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Time-varying network models

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We introduce the *exchangeable rewiring process* for modeling time-varying networks. The process fulfills fundamental mathematical and statistical properties and can be easily constructed from the novel operation of *random rewiring*. We derive basic properties of the model, including consistency under subsampling, exchangeability, and the Feller property. A reversible sub-family related to the Erdős–Rényi model arises as a special case.

Keywords: Aldous–Hoover theorem; consistency under subsampling; Erdős–Rényi random graph; exchangeable random graph; graph limit; partially exchangeable array

1. Introduction

A recent influx of academic monographs [9,11,12,19,21,28] and popular books [6,10,30] manifests a keen cultural and scientific interest in complex networks, which appeal to both applied and theoretical problems in national defense, sociology, epidemiology, computer science, statistics, and mathematics. The Erdős–Rényi random graph [13,14] remains the most widely studied network model. Its simple dynamics endow it with remarkable mathematical properties, but this simplicity overpowers any ability to replicate realistic structure. Many other network models have been inspired by empirical observations. Chief among these is the *scale-free* phenomenon, which has garnered attention since the initial observation of power law behavior for Internet statistics [16]. Celebrated is Barabási and Albert's preferential attachment model [7], whose dynamics are tied to the *rich get richer* or *Matthew effect*.[1] Citing overlooked attributes of network sampling schemes, other authors [20,31] have questioned the power law's apparent ubiquity. Otherwise, Watts and Strogatz [29] proposed a model that replicates Milgram's *small-world* phenomenon [25], the vernacular notion of *six degrees of separation* in social networks.

Networks arising in many practical settings are dynamic, they change with time. Consider a population $\{u_1, u_2, \ldots\}$ of individuals. For each $t \geq 0$, let $G_{ij}(t)$ indicate a social relationship between u_i and u_j and let $G_t := (G_{ij}(t))_{i,j\geq 1}$ comprise the indicators for the whole population at time t. For example, $G_{ij}(t)$ can indicate whether u_i and u_j are co-workers, friends, or family, have communicated by phone, email, or telegraph within the last week, month, or year, or subscribe to the same religious, political, or philosophical ideology. Within the narrow scope of social networks, the potential meanings of $G_{ij}(t)$ seem endless; expanding to other disciplines, the possible interpretations grow. In sociology, $\{G_t\}_{t\geq 0}$ records changes of social relationships

[1] "For to everyone who has will more be given, and he will have an abundance. But from the one who has not, even what he has will be taken away." (Matthew 25:29, *The Bible*, English Standard Version, 2001.)

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in a population; in other fields, the network dynamics reflect different phenomena and, therefore, can exhibit vastly different behaviors. In each case, $\{G_t\}_{t\geq 0}$ is a time-varying network.

Time-varying network models have been proposed previously in the applied statistics literature. The *Temporal Exponential Random Graph Model* (TERGM) in [17] incorporates temporal dependence into the *Exponential Random Graph Model* (ERGM). The authors highlight select properties of the TERGM, but consistency under subsampling is not among them. From the connection between sampling consistency and lack of interference, it is no surprise that the Exponential Random Graph Model is sampling consistent only under a choking restriction on its sufficient statistics [27]. McCullagh [23] argues unequivocally the importance of consistency for statistical models.

Presently, no network model both meets these logical requirements and reflects empirical observations. In this paper, rather than focus on a particular application, we discuss network modeling from first principles. We model time-varying networks by stochastic processes with a few natural invariance properties, specifically, exchangeable, consistent Markov processes.

The paper is organized as follows. In Section 2, we discuss first principles for modeling time-varying networks; in Section 3, we describe the rewiring process informally; in Section 4, we introduce the workhorse of the paper, the rewiring maps; in Sections 5 and 6, we discuss a family of time-varying network models in discrete-time; in Section 7, we extend to continuous-time; in Section 8, we show a Poisson point process construction for the rewiring process, and we use this technique to establish the Feller property; and in Section 9, we make some concluding remarks. We prove some technical lemmas and theorems in Section 10.

2. Modeling preliminaries

For now, we operate with the usual definition of a graph/network as a pair $G := (V, E)$ of vertices and edges. We delay formalities until they are needed.

Let $\mathbf{\Gamma} := \{\Gamma_t\}_{t\in T}$ be a random collection of graphs indexed by T, denoting *time*. We may think of $\mathbf{\Gamma}$ as a collection of social networks (for the same population) that changes as a result of social forces, for example, geographical relocation, broken relationships, new relationships, etc., but our discussion generalizes to other applications.

In practice, we can observe only a finite sample of individuals. Since the population size is often unknown, we assume an infinite population so that our model only depends on known quantities. Thus, each Γ_t is a graph with infinitely many vertices, of which we observe a finite sub-network $\Gamma_t^{[n]}$ with $n = 1, 2, \ldots$ vertices. Since the vertex labels play no role, we always assume sampled graphs have vertex set $[n] := \{1, \ldots, n\}$, where n is the sample size, and the population graph is infinite with vertex set $\mathbb{N} := \{1, 2, \ldots\}$, the natural numbers.

The models we consider are *Markovian*, *exchangeable*, and *consistent*.

2.1. The Markov property

The process $\mathbf{\Gamma}$ has the *Markov property* if, for every $t > 0$, its pre-t and post-t σ-fields are conditionally independent given the present state Γ_t. Put another way, the current state Γ_t incorporates all past and present information about the process, and so the future evolution depends on $\sigma\langle\Gamma_s\rangle_{s\le t}$ only through Γ_t.

It is easy to conceive of counterarguments to this assumption: in a social network, suppose there is no edge between individuals i and i' or between j and j' at time $t > 0$. Then, informally,[2] we expect the future (marginal) evolution of edges ii' and jj' to be identically distributed. But if, in the past, i and i' have been frequently connected and j and j' have not, we might infer that the latent relationships among these individuals are different and, thus, their corresponding edges should evolve differently. For instance, given their past behavior, we might expect that i and i' are more likely than j and j' to reconnect in the future.

Despite such counterarguments, the Markov property is widely used and works well in practice. Generalizations to the Markov property may be appropriate for specific applications, but they run the risk of overfitting.

2.2. Exchangeability

Structure and changes to structure drive our study of networks. Vertex labels carry no substantive meaning other than to keep track of this structure over time; thus, a suitable model is *exchangeable*, that is, its distributions are invariant under relabeling of the vertices.

For a model on finite networks (i.e., finitely many vertices), exchangeability can be induced trivially by averaging uniformly over all permutations of the vertices. But we assume an infinite population, for which the appropriate invariance is *infinite exchangeability*, the combination of exchangeability and consistency under subsampling (Section 2.3). Unlike the finite setting, infinite exchangeability cannot be imposed arbitrarily by averaging; it must be an inherent feature of the model.

2.3. Markovian consistency under subsampling

For any graph with vertex set V, there is a natural and obvious restriction to an induced subgraph with vertex set $V' \subset V$ by removing all vertices and edges that are not fully contained in V'. The assumption of *Markovian consistency*, or simply *consistency*, for a graph-valued Markov process implies that, for every $n \in \mathbb{N}$, the restriction $\mathbf{\Gamma}^{[n]}$ of $\mathbf{\Gamma}$ to the space of graphs with vertex set $[n]$ is, itself, a Markov process. Note that this property does not follow immediately from the Markov assumption for $\mathbf{\Gamma}$ because the restriction operation is a many-to-one function and, in general, a function of a Markov process need not be Markov. Also note that the behavior of the restriction $\mathbf{\Gamma}^{[n]}$ can depend on $\mathbf{\Gamma}$ through as much as its exchangeable σ-field, which depends only on the "tail" of the process.

Markovian consistency may be unjustified in some network modeling applications. This contrasts with other combinatorial stochastic process models, for example, coalescent processes [18], for which consistency is induced by an inherent *lack of interference* in the underlying scientific phenomena. Nevertheless, if we assume the network is a sample from a larger network, then

[2] We are implicitly ignoring the dependence between ii' and jj' for the sake of illustration.

consistency permits out-of-sample statistical inference [23]. Without Markovian consistency in a time-varying Markov model, the sampled process can depend on the whole (unobserved) process, leaving little hope for meaningful inference.

3. Rewiring processes: Informal description

We can envision at least two kinds of network dynamics that correspond, intuitively, to *local* and *global* structural changes. Local changes involve *only one* edge, global changes involve a *positive fraction* of edges. We say the *status* of edge ij is *on* if there is an edge between i and j; otherwise, we say the status is *off*.

A local change occurs whenever the status of exactly one edge changes, called a *single-edge update*. An easy way to generate single-edge updates is by superposition of independent rate-1 Poisson processes. For each pair $i < j$, we let $\{T_k^{ij}\}_{k\geq 1}$ be the arrival times of a rate-1 Poisson point process. At each arrival time, the status of the edge between i and j changes (either from 'off' to 'on' or the reverse). Doing this independently for each pair results in an infinite number of changes to the network in any arbitrary time interval, but only finitely many changes within each finite subnetwork. We call a process with this description a *local-edge process*; see Section 7.3.

A global change occurs whenever the status of a positive proportion of edges changes simultaneously. In practice, such an event might indicate a major external disturbance within the population, for example, spread or fear of a pandemic. Modeling such processes in continuous-time requires more preparation than the local-edge process.

For an example, consider generating a discrete-time Markov chain $\mathbf{\Gamma} := \{\Gamma_m\}_{m=0,1,2,\ldots}$ on the finite space of graphs with vertex set $[n]$. At any time m, given $\Gamma_m = G$, we can generate a transition to a new state G' as follows. Independently for each pair $i < j$, we flip a coin to determine whether to put an edge between i and j in G': if ij is *on* in G, we flip a p_1-coin; otherwise, we flip a p_0-coin. This description results in a simple, exchangeable Markov chain on finite graphs, which we call the *Erdős–Rényi rewiring chain* (Section 5.1). More general transitions are possible, for example, edges need not evolve independently. We use the next Markov chain as a running example of a discrete-time rewiring chain.

3.1. A reversible Markov chain on graphs

We fix $n \in \mathbb{N}$ and regard an undirected graph G with vertex set $[n]$ as a $\{0, 1\}$-valued symmetric matrix $(G_{ij})_{1\leq i,j\leq n}$ such that $G_{ii} = 0$ for all $i = 1, \ldots, n$; that is, we represent a graph by its adjacency matrix with $G_{ij} := \mathbf{1}\{ij \text{ is on}\}$. For any pair of graphs (G, G'), we can compute the statistic $\mathbf{n} := \mathbf{n}(G, G') := (n_{00}, n_{01}, n_{10}, n_{11})$, where for $r, s = 0, 1$,

$$n_{rs} := \sum_{1\leq i<j\leq n} \mathbf{1}\{G_{ij} = r \text{ and } G'_{ij} = s\}.$$

For example, n_{01} is the number of pairs i, j for which the status of ij changes from 0 to 1 from G to G'. We use $\mathbf{n}$ as a sufficient statistic to define the transition probability

$$P_{\alpha,\beta}^{(n)}(G, G') := \frac{\alpha^{\uparrow n_{00}}\beta^{\uparrow n_{01}}\alpha^{\uparrow n_{11}}\beta^{\uparrow n_{10}}}{(\alpha+\beta)^{\uparrow(n_{01}+n_{00})}(\alpha+\beta)^{\uparrow(n_{10}+n_{11})}},$$

where $\alpha^{\uparrow j} := \alpha(\alpha+1)\cdots(\alpha+j-1)$ and $\alpha, \beta > 0$.

The sufficient statistic $\mathbf{n}$ is invariant under joint relabeling of the vertices of (G, G') and so the transition law is exchangeable. Furthermore, $P_{\alpha,\beta}^{(n)}$ is reversible with respect to

$$\varepsilon_{\alpha+\beta,\alpha+\beta}^{(n)}(G) := \frac{(\alpha+\beta)^{\uparrow n_0}(\alpha+\beta)^{\uparrow n_1}}{(2\alpha+2\beta)^{\uparrow n}},$$

where $n_r := \sum_{1\le i<j\le n} \mathbf{1}\{G_{ij} = r\}$, $r = 0, 1$. The distribution $\varepsilon_{\alpha,\beta}^{(n)}$ arises as a mixture of Erdős–Rényi random graphs with respect to the Beta(α, β) distribution. Furthermore, $\{P_{\alpha,\beta}^{(n)}\}_{n\in\mathbb{N}}$ is a consistent collection of transition probabilities and, therefore, determines a unique transition probability (and hence Markov chain) on the space of infinite graphs with vertex set $\mathbb{N}$.

Though consistency is not immediately obvious for the above family, the savvy reader might anticipate it: the formula for $P_{\alpha,\beta}^{(n)}$ involves rising factorials (i.e., Gamma functions), which also appear in other consistent combinatorial stochastic processes, for example, the Chinese restaurant process [26] and the Beta-splitting model for fragmentation trees [3,24]. We need not prove consistency explicitly for this model; it follows from our more general construction of rewiring processes, all of which are consistent (Theorem 5.1). We discuss the above family further in Section 5.1.

3.2. A more general construction

Throughout the paper, we construct exchangeable and consistent Markov processes using a special *rewiring measure* (Section 6). In continuous-time, Markov processes can admit infinitely many jumps in arbitrarily small time intervals; however, by the consistency assumption, any edge can change only finitely often in bounded intervals. In this case, we can choose a σ-finite rewiring measure to direct the process.

4. Preliminaries and the rewiring maps

For $n = 1, 2, \ldots$, an (undirected) *graph* G with vertex set $[n]$ can be represented by its symmetric *adjacency matrix* $(G_{ij})_{1\le i,j\le n}$ for which $G_{ij} = 1$ if G has an edge between i and j, and $G_{ij} = 0$ otherwise. By convention, we always assume $G_{ii} = 0$ for all $i = 1, \ldots, n$. We write $\mathcal{G}_n$ to denote the finite collection of all graphs with vertex set $[n]$.

On $\mathcal{G}_n$, we define the following operation of *rewiring*. Let $w := (w_{ij})_{1\le i,j\le n}$ be an $n \times n$ symmetric matrix with entries in $\{0, 1\} \times \{0, 1\}$ and all diagonal entries $(0, 0)$. For convenience, we write each entry of w as a pair $w_{ij} := (w_{ij}^0, w_{ij}^1)$, $1 \le i, j \le n$. We define a map $w : \mathcal{G}_n \to \mathcal{G}_n$

by $G \mapsto G' := w(G)$, where

$$G'_{ij} := \begin{cases} w^0_{ij}, & G_{ij} = 0, \\ w^1_{ij}, & G_{ij} = 1, \end{cases} \qquad 1 \le i, j \le n. \tag{4.1}$$

More compactly, we may write $w(G) := (w_{ij}^{G_{ij}})_{1 \le i,j \le n}$. We call w a *rewiring map* and $w(G)$ the *rewiring* of G by w. We write $\mathcal{W}_n$ to denote the collection of all rewiring maps $\mathcal{G}_n \to \mathcal{G}_n$, which are in one-to-one correspondence with $n \times n$ symmetric matrices with entries in $\{0, 1\} \times \{0, 1\}$ and all diagonal entries $(0, 0)$.

The following display illustrates the rewiring operation in (4.1). Given $G \in \mathcal{G}_n$ and $w \in \mathcal{W}_n$, we obtain $w(G)$ by choosing the appropriate element of each entry of w: if $G_{ij} = 0$, we choose the *left* coordinate of w_{ij}; if $G_{ij} = 1$, we choose the *right* coordinate of w_{ij}. For example,

$$\begin{array}{c} G \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} w \\ \begin{pmatrix} (\underline{\mathbf{0}},0) & (1,\underline{\mathbf{0}}) & (0,\underline{\mathbf{1}}) & (\underline{\mathbf{0}},0) & (0,\underline{\mathbf{1}}) \\ (1,\underline{\mathbf{0}}) & (\underline{\mathbf{0}},0) & (\underline{\mathbf{1}},0) & (\underline{\mathbf{1}},1) & (1,\underline{\mathbf{0}}) \\ (0,\underline{\mathbf{1}}) & (\underline{\mathbf{1}},0) & (\underline{\mathbf{0}},0) & (0,\underline{\mathbf{1}}) & (\underline{\mathbf{0}},0) \\ (\underline{\mathbf{0}},0) & (\underline{\mathbf{1}},1) & (0,\underline{\mathbf{1}}) & (\underline{\mathbf{0}},0) & (\underline{\mathbf{1}},0) \\ (0,\underline{\mathbf{1}}) & (1,\underline{\mathbf{0}}) & (\underline{\mathbf{0}},0) & (\underline{\mathbf{1}},0) & (\underline{\mathbf{0}},0) \end{pmatrix} \end{array} \mapsto \begin{array}{c} w(G) \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

A unique symmetric $n \times n$ matrix determines each element in $\mathcal{G}_n$ and $\mathcal{W}_n$, and so there is a natural restriction operation on both spaces by taking the leading $m \times m$ submatrix, for any $m \le n$. In particular, we write

$$\begin{aligned} \mathbf{R}_{m,n} G &:= G|_{[m]} := (G_{ij})_{1 \le i,j \le m}, \qquad G \in \mathcal{G}_n, \quad \text{and} \\ w|_{[m]} &:= (w_{ij})_{1 \le i,j \le m}, \qquad w \in \mathcal{W}_n, \end{aligned} \tag{4.2}$$

to denote the restrictions of $G \in \mathcal{G}_n$ and $w \in \mathcal{W}_n$ to $\mathcal{G}_m$ and $\mathcal{W}_m$, respectively. These restriction operations lead to the notions of *infinite graphs* and *infinite rewiring maps* as infinite symmetric arrays with entries in the appropriate space, either $\{0, 1\}$ or $\{0, 1\} \times \{0, 1\}$. We write $\mathcal{G}_\infty$ to denote the space of infinite graphs, identified by a $\{0, 1\}$-valued *adjacency array*, and $\mathcal{W}_\infty$ to denote the space of infinite rewiring maps, identified by a symmetric $\{0, 1\} \times \{0, 1\}$-valued array with $(0, 0)$ on the diagonal.

Any $w \in \mathcal{W}_\infty$ acts on $\mathcal{G}_\infty$ just as in (4.1) and, for any $G \in \mathcal{G}_\infty$, the rewiring operation satisfies

$$w(G)|_{[n]} = w|_{[n]}(G|_{[n]}) \qquad \text{for all } n \in \mathbb{N}.$$

The spaces $\mathcal{G}_\infty$ and $\mathcal{W}_\infty$ are uncountable but can be equipped with the discrete σ-algebras $\sigma\langle\bigcup_{n\in\mathbb{N}} \mathcal{G}_n\rangle$ and $\sigma\langle\bigcup_{n\in\mathbb{N}} \mathcal{W}_n\rangle$, respectively, so that the restriction maps $\cdot|_{[n]}$ are measurable for every $n \in \mathbb{N}$. Moreover, both $\mathcal{G}_\infty$ and $\mathcal{W}_\infty$ come equipped with a product-discrete topology induced, for example, by the ultrametric

$$d(w, w') := 1/\max\{n \in \mathbb{N} \colon w|_{[n]} = w'|_{[n]}\}, \qquad w, w' \in \mathcal{W}_\infty. \tag{4.3}$$

The metric on $\mathcal{G}_\infty$ is analogous. Both $\mathcal{G}_\infty$ and $\mathcal{W}_\infty$ are compact, complete, and separable metric spaces. Much of our development hinges on the following proposition, whose proof is straightforward.

Proposition 4.1. *Rewiring maps are associative under composition and Lipschitz continuous in the metric* (4.3), *with Lipschitz constant* 1.

4.1. Weakly exchangeable arrays

Let $\mathcal{S}_\mathbb{N}$ denote the collection of *finite* permutations of $\mathbb{N}$, that is, permutations $\sigma : \mathbb{N} \to \mathbb{N}$ for which $\#\{i \in \mathbb{N}: \sigma(i) \neq i\} < \infty$. We call any random array $X := (X_{ij})_{i,j\geq 1}$ *weakly exchangeable* if

$$X \text{ is almost surely symmetric, that is, } X_{ij} = X_{ji} \text{ for all } i, j \text{ with probability one,}$$

and

$$X =_\mathcal{L} X^\sigma := (X_{\sigma(i)\sigma(j)})_{i,j\geq 1} \qquad \text{for all finite permutations } \sigma : \mathbb{N} \to \mathbb{N},$$

where $=_\mathcal{L}$ denotes *equality in law*. Aldous defines weak exchangeability using only the latter condition; see [2], Chapter 14, page 132. We impose symmetry for convenience – in this paper, all graphs and rewiring maps are symmetric arrays.

From the discussion in Section 2.2, we are interested in models for random graphs Γ that are *exchangeable*, meaning the adjacency matrix $(\Gamma_{ij})_{i,j\geq 1}$ is a weakly exchangeable $\{0, 1\}$-valued array. Likewise, we call a random rewiring map W *exchangeable* if its associated $\{0, 1\} \times \{0, 1\}$-valued array $(W_{ij})_{i,j\geq 1}$ is weakly exchangeable.

de Finetti's theorem represents any infinitely exchangeable sequence $Z := (Z_i)_{i\geq 1}$ in a Polish space $\mathcal{S}$ with a (non-unique) measurable function $g : [0, 1]^2 \to \mathcal{S}$ such that $Z =_\mathcal{L} Z^*$, where

$$Z_i^* := g(\alpha, \eta_i), \qquad i \geq 1, \tag{4.4}$$

for $\{\alpha; (\eta_i)_{i\geq 1}\}$ independent, identically distributed (i.i.d.) Uniform random variables on $[0, 1]$. The Aldous–Hoover theorem [1,2] extends de Finetti's representation (4.4) to weakly exchangeable $\mathcal{S}$-valued arrays: to any such array X, there exists a (non-unique) measurable function $f : [0, 1]^4 \to \mathcal{S}$ satisfying $f(\bullet, b, c, \bullet) = f(\bullet, c, b, \bullet)$ such that $X =_\mathcal{L} X^*$, where

$$X_{ij}^* := f(\alpha, \eta_i, \eta_j, \lambda_{\{i,j\}}), \qquad i, j \geq 1, \tag{4.5}$$

for $\{\alpha; (\eta_i)_{i\geq 1}; (\lambda_{\{i,j\}})_{i>j\geq 1}\}$ i.i.d. Uniform random variables on $[0, 1]$.

The function f has a statistical interpretation that reflects the structure of the random array. In particular, f decomposes the law of X_{ij}^* into individual $\lambda_{\{i,j\}}$, row η_i, column η_j, and overall α effects. The overall effect plays the role of the mixing measure in the de Finetti interpretation. If g in (4.4) is constant with respect to its first argument, that is, $g(a, \cdot) = g(a', \cdot)$ for all $a, a' \in [0, 1]$, then Z^* constructed in (4.4) is an i.i.d. sequence. Letting g vary with its first argument produces a mixture of i.i.d. sequences. A fundamental interpretation of de Finetti's theorem is:

every infinitely exchangeable sequence is a mixture of i.i.d. sequences.

Similarly, if f in (4.5) satisfies $f(a,\cdot,\cdot,\cdot) = f(a',\cdot,\cdot,\cdot)$ for all $a, a' \in [0,1]$, then X^* is *dissociated*, that is

$$X^*|_{[n]} \text{ is independent of } X^*|_{\{n+1,n+2,\dots\}} \text{ for all } n \in \mathbb{N}. \tag{4.6}$$

The Aldous–Hoover representation (4.5) spurs the sequel to de Finetti's interpretation:

every weakly exchangeable array is a mixture of dissociated arrays.

See Aldous [2], Chapter 14, for more details. We revisit the theory of weakly exchangeable arrays in Section 6.

5. Discrete-time rewiring Markov chains

Throughout the paper, we use the rewiring maps to construct Markov chains on $\mathcal{G}_\infty$. From any probability distribution ω_n on $\mathcal{W}_n$, we generate $W_1, W_2, \dots$ i.i.d. from ω_n and a random graph $\Gamma_0 \in \mathcal{G}_n$ (independently of $W_1, W_2, \dots$). We then define a Markov chain $\{\Gamma_m\}_{m=0,1,2,\dots}$ on $\mathcal{G}_n$ by

$$\Gamma_m := W_m(\Gamma_{m-1}) = (W_m \circ \cdots \circ W_1)(\Gamma_0), \qquad m \geq 1. \tag{5.1}$$

We call ω_n *exchangeable* if $W \sim \omega_n$ is an exchangeable rewiring map, that is, $W =_{\mathcal{L}} W^\sigma$ for all permutations $\sigma : [n] \to [n]$.

Proposition 5.1. *Let ω_n be an exchangeable probability measure on $\mathcal{W}_n$ and let $\mathbf{\Gamma} := \{\Gamma_m\}_{m=0,1,2,\dots}$ be as constructed in (5.1) from an exchangeable initial state Γ_0 and $W_1, W_2, \dots$ i.i.d. from ω_n. Then $\mathbf{\Gamma}$ is an exchangeable Markov chain on $\mathcal{G}_n$ with transition probability*

$$P_{\omega_n}(G, G') := \omega_n\bigl(\{W \in \mathcal{W}_n : W(G) = G'\}\bigr), \qquad G, G' \in \mathcal{G}_n. \tag{5.2}$$

Proof. The Markov property is immediate by mutual independence of $\Gamma_0, W_1, W_2, \dots$. The formula for the transition probabilities (5.2) follows by description (5.1) of $\mathbf{\Gamma}$.

We need only show that $\mathbf{\Gamma}$ is exchangeable. By assumption, Γ_0 is an exchangeable random graph on n vertices, and so its distribution is invariant under arbitrary permutation of $[n]$. Moreover, the law of $W \sim \omega_n$ satisfies $W =_{\mathcal{L}} W^\sigma$ and, for any fixed $w \in \mathcal{W}_n$ and $\sigma \in \mathcal{S}_n$, $G' := w(G)$ satisfies

$$G'^{\sigma}_{ij} := G'_{\sigma(i)\sigma(j)} = w_{\sigma(i)\sigma(j)}^{G_{\sigma(i)\sigma(j)}},$$

the ij-entry of $w^\sigma(G^\sigma)$. Therefore, $W^\sigma(G^\sigma) = W(G)^\sigma$ and, for any exchangeable graph Γ and exchangeable rewiring map W, we have

$$W(\Gamma)^\sigma = W^\sigma\bigl(\Gamma^\sigma\bigr) =_{\mathcal{L}} W(\Gamma) \qquad \text{for all } \sigma \in \mathcal{S}_n.$$

Hence, the transition law of $\mathbf{\Gamma}$ is equivariant with respect to relabeling. Since the initial state Γ_0 is exchangeable, so is the Markov chain. □

Definition 5.1. *We call $\{\Gamma_m\}_{m=0,1,2,\dots}$ an ω_n*-rewiring Markov chain.

From the discussion in Section 4, we can define an exchangeable measure $\omega^{(n)}$ on $\mathcal{W}_n$ as the restriction to $\mathcal{W}_n$ of an exchangeable probability measure ω on $\mathcal{W}_\infty$, where

$$\omega^{(n)}(W) := \omega\big(\big\{W^* \in \mathcal{W}_\infty\colon W^*|_{[n]} = W\big\}\big), \qquad W \in \mathcal{W}_n. \tag{5.3}$$

Denote by $P_\omega^{(n)}$ the transition probability measure of an $\omega^{(n)}$-rewiring Markov chain on $\mathcal{G}_n$, as defined in (5.2).

Theorem 5.1. *For any exchangeable probability measure ω on $\mathcal{W}_\infty$, $\{P_\omega^{(n)}\}_{n\in\mathbb{N}}$ is a consistent family of exchangeable transition probabilities in the sense that*

$$P_\omega^{(m)}\big(G, G'\big) = P_\omega^{(n)}\big(G^*, \mathbf{R}_{m,n}^{-1}\big(G'\big)\big), \qquad G, G' \in \mathcal{G}_m, \qquad \textit{for all } m \le n, \tag{5.4}$$

for every $G^* \in \mathbf{R}_{m,n}^{-1}(G) := \{G'' \in \mathcal{G}_n\colon G''|_{[m]} = G\}$, where $\mathbf{R}_{m,n}$ is defined in (4.2).

Proof. Proposition 5.1 implies exchangeability of $\{P_\omega^{(n)}\}_{n\in\mathbb{N}}$. It remains to show that $\{P_\omega^{(n)}\}_{n\in\mathbb{N}}$ satisfies (5.4). By (5.2),

$$P_\omega^{(n)}\big(G, G'\big) := \omega^{(n)}\big(\big\{W \in \mathcal{W}_n\colon W(G) = G'\big\}\big), \qquad G, G' \in \mathcal{G}_n.$$

Now, for any $m \le n$, fix $G, G' \in \mathcal{G}_m$ and $G^* \in \mathbf{R}_{m,n}^{-1}(G)$. Then (5.4) requires

$$\omega^{(n)}\big(\big\{W \in \mathcal{W}_n\colon W\big(G^*\big) \in \mathbf{R}_{m,n}^{-1}\big(G'\big)\big\}\big) = \omega^{(m)}\big(\big\{W \in \mathcal{W}_m\colon W(G) = G'\big\}\big),$$

which follows by definition (5.3) of $\omega^{(n)}$. To see this, note that

$$\begin{aligned}
\omega^{(n)}\big(\big\{W \in \mathcal{W}_n\colon W|_{[m]}(G) = G'\big\}\big) &= \omega\big(\big\{W \in \mathcal{W}_\infty\colon (W|_{[n]})|_{[m]}(G) = G'\big\}\big) \\
&= \omega\big(\big\{W \in \mathcal{W}_\infty\colon W|_{[m]}(G) = G'\big\}\big) \\
&= \omega^{(m)}\big(\big\{W \in \mathcal{W}_m\colon W(G) = G'\big\}\big).
\end{aligned}$$

This completes the proof. □

Remark 5.1. The consistency condition (5.4) for Markov chains is exactly the necessary and sufficient condition for a function of a Markov chain to be a Markov chain, as proven in [8]. Before describing the measure ω from Theorem 5.1 in further detail, we first show some concrete examples of rewiring chains.

5.1. The Erdős–Rényi rewiring chain

For any $0 \le p \le 1$, let ε_p denote the Erdős–Rényi measure on $\mathcal{G}_\infty$, which we define by its finite-dimensional restrictions $\varepsilon_p^{(n)}$ to $\mathcal{G}_n$ for each $n \in \mathbb{N}$,

$$\varepsilon_p^{(n)}(G) := \prod_{1\le i<j\le n} p^{G_{ij}}(1-p)^{1-G_{ij}}, \qquad G \in \mathcal{G}_n. \tag{5.5}$$

Given any pair $(p_0, p_1) \in [0,1] \times [0,1]$, the (p_0, p_1)-*Erdős–Rényi chain* has finite-dimensional transition probabilities

$$P^{(n)}_{p_0,p_1}(G, G') := \prod_{1 \le i < j \le n} p_{G_{ij}}^{G'_{ij}} (1 - p_{G_{ij}})^{1 - G'_{ij}}, \qquad G, G' \in \mathcal{G}_n. \tag{5.6}$$

Proposition 5.2. *For $0 < p_0, p_1 < 1$, the (p_0, p_1)-Erdős–Rényi rewiring chain has unique stationary distribution ε_q, with $q := p_0/(1 - p_1 + p_0)$.*

Proof. By assumption, both p_0 and p_1 are strictly between 0 and 1 and, thus, (5.5) assigns positive probability to every transition in $\mathcal{G}_n$, for every $n \in \mathbb{N}$. Therefore, each finite-dimensional chain is aperiodic and irreducible, and each possesses a unique stationary distribution $\theta^{(n)}$. By consistency of the transition probabilities $\{P^{(n)}_{p_0,p_1}\}_{n \in \mathbb{N}}$ (Theorem 5.1), the finite-dimensional stationary measures $\{\theta^{(n)}\}_{n \in \mathbb{N}}$ must be exchangeable and consistent and, therefore, they determine a unique measure θ on $\mathcal{G}_\infty$, which is stationary for P_{p_0,p_1}. Furthermore, by conditional independence of the edges of G', given G, the stationary law must be Erdős–Rényi with some parameter $q \in (0, 1)$.

In an $\varepsilon^{(n)}_q$-random graph, all edges are present or not independently with probability q. Therefore, it suffices to look at the probability of the edge between vertices labeled 1 and 2. In this case, we need to choose q so that

$$qp_1 + (1 - q)p_0 = q,$$

which implies $q = p_0/(1 - p_1 + p_0)$. □

Remark 5.2. Some elementary special cases of the (p_0, p_1)-Erdős–Rényi rewiring chain are worth noting. First, for (p_0, p_1) either $(0, 0)$ or $(1, 1)$, this chain is degenerate at either the empty graph $\mathbf{0}$ or the complete graph $\mathbf{1}$ and has unique stationary measure ε_0 or ε_1, respectively. On the other hand, when $(p_0, p_1) = (0, 1)$, the chain is degenerate at its initial state and so its initial distribution is stationary. However, if $(p_0, p_1) = (1, 0)$, then the chain alternates between its initial state G and its complement $\bar{G} := (\bar{G}_{ij})_{i,j \ge 1}$, where $\bar{G}_{ij} := 1 - G_{ij}$ for all $i, j \ge 1$; in this case, the chain is periodic and does not have a unique stationary distribution. We also note that when $(p_0, p_1) = (p, p)$ for some $p \in (0, 1)$, the chain is simply an i.i.d. sequence of ε_p-random graphs with stationary distribution ε_q, where $q = p/(1 - p + p) = p$, as it must.

For $\alpha, \beta > 0$, we define the *mixed Erdős–Rényi rewiring chain* through $\varepsilon^{(n)}_{\alpha,\beta}$, the mixture of $\varepsilon^{(n)}_p$-laws with respect to the Beta law with parameter (α, β). Writing

$$\mathscr{B}_{\alpha,\beta}(\mathrm{d}p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha - 1}(1 - p)^{\beta - 1}\,\mathrm{d}p,$$

we derive

$$\begin{aligned}\varepsilon^{(n)}_{\alpha,\beta}(G) &:= \int_{[0,1]} \varepsilon^{(n)}_p(G) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\,\mathrm{d}p \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+n_1)\Gamma(\beta+n_0)}{\Gamma(\alpha+\beta+n)} \int_{[0,1]} \mathcal{B}_{\alpha+n_1,\beta+n_0}(\mathrm{d}p) \\ &= \frac{\alpha^{\uparrow n_1}\beta^{\uparrow n_0}}{(\alpha+\beta)^{\uparrow n}},\end{aligned}$$

where $n_r := \sum_{1\le i<j\le n}\{G_{ij} = r\}$, $r = 0, 1$, and $\alpha^{\uparrow n} = \alpha(\alpha+1)\cdots(\alpha+n-1)$. For $\alpha_0, \beta_0, \alpha_1, \beta_1 > 0$, we define *mixed Erdős–Rényi transition probabilities* by

$$P^{(n)}_{(\alpha_0,\beta_0),(\alpha_1,\beta_1)}\bigl(G, G'\bigr) := \frac{\alpha_0^{\uparrow n_{01}}\beta_0^{\uparrow n_{00}}\alpha_1^{\uparrow n_{11}}\beta_1^{\uparrow n_{10}}}{(\alpha_0+\beta_0)^{\uparrow(n_{00}+n_{01})}(\alpha_1+\beta_1)^{\uparrow(n_{10}+n_{11})}}, \qquad G, G' \in \mathcal{G}_n. \tag{5.7}$$

An interesting special case takes $(\alpha_0, \beta_0) = (\beta, \alpha)$ and $(\alpha_1, \beta_1) = (\alpha', \beta)$ for $\alpha, \alpha', \beta > 0$. In this case, (5.7) becomes

$$P^{(n)}_{(\beta,\alpha),(\alpha',\beta)}\bigl(G, G'\bigr) = \frac{\alpha^{\uparrow n_{00}}\beta^{\uparrow n_{01}}\alpha'^{\uparrow n_{11}}\beta^{\uparrow n_{10}}}{(\alpha+\beta)^{\uparrow n_0}(\alpha'+\beta)^{\uparrow n_1}}, \qquad G, G' \in \mathcal{G}_n.$$

Proposition 5.3. *$P^{(n)}_{(\beta,\alpha),(\alpha',\beta)}$ is reversible with respect to $\varepsilon^{(n)}_{\alpha+\beta,\alpha'+\beta}$.*

Proof. For fixed $G, G' \in \mathcal{G}_n$, we write $n_{rs} := \sum_{i<j}\mathbf{1}\{G_{ij} = r \text{ and } G'_{ij} = s\}$ and $n'_{rs} := \sum_{i<j}\mathbf{1}\{G'_{ij} = r \text{ and } G_{ij} = s\}$. Note that $n'_{rs} = n_{sr}$. Therefore, we have

$$\begin{aligned}\varepsilon^{(n)}_{\alpha+\beta,\alpha'+\beta}(G)P^{(n)}_{(\beta,\alpha),(\alpha',\beta)}\bigl(G, G'\bigr) &= \frac{\alpha^{\uparrow n_{00}}\beta^{\uparrow n_{01}}\alpha'^{\uparrow n_{11}}\beta^{\uparrow n_{10}}}{(\alpha+2\beta+\alpha')^{\uparrow n}} \\ &= \frac{\alpha^{\uparrow n'_{00}}\beta^{\uparrow n'_{10}}\alpha'^{\uparrow n'_{11}}\beta^{\uparrow n'_{01}}}{(\alpha+2\beta+\alpha')^{\uparrow n}} \\ &= \varepsilon^{(n)}_{\alpha+\beta,\alpha'+\beta}\bigl(G'\bigr)P^{(n)}_{(\beta,\alpha),(\alpha',\beta)}\bigl(G', G\bigr),\end{aligned}$$

establishing detailed balance and, thus, reversibility. □

A mixed Erdős–Rényi Markov chain is directed by

$$\omega(\mathrm{d}W) := \int_{[0,1]\times[0,1]} \omega_{p_0,p_1}(\mathrm{d}W)(\mathcal{B}_{\alpha_0,\beta_0}\otimes\mathcal{B}_{\alpha_1,\beta_1})(\mathrm{d}p_0, \mathrm{d}p_1), \qquad W \in \mathcal{W}_\infty,$$

where ω_{p_0,p_1} is determined by its finite-dimensional distributions

$$\omega^{(n)}_{p_0,p_1}(W) := \prod_{1\le i<j\le n} p_0^{W^0_{ij}}(1-p_0)^{1-W^0_{ij}} p_1^{W^1_{ij}}(1-p_1)^{1-W^1_{ij}}, \qquad W \in \mathcal{W}_n,$$

for $0 < p_0, p_1 < 1$, for every $n \in \mathbb{N}$.

In the next section, we see that a representation of the directing measure ω as a mixture of simpler measures holds more generally. Notice that $W \sim \omega_{p_0,p_1}$ is dissociated for all fixed $(p_0, p_1) \in (0,1) \times (0,1)$. By the Aldous–Hoover theorem, we can express any exchangeable measure on $\mathcal{W}_\infty$ as a mixture of dissociated measures.

6. Exchangeable rewiring maps and their rewiring limits

To more precisely describe the mixing measure ω, we extend the theory of *graph limits* to its natural analog for rewiring maps. We first review the related theory of graph limits, as surveyed by Lovász [21].

6.1. Graph limits

A graph limit is a statistic that encodes a lot of structural information about an infinite graph. In essence, the graph limit of an exchangeable random graph contains all relevant information about its distribution.

For any injection $\psi : [m] \to [n]$, $m \le n$, and $G \in \mathcal{G}_n$, we define $G^\psi := (G_{\psi(i)\psi(j)})_{1 \le i,j \le m}$. In words, G^ψ is the subgraph G induces on $[m]$ by the vertices in the range of ψ. Given $G \in \mathcal{G}_n$ and $F \in \mathcal{G}_m$, we define $\operatorname{ind}(F, G)$ to equal the number of injections $\psi : [m] \to [n]$ such that $G^\psi = F$. Intuitively, $\operatorname{ind}(F, G)$ is the number of "copies" of F in G, which we normalize to obtain the *density* of F in G,

$$t(F, G) := \frac{\operatorname{ind}(F, G)}{n^{\downarrow m}}, \qquad F \in \mathcal{G}_m, G \in \mathcal{G}_n, \tag{6.1}$$

where $n^{\downarrow m} := n(n-1)\cdots(n-m+1)$ is the number of unique injections $\psi : [m] \to [n]$. The *limiting density* of F in any infinite graph $G \in \mathcal{G}_\infty$ is

$$t(F, G) := \lim_{n\to\infty} t(F, G|_{[n]}), \qquad F \in \mathcal{G}_m, \qquad \text{if it exists.} \tag{6.2}$$

The collection $\mathcal{G}^* := \bigcup_{m\in\mathbb{N}} \mathcal{G}_m$ is countable and so we can define the *graph limit* of $G \in \mathcal{G}_\infty$ by

$$|G| := \bigl(t(F, G)\bigr)_{F\in\mathcal{G}^*}, \tag{6.3}$$

provided $t(F, G)$ exists for all $F \in \mathcal{G}^*$. Any graph limit is an element in $[0,1]^{\mathcal{G}^*}$, which is compact under the metric

$$\rho\bigl(x, x'\bigr) := \sum_{n\in\mathbb{N}} 2^{-n} \sum_{F\in\mathcal{G}_n} \bigl|x_F - x'_F\bigr|, \qquad x, x' \in [0,1]^{\mathcal{G}^*}. \tag{6.4}$$

The space of graph limits is a compact subset of $[0,1]^{\mathcal{G}^*}$, which we denote by $\mathcal{D}^*$. We implicitly equip $[0,1]^{\mathcal{G}^*}$ with its Borel σ-field and $\mathcal{D}^*$ with its trace σ-field.

Any $D \in \mathcal{D}^*$ is a sequence $(D_F)_{F \in \mathcal{G}^*}$, where $D(F) := D_F$ denotes the coordinate of D corresponding to $F \in \mathcal{G}^*$. In this way, any $D \in \mathcal{D}^*$ determines a probability measure $\gamma_D^{(n)}$ on $\mathcal{G}_n$, for every $n \in \mathbb{N}$, by

$$\gamma_D^{(n)}(G) := D(G), \qquad G \in \mathcal{G}_n. \tag{6.5}$$

Furthermore, the collection $(\gamma_D^{(n)})_{n\in\mathbb{N}}$ is consistent and exchangeable on $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ and, by Kolmogorov's extension theorem, determines a unique exchangeable measure γ_D on $\mathcal{G}_\infty$, for which γ_D-almost every $G \in \mathcal{G}_\infty$ has $|G| = D$.

Conversely, combining the Aldous–Hoover theorem for weakly exchangeable arrays ([2], Theorem 14.21) and Lovász–Szegedy theorem of graph limits ([22], Theorem 2.7), any exchangeable random graph Γ is governed by a mixture of γ_D measures. In particular, to any exchangeable random graph Γ, there exists a unique probability measure Δ on $\mathcal{D}^*$ such that $\Gamma \sim \gamma_\Delta$, where

$$\gamma_\Delta(\cdot) := \int_{\mathcal{D}^*} \gamma_D(\cdot)\Delta(\mathrm{d}D). \tag{6.6}$$

6.2. Rewiring limits

Since $\{0,1\} \times \{0,1\}$ is a finite space, the Aldous–Hoover theorem applies to exchangeable rewiring maps. Following Section 6.1, we define the *density* of $V \in \mathcal{W}_m$ in $W \in \mathcal{W}_n$ by

$$t(V,W) := \frac{\mathrm{ind}(V,W)}{n^{\downarrow m}}, \tag{6.7}$$

where $\mathrm{ind}(V,W)$ equals the number of injections $\psi : [m] \to [n]$ for which $W^\psi = V$. For an infinite rewiring map $W \in \mathcal{W}_\infty$, we define

$$t(V,W) := \lim_{n\to\infty} t(V, W|_{[n]}), \qquad W \in \mathcal{W}_m, \qquad \text{if it exists.}$$

As for graphs, the collection $\mathcal{W}^* := \bigcup_{m\in\mathbb{N}} \mathcal{W}_m$ is countable and so we can define the *rewiring limit* of $W \in \mathcal{W}_\infty$ by

$$|W| := \bigl(t(V,W)\bigr)_{V\in\mathcal{W}^*}, \tag{6.8}$$

provided $t(V,W)$ exists for all $V \in \mathcal{W}^*$.

We write $\mathcal{V}^* \subset [0,1]^{\mathcal{W}^*}$ to denote the compact space of rewiring limits and $\upsilon_V = \upsilon(V)$ to denote the coordinate of $\upsilon \in \mathcal{V}^*$ corresponding to $V \in \mathcal{W}^*$. We equip $\mathcal{V}^*$ with the metric

$$\rho(\upsilon,\upsilon') := \sum_{n\in\mathbb{N}} 2^{-n} \sum_{V\in\mathcal{W}_n} \bigl|\upsilon_V - \upsilon'_V\bigr|, \qquad \upsilon,\upsilon' \in \mathcal{V}^*. \tag{6.9}$$

Lemma 6.1. *Every $\upsilon \in \mathcal{V}^*$ satisfies*

- $\upsilon(V) = \sum_{\{V^*\in\mathcal{W}_{n+1}:V^*|_{[n]}=V\}} \upsilon(V^*)$ *for every $V \in \mathcal{W}_n$, for all $n \in \mathbb{N}$, and*
- $\sum_{V\in\mathcal{W}_n} \upsilon(V) = 1$ *for every $n \in \mathbb{N}$.*

Proof. By definition of $\mathcal{V}^*$, we may assume that υ is the rewiring limit $|W^*|$ of some $W^* \in \mathcal{W}_\infty$ so that $\upsilon(V) = t(V, W^*)$, for every $V \in \mathcal{W}^*$. From the definition of the rewiring limit (6.8),

$$\sum_{W \in \mathcal{W}_m} \upsilon(W) = \sum_{W \in \mathcal{W}_m} \lim_{n\to\infty} \frac{\operatorname{ind}(W, W^*|_{[n]})}{n^{\downarrow m}} = \lim_{n\to\infty} \sum_{W \in \mathcal{W}_m} \frac{\operatorname{ind}(W, W^*|_{[n]})}{n^{\downarrow m}} = 1,$$

where the interchange of sum and limit is justified by the Bounded Convergence theorem because $0 \le \operatorname{ind}(W, W^*|_{[n]})/n^{\downarrow m} \le 1$ for all $W \in \mathcal{W}_m$. Also, for every $m \le n$ and $W \in \mathcal{W}_m$, we have

$$\begin{aligned}
\sum_{\{W' \in \mathcal{W}_n : W'|_{[m]} = W\}} \upsilon(W') &= \sum_{\{W' \in \mathcal{W}_n : W'|_{[m]} = W\}} \lim_{k\to\infty} \frac{\operatorname{ind}(W', W^*|_{[k]})}{k^{\downarrow n}} \\
&= \lim_{k\to\infty} \sum_{\{W' \in \mathcal{W}_n : W'|_{[m]} = W\}} \frac{\operatorname{ind}(W', W^*|_{[k]})}{k^{\downarrow n}} \\
&= \lim_{k\to\infty} \frac{\operatorname{ind}(W, W^*|_{[k]})}{k^{\downarrow m}} \\
&= \upsilon(W).
\end{aligned}$$

This follows by the definition of $\operatorname{ind}(\cdot,\cdot)$ and also because, for any $\psi : [m] \to [k]$ there are $k^{\downarrow n}/k^{\downarrow m}$ injections $\psi' : [n] \to [k]$ such that ψ' coincides with ψ on $[m]$. □

Lemma 6.2. *$(\mathcal{V}^*, \rho)$ is a compact metric space.*

Theorem 6.1. *Let W be a dissociated exchangeable rewiring map. Then, with probability one, $|W|$ exists and is nonrandom.*

We delay the proofs of Lemma 6.2 and Theorem 6.1 until Section 10.

Corollary 6.1. *Let $W \in \mathcal{W}_\infty$ be an exchangeable random rewiring map. Then $|W|$ exists almost surely.*

Proof. By Theorem 6.1, every dissociated rewiring map possesses a nonrandom rewiring limit almost surely. By the Aldous–Hoover theorem, W is a mixture of dissociated rewiring maps and the conclusion follows. □

By Lemma 6.1, any $\upsilon \in \mathcal{V}^*$ determines a probability measure Ω_υ on $\mathcal{W}_\infty$ in a straightforward way: for each $n \in \mathbb{N}$, we define $\Omega_\upsilon^{(n)}$ as the probability distribution on $\mathcal{W}_n$ with

$$\Omega_\upsilon^{(n)}(w) := \upsilon(w), \qquad w \in \mathcal{W}_n. \tag{6.10}$$

Proposition 6.1. *For any $\upsilon \in \mathcal{V}^*$, $\{\Omega_\upsilon^{(n)}\}_{n\in\mathbb{N}}$ is a collection of exchangeable and consistent probability distributions on $\{\mathcal{W}_n\}_{n\in\mathbb{N}}$. In particular, $\{\Omega_\upsilon^{(n)}\}_{n\in\mathbb{N}}$ determines a unique exchangeable probability measure Ω_υ on $\mathcal{W}_\infty$ for which Ω_υ-almost every $w \in \mathcal{W}_\infty$ has $|w| = \upsilon$.*

Proof. By Lemma 6.1, the collection $\{\Omega_\upsilon^{(n)}\}_{n\in\mathbb{N}}$ in (6.10) is a consistent family of probability distributions on $\{\mathcal{W}_n\}_{n\in\mathbb{N}}$. Exchangeability follows because $\mathrm{ind}(w, W^*|_{[n]})$ is invariant under relabeling of w, that is, $\mathrm{ind}(w, W^*) = \mathrm{ind}(w^\sigma, W^*|_{[n]}^{\sigma'})$ for all permutations $\sigma \in \mathcal{S}_m$ and $\sigma' \in \mathcal{S}_n$. By Kolmogorov's extension theorem, $\{\Omega_\upsilon^{(n)}\}_{n\in\mathbb{N}}$ determines a unique measure Ω_υ on the limit space $\mathcal{W}_\infty$. Finally, $W \sim \Omega_\upsilon$ is dissociated and so, by Theorem 6.1, $|W| = \upsilon$ almost surely. □

We call Ω_υ in Proposition 6.1 a *rewiring measure* directed by υ. For any measure Υ on $\mathcal{V}^*$, we define the Υ-mixture of rewiring measures by

$$\Omega_\Upsilon(\cdot) := \int_{\mathcal{V}^*} \Omega_\upsilon(\cdot)\Upsilon(\mathrm{d}\upsilon). \tag{6.11}$$

Corollary 6.2. *To any exchangeable rewiring map W, there exists a unique probability measure Υ on $\mathcal{V}^*$ such that $W \sim \Omega_\Upsilon$.*

Proof. This follows by the Aldous–Hoover theorem and Proposition 6.1. □

From Theorem 6.1 and Proposition 6.1, any probability measure Υ on $\mathcal{V}^*$ corresponds to an Ω_Υ-rewiring chain as in Theorem 5.1.

7. Continuous-time rewiring processes

We now refine our discussion to rewiring chains in continuous-time, for which infinitely many transitions can "bunch up" in arbitrarily small intervals, but individual edges jump only finitely often in bounded intervals.

7.1. Exchangeable rewiring process

Henceforth, we write $\mathbf{id}$ to denote the identity $\mathcal{G}_\infty \to \mathcal{G}_\infty$ and, for $n \in \mathbb{N}$, we write $\mathbf{id}_n$ to denote the identity $\mathcal{G}_n \to \mathcal{G}_n$. Let ω be an exchangeable measure on $\mathcal{W}_\infty$ such that

$$\omega(\{\mathbf{id}\}) = 0 \quad \text{and} \quad \omega(\{W \in \mathcal{W}_\infty \colon W|_{[n]} \neq \mathbf{id}_n\}) < \infty \qquad \text{for every } n \geq 2. \tag{7.1}$$

Similar to our definition of P_ω in Section 5, we use ω to define the transition rates of continuous-time ω-rewiring chain. Briefly, we assume $\omega(\{\mathbf{id}\}) = 0$ because the identity map $\mathcal{G}_\infty \to \mathcal{G}_\infty$ is immaterial for continuous-time processes. The finiteness assumption on the right of (7.1) ensures that the paths of the finite restrictions are càdlàg.

For each $n \in \mathbb{N}$, we write $\omega^{(n)}$ to denote the restriction of ω to $\mathcal{W}_n$ and define

$$q_\omega^{(n)}(G, G') := \begin{cases} \omega^{(n)}(\{W \in \mathcal{W}_n \colon W(G) = G'\}), & G \neq G' \in \mathcal{G}_n, \\ 0, & G = G' \in \mathcal{G}_n. \end{cases} \tag{7.2}$$

Proposition 7.1. *For each $n \in \mathbb{N}$, $q_\omega^{(n)}$ is a finite, exchangeable conditional measure on $\mathcal{G}_n$. Moreover, the collection $\{q_\omega^{(n)}\}_{n\geq 2}$ satisfies*

$$q_\omega^{(m)}(G, G') = q_\omega^{(n)}(G^*, \mathbf{R}_{m,n}^{-1}(G')), \qquad G \neq G' \in \mathcal{G}_m, \tag{7.3}$$

for all $G^* \in \mathbf{R}_{m,n}^{-1}(G)$, for all $m \leq n$, for every $n \in \mathbb{N}$, where $\mathbf{R}_{m,n}$ is the restriction map $\mathcal{G}_n \to \mathcal{G}_m$ defined in (4.2).

Proof. Finiteness of $q_\omega^{(n)}$ follows from (7.1) since, for every $G \in \mathcal{G}_n$,

$$q_\omega^{(n)}(G, \mathcal{G}_n) = q_\omega^{(n)}(G, \mathcal{G}_n \setminus \{G\}) = \omega^{(n)}(\{W \in \mathcal{W}_n \colon W(G) \neq G\}) \leq \omega^{(n)}(\{W \neq \mathbf{id}_n\}) < \infty.$$

Exchangeability of $q_\omega^{(n)}$ follows by Proposition 5.1 and exchangeability of ω. Consistency of $\{q_\omega^{(n)}\}_{n\geq 2}$ results from Lipschitz continuity of rewiring maps (Proposition 4.1) and consistency of the finite-dimensional marginals $\{\omega^{(n)}\}_{n\in\mathbb{N}}$ associated to ω: for fixed $G \neq G' \in \mathcal{G}_m$ and $G^* \in \mathbf{R}_{m,n}^{-1}(G)$,

$$\begin{aligned}
q_\omega^{(n)}(G^*, \mathbf{R}_{m,n}^{-1}(G')) &= \sum_{G'' : G''|_{[m]} = G'} q_\omega^{(n)}(G^*, G'') \\
&= \sum_{G'' : G''|_{[m]} = G'} \omega^{(n)}(\{W \in \mathcal{W}_n \colon W(G^*) = G''\}) \\
&= \omega^{(n)}(\{W \in \mathcal{W}_n \colon W|_{[m]}(G) = G'\}) \\
&= \omega^{(m)}(\{W \in \mathcal{W}_m \colon W(G) = G'\}) \\
&= q_\omega^{(m)}(G, G').
\end{aligned}$$

$\square$

From $\{q_\omega^{(n)}\}_{n\in\mathbb{N}}$, we define a collection of infinitesimal jump rates $\{Q_\omega^{(n)}\}_{n\in\mathbb{N}}$ by

$$Q_\omega^{(n)}(G, G') := \begin{cases} q_\omega^{(n)}(G, G'), & G' \neq G, \\ -q_\omega^{(n)}(G, \mathcal{G}_n \setminus \{G\}), & G' = G. \end{cases} \tag{7.4}$$

Corollary 7.1. *The infinitesimal generators $\{Q_\omega^{(n)}\}_{n\in\mathbb{N}}$ are exchangeable and consistent and, therefore, define the infinitesimal jump rates Q_ω of an exchangeable Markov process on $\mathcal{G}_\infty$.*

Proof. Consistency when $G' \neq G$ was already shown in Proposition 7.1. We must only show that $Q_\omega^{(n)}$ is consistent for $G' = G$. Fix $n \in \mathbb{N}$ and $G \in \mathcal{G}_n$. Then, for any $G^* \in \mathbf{R}_{n,n+1}^{-1}(G)$, we have

$$\begin{aligned}
Q_\omega^{(n+1)}(G^*, \mathbf{R}_{n,n+1}^{-1}(G)) &= -q_\omega^{(n+1)}(G^*, \mathcal{G}_{n+1} \setminus \{G^*\}) + \sum_{G'' \in \mathbf{R}_{n,n+1}^{-1}(G) : G'' \neq G^*} q_\omega^{(n+1)}(G^*, G'') \\
&= -q_\omega^{(n+1)}(G^*, \mathcal{G}_{n+1} \setminus \mathbf{R}_{n,n+1}^{-1}(G))
\end{aligned}$$

$$= -q_{\omega}^{(n)}\big(G, \mathcal{G}_n \setminus \{G\}\big)$$
$$= Q_{\omega}^{(n)}(G, G). \qquad \square$$

In Section 3, we mentioned local and global discontinuities for graph-valued processes. In the next two sections, we formally incorporate these discontinuities into a continuous-time rewiring process: in Section 7.2, we extend the notion of random rewiring from discrete-time; in Section 7.3, we introduce transitions for which, at the time of a jump, only a single edge in the network changes. Over time, the local changes can accumulate to cause a non-trivial change to network structure.

7.2. Global jumps: Rewiring

In this section, we specialize to the case where $\omega = \Omega_\Upsilon$ for some measure Υ on $\mathcal{V}^*$ satisfying

$$\Upsilon\big(\{\mathbf{I}\}\big) = 0 \quad \text{and} \quad \int_{\mathcal{V}^*} \big(1 - \upsilon_*^{(2)}\big)\Upsilon(\mathrm{d}\upsilon) < \infty, \tag{7.5}$$

where $\mathbf{I}$ is the rewiring limit of $\mathbf{id} \in \mathcal{W}_\infty$ and $\upsilon_*^{(n)} := \upsilon(\mathbf{id}_n)$ is the entry of υ corresponding to $\mathbf{id}_n$, for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we write $q_\Upsilon^{(n)}$ to denote $q_\omega^{(n)}$ for $\omega = \Omega_\Upsilon$, and likewise for the infinitesimal generator $Q_\Upsilon^{(n)}$.

Lemma 7.1. *For Υ satisfying (7.5), the rewiring measure Ω_Υ satisfies (7.1).*

Proof. By Theorem 6.1, $\Upsilon(\{\mathbf{I}\}) = 0$ implies $\Omega_\Upsilon(\{\mathbf{id}\}) = 0$. We need only show that $\int_{\mathcal{V}^*}(1 - \upsilon_*^{(2)})\Upsilon(\mathrm{d}\upsilon) < \infty$ implies $\Omega_\Upsilon^{(n)}(\{W \in \mathcal{W}_n\colon W \neq \mathbf{id}_n\}) < \infty$ for every $n \geq 2$. For any $\upsilon \in \mathcal{V}^*$,

$$\begin{aligned}
\Omega_\upsilon\big(\{W \in \mathcal{W}_\infty\colon W|_{[n]} \neq \mathbf{id}_n\}\big) &= \Omega_\upsilon\Bigg(\bigcup_{1\leq i<j\leq n} \{W \in \mathcal{W}_\infty\colon W|_{\{i,j\}} \neq \mathbf{id}_{\{i,j\}}\}\Bigg) \\
&\leq \sum_{1\leq i<j\leq n} \Omega_\upsilon\big(\{W \in \mathcal{W}_\infty\colon W|_{\{i,j\}} \neq \mathbf{id}_{\{i,j\}}\}\big) \\
&= \sum_{1\leq i<j\leq n} \Omega_\upsilon^{(2)}\big(\mathcal{W}_2 \setminus \{\mathbf{id}_2\}\big) \\
&= \frac{n(n-1)}{2}\big(1 - \upsilon_*^{(2)}\big).
\end{aligned}$$

Hence, by (7.5),

$$\Omega_\Upsilon\big(\{W \in \mathcal{W}_\infty\colon W|_{[n]} \neq \mathbf{id}_n\}\big) \leq \int_{\mathcal{V}^*} \frac{n(n-1)}{2}\big(1 - \upsilon_*^{(2)}\big)\Upsilon(\mathrm{d}\upsilon) < \infty,$$

for every $n \geq 2$. $\square$

Proposition 7.2. *For each* $n \in \mathbb{N}$, $q_{\Upsilon}^{(n)}$ *is a finite, exchangeable conditional measure on* $\mathcal{G}_n$. *Moreover,* $\{q_{\Upsilon}^{(n)}\}_{n\in\mathbb{N}}$ *satisfies*

$$q_{\Upsilon}^{(m)}(G, G') = q_{\Upsilon}^{(n)}(G^*, \mathbf{R}_{m,n}^{-1}(G')), \qquad G \neq G' \in \mathcal{G}_m, \qquad \textit{for all } G^* \in \mathbf{R}_{m,n}^{-1}(G).$$

Proof. This follows directly from Propositions 6.1, 7.1, and Lemma 7.1. □

We may, therefore, define an infinitesimal generator for a Markov chain on $\mathcal{G}_n$ by

$$Q_{\Upsilon}^{(n)}(G, G') := \begin{cases} q_{\Upsilon}^{(n)}(G, G'), & G' \neq G, \\ -q_{\Upsilon}^{(n)}(G, \mathcal{G}_n \setminus \{G\}), & G' = G. \end{cases} \tag{7.6}$$

Theorem 7.1. *For each* Υ *satisfying* (7.5), *there exists an exchangeable Markov process* $\mathbf{\Gamma}$ *on* $\mathcal{G}_\infty$ *with finite-dimensional transition rates as in* (7.6).

We call $\mathbf{\Gamma}$ in Theorem 7.1 a *rewiring process* directed by Υ, or with rewiring measure Ω_Υ.

7.3. Local jumps: Isolated updating

For $i' > j' \geq 1$ and $k = 0, 1$, let $R_{i'j'}^k$ denote the rewiring map $\mathcal{W}_\infty \to \mathcal{W}_\infty$ that acts by mapping $G \mapsto G' := R_{i'j'}^k(G)$,

$$G'_{ij} := \begin{cases} G_{ij}, & ij \neq i'j', \\ k, & ij = i'j'. \end{cases} \tag{7.7}$$

In words, R_{ij}^k puts an edge between i and j (if $k = 1$) or no edge between i and j (if $k = 0$) and keeps every other edge fixed.

For fixed $n \in \mathbb{N}$, let $\mathbf{0}_n \in \mathcal{G}_n$ denote the *empty graph*, that is, the graph with no edges. We generate a continuous-time process $\mathbf{\Gamma}_0 := \{\Gamma_0(t)\}_{t\geq 0}$ on $\mathcal{G}_n$ as follows. First, we specify a constant $\mathbf{c}_0 > 0$ and, independently for each pair $\{i, j\} \in [n] \times [n]$, $i < j$, we generate i.i.d. random variables T_{ij} from the Exponential distribution with rate parameter $\mathbf{c}_0$. Given $\{T_{ij}\}_{i<j}$, we define $\mathbf{\Gamma}_0$ by

$$i \sim_{\Gamma_0(t)} j \quad \Longleftrightarrow \quad T_{ij} < t,$$

where $i \sim_G j$ denotes an edge between i and j in G. Clearly, $\mathbf{\Gamma}_0$ is exchangeable and converges to a unique stationary distribution $\delta_{\mathbf{1}_n}$, the point mass at the complete graph $\mathbf{1}_n$. Moreover, the distribution of T_*, the time until absorption in $\mathbf{1}_n$, is simply the law of the maximum of $n(n-1)/2$ i.i.d. Exponential random variables with rate parameter $\mathbf{c}_0$.

Conversely, we could consider starting in $\mathbf{1}_n$, the *complete graph*, and generating the above process in reverse. In this case, we specify $\mathbf{c}_1 > 0$ and let $\{T_{ij}\}_{i<j}$ be an i.i.d. collection of Exponential random variables with rate parameter $\mathbf{c}_1$. We construct $\mathbf{\Gamma}_1 := \{\Gamma_1(t)\}_{t\geq 0}$, given $\{T_{ij}\}_{i<j}$, by

$$i \sim_{\Gamma_1(t)} j \quad \Longleftrightarrow \quad T_{ij} > t.$$

For $\mathbf{c}_1 = \mathbf{c}_0$, this process evolves exactly as the *complement* of $\mathbf{\Gamma}_0$, that is,

$$\mathbf{\Gamma}_1 =_{\mathcal{L}} \bar{\mathbf{\Gamma}}_0,$$

where $\bar{\mathbf{\Gamma}}_0 := \{\bar{\Gamma}_0(t)\}_{t\geq 0}$ is defined by

$$i \sim_{\bar{\Gamma}_0(t)} j \quad \Longleftrightarrow \quad i \nsim_{\Gamma_0(t)} j,$$

for all $i \neq j$ and all $t \geq 0$.

It is natural to consider the superposition of $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$, which we call a $(\mathbf{c}_0, \mathbf{c}_1)$*-local-edge process*. Let $\mathbf{c}_0, \mathbf{c}_1 \geq 0$ and let δ_{ij}^k denote the point mass at the single-edge update map R_{ij}^k. Following Section 7.1, we define

$$\Omega_{\mathbf{c}_0,\mathbf{c}_1} := \mathbf{c}_0 \sum_{i<j} \delta_{ij}^0 + \mathbf{c}_1 \sum_{i<j} \delta_{ij}^1. \tag{7.8}$$

Lemma 7.2. *For* $\mathbf{c}_0, \mathbf{c}_1 \geq 0$, $\Omega_{\mathbf{c}_0,\mathbf{c}_1}$ *defined in* (7.8) *satisfies* (7.1).

Proof. Since $\Omega_{\mathbf{c}_0,\mathbf{c}_1}$ only charges the single-edge update maps, it is clear that it assigns zero mass to the identity map. Also, for any $n \in \mathbb{N}$, the restriction of R_{ij}^k to $\mathcal{W}_n$ coincides with the identity $\mathcal{G}_n \to \mathcal{G}_n$ except when $1 \leq i < j \leq n$. Hence,

$$\Omega_{\mathbf{c}_0,\mathbf{c}_1}^{(n)}\bigl(\{W \in \mathcal{W}_n\colon\ W \neq \mathbf{id}_n\}\bigr) = \frac{n(n-1)}{2}(\mathbf{c}_0 + \mathbf{c}_1) < \infty,$$

for every $n \geq 2$. □

Corollary 7.2. *For any* $\mathbf{c}_0, \mathbf{c}_1 \geq 0$, *there exists an exchangeable Markov process on* $\mathcal{G}_\infty$ *with jump rates given by* $\Omega_{\mathbf{c}_0,\mathbf{c}_1}$.

Proof. For every $n \in \mathbb{N}$, the total jump rate out of any $G \in \mathcal{G}_n$ can be no larger than

$$\frac{n(n-1)}{2}(\mathbf{c}_0 \vee \mathbf{c}_1) < \infty,$$

and so the finite-dimensional hold times are almost surely positive and the process on $\mathcal{G}_n$ has càdlàg sample paths. The Markov property and exchangeability follow by independence of the Exponential hold times $\{T_{ij}\}_{1\leq i<j\leq n}$ and Corollary 7.1. Consistency is apparent by the construction from independent Poisson point processes. This completes the proof. □

Definition 7.1. *For any measure* Υ *satisfying* (7.5), $\mathbf{c}_0, \mathbf{c}_1 \geq 0$, *we call a rewiring process with jump measure* $\omega = \Omega_\Upsilon + \Omega_{\mathbf{c}_0,\mathbf{c}_1}$ *an* $(\Upsilon, \mathbf{c}_0, \mathbf{c}_1)$-rewiring process.

From our discussion in this section, the $(\Upsilon, \mathbf{c}_0, \mathbf{c}_1)$-rewiring process exists for any choice of Υ satisfying (7.5) and $\mathbf{c}_0, \mathbf{c}_1 \geq 0$. Individually, Ω_Υ and $\Omega_{\mathbf{c}_0,\mathbf{c}_1}$ satisfy (7.1) and, thus, so does $\omega := \Omega_\Upsilon + \Omega_{\mathbf{c}_0,\mathbf{c}_1}$. Furthermore, the family of $(\Upsilon, \mathbf{c}_0, \mathbf{c}_1)$-rewiring processes is Markovian, exchangeable, and consistent.

8. Simulating rewiring processes

We can construct an $(\Upsilon, \mathbf{c}_0, \mathbf{c}_1)$-rewiring process from a Poisson point process. For $\omega := \Omega_\Upsilon + \Omega_{\mathbf{c}_0,\mathbf{c}_1}$, where Υ satisfies (7.5) and $\mathbf{c}_0, \mathbf{c}_1 \geq 0$, let $\mathbf{W} := \{(t, W_t)\} \subset \mathbb{R}^+ \times \mathcal{W}_\infty$ be a Poisson point process with intensity $\mathrm{d}t \otimes \omega$. To begin, we take Γ_0 to be an exchangeable random graph and, for each $n \in \mathbb{N}$, we define $\mathbf{\Gamma}^{[n]} := (\Gamma_t^{[n]})_{t\geq 0}$ on $\mathcal{G}_n$ by $\Gamma_0^{[n]} := \Gamma_0|_{[n]}$ and

- if $t > 0$ is an atom time of $\mathbf{W}$ such that $W_t^{[n]} := W_t|_{[n]} \neq \mathbf{id}_n$, then we put $\Gamma_t^{[n]} := W_t^{[n]}(\Gamma_{t-}^{[n]})$;
- otherwise, we put $\Gamma_t^{[n]} = \Gamma_{t-}^{[n]}$.

Proposition 8.1. *For each $n \in \mathbb{N}$, $\mathbf{\Gamma}^{[n]}$ is a Markov chain on $\mathcal{G}_n$ with infinitesimal jump rates $Q_\omega^{(n)}$ in* (7.4).

Proof. We can define $\mathbf{W}^{[n]} := \{(t, W_t^{[n]})\} \subset \mathbb{R}^+ \times \mathcal{W}_n$ from $\mathbf{W}$ by removing any atom times for which $W_t^{[n]} := W_t|_{[n]} = \mathbf{id}_n$, and otherwise putting $W_t^{[n]} := W_t|_{[n]}$. By the thinning property of Poisson point processes, $\mathbf{W}^{[n]}$ is a Poisson point process with intensity $\mathrm{d}t \otimes \omega_n$, where

$$\omega_n(\cdot) := \omega^{(n)}\bigl(\cdot \setminus \{\mathbf{id}_n\}\bigr).$$

Given $\Gamma_t^{[n]} = G$, the jump rate to state $G' \neq G$ is

$$\omega_n\bigl(\bigl\{W \in \mathcal{W}_n \colon\ W(G) = G'\bigr\}\bigr) = Q_\omega^{(n)}\bigl(G, G'\bigr),$$

and the conclusion follows. □

Theorem 8.1. *For any ω satisfying* (7.1), *the ω-rewiring process on $\mathcal{G}_\infty$ exists and can be constructed from a Poisson point process with intensity* $\mathrm{d}t \otimes \omega$ *as above.*

Proof. Let $\mathbf{W}$ be a Poisson point process with intensity $\mathrm{d}t \otimes \omega$ and construct $\{\mathbf{\Gamma}^{[n]}\}_{n\in\mathbb{N}}$ from the thinned processes $\{\mathbf{W}^{[n]}\}_{n\in\mathbb{N}}$ determined by $\mathbf{W}$. By Proposition 8.1, each $\mathbf{\Gamma}^{[n]}$ is an exchangeable Markov chain governed by $Q_\omega^{(n)}$. Moreover, $\{\mathbf{\Gamma}^{[n]}\}_{n\in\mathbb{N}}$ is compatible by construction, that is, $\Gamma_t^{[m]} = \mathbf{R}_{m,n}\Gamma_t^{[n]}$ for all $t \geq 0$, for all $m \leq n$; hence, $\{\mathbf{\Gamma}^{[n]}\}_{n\in\mathbb{N}}$ defines a process $\mathbf{\Gamma}$ on $\mathcal{G}_\infty$. As we have shown previously, the infinitesimal rates given by $\{Q_\omega^{(n)}\}_{n\in\mathbb{N}}$ are consistent and exchangeable; hence, $\mathbf{\Gamma}$ has infinitestimal generator Q_ω and is an ω-rewiring process. □

8.1. The Feller property

Any Markov process $\mathbf{\Gamma}$ on $\mathcal{G}_\infty$ is characterized by its semigroup $(\mathbf{P}_t)_{t\geq 0}$, defined as an operator on the space of continuous, bounded functions $h : \mathcal{G}_\infty \to \mathbb{R}$ by

$$\mathbf{P}_t h(G) := \mathbb{E}_G h(\Gamma_t), \qquad G \in \mathcal{G}_\infty, \tag{8.1}$$

where $\mathbb{E}_G$ denotes the expectation operator with respect to the initial distribution $\delta_G(\cdot)$, the point mass at G. We say $\mathbf{\Gamma}$ has the *Feller property* if, for all bounded, continuous functions $h : \mathcal{G}_\infty \to \mathbb{R}$, its semigroup satisfies

- $\mathbf{P}_t h(G) \to h(G)$ as $t \downarrow 0$ for all $G \in \mathcal{G}_\infty$, and
- $G \mapsto \mathbf{P}_t h(G)$ is continuous for all $t \geq 0$.

Theorem 8.2. *The semigroup $(\mathbf{P}_t^\omega)_{t\geq 0}$ of any ω-rewiring process enjoys the Feller property.*

Proof. To show the first point in the Feller property, we let $G \in \mathcal{G}_\infty$ and $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ be an ω-rewiring process with initial state $\Gamma_0 = G$ and directing measure ω satisfying (7.1). We define

$$\mathcal{F} := \bigl\{h : \mathcal{G}_\infty \to \mathbb{R} \mid \text{there exists } n \in \mathbb{N} \text{ such that } G|_{[n]} = G'|_{[n]} \Rightarrow h(G) = h\bigl(G'\bigr)\bigr\}.$$

By (7.1) and finiteness of $\mathcal{G}_n$, $\Gamma_t^{[n]} \to G|_{[n]}$ in probability as $t \downarrow 0$, for every $n \in \mathbb{N}$. Thus, for any $h \in \mathcal{F}$, let $N \in \mathbb{N}$ be such that

$$d\bigl(G, G'\bigr) \leq 1/N \quad \Longrightarrow \quad h(G) = h\bigl(G'\bigr).$$

Then $\Gamma_t^{[N]} \to G|_{[N]}$ in probability as $t \downarrow 0$ and, therefore, $\mathbf{P}_t h(G) \to h(G)$ by the Bounded Convergence theorem. Right-continuity at zero for all bounded, continuous $h : \mathcal{G}_\infty \to \mathbb{R}$ follows by the Stone–Weierstrass theorem.

For the second point, let $G, G' \in \mathcal{G}_\infty$ have $d(G, G') \leq 1/n$ for some $n \in \mathbb{N}$ and construct $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ from the same Poisson point process $\mathbf{W}$ but with initial states $\Gamma_0 = G$ and $\Gamma'_0 = G'$. By Lipschitz continuity of the rewiring maps (Proposition 4.1), $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ can never be more than distance $1/n$ apart, for all $t \geq 0$. Continuity of $\mathbf{P}_t^\omega$, for each $t \geq 0$, follows. □

By the Feller property, any ω-rewiring process has a càdlàg version and its jumps are characterized by an infinitesimal generator. In Section 7, we described the infinitesimal generator through its finite restrictions. Ethier and Kurtz [15] give an extensive treatment of the general theory of Feller processes.

9. Concluding remarks

We have presented a family of time-varying network models that is Markovian, exchangeable, and consistent, natural statistical properties that impose structure without introducing logical pitfalls. External to statistics, exchangeable models are flawed: they produce *dense* graphs when conventional wisdom suggests real-world networks are *sparse*. The Erdős–Rényi model's storied history cautions against dismay. Though it replicates little real-world network structure, the Erdős–Rényi model has produced a deluge of insight for graph-theoretic structures and is a paragon of the utility of the probabilistic method [5]. While our discussion is specific to exchangeable processes, the general descriptions in Sections 5 and 7 can be used to construct processes that are not exchangeable, and possibly even sparse.

The most immediate impact of the rewiring process may be for analyzing information spread on dynamic networks. Under the heading of *Finite Markov Information Exchange* (FMIE) processes, Aldous [4] recently surveyed interacting particle systems models for social network dynamics. Informally, FMIE processes model a random spread of information on a network. Some of the easiest to describe FMIE processes coincide with well-known interacting particle systems, such as the Voter and Contact processes; others mimic certain social behaviors, for example, *Fashionista* and *Compulsive Gambler*.

Simulation is a valuable practical tool for developing intuition about intractable problems. Aldous's expository account contains some hard open problems for time-invariant networks. Considering these same questions on dynamic networks seems an even greater challenge. Despite these barriers, policymakers and scientists alike desire to understand how trends, epidemics, and other information spread on networks. The Poisson point process construction in Section 8 could be fruitful for deriving practical answers to these problems.

10. Technical proofs

In this section, we prove some technical results from our previous discussion.

10.1. Proof of Lemma 6.2

We now show that $(\mathcal{V}^*, \rho)$ is a compact metric space. Recall that $\mathcal{V}^*$ is equipped with the metric

$$\rho(\upsilon, \upsilon') = \sum_{n\in\mathbb{N}} 2^{-n} \sum_{V\in\mathcal{W}_n} \left|\upsilon_V - \upsilon'_V\right|, \qquad \upsilon, \upsilon' \in \mathcal{V}^*.$$

Since $[0, 1]^{\mathcal{W}^*}$ is compact in this metric, it suffices to show that $\mathcal{V}^*$ is a closed subset of $[0, 1]^{\mathcal{W}^*}$.

By Lemma 6.1, every $\upsilon \in \mathcal{V}^*$ satisfies

$$\upsilon(W) = \sum_{W^*\in\mathcal{W}_{n+1}:W^*|_{[n]}=W} \upsilon\left(W^*\right) \qquad \text{for every } W \in \mathcal{W}_n$$

and

$$\sum_{W\in\mathcal{W}_n} \upsilon(W) = 1,$$

for all $n \in \mathbb{N}$. Then, for any $x \in [0, 1]^{\mathcal{W}^*} \setminus \mathcal{V}^*$, there must be some $N \in \mathbb{N}$ for which

$$\varepsilon_x := \sum_{W\in\mathcal{W}_N} \left| x^{(N)}(W) - \sum_{W^*|_{[N]}=W} x^{(N+1)}\left(W^*\right) \right| > 0.$$

For any $\delta > 0$, let $B(x,\delta) := \{x' \in [0,1]^{\mathcal{W}^*} : \rho(x,x') < \delta\}$ denote the δ-ball around x. Now, take any $x' \in B(x, 2^{-N-2}\varepsilon_x)$. By this assumption, $\rho(x,x') \le 2^{-N-2}\varepsilon_x$ and so

$$2^{-N} \sum_{W \in \mathcal{W}_N} \left| x^{(N)}(W) - x'^{(N)}(W) \right| + 2^{-N-1} \sum_{W^* \in \mathcal{W}_{N+1}} \left| x^{(N+1)}\left(W^*\right) - x'^{(N+1)}\left(W^*\right) \right| \le 2^{-N-2}\varepsilon_x;$$

whence,

$$\sum_{W \in \mathcal{W}_N} \left| x^{(N)}(W) - x'^{(N)}(W) \right| \le \frac{1}{4}\varepsilon_x \quad \text{and}$$
$$\sum_{W^* \in \mathcal{W}_{N+1}} \left| x^{(N+1)}\left(W^*\right) - x'^{(N+1)}\left(W^*\right) \right| \le \frac{1}{2}\varepsilon_x.$$

By the triangle inequality, we have

$$\begin{aligned}
\varepsilon_x &= \sum_{W \in \mathcal{W}_N} \left| x^{(N)}(W) - \sum_{W^* : W^*|_{[N]}} x^{(N+1)}\left(W^*\right) \right| \\
&\le \sum_{W \in \mathcal{W}_N} \left| x^{(N)}(W) - x'^{(N)}(W) \right| + \sum_{W \in \mathcal{W}_N} \left| \sum_{W^* : W^*|_{[N]} = W} \left(x^{(N+1)}\left(W^*\right) - x'^{(N+1)}\left(W^*\right)\right) \right| \\
&\quad + \sum_{W \in \mathcal{W}_N} \left| x'^{(N)}(W) - \sum_{W^* : W^*|_{[N]} = W} x'^{(N+1)}\left(W^*\right) \right| \\
&\le \varepsilon_x/4 + \sum_{W \in \mathcal{W}_N} \sum_{W^* : W^*|_{[N]} = W} \left| x^{(N+1)}\left(W^*\right) - x'^{(N+1)}\left(W^*\right) \right| \\
&\quad + \sum_{W \in \mathcal{W}_N} \left| x'^{(N)}(W) - \sum_{W^* : W^*|_{[N]} = W} x'^{(N+1)}\left(W^*\right) \right| \\
&\le \varepsilon_x/4 + \varepsilon_x/2 + \sum_{W \in \mathcal{W}_N} \left| x'^{(N)}(W) - \sum_{W^* : W^*|_{[N]} = W} x'^{(N+1)}\left(W^*\right) \right|.
\end{aligned}$$

Therefore,

$$\sum_{W \in \mathcal{W}_N} \left| x'^{(N)}(W) - \sum_{W^* : W^*|_{[N]} = W} x'^{(N+1)}\left(W^*\right) \right| \ge \varepsilon_x/4 > 0,$$

which implies $x' \in [0,1]^{\mathcal{W}^*} \setminus \mathcal{V}^*$, meaning $[0,1]^{\mathcal{W}^*} \setminus \mathcal{V}^*$ is open and $\mathcal{V}^*$ is closed. Since $[0,1]^{\mathcal{W}^*}$ is compact, so is $\mathcal{V}^*$. This completes the proof.

10.2. Proof of Theorem 6.1

Assume that W is an exchangeable and dissociated rewiring map. By the Aldous–Hoover theorem, we can assume W is constructed from a measurable function $f : [0,1]^4 \to \{0,1\} \times \{0,1\}$ for which (i) $f(a,\cdot,\cdot,\cdot) = f(a',\cdot,\cdot,\cdot)$ and (ii) $f(\cdot,b,c,\cdot) = f(\cdot,c,b,\cdot)$. More precisely, we assume $W_{ij} = f(\alpha, \eta_i, \eta_j, \lambda_{\{i,j\}})$, for each $i, j \geq 1$, where $\{\alpha; (\eta_i)_{i\geq 1}; (\lambda_{\{i,j\}})_{i<j}\}$ are i.i.d. Uniform random variables on $[0,1]$. Conditional on $\alpha = a$, we define the quantity

$$t_a(V, W) := P\{W|_{[m]} = V \mid \alpha = a\}, \qquad V \in \mathcal{W}_m, m \in \mathbb{N},$$

which, by the fact that W is dissociated, is independent of a; hence, we define the non-random quantity

$$t(V, W) := E\bigl(\mathbf{1}\{W|_{[m]} = V\} \mid \alpha\bigr) = P\{W|_{[m]} = V\}.$$

Recall, from Section 6.2, the definition

$$t(V, W|_{[n]}) := \frac{\operatorname{ind}(V, W|_{[n]})}{n^{\downarrow m}} := \frac{1}{n^{\downarrow m}} \sum_{\text{injections}\,\psi:[m]\to[n]} \mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\}, \qquad n \in \mathbb{N}.$$

For every $n \geq 1$, we also define

$$M_{k,n} := \frac{1}{n^{\downarrow m}} \sum_{\text{injections}\,\psi:[m]\to[n]} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[k]}\bigr), \qquad k = 0, 1, \ldots, n.$$

In particular, for every $n \in \mathbb{N}$, we have

$$M_{0,n} = \frac{1}{n^{\downarrow m}} \sum_{\text{injections}\,\psi:[m]\to[n]} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[0]}\bigr) = t(V, W)$$

and

$$M_{n,n} = \frac{1}{n^{\downarrow m}} \sum_{\text{injections}\,\psi:[m]\to[n]} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[n]}\bigr) = t(V, W|_{[n]}).$$

We wish to show that $t(V, W|_{[n]}) \to t(V, W)$ almost surely, for every $V \in \mathcal{W}_m$, $m \in \mathbb{N}$. To do this, we first show that $(M_{0,n}, M_{1,n}, \ldots, M_{n,n})$ is a martingale with respect to its natural filtration, for every $n \in \mathbb{N}$. We can then appeal to Azuma's inequality and the Borel–Cantelli lemma to show that $M_{n,n} \to t(V, W)$ as $n \to \infty$.

Note that

$$M_{k,n} = \frac{1}{n^{\downarrow m}} \sum_{\text{injections}\,\psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} E\bigl(\mathbf{1}\{W|_{[n]} = w\} \mid W|_{[k]}\bigr)\mathbf{1}\bigl\{w^{\psi} = V\bigr\}$$

and

$$E(M_{k+1,n} \mid M_{k,n}) = E\bigl(E(M_{k+1,n} \mid M_{k,n}, W|_{[k]}) \mid M_{k,n}\bigr).$$

On the inside, we have

$$
\begin{aligned}
&E(M_{k+1,n} \mid M_{k,n}, W|_{[k]}) \\
&= E\biggl(\frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} \mathbf{1}\bigl\{w^{\psi} = V\bigr\} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = w\bigr\} \mid W|_{[k]}\bigr) \Bigm| M_{k,n}, W|_{[k]}\biggr) \\
&= \frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} \mathbf{1}\bigl\{w^{\psi} = V\bigr\} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = w\bigr\} \mid W|_{[k]}\bigr);
\end{aligned}
$$

whence,

$$
\begin{aligned}
&E\bigl(E(M_{k+1,n} \mid M_{k,n}, W|_{[k]}) \mid M_{k,n}\bigr) \\
&= E\biggl(\frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} \mathbf{1}\bigl\{w^{\psi} = V\bigr\} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = w\bigr\} \mid W|_{[k]}\bigr) \Bigm| M_{k,n}\biggr) \\
&= \frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} \mathbf{1}\bigl\{w^{\psi} = V\bigr\} E\bigl(E\bigl(\mathbf{1}\{W|_{[n]} = w\} \mid W|_{[k]}\bigr) \mid M_{k,n}\bigr) \\
&= \frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \sum_{w\in\mathcal{W}_n} \mathbf{1}\bigl\{w^{\psi} = V\bigr\} E\bigl(\mathbf{1}\{W|_{[n]} = w\} \mid M_{k,n}\bigr) \\
&= M_{k,n}.
\end{aligned}
$$

Therefore, $(M_{k,n})_{k=0,1,\dots,n}$ is a martingale for every $n \in \mathbb{N}$. Furthermore, for every $k = 0, 1, \dots, n-1$,

$$
\begin{aligned}
&|M_{k+1,n} - M_{k,n}| \\
&= \frac{1}{n^{\downarrow m}} \biggl| \sum_{\text{injections } \psi:[m]\to[n]} E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[k+1]}\bigr) - E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[k]}\bigr) \biggr| \\
&\le \frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \bigl| E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[k+1]}\bigr) - E\bigl(\mathbf{1}\bigl\{W|_{[n]}^{\psi} = V\bigr\} \mid W|_{[k]}\bigr) \bigr| \\
&\le m(n-1)^{\downarrow(m-1)}/n^{\downarrow m} \\
&\le m/n,
\end{aligned}
$$

since $E(\mathbf{1}\{W|_{[n]}^{\psi} = V\} \mid W|_{[k+1]}) - E(\mathbf{1}\{W|_{[n]}^{\psi} = V\} \mid W|_{[k]}) = 0$ whenever ψ does not map an element to $k+1$. The conditions for Azuma's martingale inequality are thus satisfied and we have, for every $\varepsilon > 0$,

$$
P\bigl\{|M_{n,n} - M_{0,n}| > \varepsilon\bigr\} \le 2\exp\biggl\{-\frac{\varepsilon^2 n}{2m^2}\biggr\} \qquad \text{for every } n \in \mathbb{N}.
$$

Thus,

$$\sum_{n=1}^{\infty} P\{|M_{n,n} - t(V, W)| > \varepsilon\} \leq 2\sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon^2 n}{2m^2}\right\} < \infty,$$

and we conclude, by the Borel–Cantelli lemma, that

$$\begin{aligned}&\limsup_{n\to\infty}\{|t(V, W|_{[n]}) - t(V, W)| > \varepsilon\}\\ &\quad = \{|t(V, W|_{[n]}) - t(V, W)| > \varepsilon \text{ for infinitely many } n \in \mathbb{N}\}\end{aligned}$$

has probability zero. It follows that $\lim_{n\to\infty} t(V, W|_{[n]}) = t(V, W)$ exists with probability one for every $V \in \bigcup_{m\in\mathbb{N}} \mathcal{W}_m$. Therefore, with probability one, the rewiring limit $(t(V, W))_{V\in\mathcal{W}^*}$ exists. We have already shown, by the assumption that W is dissociated, that $t(V, W)$ is non-random for every $V \in \bigcup_{m\in\mathbb{N}} \mathcal{W}_m$; hence, the limit $(t(V, W))_{V\in\mathcal{W}^*}$ is non-random. This completes the proof.

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