the above equation (4.6) tend to give us the same trajectories as we consider finer and finer steps. In particular when $\Delta t = \Delta x$ the two approaches give us the exact same trajectories. But the difference between the two respective methods, given that the coefficient functions are sufficiently well behaved, is that one of them only gives us the boundary solution of the HSPDE, whereas the other one solves the HSPDE on a triangular grid.

This is relevant in the setting of Example 3.1, in the context of simulating the joint spot-forward dynamics. Another advantage of employing the finite difference, is that given the values $Y(t)(0)$ and $Y(t)(\Delta x)$ for a particular $t > t_0$ we easily obtain the next value $Y(t+\Delta t)(0)$ by means of the finite difference scheme (4.1). However if we employ numerical integration we cannot use this information to calculate the next step $Y(t+\Delta t)(0)$, we need to do a complete re-integration in the time domain.

We have the following convergence result, which can be used to determine whether or not the finite difference scheme (4.1) is convergent in $L^2(\mathbb{P})$ for a particular HSPDE and to determine the convergence rate. We shall only consider HSPDEs with initial value and coefficient functions that are uniformly Lipschitz in the sense of Lemma 4.2. That it we assume that

$$\begin{aligned}\mathbb{E}\big[\big|Y_0(x_1)-Y_0(x_2)\big|^2\big] \vee \mathbb{E}\big[\big|\alpha(s)(x_1)-\alpha(s)(x_2)\big|^2\big] \vee \mathbb{E}\big[\big|\beta(s)(x_1)-\beta(s)(x_2)\big|^2\big] \\ \le L|x_1-x_2|^2\end{aligned} \tag{4.7}$$

hold for all $s \in [t_0, t]$, $x_1, x_2 \ge 0$ and a constant $L > 0$.

Proposition 4.3. *Consider the finite difference scheme* (4.1) *under the representation* (4.5), *where the initial value and the coefficient functions satisfy the Lipschitz condition* (4.7). *Suppose furthermore that the coefficient functions are independent of the driving martingale process. Then if* $t_n = t_0 + n\Delta t$ *and* $x_j = j\Delta x$, *for* $n, j \ge 0$, *it holds that*

$$\begin{aligned}\mathbb{E}\big[\big|y_j^n - Y(t_n)(x_j)\big|^2\big] \le C_1(n)(\Delta x - \Delta t) + C_2(n)\Delta t^2 \\ + C_3(n)\mathbb{E}\Big[\sup_{0\le s-r<\Delta t}\big|S(t_n-s)\big(\alpha(r)-\alpha(s)\big)\big|^2\Big] \\ + C_4(n)\mathbb{E}\Big[\sup_{0\le s-r<\Delta t}\big|S(t_n-s)\big(\beta(r)-\beta(s)\big)\big|^2\Big],\end{aligned}$$

where

$$\begin{aligned}C_1(n) &= 3L(t_n-t_0)\big\{1+4(t_n-t_0)^2+4\mathbb{E}\big[\langle M\rangle(t_n)\big]\big\},\\ C_2(n) &= 12L\big\{(t_n-t_0)^2+\mathbb{E}\big[\langle M\rangle(t_n)\big]\big\},\\ C_3(n) &= 12L(t_n-t_0)^2 \quad \textit{and} \quad C_4(n) = 12\mathbb{E}\big[\langle M\rangle(t_n)\big].\end{aligned}$$

Proof. First, notice that

$$\mathbb{E}\big[\big|T^{N-1-i}Y_0 - S(t-t_0)Y_0\big|^2\big] \le L(t-t_0)(\Delta x - \Delta t)$$

follows directly from Lemma 4.2. Since M is square integrable and independent to β it holds by Itô isometry and Lemma 4.2 that

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} T^{N-1-i}\beta(t_i)\Delta M^i - \sum_{i=0}^{N-1} S(t-t_{i+1})\beta(t_i)\Delta M^i\right|^2\right] \\
&\quad = \mathbb{E}\left[\int_{t_0}^{t}\sum_{i=0}^{N-1}\big(T^{N-1-i}\beta(t_i) - S(t-t_{i+1})\beta(t_i)\big)^2 1_{[t_i,t_{i+1})}(s)\,\mathrm{d}\langle M\rangle(s)\right] \\
&\quad = \sum_{i=0}^{N-1}\mathbb{E}\big[\big(T^{N-1-i}\beta(t_i) - S(t-t_{i+1})\beta(t_i)\big)^2\big]\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\mathrm{d}\langle M\rangle(s)\right] \\
&\quad \le L(\Delta x - \Delta t)\sum_{i=0}^{N-1}(t-t_{i+1})\mathbb{E}\big[\langle M\rangle(t_{i+1}) - \langle M\rangle(t_i)\big] \le L(t-t_0)\mathbb{E}\big[\langle M\rangle(t)\big](\Delta x - \Delta t).
\end{aligned}$$

Furthermore by Lipschitz continuity and independence of M and β we get that

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} S(t-t_{i+1})\beta(t_i)\Delta M^i - \int_{t_0}^{t} S(t-s)\beta(s)\,\mathrm{d}M(s)\right|^2\right] \\
&\quad = \sum_{i=0}^{N-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\big(S(t-t_{i+1})\beta(t_i) - S(t-s)\beta(s)\big)^2\,\mathrm{d}\langle M\rangle(s)\right] \\
&\quad \le \sum_{i=0}^{N-1}\mathbb{E}\Big[\sup_{s\in[t_i,t_{i+1})}\big(S(t-t_{i+1})\beta(t_i) - S(t-s)\beta(s)\big)^2\Big]\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\mathrm{d}\langle M\rangle(s)\right] \\
&\quad \le 2\sum_{i=0}^{N-1}\mathbb{E}\Big[\sup_{s\in[t_i,t_{i+1})}\Big(\big(S(t-t_{i+1})\beta(t_i) - S(t-s)\beta(t_i)\big)^2 + \big(S(t-s)\beta(t_i) - S(t-s)\beta(s)\big)^2\Big)\Big] \\
&\qquad\qquad \times \mathbb{E}\left[\int_{t_i}^{t_{i+1}}\mathrm{d}\langle M\rangle(s)\right] \\
&\quad \le 2\mathbb{E}\big[\langle M\rangle(t)\big]\Big(L\Delta t^2 + \mathbb{E}\Big[\sup_{0\le s-r<\Delta t}\big|S(t-s)\big(\beta(r) - \beta(s)\big)\big|^2\Big]\Big).
\end{aligned}$$

Putting the above inequalities together and employing the elementary inequality $(x+y)^2 \le 2(x^2+y^2)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i=0}^{N-1} T^{N-1-i}\beta(t_i)\Delta M^i - \int_{t_0}^{t} S(t-s)\beta(s)\,\mathrm{d}M(s)\right|^2\right] \\
&\quad \le 4\mathbb{E}\big[\langle M\rangle(t)\big]\Big(L(t-t_0)(\Delta x - \Delta t) + L\Delta t^2 + \mathbb{E}\Big[\sup_{0\le s-r<\Delta t}\big|S(t-s)\big(\beta(r) - \beta(s)\big)\big|^2\Big]\Big).
\end{aligned}$$

Now

$$\mathbb{E}\left[\left|\sum_{i=0}^{N-1} T^{N-1-i}\alpha(t_i)\Delta t - \int_{t_0}^{t} S(t-s)\alpha(s)\,\mathrm{d}s\right|^2\right]$$
$$\leq 4(t-t_0)^2\Big(L(t-t_0)(\Delta x - \Delta t) + L\Delta t^2 + \mathbb{E}\Big[\sup_{0\leq s-r<\Delta t}\big|S(t-s)\big(\alpha(r)-\alpha(s)\big)\big|^2\Big]\Big)$$

follows in a similar manner, replacing the Itô isometry argument by the Cauchy–Schwarz inequality. The proof is completed by employing the representation in Lemma 4.1, the elementary inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$, and collecting the resulting terms. □

Now let us finish the section by coming back to LSS processes. In this case, we are only concerned with HSPDEs that have coefficient functions which can be separated into a stochastic part and a deterministic part. That is, HSPDEs that have coefficient functions on the following form:

$$\alpha(t) = pa(t-) \quad \text{and} \quad \beta(t) = g\sigma(t-), \tag{4.8}$$

where $p, g \in F$ are Lipschitz continuous functions with a joint Lipschitz constant $L > 0$, and $\{a(t)\}_{t\geq t_0}$ and $\{\sigma(t)\}_{t\geq t_0}$ are predictable and adapted stochastic processes that satisfy (2.2). We shall moreover require that

$$|g|^2 \vee |p|^2 < K \tag{4.9}$$

for a constant $K \geq 1$. Indeed for our function space equipped with a supremum norm these assumptions guarantee that the corresponding HSPDE has a well-defined mild solution.

Corollary 4.4. *Consider the finite difference scheme* (4.1) *under the representation* (4.5), *where the initial value and the coefficient functions satisfy the Lipschitz condition* (4.7). *Suppose furthermore that the coefficient functions are independent of the driving martingale process, and that* (4.8) *and* (4.9) *hold. Then if $t_n = t_0 + n\Delta t$ and $x_j = j\Delta x$, for $n, j \geq 0$, it holds that*

$$\begin{aligned}
&\mathbb{E}\big[\big|y_j^n - Y(t_n)(x_j)\big|^2\big] \\
&\quad\leq C_1(n)(\Delta x - \Delta t) + C_2(n)\Delta t^2 \\
&\qquad + C_3(n)\mathbb{E}\Big[\sup_{0\leq s-r<\Delta t}\big|a(r)-a(s)\big|^2\Big] + C_4(n)\mathbb{E}\Big[\sup_{0\leq s-r<\Delta t}\big|\sigma(r)-\sigma(s)\big|^2\Big],
\end{aligned}$$

where

$$\begin{aligned}
C_1(n) &= 3L(t_n-t_0)\big\{1+4(t_n-t_0)^2+4\mathbb{E}\big[\langle M\rangle(t_n)\big]\big\},\\
C_2(n) &= 12L\big\{(t_n-t_0)^2+\mathbb{E}\big[\langle M\rangle(t_n)\big]\big\},\\
C_3(n) &= 12KL(t_n-t_0)^2 \quad \textit{and} \quad C_4(n) = 12K\mathbb{E}\big[\langle M\rangle(t_n)\big].
\end{aligned}$$

In particular if M is a Lévy process then $\mathbb{E}[\langle M\rangle(t)] = Ct$ for a constant $C \geq 0$.

5. Numerical examples

In this section, we present some numerical examples to illustrate the finite difference scheme and our convergence results in the previous section. As an example, consider

$$g(u)=\frac{a}{u+b}\mathrm{e}^{-\alpha u}, \tag{5.1}$$

where $a, b>0$ and $\alpha \geq 0$. This is a blend of the kernel function suggested by Bjerksund *et al.* [7] and the OU process, and thus constitutes a potential kernel function for applications in electricity. Returning to Example 3.1, for a fixed grid in time $t_0<t_1<\cdots<t_N$ and space $0=x_0<x_1<\cdots<x_J$ with fixed increments Δt and Δx respectively, consider simulating the random field

$$Y(t,x)=\int_0^t g(t-s+x)\sigma\left(s^-\right)\mathrm{d}B(s), \tag{5.2}$$

where g represents the kernel function (5.1), B is standard Brownian motion and $\sigma^2(t)=Z(t)$, where

$$Z(t)=\int_{-\infty}^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}U(s), \tag{5.3}$$

and U is a subordinator process. Now simulating (5.2) on a rectangular grid with the finite difference method is much more efficient than using numerical integration to calculate each trajectory for a fixed x. As an example of that we implemented the finite difference method in Matlab for the rectangular grid where $t_0=0$, $t_N=1$, $x_J=2$, $\Delta t=\Delta x=0.01$, $\lambda=0.01$ and U is an inverse Gaussian process with parameters $\delta=15$, $\gamma=1$, and the kernel function (5.1) has parameters $a=b=1, \alpha=0.01$. See Figure 1 for a plot of the relative error between the boundary of the finite difference method and the numerical integration method, and the field obtained. For reference, we simulated the same rectangular grid by means of numerical integration for each fixed x. Using the tic, toc Matlab function, we measured the efficiency of the respective methods in terms of speed. Unsurprisingly the finite difference method was faster, using 0.0731 sec, whereas the numerical integration method used 0.3536 sec (the experiments were performed on a standard laptop computer).

Finally, we remark that it is also easy to estimate the error from estimating the volatility as follows: For a constant $C=\mathbb{E}[U^2(1)]\geq 0$ and $r>s$ it holds that

$$\begin{aligned}
\mathbb{E}\left[\left|\sigma(s)-\sigma(r)\right|^2\right] &= \mathbb{E}\left[Z(s)+Z(r)-2\left(Z(s)Z(r)\right)^{1/2}\right] \\
&= C\left(\int_{-\infty}^s \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}u+\int_{-\infty}^r \mathrm{e}^{-\lambda(r-u)}\,\mathrm{d}u\right) \\
&\quad -2\mathbb{E}\left[\left(\int_{-\infty}^s \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)\int_{-\infty}^r \mathrm{e}^{-\lambda(r-u)}\,\mathrm{d}U(u)\right)^{1/2}\right],
\end{aligned}$$

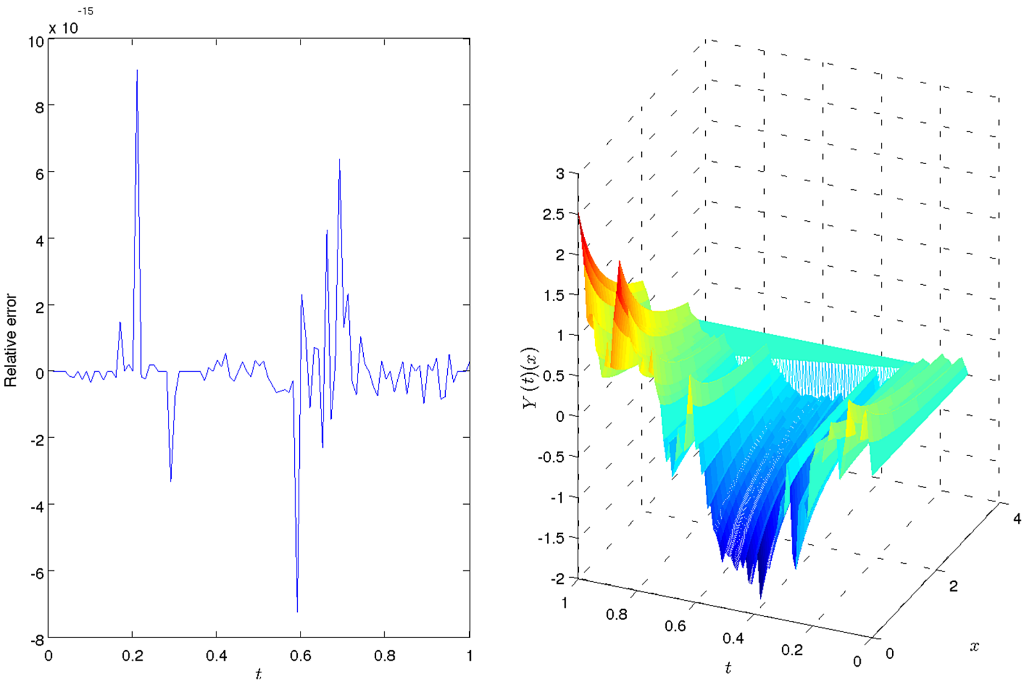

Figure 1. Left: the relative error of the boundary path of (5.2) ($x = 0$) where g is given by (5.1), obtained by numerical integration versus the finite difference scheme. Right: the field (5.2) where g is given by (5.1), obtained by the finite difference method, on a rectangular grid with step sizes $\Delta t = \Delta x = 0.01$.

and by non-negativity of the stochastic integral driven by a subordinator it holds that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)\int_{-\infty}^{r} \mathrm{e}^{-\lambda(r-u)}\,\mathrm{d}U(u)\right)^{1/2}\right]\\
&\quad= \mathrm{e}^{-\lambda(r-s)/2}\mathbb{E}\left[\left(\int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)\left(\int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)+\int_{s}^{r} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)\right)\right)^{1/2}\right]\\
&\quad\geq \mathrm{e}^{-\lambda(r-s)/2}\mathbb{E}\left[\int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}U(u)\right].
\end{aligned}$$

So for $r > s$, we may conclude that

$$\begin{aligned}
\mathbb{E}\left[\left|\sigma(s)-\sigma(r)\right|^2\right] &\leq C\left(\int_{-\infty}^{r} \mathrm{e}^{-\lambda(r-u)}\,\mathrm{d}u - \left(2\mathrm{e}^{-\lambda(r-s)/2}-1\right)\int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-u)}\,\mathrm{d}u\right)\\
&= \frac{2C}{\lambda}\left(1-\mathrm{e}^{-\lambda(r-s)/2}\right),
\end{aligned}$$

and thus by taking supremum we conclude that

$$\sup_{|s-r|<\Delta t} \mathbb{E}\big[\big|\sigma(s)-\sigma(r)\big|^2\big] \leq \frac{2C}{\lambda}\big(1-\mathrm{e}^{-\lambda\Delta t/2}\big).$$

Having benchmarked our method of obtaining space time fields against the more straightforward approach of numerical integration, we would like to point out that our method has a variety of potential applications. One might for example consider the problem of simulating fractional Brownian motion (see, e.g., Biagini *et al.* [6]). Recall that for a given Hurst parameter $H \in (0, 1)$ fractional Brownian motion can be written as

$$B^H(t) = \frac{1}{\Gamma(H+1/2)}\bigg(\int_{-\infty}^{t}(t-s)^{H-1/2}\,\mathrm{d}B(s) - \int_{-\infty}^{0}(-s)^{H-1/2}\,\mathrm{d}B(s)\bigg),$$

for $t \in \mathbb{R}$. Now notice that the kernel function $g(u) = u^{H-1/2}$ is not Lipschitz at the origin. Thus, we can not apply our convergence result 4.3 directly. However, we may for a given $\varepsilon > 0$ define an approximative kernel function

$$h_\varepsilon(u) = \begin{cases} g(u), & \text{if } u \geq \varepsilon, \\ g(\varepsilon), & \text{if } u \in [0, \varepsilon], \end{cases}$$

and employ Lemma 2.2 find that

$$\|g - h_\varepsilon\|^2_{L^2(\mathbb{R}_+)} \leq (2 + 1/H)\varepsilon^{2H}.$$

So unsurprisingly this estimate is better for H closer to one than the origin. Hence we may again employ Lemma 2.2 together with Proposition 4.3 to control simulation errors when employing the finite difference scheme with the kernel function h_ε to simulate a trajectory of fractional Brownian motion for a given Hurst parameter H.

6. Conclusion

We have defined, and analysed, a finite difference method for simulating mild solutions of a particular HSPDE. Further we have described how VMV processes may be viewed as mild solutions of these particular HSPDEs, and thus obtained an alternative to numerical integration for simulating VMV processes. Finally, we have seen in experiments that our finite difference method is more time efficient than numerical integration for simulating a space time random field LSS process driven by non-exponential kernel functions. Our examples also include the simulation of fractional Brownian/Lévy random fields. We remark that the finite difference scheme may also be applied for the simulation of forward rates in the Musiela parametrisation of the Heath–Jarrow–Morton modelling approach in fixed-income markets (see [12]). In future studies, we will extend our HSPDE approach to the simulation of so-called ambit fields (see [1]).

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