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Exponential rate of convergence in current reservoirs

ANNA DE MASI[1], ERRICO PRESUTTI[2], DIMITRIOS TSAGKAROGIANNIS[3] and MARIA EULALIA VARES[4]

[1]*Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università di L'Aquila, Via Vetoio, 1 67100 L'Aquila, Italy. E-mail: demasi@univaq.it*
[2]*Gran Sasso Science Institute, Viale Francesco Crispi, 7 67100 L'Aquila, Italy. E-mail: errico.presutti@gmail.com*
[3]*Department of Mathematics, University of Sussex, Pevensey 2 Building, Falmer Campus, Brighton BN1 9QH, UK. E-mail: D.Tsagkarogiannis@sussex.ac.uk*
[4]*Instituto de Matemática, Universidade Federal do Rio de Janeiro, Av. Athos da S. Ramos 149, 21941-909, Rio de Janeiro, RJ, Brazil. E-mail: eulalia@im.ufrj.br*

In this paper, we consider a family of interacting particle systems on $[-N, N]$ that arises as a natural model for current reservoirs and Fick's law. We study the exponential rate of convergence to the stationary measure, which we prove to be of the order N^{-2}.

Keywords: exponential convergence to the stationary measure; interacting particle systems

1. Introduction

In this paper, we study a family of interacting particle systems whose state space is $\{0, 1\}^{[-N,N]}$. For each N, the dynamics is a Markov process with generator $L = L_0 + L_b$, L_0 the generator of the stirring process (see (2.1) below), L_b the generator of a birth-death process whose events are localized in a neighborhood of the end-points; see (3.1).

In particular, we focus on the case when around N there are only births while around $-N$ there are only deaths. The system is then "unbalanced" and in the stationary measure μ_N^{st} there is a non-zero steady current of particles flowing from right to left. This system is designed to model the Fick's law which relates the current to the density gradient.

In statistical mechanics, non-equilibrium is not as well understood as equilibrium, hence the interest, from a physical viewpoint, to look at systems which are stationary yet in non-equilibrium: in our case, the stationary process is in fact non-reversible and the stationary measure μ_N^{st} not Gibbsian.

There is a huge literature on stationary non-equilibrium measures, in particular, on their large deviations, as they are related to "out of equilibrium thermodynamics" (see, for instance, [1–3,8]). Our goal is to study the exponential rate at which the dynamics converges to the stationary measure, and how it depends on the system size. Spectral gaps have been well studied in the reversible or Gibbsian set-up, both for stirring and for more general interacting particle systems (see, for instance, [10]). The techniques used in those situations, however, do not seem to apply to our non-equilibrium model. We shall rather rely on stochastic inequalities and coupling methods, thus reducing the problem to that of bounding the extinction time of the set of discrepancies between two coupled evolutions. The case of a single discrepancy can be regarded as an environment dependent random walk with death rate which also depends on the environment. Its extinction time has been studied in [7] and, as we shall see here, is closely related to the exponential rate of convergence in our model.

The main part of this paper refers to the case of "current reservoirs" (where L_b should have a factor $1/N$). Much simpler is the case when L_b fixes the different densities at the boundaries, whose analysis is carried out sketchily in the next section simply as an introduction.

2. Density reservoirs

We consider in this section the Markov process on $\{0,1\}^{[-N,N]}$ with generator $L = L_0 + L'$, where denoting by η the elements of $\{0,1\}^{[-N,N]}$,

$$L_0 f(\eta) := \frac{1}{2} \sum_{x=-N}^{N-1} \left[f\left(\eta^{(x,x+1)}\right) - f(\eta) \right] \tag{2.1}$$

with $\eta^{(x,x+1)}(x) = \eta(x+1)$, $\eta^{(x,x+1)}(x+1) = \eta(x)$ and $\eta^{(x,x+1)}(\cdot) = \eta(\cdot)$ elsewhere

$$\begin{aligned} L' f(\eta) = {} & \rho_+ \left[f\left(\eta^{(+,N)}\right) - f(\eta) \right] + (1-\rho_+) \left[f\left(\eta^{(-,N)}\right) - f(\eta) \right] \\ & + \rho_- \left[f\left(\eta^{(+,-N)}\right) - f(\eta) \right] + (1-\rho_-) \left[f\left(\eta^{(-,-N)}\right) - f(\eta) \right], \end{aligned}$$

where $1 \ge \rho_+ > \rho_- \ge 0$ and $\eta^{+,x}(x) = 1$, $\eta^{+,x}(y) = \eta(y)$, $y \neq x$; analogously, $\eta^{-,x}(x) = 0$, $\eta^{-,x}(y) = \eta(y)$, $y \neq x$.

The process corresponding to L' alone leaves unchanged the occupations at $|x| < N$ while the equilibrium probabilities of occupation at $\pm N$ are equal to $\rho_\pm$. Since $\rho_+ > \rho_-$, this creates a density gradient and the full process with generator $L = L_0 + L'$ describes the particles flux determined by the density gradient. The process is uniformly Döblin, in particular, there is a unique stationary measure μ_N^{st} to which the process converges exponentially fast. The averages $\mu_N^{\mathrm{st}}[\eta(x)]$ describe a linear density profile in agreement with Fick's law. Fluctuations in the stationary regime are well characterized ([11], and the large deviations as well, [8]).

Denote by μ_N the initial distribution and by $\mu_N S_t$ the distribution at time t (i.e., the law at time t of the process with generator L starting from μ_N). Then, since the process is uniformly Döblin, for any positive integer N there are strictly positive constants c_N and b_N so that

$$\left\| \mu_N S_t - \mu_N^{\mathrm{st}} \right\| \le c_N \mathrm{e}^{-b_N t} \qquad \text{for any } \mu_N \text{ and } t > 0, \tag{2.2}$$

where for any signed measure λ on $\{0,1\}^{[-N,N]}$

$$\|\lambda\| = \sum_{\eta} \big|\lambda(\eta)\big|. \tag{2.3}$$

We now prove the following.

Theorem 2.1. *There are c and $b>0$ independent of N so that for any initial measure μ_N and all $t>0$*

$$\big\|\mu_N S_t - \mu_N^{\mathrm{st}}\big\| \le cN\mathrm{e}^{-bN^{-2}t}. \tag{2.4}$$

Proof. Let

$$\mathcal{X}_N = \big\{\underline{\eta} = \big(\eta^{(1)}, \eta^{(2)}\big) \in \big(\{0,1\}\times\{0,1\}\big)^{[-N,N]} : \eta_{\neq}(x) := \eta^{(1)}(x) - \eta^{(2)}(x) \ge 0, \forall x\big\}, \tag{2.5}$$

and, for $f:\mathcal{X}_N \to \mathbb{R}$,

$$\begin{aligned}
\mathcal{L}_0 f(\underline{\eta}) &:= \frac{1}{2}\sum_{x=-N}^{N-1} \big[f\big(\underline{\eta}^{(x,x+1)}\big) - f(\underline{\eta})\big],\\
\mathcal{L}' f(\underline{\eta}) &= \rho_+\big[f\big(\underline{\eta}^{(+,N)}\big) - f(\underline{\eta})\big] + (1-\rho_+)\big[f\big(\underline{\eta}^{(-,N)}\big) - f(\underline{\eta})\big]\\
&\quad + \rho_-\big[f\big(\underline{\eta}^{(+,-N)}\big) - f(\underline{\eta})\big] + (1-\rho_-)\big[f\big(\underline{\eta}^{(-,-N)}\big) - f(\underline{\eta})\big],
\end{aligned}$$

where $\underline{\eta}^{(+,x)}(x) = (1,1)$, $\underline{\eta}^{(-,x)}(x) = (0,0)$, and coincide with $\underline{\eta}$ elsewhere, $x = \pm N$.

It is easy to see that $\mathcal{L}_0$ and $\mathcal{L}'$ define Markov generators on $\mathcal{X}_N$. Moreover, when acting on functions that depend on only one of the two entries, $\eta^{(1)}$ or $\eta^{(2)}$, of $\underline{\eta}$, we see that $\mathcal{L}_0 + \mathcal{L}'$ coincide with L, and so it defines a coupling between the processes with generator L starting from two comparable configurations $\eta^{(1)}$ and $\eta^{(2)}$ ($\eta^{(1)}(x) \ge \eta^{(2)}(x)$ for all x), showing that the L-evolution is attractive in the sense of [9] (i.e., preserves order). In particular, we may take $\eta^{(1)} \equiv 1$ and $\eta^{(2)} \equiv 0$ the configurations that are identically 1 and, respectively, 0. Moreover, $\mathcal{L}_0$ leaves unchanged the number of discrepancies which instead may decrease under the action of $\mathcal{L}'$. Write $\mathbf{P}$ for the law of the process starting from $\eta^{(1)} \equiv 1$ and $\eta^{(2)} \equiv 0$ and call $\pi(x,t) = \mathbf{P}[\eta_{\neq}(x,t) = 1]$. We then have, recalling that $\pi(x,0) = 1$ for all x,

$$\pi(x,t) = 1 - \int_0^t \big(p_s(x,N)\pi(N,t-s) + p_s(x,-N)\pi(-N,t-s)\big)\,\mathrm{d}s, \tag{2.6}$$

where $p_s(x,y)$ is the probability under the stirring process (with only one particle) of going from x to y in a time s; this is the same as the probability of a simple random walk whose jumps outside $[-N,N]$ are suppressed. Indeed, (2.6) follows at once from the integration by parts formula for the semigroup $\mathcal{S}_t$ generated by $\mathcal{L}_0 + \mathcal{L}'$, with $\mathcal{S}_t^0$ the semigroup generated by $\mathcal{L}_0$, and recalling that the effect of $\mathcal{L}'$ is to kill discrepancies at N and $-N$ with rate 1:

$$\mathcal{S}_t(f) = S_t^0(f) + \int_0^t \mathcal{S}_{t-s}\big(\mathcal{L}' S_s^0 f\big)\,\mathrm{d}s,$$

where $f : \mathcal{X}_N \to \mathbb{R}$. From (2.6), we see that

$$\pi(x,t) = E_x\bigl[\mathrm{e}^{-T^*(t)}\bigr],$$

where E_x is the expectation of the process with transition probabilities $p_s(x,y)$ and

$$T^*(t) = \int_0^t (\mathbf{1}_{x_s=N} + \mathbf{1}_{x_s=-N})\,\mathrm{d}s$$

is the time spent at $\{-N, N\}$ during $[0,t]$. Indeed,

$$E_x\bigl[\mathrm{e}^{-T^*(t)}\bigr] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^t \cdots \int_0^t \mathrm{d}s_1 \cdots \mathrm{d}s_n\, E_x\Biggl[\prod_{i=1}^{n} \{\mathbf{1}_{x_{s_i}=N} + \mathbf{1}_{x_{s_i}=-N}\}\Biggr]$$

which is the same series which is obtained by iterating (2.6).

We shall prove that

$$E_x\bigl[\mathrm{e}^{-T^*(t)}\bigr] \le c\mathrm{e}^{-bN^{-2}t} \tag{2.7}$$

which will then imply

$$\sum_{x=-N}^{N} \mathbf{P}\bigl[\eta_{\neq}(x,t) = 1\bigr] \le Nc\mathrm{e}^{-bN^{-2}t}$$

and so (2.4), because $\mu_N S_t$ and μ_N^{st} are squeezed in between the laws of the marginals of the coupled process.

Proof of (2.7). By an iterative argument, it is enough to show that

$$\sup_{x\in[-N,N]} E_x\bigl[\mathrm{e}^{-\tau}\bigr] \le p < 1, \qquad \tau := T^*\bigl(N^2\bigr).$$

But

$$\inf_{x\in[-N,N]} P_x[\tau \ge 1] \ge \delta > 0 \tag{2.8}$$

as the probability of reaching $\{-N, N\}$ by time $N^2 - 1$ is bounded from below uniformly in the starting point and the probability of not moving for a unit time interval is also bounded away from 0. By (2.8),

$$\begin{aligned} E_x\bigl[\mathrm{e}^{-\tau}\bigr] &= E_x\bigl[\mathrm{e}^{-\tau}; \tau < 1\bigr] + E_x\bigl[\mathrm{e}^{-\tau}; \tau \ge 1\bigr] \\ &\le 1 - P_x[\tau \ge 1] + P_x[\tau \ge 1]\mathrm{e}^{-1} \le 1 - \delta\bigl(1 - \mathrm{e}^{-1}\bigr). \end{aligned}$$

□

3. Main result

In this paper, we study the process with generator $L = L_0 + L_b$, L_0 as in (2.1), $L_b = L_{b,+} + L_{b,-}$ describes births and deaths near the boundaries. Namely, denoting by η the elements of

$\{0,1\}^{[-N,N]}$ and by f functions on $\{0,1\}^{[-N,N]}$,

$$\begin{aligned} L_{b,\pm} f(\eta) &:= \frac{j}{2N} \sum_{x \in I_\pm} D_\pm \eta(x) \big[f\big(\eta^{(x)}\big) - f(\eta) \big], \\ D_+ \eta(x) &= \big(1-\eta(x)\big)\eta(x+1)\cdots\eta(N), \\ D_- \eta(x) &= \eta(x)\big(1-\eta(x-1)\big)\cdots\big(1-\eta(-N)\big), \end{aligned} \tag{3.1}$$

where $j > 0$ is a parameter of the model, $I_+ = \{N-1, N\}$ and $I_- = \{-N, -N+1\}$ (in [4–6] $I_\pm$ consist of K sites, here we restrict to $K = 2$ only for notational simplicity). Thus $L_{b,+}$ adds a particle at rate $\frac{j}{2N}$ in the last empty site (if any) in I_+ while at the same rate $L_{b,-}$ takes out the first particle (if any) in I_-.

Motivations for this model can be found in previous papers, [4–6], where we have studied the hydrodynamic behavior of the system and the profile of the stationary measure as $N \to \infty$. The analysis in the above papers does not say what happens for the process after the hydrodynamical regime, that is, at times longer than N^2. This is the aim of the current paper where we study the time scale for reaching the stationary regime.

We use the same notation as in the previous section with $S_t = \mathrm{e}^{Lt}$ and $\mu_N S_t$, $t \ge 0$, the law at time t of the process with generator L starting from μ_N:

$$\mu_N S_t[f] = \mu_N\big[\mathrm{e}^{Lt} f\big] = \mu_N\big[S_t(f)\big]. \tag{3.2}$$

If $j = 0$, that is, $L = L_0$ the sets $\{\sum \eta(x) = M\}$, $0 \le M \le 2N+1$, are invariant so that the process is not even ergodic. However, the presence of L_b, even if "small" due to the rate $j/2N$, changes drastically the long time behavior of the system and it is therefore crucial in the computation of the spectral gap. Our process, like the one in the previous section, is uniformly Döblin; there is therefore a unique stationary measure μ_N^{st} and (2.2) holds in the present context as well. We prove the analogue of Theorem 2.1.

Theorem 3.1. *There are c and $b > 0$ independent of N so that*

$$\big\|\mu_N S_t - \mu_N^{\mathrm{st}}\big\| \le cN\mathrm{e}^{-bN^{-2}t}, \qquad \textit{for all initial measures } \mu_N \textit{ and all } t > 0.$$

Theorem 3.1 is the main result in this paper and it will be proved in the next sections.

The rate N^{-2} in the exponent in (3.3) cannot be improved, as can be easily seen by bounding from below the probability that an initially existing discrepancy does not disappear by the time N^2.

The result is in several respects surprising: the spectral gap in fact scales as N^{-2} just like in the stirring process (i.e., with $j = 0$) restricted to any of the invariant subspaces $\{\eta : \sum \eta(x) = M\}$. The result says that in a time of the same order the full process manages to equilibrate among all the above subsets according to μ_N^{st}; also, the time for this to happen scales in the same way as for the process of the previous section, where however the birth-death events are not scaled down with N as in Theorem 3.1.

We do not have sharp information on μ_N^{st}. In [6], we have proved that the set $\mathcal{M}$ of all probability measures on $\{0,1\}^{[-N,N]}$ shrinks after a time of order N^2 to a smaller set $\mathcal{M}_N$ but we have no information on the way it further shrinks at later times. All measures in $\mathcal{M}_N$ are close to a product measure γ_N, meaning that the expectation of products $\eta(x_1)\cdots\eta(x_n)$ are close (the accuracy increasing with N) to those of γ_N, for all n-tuples of distinct sites x_i; n is given, but it can be taken larger and larger as N increases. We also know that the expectations $\gamma_N[\eta(x)]$ are close to $\rho^{\mathrm{st}}(x/N)$, where $\rho^{\mathrm{st}}(r)$, $r\in[-1,1]$, is the stationary solution of the limit hydrodynamic equation; it is an increasing linear function and $\rho^{\mathrm{st}}(-1)=1-\rho^{\mathrm{st}}(1)>0$.

We thus know that μ_N^{st} is close (in the above sense) to the product measure γ_N, but that is all, which does not seem detailed enough to apply the usual techniques for the investigation of the spectral gap using equilibrium estimates. We proceed differently, and our proof of Theorem 3.1 follows along the lines of the much simpler Theorem 2.1. It relies on a careful analysis of the time evolution, exploiting stochastic inequalities, as in the previous section. We thus consider a coupled process on $\mathcal{X}_N$ (see (2.5)), which again starts from $\eta^{(1)}(x,0)=1$ and $\eta^{(2)}(x,0)=0$ for all $x\in[-N,N]$. The process is defined in such a way that the marginal distributions of $\eta^{(1)}$ and $\eta^{(2)}$ have the law of process with generator L. By the definition of $\mathcal{X}_N$, $\eta^{(1)}\ge\eta^{(2)}$ at all times (order is preserved) and the proof of Theorem 3.1 follows from an estimate on the extinction time of the "discrepancy configuration" $\eta_{\neq}=\eta^{(1)}-\eta^{(2)}$. We shall in fact prove that there are c and $b>0$ independent of N so that

$$\sum_{x=-N}^{N}\mathbf{P}\big[\eta_{\neq}(x,t)=1\big]\le cN\mathrm{e}^{-bN^{-2}t}. \tag{3.3}$$

4. The coupled process

Throughout the sequel, we shall use the following.

Notation. $\varepsilon:=N^{-1}$; *for* $\underline{\eta}=(\eta^{(1)},\eta^{(2)})\in\mathcal{X}_N$ *as defined in* (2.5), *and* $x\in[-N,N]$,

$$\begin{aligned}\eta_{\neq}(x)&=\eta^{(1)}(x)-\eta^{(2)}(x),\\ \eta_1(x)&=\eta^{(1)}(x)\eta^{(2)}(x),\\ \eta_0(x)&=\big(1-\eta^{(1)}(x)\big)\big(1-\eta^{(2)}(x)\big),\end{aligned} \tag{4.1}$$

$\eta_{\neq},\eta_1,\eta_0$ *are all in* $\{0,1\}^{[-N,N]}$ *and* $\eta_{\neq}+\eta_1+\eta_0\equiv 1$. *Thus*, (4.1) *establishes a one-to-one correspondence between* $\mathcal{X}_N$ *and* $\{\neq,1,0\}^{[-N,N]}$. *By an abuse of notation, we shall denote again by* $\underline{\eta}$ *the elements of* $\{\neq,1,0\}^{[-N,N]}$, *thinking of* $\eta_{\neq},\eta_1,\eta_0$ *as functions of* $\underline{\eta}$. *We may then say that a* $\neq$, 1 *or* 0-*particle is at* x *according to the value of* $\underline{\eta}(x)$.

Definition. *Call* L_0' *the stirring generator acting on functions on* $\mathcal{X}_N$ (*defined as in* (2.1) *with* η *replaced by* $\underline{\eta}$) *and let* $L_c=L_0'+\frac{j}{2N}L_1$, $L_1=L_r+L_l$, *be the generator acting on functions on*

$\mathcal{X}_N$, where $L_r f$ is defined as

$$
\begin{aligned}
L_r f(\underline{\eta}) = & \sum_{i=N-1}^{N} D(\underline{\eta}, i)\big[f\big(\underline{\eta}^{\neq,1,i}\big) - f(\underline{\eta})\big] \\
& + A(\underline{\eta}, N)\big[f\big(\underline{\eta}^{\neq,1,N;0,\neq,N-1}\big) - f(\underline{\eta})\big] + \sum_{i=N-1}^{N} B(\underline{\eta}, i)\big[f\big(\underline{\eta}^{0,1,i}\big) - f(\underline{\eta})\big]
\end{aligned}
\tag{4.2}
$$

and where $\underline{\eta}^{a,b,i}$ changes from a to b the value of $\underline{\eta}$ at site i if $\underline{\eta}(i) = a$, and $\underline{\eta}^{a,b,i} = \underline{\eta}$ otherwise, and $\underline{\eta}^{\neq,1,N;0,\neq,N-1} = (\underline{\eta}^{\neq,1,N})^{0,\neq,N-1}$,

$$
\begin{aligned}
& D(\underline{\eta}, N) = \eta_{\neq}(N)\big[1 - \eta_0(N-1)\big], \qquad D(\eta, N-1) = \eta_{\neq}(N-1)\eta_1(N), \\
& A(\underline{\eta}, N) = \eta_{\neq}(N)\eta_0(N-1), \\
& B(\underline{\eta}, N) = \eta_0(N), \qquad B(\eta, N-1) = \eta_0(N-1)\eta_1(N).
\end{aligned}
$$

Thus, L_r describes three types of events all occurring in I_+:

- D-events: a $\neq$-particle becomes a 1-particle.
- A events: a $\neq$-particle becomes a 1-particle and simultaneously a 0-particle becomes a $\neq$-particle.
- B-events: a 0-particle becomes a 1-particle.

L_l is defined analogously by changing I_+ into I_- and η_0 with η_1. One can easily check that

$$
L_c f = Lg, \qquad \text{whenever } f(\underline{\eta}) = g\big(\eta^{(i)}\big), i = 1, 2, \tag{4.3}
$$

L the generator in Section 3. Thus, the process generated by L_c is a coupling of two processes both with generator L and L preserves order (this is just the standard basic coupling, as in [9]; see also Proposition 3.1 of [6]).

5. Graphical construction

Following the so-called Harris graphical construction, we realize the coupled process in a probability space $(\Omega, \mathcal{F}, P)$ where several independent Poisson processes are defined.

Definition. *The probability space $(\Omega, \mathcal{F}, P)$. The elements $\omega \in \Omega$ have the form*

$$
\omega = \big(\underline{t}^{(x)}, x \in [-N, N-1]; \underline{t}^{(A,\pm N)}\underline{t}^{(D,\pm N)}; \underline{t}^{(D,\pm(N-1))}; \underline{t}^{(B,\pm N)}; \underline{t}^{(B,\pm(N-1))}\big),
$$

where each entry is a sequence in $\mathbb{R}_+$ whose elements are interpreted as times. Under P, the entries are independent Poisson processes: each one of the $\underline{t}^{(x)}$ has intensity $1/2$, and all the others have each intensity $\varepsilon j/2$.

With probability 1, all times are different from each other and there are finitely many events in a compact. For any such $\omega \in \Omega$, we construct piecewise constant functions $\eta_1(x, t; \omega)$,

$\eta_0(x,t;\omega)$, $\eta_{\neq}(x,t;\omega)$, as follows. The jump times are a subset of the events in the above Poisson processes, more specifically at the times $t = t_n^{(x)}$ we exchange the content of the sites x and $x+1$ (i.e., we do a stirring at $(x,x+1)$); the other jumps are:

- At the times $t = t_n^{(A,\pm N)}$, the configuration is updated only if $\eta_{\neq}(\pm N, t^-) = 1$, $\eta_0(\pm(N-1), t^-) = 1$ and the new configuration has $\eta_{\neq}(\pm(N-1), t^+) = 1$ and $\eta_1(\pm N, t^+) = 1$; the values at other sites remain unchanged.
- At the times $t = t_n^{(D,\pm N)}$, the configuration is updated only if $\eta_{\neq}(\pm N, t^-) = 1$ and $\eta_0(\pm(N-1), t^-) = 0$, the new configuration has $\eta_1(\pm N, t^+) = 1$; the values at other sites unchanged.
- At the times $t = t_n^{(D,\pm(N-1))}$, the configuration is updated only if $\eta_{\neq}(\pm(N-1), t^-) = 1$ and $\eta_1(\pm N, t^-) = 1$; the new configuration has $\eta_1(\pm(N-1), t^+) = 1$; the values at other sites unchanged.
- At the times $t = t_n^{(B,\pm N)}$, the configuration is updated only if $\eta_0(\pm N, t^-) = 1$; the new configuration has $\eta_1(\pm N, t^+) = 1$; the values at other sites unchanged.
- At the times $t = t_n^{(B,\pm(N-1))}$, the configuration is updated only if $\eta_1(N, t^-) = 1$ and $\eta_0(\pm(N-1), t^-) = 1$; the new configuration has $\eta_1(\pm(N-1), t^+) = 1$; the values at other sites unchanged.

We take initially $\eta_{\neq}(x,0) = 1$ for all x, then the variables $\underline{\eta}(x,t;\omega)$ defined as above on (Ω, P) have the law of the coupled process defined in Section 4.

Definition. *Labeling the discrepancies. By realizing the process in the space* $(\Omega, \mathcal{F}, P)$, *we can actually follow the discrepancies in time. Indeed consider the discrepancy initially at a site* $z \in [-N, N]$. *Then the discrepancy will move following the marks of* ω. *Namely, it moves at the stirring times, that is, it jumps from* x *to* $x+1$ (*or from* $x+1$ *to* x) *at the times* $t \in \underline{t}^{(x)}$. *Moreover, it jumps from* N *to* $N-1$ *at the times in* $\underline{t}^{(A,N)}$ (*if* $\eta_0(N-1) = 1$) *and analogously from* $-N$ *to* $-N+1$ *at the times in* $\underline{t}^{(A,-N)}$ (*if* $\eta_1(-N+1) = 1$). *Finally, we say that the discrepancy dies* (*and goes to the state* $\varnothing$) *at the times* $\underline{t}^{(D,\pm N)}$, $\underline{t}^{(D,\pm(N-1))}$ (*if the conditions for the event are satisfied, as explained in the previous paragraphs*).

We thus label the initial discrepancies by assigning with uniform probability a label in $\{1, \dots, 2N+1\}$ to each site in $[-N, N]$ and call $(z_1, \dots, z_{2N+1})$ the sites corresponding to the labels $1, \dots, 2N+1$. This is done independently of ω and by an abuse of notation we still denote by P the joint law of ω and the labeling. Since initially all sites are occupied by discrepancies, we may interpret z_i as the position at time 0 of the discrepancy with label i. In particular at time 0, the probability that $z_i = x$ is equal to $1/(2N+1)$. Given $\omega \in \Omega$, we follow the motion of the labeled discrepancies as described above and define accordingly the variables $z_i(t,\omega)$ which take values in $\{[-N, N] \cup \varnothing\}$. Thus, the set $Z(t,\omega)$ of all $z_i(t,\omega) \neq \varnothing$ is equal to $\{x : \eta_{\neq}(x,t;\omega) = 1\}$, so that

$$\begin{aligned} P\Big[\sum_x \eta_{\neq}(x,t) > 0\Big] &= P\big[\text{there is } i : z_i(t,\omega) \neq \varnothing\big] \le \sum_i P\big[z_i(t,\omega) \neq \varnothing\big] \\ &= (2N+1) P\big[z_1(t,\omega) \neq \varnothing\big], \end{aligned} \tag{5.1}$$

the last equality by symmetry.

Obviously, $P[z_1(t,\omega) \neq \varnothing]$ does not depend on the labels of the other z-particles so that we may and shall describe the system in terms of a random walk $z_t = z_1(t,\omega)$ in a random environment $\eta_t \in \{\neq, 0, 1\}^{[-N,N]\setminus z_t}$ when $z_t \neq \varnothing$ (i.e., it is alive); when $z_t = \varnothing$ then $\eta_t \in \{\neq, 0, 1\}^{[-N,N]}$, but since we want to study $P[z_1(t,\omega) \neq \varnothing]$ what happens after the death of z is not relevant.

We have reduced the problem to the analysis of the extinction time of a random walk in a random environment: the problem looks now very similar to the one considered in [7], the only difference being that the environment has a more complex structure with three rather than two states per site. But the procedure is essentially the same as we briefly sketch in the sequel.

6. The auxiliary random walk process

Once the initial condition (z, η^*) has been fixed, we can consider an auxiliary time dependent Markov process $(\tilde{z}_t)$ as in [7], whose extinction time has the same law as that of the true process $(z_1(t))$ of the previous section. The transition rates for $\tilde{z}_t$ are given by the conditional expectation of the transition rates of $(z_1(\cdot))$ conditioned on $z_1(t)$. Thus, they depend on the law of the full process, and hence on the initial datum (z, η^*). This time dependent generator $\mathcal{L}_t$ is given in (6.3) below, and satisfies

$$\tilde{E}_z\big[\mathcal{L}_t f(\tilde{z}_t)\big] = E_{z,\eta^*}\big[L\phi\big(z_1(t), \eta_t\big)\big] = \frac{\mathrm{d}}{\mathrm{d}t} E_{z,\eta^*}\big[\phi\big(z_1(t), \eta_t\big)\big],$$

where $\phi(z,\eta) = f(z)$ and $f : \Lambda_N \cup \varnothing \to \mathbb{R}$.

Since

$$\begin{aligned} L_r\phi = {} & \frac{j}{2N}\big\{\mathbf{1}_{z=N}\big(1-\eta_0(N-1)\big)\big[f(\varnothing) - f(N)\big] \\ & \quad + \mathbf{1}_{z=N-1}\eta_1(N)\big[f(\varnothing) - f(N-1)\big]\big\} \\ & + \frac{j}{2N}\mathbf{1}_{z=N}\eta_0(N-1)\big[f(N-1) - f(N)\big], \\ L_l\phi = {} & \frac{j}{2N}\big\{\mathbf{1}_{z=-N}\big(1-\eta_1(-N+1)\big)\big[f(\varnothing) - f(-N)\big] \\ & \quad + \mathbf{1}_{z=-N+1}\eta_0(-N)\big[f(\varnothing) - f(-N+1)\big]\big\} \\ & + \frac{j}{2N}\mathbf{1}_{z=-N}\eta_1(-N+1)\big[f(-N+1) - f(-N)\big], \end{aligned}$$

we set

$$\begin{aligned} d(N,t) &= \frac{j}{2N} E_{z_0,\eta^*}\big[1 - \eta_0(N-1,t) | z_t = N\big], \\ d(N-1,t) &= \frac{j}{2N} E_{z_0,\eta^*}\big[\eta_1(N,t) | z_t = N-1\big], \end{aligned}$$

$$
\begin{aligned}
d(-N,t) &= \frac{j}{2N} E_{z_0,\eta^*}\big[\big(1-\eta_1(-N+1,t)\big)|z_t=-N\big],\\
d(-N+1,t) &= \frac{j}{2N} E_{z_0,\eta_0}\big[\eta_0(-N,t)|z_t=-N+1\big],
\end{aligned} \tag{6.1}
$$

$$
\begin{aligned}
a(N,t) &= \frac{j}{2N} E_{z_0,\eta^*}\big[\eta_0(N-1,t)|z_t=N\big],\\
a(-N,t) &= \frac{j}{2N} E_{z_0,\eta^*}\big[\eta_1(-N+1,t)|z_t=-N\big],
\end{aligned} \tag{6.2}
$$

and $d(z,t)=0$ if $|z|<N-1$. Thus, for $t\geq 0$, we have

$$
\begin{aligned}
\mathcal{L}_t f(z) = \mathcal{L}^0 f(z) + d(z,t)\big[f(\varnothing)-f(z)\big] + \mathbf{1}_{z=N} a(N,t)\big[f(N-1)-f(N)\big]\\
+ \mathbf{1}_{z=-N} a(-N,t)\big[f(-N+1)-f(-N)\big].
\end{aligned} \tag{6.3}
$$

The process $\tilde{z}_t$ is a simple random walk with extra jumps from N to $N-1$ and $-N$ to $-N+1$ with time-dependent intensity $a(\pm N,t)$; moreover, it has death rate $d(z,t)$ (rate to go to $\varnothing$). Observe that

$$
d(z,t) \geq \frac{j}{2N} E_{z_0,\eta^*}\big[\eta_1(N-1,t)|z_t=N\big]\mathbf{1}_{z=N},
$$

and the analysis becomes very similar to the case treated in [7]. From the same argument leading to Theorem 1 therein, we have that for any initial configuration η^* and z_0:

$$
P\big[z_1(t)\neq\varnothing\big] \leq c\mathrm{e}^{-bN^{-2}t},
$$

which completes the proof.

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