Bernoulli **21**(2), 2015, 1166–1199
DOI: 10.3150/14-BEJ601

Backward stochastic variational inequalities on random interval

LUCIAN MATICIUC[1,2,*] and AUREL RĂŞCANU[1,3,**]

[1]*Faculty of Mathematics, "Alexandru Ioan Cuza" University, Carol I Blvd., no. 11, Iasi, 700506, Romania. E-mail:* [*]*lucian.maticiuc@uaic.ro;* [**]*aurel.rascanu@uaic.ro*
[2]*Department of Mathematics, "Gheorghe Asachi" Technical University, Carol I Blvd., no. 11, Iasi, 700506, Romania*
[3]*"Octav Mayer" Mathematics Institute of the Romanian Academy, Iasi branch, Carol I Blvd., no. 8, Iasi, 700506, Romania*

The aim of this paper is to study, in the infinite dimensional framework, the existence and uniqueness for the solution of the following multivalued generalized backward stochastic differential equation, considered on a random, possibly infinite, time interval:

$$\begin{cases} -\mathrm{d}Y_t + \partial_y \Psi(t, Y_t)\,\mathrm{d}Q_t \ni \Phi(t, Y_t, Z_t)\,\mathrm{d}Q_t - Z_t\,\mathrm{d}W_t, & 0 \le t < \tau, \\ Y_\tau = \eta, \end{cases}$$

where τ is a stopping time, Q is a progressively measurable increasing continuous stochastic process and $\partial_y \Psi$ is the subdifferential of the convex lower semicontinuous function $y \longmapsto \Psi(t, y)$.

As applications, we obtain from our main results applied for suitable convex functions, the existence for some backward stochastic partial differential equations with Dirichlet or Neumann boundary conditions.

Keywords: backward stochastic differential equations; subdifferential operators; stochastic variational inequalities; stochastic partial differential equations

1. Introduction

In this paper, we are interested to prove the existence and uniqueness of a triple (Y, Z, K) which is the solution for the following generalized backward stochastic variational inequality (BSVI for short) considered in the Hilbert space framework:

$$\begin{cases} Y_t + \displaystyle\int_{t\wedge\tau}^{\tau} \mathrm{d}K_s = \eta + \int_{t\wedge\tau}^{\tau} \big[F(s, Y_s, Z_s)\,\mathrm{d}s + G(s, Y_s)\,\mathrm{d}A_s\big] - \int_{t\wedge\tau}^{\tau} Z_s\,\mathrm{d}W_s, & \text{a.s.}, \\ \mathrm{d}K_t \in \partial\varphi(Y_t)\,\mathrm{d}t + \partial\psi(Y_t)\,\mathrm{d}A_t, \qquad \forall t \ge 0, \end{cases} \tag{1.1}$$

where $\{W_t : t \ge 0\}$ is a cylindrical Wiener process, $\partial\varphi$, $\partial\psi$ are the subdifferentials of a convex lower semicontinuous functions φ, ψ, $\{A_t : t \ge 0\}$ is a progressively measurable increasing continuous stochastic process, and τ is a stopping time.

In fact, we will define and prove the existence of the solution for an equivalent form of (1.1):

$$\begin{cases} Y_t + \displaystyle\int_t^{\infty} \mathrm{d}K_s = \eta + \int_t^{\infty} \Phi(s, Y_s, Z_s)\,\mathrm{d}Q_s - \int_t^{\infty} Z_s\,\mathrm{d}W_s, & \text{a.s.}, t \ge 0, \\ \mathrm{d}K_t \in \partial_y \Psi(t, Y_t)\,\mathrm{d}Q_t, \qquad \text{on } [0, \infty), \end{cases} \tag{1.2}$$

with Q, Φ and Ψ adequately defined. The notation $\mathrm{d}K_t \in \partial_y \Psi(t, Y_t)\,\mathrm{d}Q_t$ means that Y is a continuous stochastic process and for any continuous stochastic process $X : \mathbb{R}_+ \to H$ and any $0 \le s_1 \le s_2$, the bounded variation of K on $[s_1, s_2]$ is finite and the following inequality holds:

$$\int_{s_1}^{s_2} \langle X_r - Y_r, \mathrm{d}K_r \rangle + \int_{s_1}^{s_2} \Psi(r, Y_r)\,\mathrm{d}Q_r \le \int_{s_1}^{s_2} \Psi(r, Y_r)\,\mathrm{d}Q_r, \qquad \text{a.s.}$$

The study of the backward stochastic differential equations (BSDEs for short) in the finite dimensional case (equation of type (1.1) with A and φ equal to 0) was initiated by Pardoux and Peng [16] (see also Pardoux and Peng [15]). The authors have proved the existence and the uniqueness of the solution for the BSDE on fixed time interval, under the assumption of Lipschitz continuity of F with respect to y and z and square integrability of η and $F(t, 0, 0)$. The case of BSDEs on random time interval (possibly infinite), under weaker assumptions on the data, have been treated by Darling and Pardoux [5], where it is obtained, as application, the existence of a continuous viscosity solution to the elliptic partial differential equations (PDEs) with Dirichlet boundary conditions.

The more general case of scalar BSDEs with one-sided reflection and associated optimal control problems was considered by El Karoui, Kapoudjian, Pardoux, Peng and Quenez [8] and with two-sided reflection associated with stochastic game problem by Cvitanić and Karatzas [4].

When the obstacles are fixed, the reflected BSDE become a particular case of BSVI of type (1.1), by taking Ψ as convex indicator of the interval defined by obstacles. We must mention that the solution of a BSVI belongs to the domain of the operator $\partial\Psi$ and it is reflected at the boundary of this.

The standard work on BSVI in the finite dimensional case is that of Pardoux and Răşcanu [17], where it is proved the existence and uniqueness of the solution (Y, Z, K) for BSVI (1.1) with $A \equiv 0$, under the following assumptions on F: monotonicity with respect to y (in the sense that $\langle y' - y, F(t, y', z) - F(t, y, z)\rangle \le \alpha |y' - y|^2$), Lipschitzianity with respect to z and a sublinear growth for $F(t, y, 0)$. Moreover, it is shown that, unlike the forward case, the process K is absolute continuous with respect to $\mathrm{d}t$. In Pardoux and Răşcanu [18], the same authors extend these results to the Hilbert spaces framework. Afterwards, various particular cases of BSVI (1.1) were the subject of many articles: Maticiuc and Răşcanu [11], Maticiuc, Răşcanu and Zălinescu [12], Maticiuc and Rotenstein [13], Maticiuc and Nie [9] (where the backward equations are studied in the frame of fractional stochastic calculus) and Diomande and Maticiuc [7] (where the generator F at the moment t is allowed to depend on the past values on $[0, t]$ of the solution (Y, Z)).

Our paper generalizes the existence and uniqueness results from Pardoux and Răşcanu [18] by considering random time interval $[0, \tau]$ and the Lebesgue–Stieltjes integral terms, and by assuming a weaker boundedness condition for the generator Φ (instead of the sublinear growth), that is,

$$\mathbb{E}\left(\int_0^T \Phi_\rho^\#(s)\,\mathrm{d}s\right)^p < \infty, \qquad \text{where } \Phi_\rho^\#(t) := \sup_{|y| \le \rho} \left|\Phi(t, y, 0)\right|. \tag{1.3}$$

We mention that, since τ is a stopping time, the presence of the process A is justified by the possible applications of equation (1.1) in proving probabilistic interpretation for the solution of elliptic multivalued partial differential equations with Neumann boundary conditions on a domain from $\mathbb{R}^d$. The stochastic approach of the existence problem for finite dimensional multivalued parabolic PDEs, was considered by Maticiuc and Răşcanu [11].

Concerning assumption (1.3), we recall that, in the case of finite dimensional BSDE, Pardoux [14] has used a similar condition, in order to prove the existence of a solution in L^2. His result was generalized by Briand, Delyon, Hu, Pardoux and Stoica [3], where it is proved the existence in L^p of the solution for BSDEs considered both with fixed and random terminal time. We mention that the assumptions from our paper are, broadly speaking, similar to those of Briand, Delyon, Hu, Pardoux and Stoica [3].

The article is organized as follows: in the next section a brief summary of infinite dimensional stochastic integral and the assumptions are given. Section 3 is devoted to the proof of the existence and uniqueness of a strong solution for (1.2). In the Section 4, is a new type of solution (called variational weak solution) and it is also proves the existence and uniqueness result. In Section 4 are obtained, as applications, the existence of the solution for various type of backward stochastic partial differential equations with boundary conditions. The Appendix contains, following Pardoux and Răşcanu [19], some results useful throughout the paper.

2. Preliminaries

2.1. Infinite dimensional framework

In the beginning of this subsection, we give a brief exposition of the stochastic integral with respect to a Wiener process defined on a Hilbert space. For a deeper discussion concerning the notion of cylindrical Wiener process and the construction of the stochastic integral, we refer reader to Da Prato and Zabczyk [6].

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set $\mathcal{N}_{\mathbb{P}} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$, a right continuous and complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and two real separable Hilbert spaces H, H_1.

Let us denote by $\mathcal{S}_H^p[0,T]$, $p \geq 0$, the complete metric space of continuous progressively measurable stochastic process (p.m.s.p.) $X : \Omega \times [0,T] \to H$ with the metric given by

$$\rho_p(X,\tilde{X}) = \begin{cases} \Big(\mathbb{E} \sup\limits_{t\in[0,T]} |X_t - \tilde{X}_t|^p\Big)^{1\wedge 1/p} < \infty, & \text{if } p > 0, \\ \mathbb{E}\Big(1 \wedge \sup\limits_{t\in[0,T]} |X_t - \tilde{X}_t|\Big) < \infty, & \text{if } p = 0, \end{cases}$$

and by $\mathcal{S}_H^p$ the space of p.m.s.p. $X : \Omega \times [0,\infty) \to H$ such that, for all $T > 0$, the restriction $X|_{[0,T]} \in \mathcal{S}_H^p[0,T]$. To shorten notation, we continue to write $\mathcal{S}^p$ for $\mathcal{S}_H^p$. Remark that $\mathcal{S}_H^p[0,T]$ is a Banach space for $p \geq 1$.

By $\mathrm{M}^p(\Omega \times [0,T]; H)$, $p \geq 1$, we denote the Banach space of the continuous stochastic processes M such that $\mathbb{E}(|M(t)|^p) < \infty$, $\forall t \in [0,T]$, $M(0) = 0$ a.s., and $\mathbb{E}^{\mathcal{F}_s}(M_t) = M_s$, a.s. for all $0 \leq s \leq t \leq T$. The norm is defined by $\|M\|_{\mathrm{M}^p} = [\mathbb{E}(|M(T)|^p)]^{1/p}$. If $p > 1$, then $\mathrm{M}^p(\Omega \times [0,T]; H)$ is a closed linear subspace of $\mathcal{S}_H^p[0,T]$.

Let $W = \{W_t(a) : t \geq 0, a \in H_1\} \subset L^0(\Omega, \mathcal{F}, \mathbb{P})$ be a Gaussian family of real-valued random variables with zero mean and the covariance function given by $\mathbb{E}[W_t(a)W_s(b)] = (t \wedge s)\langle a, b\rangle_{H_1}$, $s, t \geq 0$, $a, b \in H_1$. We call W a H_1-Wiener process if, for all $t \geq 0$,

(i) $\mathcal{F}_t^W := \sigma\{W_s(a) : s \in [0,t], a \in H_1\} \vee \mathcal{N}_{\mathbb{P}} \subset \mathcal{F}_t$,
(ii) $W_{t+h}(a) - W_t(a)$ is independent of $\mathcal{F}_t$, for all $h > 0$, $a \in H_1$.

Let $\{e_i\}_{i\in\mathbb{N}^*}$ be an orthonormal and complete basis in H_1. We introduce the separable Hilbert space $L_2(H_1; H)$ of Hilbert–Schmidt operators from H_1 to H, that is, the space of linear operators $Z : H_1 \to H$ such that $|Z|^2_{L_2(H_1;H)} = \sum_{i=1}^{\infty} |Ze_i|_H^2 = \mathrm{Tr}(Z^*Z) < \infty$. It will cause no confusion if we use $|Z|$ to designate the norm in $L_2(H_1; H)$.

The sequence $W^i = \{W_t^i := W_t(e_i) : t \in [0,T]\}$, $i \in \mathbb{N}^*$, defines a family of real-valued Wiener processes mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$.

If H_1 is finite dimensional space then we have the representation $W_t = \sum_i W_t^i$, $t \geq 0$, but, in general case, this series does not converge in H_1, but rather in a larger space H_2 such that $H_1 \subset H_2$ with the injection $J : H_1 \to H_2$ being a Hilbert–Schmidt operator. Moreover, $W \in \mathrm{M}^2(\Omega \times [0,T]; H_2)$.

For $0 < T \leq \infty$, we will denote by $\Lambda^p_{L_2(H_1,H)}(0,T)$, $p \geq 0$, the space $L^p_{ad}(\Omega \times (0,T); L_2(H_1, H))$, that is, the complete metric space of progressively measurable stochastic processes $Z : \Omega \times (0,T) \to L_2(H_1, H)$ with metric of convergence

$$d_p(Z, \tilde{Z}) = \begin{cases} \left(\mathbb{E}\left(\int_0^T |Z_s - \tilde{Z}_s|^2 \, \mathrm{d}s\right)^{p/2}\right)^{1\wedge 1/p} < \infty, & \text{if } p > 0, \\ \mathbb{E}\left(1 \wedge \left(\int_0^T |Z_s - \tilde{Z}_s|^2 \, \mathrm{d}s\right)^{1/2}\right) < \infty, & \text{if } p = 0. \end{cases}$$

The space $\Lambda^p_{L_2(H_1,H)}(0,T)$ is a Banach space for $p \geq 1$ with norm $\|Z\|_{\Lambda^p} = d_p(Z, 0)$. From now on, for simplicity of notation, we write $\Lambda^p(0,T)$ instead of $\Lambda^p_{L_2(H_1,H)}(0,T)$ (when no confusion can arise).

Let us denote by Λ^p the space of measurable stochastic processes $X : \Omega \times [0,\infty) \to H$ such that, for all $T > 0$, the restriction $X|_{[0,T]} \in \Lambda^p(0,T)$.

For any $Z \in \Lambda^2$ let the stochastic integral $\mathrm{I}(Z)(t) = \int_0^t Z_s \, \mathrm{d}W_s := \sum_{i=1}^{\infty} \int_0^t Z_s(e_i) \, \mathrm{d}W_s(e_i)$, $t \in [0,T]$, where $\{e_i\}_i$ is an orthonormal basis in H_1. Note that the introduced stochastic integral does not depend on the choice of the orthonormal basis on H_1. By the standard localization procedure, we can extend this integral as a linear continuous operator $\mathrm{I} : \Lambda^p(0,T) \to \mathcal{S}^p[0,T]$, $p \geq 0$, and it has the following properties:

Proposition 2.1. *Let* $Z \in \Lambda^p(0,T)$. *Then*

(i) $\mathbb{E}\mathrm{I}(Z)(t) = 0$, $\forall t \in [0,T]$, *if* $p \geq 1$,
(ii) $\mathbb{E}|\mathrm{I}(Z)(T)|^2 = \|Z\|^2_{\Lambda^2}$, *if* $p \geq 2$,
(iii) $\frac{1}{c_p}\|Z\|^p_{\Lambda^p} \leq \mathbb{E}\sup_{t\in[0,T]} |\mathrm{I}(Z)(t)|^p \leq c_p \|Z\|^p_{\Lambda^p}$, *if* $p > 0$ (*Burkholder–Davis–Gundy inequality*),
(iv) $\mathrm{I}(Z) \in \mathrm{M}^p(\Omega \times [0,T]; H)$, *if* $p \geq 1$.

From now on, we shall consider that the original filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is replaced by the filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$ generated by the Wiener process. The following Hilbert space version of the martingale representation theorem, extended to a random interval, holds the following proposition.

Proposition 2.2. *Let $\tau:\Omega\to[0,\infty]$ be a stopping time, $p>1$ and $\eta:\Omega\to H$ be a $\mathcal{F}_\tau$-measurable random variable such that $\mathbb{E}|\eta|^p<\infty$. Then*

1. *there exists a unique stochastic process $\zeta\in\Lambda^p(0,\infty)$ such that $\eta=\mathbb{E}\eta+\int_0^\tau\zeta_s\,\mathrm{d}W_s$ and $\zeta_t=\mathbb{1}_{[0,\tau]}(t)\zeta_t$, $\forall t\geq 0$, or equivalently,*
2. *there exists a unique pair $(\xi,\zeta)\in\mathcal{S}^p\times\Lambda^p(0,\infty)$ such that*

$$\xi_t=\eta-\int_{t\wedge\tau}^{\tau}\zeta_s\,\mathrm{d}W_s,\qquad a.s.,\ t\geq 0,\tag{2.1}$$

or equivalently,

3. *there exists a unique pair $(\xi,\zeta)\in\mathcal{S}^p\times\Lambda^p(0,\infty)$ such that $\xi_t=\eta-\int_t^\infty\zeta_s\,\mathrm{d}W_s$, a.s., $t\geq 0$ and $\xi_t=\mathbb{E}^{\mathcal{F}_t}\eta=\mathbb{E}^{\mathcal{F}_{t\wedge\tau}}\eta$ and $\zeta_t=\mathbb{1}_{[0,\tau]}(t)\zeta_t$, $t\geq 0$.*

2.2. Assumptions and definitions

In order to study equation (1.1), or the equivalent form (1.2), we introduce the next assumptions:

(A_1) *The parameter $p\geq 2$;*
(A_2) *The random variable $\tau:\Omega\to[0,\infty]$ is a stopping time;*
(A_3) *The random variable $\eta:\Omega\to H$ is $\mathcal{F}_\tau$-measurable such that $\mathbb{E}|\eta|^p<\infty$ and the stochastic process $(\xi,\zeta)\in\mathcal{S}^p\times\Lambda^p(0,\infty)$ is the unique pair associated to η such that we have the martingale representation formula (2.1);*
(A_4) *The process $\{A_t:t\geq 0\}$ is a progressively measurable increasing continuous stochastic process such that $A_0=0$;*
(A_5) *The functions $F:\Omega\times[0,\infty)\times H\times L_2(H_1,H)\to H$ and $G:\Omega\times[0,\infty)\times H\to H$ are such that*

$$\begin{cases} F(\cdot,\cdot,y,z),\ G(\cdot,\cdot,y) \text{ are p.m.s.p., for all}(y,z)\in H\times L_2(H_1,H),\\ F(\omega,t,\cdot,\cdot),\ G(\omega,t,\cdot) \text{ are continuous functions a.e.,}\end{cases}$$

and $\int_0^T F_\rho^\#(s)\,\mathrm{d}s+\int_0^T G_\rho^\#(s)\,\mathrm{d}A_s<\infty$, $\forall\rho,T\geq 0$, $\mathbb{P}$-a.s., where $F_\rho^\#(s)=\sup_{|y|\leq\rho}|F(s,y,0)|$ and $G_\rho^\#(s)=\sup_{|y|\leq\rho}|G(s,y)|$.

Moreover, there exist two p.m.s.p. $\mu,\nu:\Omega\times[0,\infty)\to\mathbb{R}$ such that $\int_0^T|\mu_t|^2\,\mathrm{d}t<\infty$ and $\int_0^T|\nu_t|^2\,\mathrm{d}A_t<\infty$, for all $T>0$, $\mathbb{P}$-a.s., and there exists $\ell\geq 0$, such that, for all $y,y'\in H$, $z,z'\in L_2(H_1,H)$,

$$\begin{aligned}\langle y'-y,F(t,y',z)-F(t,y,z)\rangle&\leq\mathbb{1}_{[0,\tau]}(t)\mu_t|y'-y|^2,\\ \langle y'-y,G(t,y')-G(t,y)\rangle&\leq\mathbb{1}_{[0,\tau]}(t)\nu_t|y'-y|^2,\\ |F(t,y,z')-F(t,y,z)|&\leq\mathbb{1}_{[0,\tau]}(t)\ell|z'-z|.\end{aligned}\tag{2.2}$$

Let us introduce the function

$$Q_t(\omega):=t+A_t(\omega)$$

and let $\{\alpha_t : t \geq 0\}$ be the a real positive p.m.s.p. (given by Radon–Nikodym's representation theorem) such that $\alpha \in [0,1]$ and $\mathrm{d}t = \alpha_t \, \mathrm{d}Q_t$ and $\mathrm{d}A_t = (1-\alpha_t)\,\mathrm{d}Q_t$.

Let

$$\Phi(\omega,t,y,z) := \mathbb{1}_{[0,\tau(\omega)]}(t)\big[\alpha_t(\omega)F(\omega,t,y,z) + \big(1-\alpha_t(\omega)\big)G(\omega,t,y)\big],$$

in which case (2.2) yields

$$\begin{aligned}\big\langle y'-y, \Phi\big(t,y',z\big) - \Phi(t,y,z)\big\rangle &\leq \mathbb{1}_{[0,\tau]}(t)\big[\mu_t\alpha_t + \nu_t(1-\alpha_t)\big]\big|y'-y\big|^2,\\ \big|\Phi\big(t,y,z'\big) - \Phi(t,y,z)\big| &\leq \mathbb{1}_{[0,\tau]}(t)\ell\alpha_t\big|z'-z\big|.\end{aligned}$$

For $a > 1$, let

$$\begin{aligned} V_t &= \int_0^t \mathbb{1}_{[0,\tau]}(s)\bigg[\bigg(\mu_s + \frac{a}{2}\ell^2\bigg)\alpha_s + \nu_s(1-\alpha_s)\bigg]\mathrm{d}Q_s \\ &= \int_0^t \mathbb{1}_{[0,\tau]}(s)\bigg[\bigg(\mu_s + \frac{a}{2}\ell^2\bigg)\mathrm{d}s + \nu_s\,\mathrm{d}A_s\bigg]. \end{aligned} \tag{2.3}$$

We can give now some a priori estimates concerning the solution of (1.1).

Lemma 2.1. *Let* $(Y,Z), (\tilde{Y},\tilde{Z}) \in \mathcal{S}^0[0,T] \times \Lambda^0(0,T)$. *Under assumption* (A_5) *the following inequalities hold*, *in the sense of signed measures on* $[0,\infty)$,

$$\big\langle Y_s, \Phi(s,Y_s,Z_s)\,\mathrm{d}Q_s\big\rangle \leq |Y_s|\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s + |Y_s|^2\,\mathrm{d}V_s + \frac{1}{2a}|Z_s|^2\,\mathrm{d}s \tag{2.4}$$

and

$$\big\langle Y_s - \tilde{Y}_s, \Phi(s,Y_s,Z_s) - \Phi(s,\tilde{Y}_s,\tilde{Z}_s)\big\rangle \mathrm{d}Q_s \leq |Y_s - \tilde{Y}_s|^2\,\mathrm{d}V_s + \frac{1}{2a}|Z_s - \tilde{Z}_s|^2\,\mathrm{d}s. \tag{2.5}$$

Proof. The inequalities can be obtained by standard calculus (applying the monotonicity and Lipschitz property of function Φ). □

(A_6) $\varphi,\psi : H \to [0,+\infty]$ *are proper convex lower semicontinuous* (*l.s.c.*) *functions such that* $\varphi(0) = \psi(0) = 0$ (*consequently* $0 \in \partial\varphi(0) \cap \partial\psi(0)$).

Let us define

$$\Psi(\omega,t,y) := \mathbb{1}_{[0,\tau(\omega)]}(t)\big[\alpha_t(\omega)\varphi(y) + \big(1-\alpha_t(\omega)\big)\psi(y)\big].$$

We recall now that the multivalued subdifferential operator $\partial\varphi$ is the maximal monotone operator

$$\partial\varphi(y) := \big\{\hat{y} \in H : \langle \hat{y}, v-y\rangle + \varphi(y) \leq \varphi(v), \forall v \in H\big\}.$$

We define $\mathrm{Dom}(\varphi) = \{y \in H : \varphi(y) < \infty\}$ and $\mathrm{Dom}(\partial\varphi) = \{y \in H : \partial\varphi(y) \neq \emptyset\} \subset \mathrm{Dom}(\varphi)$ and by $(y,\hat{y}) \in \partial\varphi$ we understand that $y \in \mathrm{Dom}(\partial\varphi)$ and $\hat{y} \in \partial\varphi(y)$. We know that $\mathrm{int}(\mathrm{Dom}(\varphi)) = \mathrm{int}(\mathrm{Dom}(\partial\varphi))$ and $\overline{\mathrm{Dom}(\varphi)} = \overline{\mathrm{Dom}(\partial\varphi)}$.

Definition 2.2. *If $k:[0,\infty)\to H$ is a locally bounded variation function, $a:[0,\infty)\to\mathbb{R}$ is a real increasing function, $y:[0,\infty)\to H$ is a continuous function and φ is like in* (A_6), *then notation $\mathrm{d}k_t\in\partial\varphi(y_t)\,\mathrm{d}a_t$ means that for any continuous function $x:[0,\infty)\to H$, it holds*

$$\int_t^s \langle x_r-y_r,\mathrm{d}k_r\rangle+\int_t^s\varphi(y_r)\,\mathrm{d}a_r\le\int_t^s\varphi(x_r)\,\mathrm{d}a_r,\qquad 0\le t\le s. \tag{2.6}$$

Now we are able to introduce the rigorous definition of a solution for equation (1.1). First, using definitions of Q, Φ and Ψ, respectively, we can rewrite (1.1) in the form

$$\begin{cases} Y_t+\displaystyle\int_t^\infty \mathrm{d}K_s=\eta+\int_t^\infty\Phi(s,Y_s,Z_s)\,\mathrm{d}Q_s-\int_t^\infty Z_s\,\mathrm{d}W_s, & \text{a.s., } t\ge 0,\\ \mathrm{d}K_t\in\partial_y\Psi(t,Y_t)\,\mathrm{d}Q_t=\partial\varphi(Y_t)\,\mathrm{d}t+\partial\psi(Y_t)\,\mathrm{d}A_t, & \text{on } [0,\infty). \end{cases} \tag{2.7}$$

Definition 2.3. *We call $(Y_t,Z_t,K_t)_{t\ge0}$ a solution of* (2.7) *if K has locally bounded variation and $(Y,Z)\in S^0\times\Lambda^0$ with $(Y_t,Z_t)=(\xi_t,\zeta_t)=(\eta,0)$ for $t>\tau$ such that*

(i) $\int_0^T|\Phi(s,Y_s,Z_s)|\,\mathrm{d}Q_s<\infty$, $\mathbb{P}$-*a.s., for all* $T\ge0$,
(ii) $\mathrm{d}K_t\in\partial_y\Psi(t,Y_t)\,\mathrm{d}Q_t$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}Q_t$-*a.e.*,
(iii) $e^{2V_T}|Y_T-\xi_T|^2+\int_T^\infty e^{2V_s}|Z_s-\zeta_s|^2\,\mathrm{d}s\xrightarrow{prob.}0$, *as* $T\to\infty$ (*where V is given by* (2.3)) *and*
(iv) $Y_t+\int_t^T\mathrm{d}K_s=Y_T+\int_t^T\Phi(s,Y_s,Z_s)\,\mathrm{d}Q_s-\int_t^T Z_s\,\mathrm{d}W_s$, *a.s.*, $\forall 0\le t\le T$.

Let $\varepsilon>0$ and the Moreau–Yosida regularization of φ given by $\varphi_\varepsilon(y)=\inf\{\frac{1}{2\varepsilon}|y-v|^2+\varphi(v):v\in H\}$, which is a C^1 convex function. We mention some properties (see Brézis [2], and Pardoux and Răşcanu [17] for the last one): for all $x,y\in H$

$$\begin{aligned} &\text{(a)}\quad \varphi_\varepsilon(x)=\frac{\varepsilon}{2}\big|\nabla\varphi_\varepsilon(x)\big|^2+\varphi\big(x-\varepsilon\nabla\varphi_\varepsilon(x)\big),\\ &\text{(b)}\quad \nabla\varphi_\varepsilon(x)=\partial\varphi_\varepsilon(x)\in\partial\varphi\big(x-\varepsilon\nabla\varphi_\varepsilon(x)\big),\\ &\text{(c)}\quad \big|\nabla\varphi_\varepsilon(x)-\nabla\varphi_\varepsilon(y)\big|\le\frac{1}{\varepsilon}|x-y|,\\ &\text{(d)}\quad \big\langle\nabla\varphi_\varepsilon(x)-\nabla\varphi_\varepsilon(y),x-y\big\rangle\ge0,\\ &\text{(e)}\quad \big\langle\nabla\varphi_\varepsilon(x)-\nabla\varphi_\delta(y),x-y\big\rangle\ge-(\varepsilon+\delta)\big\langle\nabla\varphi_\varepsilon(x),\nabla\varphi_\delta(y)\big\rangle. \end{aligned} \tag{2.8}$$

We introduce the *compatibility conditions* between φ,ψ (which have previously been used in Maticiuc and Răşcanu [11]):

(A_7) *For all $\varepsilon>0$, $t\ge0$, $y\in H$, $z\in L_2(H_1,H)$*

$$\begin{aligned} &\text{(i)}\quad \big\langle\nabla\varphi_\varepsilon(y),\nabla\psi_\varepsilon(y)\big\rangle\ge0,\\ &\text{(ii)}\quad \big\langle\nabla\varphi_\varepsilon(y),G(t,y)+\nu_t^-y\big\rangle\le\big|\nabla\psi_\varepsilon(y)\big|\big|G(t,y)+\nu_t^-y\big|,\qquad \mathbb{P}\text{-a.s.},\\ &\text{(iii)}\quad \big\langle\nabla\psi_\varepsilon(y),F(t,y,z)+\mu_t^-y\big\rangle\le\big|\nabla\varphi_\varepsilon(y)\big|\big|F(t,y,z)+\mu_t^-y\big|,\qquad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{2.9}$$

where $\mu^-=-\min\{\mu,0\}$ and $\nu^-=-\min\{\nu,0\}$.

Example 2.4. Let $H = \mathbb{R}$.

A. Clearly, since $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are increasing monotone, we see that, if $y(G(t,y)+\nu_t^- y) \le 0$ and $y(F(t,y,z)+\mu_t^- y) \le 0$, $\forall t, y, z$, then compatibility assumptions (2.9) are satisfied.

B. If $\varphi, \psi : \mathbb{R} \to (-\infty, +\infty]$ are the convexity indicator functions, that is,

$$\varphi(y) = \begin{cases} 0, & \text{if } y \in [a_1, a_2], \\ +\infty, & \text{if } y \notin [a_1, a_2], \end{cases} \quad \text{and} \quad \psi(y) = \begin{cases} 0, & \text{if } y \in [b_1, b_2], \\ +\infty, & \text{if } y \notin [b_1, b_2], \end{cases}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are such that $0 \in [a_1, a_2] \cap [b_1, b_2]$ (see assumption (A_6)), then $\nabla\varphi_\varepsilon(y) = \frac{1}{\varepsilon}[(y-a_2)^+ - (a_1 - y)^+]$ and similar for $\nabla\psi_\varepsilon$.

Since (A_7)(i) is fulfilled, the compatibility assumptions become $G(t,y) + \nu_t^- y \ge 0$, for $y \le a_1$ and $G(t,y) + \nu_t^- y \le 0$, for $y \ge a_2$, and, respectively, $F(t,y,z) + \mu_t^- y \ge 0$, for $y \le b_1$ and $F(t,y,z) + \mu_t^- y \le 0$, for $y \ge b_2$.

The last assumption is the following:

(A_8) *There exist the p.m.s.p.* $\tilde{\mu}, \tilde{\nu} : \Omega \times [0,\infty) \to \mathbb{R}$ *with* $\tilde{\mu} \ge \max\{\mu, \frac{1}{2}\mu\}$ *and* $\tilde{\nu} \ge \max\{\nu, \frac{1}{2}\nu\}$, *such that* $\int_0^T (|\tilde{\mu}_t|^2\,\mathrm{d}t + |\tilde{\nu}_t|^2\,\mathrm{d}A_t) < \infty, \forall T > 0, \mathbb{P}$*-a.s. and, using notation*

$$\tilde{V}_t := \int_0^t \mathbb{1}_{[0,\tau]}(s)\Big[\Big(\tilde{\mu}_s + \frac{a}{2}\ell^2\Big)\,\mathrm{d}s + \tilde{\nu}_s\,\mathrm{d}A_s\Big], \tag{2.10}$$

we suppose

$$\begin{aligned} &\text{(i)} \quad \mathbb{E}\big[\mathrm{e}^{2\sup_{s\in[0,\tau]}\tilde{V}_s}\big(\varphi(\eta)+\psi(\eta)\big)\big] < \infty, \\ &\text{(ii)} \quad \mathbb{E}\big(\mathrm{e}^{p\sup_{s\in[0,\tau]}\tilde{V}_s}|\eta|^p\big) + \mathbb{E}\big(Q_T^p\big) < \infty, \qquad \forall T > 0, \\ &\text{(iii)} \quad \mathbb{E}\Big(\int_0^\tau \mathrm{e}^{2\tilde{V}_s}\Psi(s,\xi_s)\,\mathrm{d}Q_s\Big)^{p/2} + \mathbb{E}\Big(\int_0^\tau \mathrm{e}^{\tilde{V}_s}\big|\Phi(s,\xi_s,\zeta_s)\big|\,\mathrm{d}Q_s\Big)^p < \infty, \end{aligned} \tag{2.11}$$

and the locally boundedness conditions:

$$\begin{aligned} &\text{(iv)} \quad \mathbb{E}\Big(\int_0^T \mathrm{e}^{\tilde{V}_s}\sup_{|y|\le\rho}\big|F\big(s,\mathrm{e}^{-\tilde{V}_s}y,0\big) - \tilde{\mu}_s y\big|\,\mathrm{d}s\Big)^p \\ &\qquad + \mathbb{E}\Big(\int_0^T \mathrm{e}^{\tilde{V}_s}\sup_{|y|\le\rho}\big|G\big(s,\mathrm{e}^{-\tilde{V}_s}y\big) - \tilde{\nu}_s y\big|\,\mathrm{d}A_s\Big)^p < \infty, \qquad \forall T, \rho > 0, \\ &\text{(v)} \quad \mathbb{E}\int_0^\tau \mathrm{e}^{2\tilde{V}_s}\sup_{|y|\le\rho}\big|F\big(s,\mathrm{e}^{-\tilde{V}_s}y,0\big)\big|^2\,\mathrm{d}s \\ &\qquad + \mathbb{E}\int_0^\tau \mathrm{e}^{2\tilde{V}_s}\sup_{|y|\le\rho}\big|G\big(s,\mathrm{e}^{-\tilde{V}_s}y\big)\big|^2\,\mathrm{d}A_s < \infty, \qquad \forall \rho > 0. \end{aligned} \tag{2.12}$$

Remark 2.1. We point out that the purpose of defining of the new process $\tilde{V}$ is due to the computations; see, e.g., inequalities (3.7) and (3.19) from the proof of the first main theorem, where it is necessary to have a new process $\tilde{V}$ such that $\mathrm{d}V_t \le \mathrm{d}\tilde{V}_t$ and $\frac{1}{2}\,\mathrm{d}V_t \le \mathrm{d}\tilde{V}_t$ on $[0,\infty)$.

Remark 2.2. It can be choose in (A_8), in particular, $\tilde{\mu}$ and $\tilde{\nu}$ such that $\tilde{\mu} = \mu^+ = \max\{\mu, 0\}$ and $\tilde{\nu} = \nu^+ = \max\{\nu, 0\}$. In this case $\tilde{V}$ defined by (2.10) will become non-decreasing, hence $\sup_{s\in[0,\tau]} \tilde{V}_s = \tilde{V}_\tau$ and (2.11) and (2.12) will be simplified.

We prefer to keep inequalities $\tilde{\mu} \ge \max\{\mu, \frac{1}{2}\mu\}$ and $\tilde{\nu} \ge \max\{\nu, \frac{1}{2}\nu\}$ in this form because we allow to $\tilde{\mu}$ and $\tilde{\nu}$ to be negative and therefore to enlarge the class of the generators F and G who satisfy (2.11) and (2.12) (and also we not restrict the class of the final data η).

3. Main result: The existence of the strong solution

We present first the definition of a solution in the strong case when the process K is absolutely continuous with respect to $\mathrm{d}Q$ (i.e., $\mathrm{d}K_t = U_t\,\mathrm{d}Q_t$ on $[0,\infty)$).

Definition 3.1. *We call $(Y_t, Z_t, U_t)_{t\ge0}$ a strong solution of (2.7) if there exist two p.m.s.p. U^1, U^2 and $U_t := \mathbb{1}_{[0,\tau]}(t)[\alpha_t U_t^1 + (1-\alpha_t)U_t^2]$, such that (Y, Z, K) is a solution of (2.7) with $K_t = \int_0^t U_s\,\mathrm{d}Q_s$ and*

$$\begin{aligned} &\text{(i)} \quad \int_0^T |U_s|\,\mathrm{d}Q_s < \infty, \qquad \mathbb{P}\text{-}a.s., \textit{for all } T \ge 0,\\ &\text{(ii)} \quad U_t^1 \in \partial\varphi(Y_t), \qquad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-}a.e.,\ U_t^2 \in \partial\psi(Y_t),\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}A_t\text{-}a.e.,\\ &\text{(iii)} \quad \mathbb{E}\big(\mathrm{e}^{2V_T}|Y_T - \xi_T|^2\big) + \mathbb{E}\int_T^\infty \mathrm{e}^{2V_s}|Z_s - \zeta_s|^2\,\mathrm{d}s \to 0, \qquad \textit{as } T \to \infty, \end{aligned} \tag{3.1}$$

where V is given by (2.3).

Remark 3.1. If there exists $C > 0$ such that $\sup_{s\in[0,\tau]} |V_s| \le C$, $\mathbb{P}$-a.s., then the condition (3.1)(iii) is equivalent to $\mathbb{E}|Y_T - \eta|^2 + \mathbb{E}\int_T^\infty |Z_s|^2\,\mathrm{d}s \to 0$, as $T \to \infty$.

We can now formulate the first main result. In order to obtain the absolute continuity with respect to $\mathrm{d}Q_t$ of the process K (as in Definition 3.1) it is necessary to impose a supplementary assumption:

(A_9) *There exists $R_0 > 0$ such that, for all $t \ge 0$,*

$$\mathbb{E}^{\mathcal{F}_t}\big(\mathrm{e}^{2\sup_{s\ge t}\tilde{V}_s}|\eta|^2\big) + \mathbb{E}^{\mathcal{F}_t}\bigg(\int_{t\wedge\tau}^\tau \mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^2 \le R_0, \qquad \text{a.s.} \tag{3.2}$$

Remark 3.2. We mention that without this assumption we are not able to prove, among other, that there exist two processes U^1 and U^2 such that $K_t = \int_0^t \mathbb{1}_{[0,\tau]}(t)[U_t^1\,\mathrm{d}t + U_t^2\,\mathrm{d}A_t]$ (see step F from the proof of the next theorem).

Theorem 3.2. *Let assumptions* (A_1)–(A_9) *be satisfied. Then the backward stochastic variational inequality* (2.7) *has a unique solution* (Y, Z, U) *such that for all* $T \geq 0$,

$$\mathbb{E} \sup_{s\in[0,T]} e^{p\tilde{V}_s}|Y_s|^p < \infty \tag{3.3}$$

and

$$Y_t + \int_t^T U_s \, dQ_s = Y_T + \int_t^T \Phi(s, Y_s, Z_s) \, dQ_s - \int_t^T Z_s \, dW_s, \qquad a.s., \forall t \in [0, T]. \tag{3.4}$$

Moreover, for all $2 \leq q \leq p$, *there exists a constant* $C = C(a, q) > 0$ *such that, for all* $t \geq 0$, $\mathbb{P}$-*a.s.*

$$\begin{aligned}
&\text{(a)} \quad e^{q\tilde{V}_t}|Y_t|^q + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2\tilde{V}_s}|Z_s|^2 \, ds\right)^{q/2} \\
&\qquad \leq C\mathbb{E}^{\mathcal{F}_t}\left[e^{q\sup_{s\geq t}\tilde{V}_s}|\eta|^q + \left(\int_t^\infty e^{\tilde{V}_s}\left|\Phi(s, 0, 0)\right| dQ_s\right)^q\right], \\
&\text{(b)} \quad e^{q\tilde{V}_t}|Y_t - \xi_t|^q + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2\tilde{V}_s}|Z_s - \zeta_s|^2 \, ds\right)^{q/2} \\
&\qquad \leq C\mathbb{E}^{\mathcal{F}_t}\left[\left(\int_t^\infty e^{2\tilde{V}_s}\Psi(s, \xi_s) \, dQ_s\right)^{q/2} + \left(\int_t^\infty e^{\tilde{V}_s}\left|\Phi(s, \xi_s, \zeta_s)\right| dQ_s\right)^q\right], \\
&\text{(c)} \quad \mathbb{E}\left[e^{2\tilde{V}_t}\left(\varphi(Y_t) + \psi(Y_t)\right)\right] \leq \mathbb{E}\left[e^{2\tilde{V}_\infty}\left(\varphi(\eta) + \psi(\eta)\right)\right], \\
&\text{(d)} \quad \lim_{T\to\infty} \mathbb{E}\left[e^{p\tilde{V}_T}|Y_T - \xi_T|^p + \left(\int_T^\infty e^{2\tilde{V}_s}|Z_s - \zeta_s|^2 \, ds\right)^{p/2}\right] = 0
\end{aligned} \tag{3.5}$$

and

$$\text{(e)} \quad \mathbb{E}\int_0^\tau \left[e^{2\tilde{V}_s}\left(\left|U_s^1\right|^2 ds + \left|U_s^2\right|^2 dA_s\right)\right] < \infty. \tag{3.6}$$

Proof. If (Y, Z), $(\bar{Y}, \bar{Z})$ are two solutions, in the sense of Definition 3.1, that satisfy (3.3), then $\mathbb{E}\sup_{s\in[0,T]} e^{p\tilde{V}_s}|Y_s - \bar{Y}_s|^p < \infty$. From (2.5), satisfied by the process $Y_s - \bar{Y}_s$, we conclude that

$$\left\langle Y_s - \bar{Y}_s, \Phi(s, Y_s, Z_s) - \Phi(s, \bar{Y}_s, \bar{Z}_s) - U_s + \bar{U}_s\right\rangle dQ_s \leq |Y_s - \bar{Y}_s|^2 \, d\tilde{V}_s + \frac{1}{2a}|Z_s - \tilde{Z}_s|^2 \, ds, \tag{3.7}$$

since $\langle Y_s - \bar{Y}_s, U_s - \bar{U}_s\rangle \geq 0$, for $U_s \in \partial_y\Psi(s, Y_s)$ and $\bar{U}_s \in \partial_y\Psi(s, \bar{Y}_s)$, and $dV_s \leq d\tilde{V}_s$ on $[0, \tau]$.

Applying Proposition A.1 from the Appendix, it follows that there exists $C = C(a, p) > 0$ such that

$$\mathbb{E} \sup_{s\in[0,T]} e^{p\tilde{V}_s}|Y_s - \bar{Y}_s|^p + \mathbb{E}\left(\int_0^T e^{2\tilde{V}_s}|Z_s - \bar{Z}_s|^2 \, ds\right)^{p/2} \leq C\mathbb{E}\left(e^{p\tilde{V}_T}|Y_T - \bar{Y}_T|^p\right) \xrightarrow[T\to\infty]{} 0,$$

and the uniqueness is proved.

The proof of the existence will be split into several steps.

A. *Approximating problem.* Let $n \in \mathbb{N}^*$ and $\varepsilon = 1/n$. We consider the approximating stochastic equation

$$\begin{aligned} &Y_t^n + \int_t^\infty \mathbb{1}_{[0,n]}(s) \nabla_y \Psi^n\big(s, Y_s^n\big)\, \mathrm{d}Q_s \\ &\quad = \eta + \int_t^\infty \mathbb{1}_{[0,n]}(s) \Phi\big(s, Y_s^n, Z_s^n\big)\, \mathrm{d}Q_s - \int_t^\infty Z_s^n \,\mathrm{d}W_s, \qquad \mathbb{P}\text{-a.s., } \forall t \geq 0, \end{aligned} \tag{3.8}$$

or equivalent, $\mathbb{P}$-a.s.,

$$\begin{cases} Y_t^n + \displaystyle\int_t^n \nabla_y \Psi^n\big(s, Y_s^n\big)\, \mathrm{d}Q_s \\ \quad = \mathbb{E}^{\mathcal{F}_n} \tilde{\eta} + \displaystyle\int_t^n \Phi\big(s, Y_s^n, Z_s^n\big)\, \mathrm{d}Q_s - \int_t^n Z_s^n \,\mathrm{d}W_s, \qquad \forall t \in [0,n], \\ \big(Y_t^n, Z_t^n\big) = (\xi_t, \zeta_t), \qquad \forall t > n, \end{cases} \tag{3.9}$$

with $\Psi^n(\omega, s, y) := \mathbb{1}_{[0,\tau(\omega)]}(s)[\alpha_s(\omega)\varphi_{1/n}(y) + (1-\alpha_s(\omega))\psi_{1/n}(y)]$.

We notice that $\Phi_n(t,y,z) := \mathbb{1}_{[0,n]}(t)(\Phi(t,y,z) - \nabla_y \Psi^n(t,y))$ satisfies inequalities

$$\begin{aligned} \big\langle y' - y, \Phi_n\big(t, y', z\big) - \Phi_n(t,y,z)\big\rangle &\leq \mathbb{1}_{[0,n\wedge\tau]}(t)\big[(\mu_t - n)\alpha_t + (\nu_t - n)(1-\alpha_t)\big]\big|y' - y\big|^2 \\ &\leq \mathbb{1}_{[0,n\wedge\tau]}(t)\big[\tilde{\mu}_t \alpha_t + \tilde{\nu}_t(1-\alpha_t)\big]\big|y' - y\big|^2 \end{aligned}$$

and $|\Phi_n(t,y,z') - \Phi_n(t,y,z)| \leq \mathbb{1}_{[0,n\wedge\tau]}(t)\ell\alpha_t|z' - z|$, since $\mu_s \leq \tilde{\mu}_s$ and $\nu_s \leq \tilde{\nu}_s$ on $[0,\infty)$.

The corresponding process $\tilde{V}_t^n$ (see definitions (2.3) and (2.10)) is given by

$$\tilde{V}_t^n = \int_0^t \mathbb{1}_{[0,n\wedge\tau]}(s)\bigg[\bigg(\tilde{\mu}_s + \frac{a}{2}\ell^2\bigg)\mathrm{d}s + \tilde{\nu}_s \,\mathrm{d}A_s\bigg].$$

Obviously, $\tilde{V}_t^n = \tilde{V}_{t\wedge n}$, $\forall t \geq 0$.

Applying Proposition A.2 from the Appendix, with Φ replaced with Φ_n, we deduce that equation (3.9) has a unique solution (Y^n, Z^n) such that

$$\mathbb{E} \sup_{s\in[0,n]} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n\big|^p + \mathbb{E}\bigg(\int_0^n \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n\big|^2 \,\mathrm{d}s\bigg)^{p/2} < \infty,$$

and, using (A.2), it can be prove that

$$\mathbb{E} \sup_{s\in[0,T]} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n\big|^p + \mathbb{E}\bigg(\int_0^T \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n\big|^2 \,\mathrm{d}s\bigg)^{p/2} < \infty, \qquad \text{for all } T \geq 0.$$

B. *Boundedness of Y^n and Z^n.* Since φ^n, ψ^n are convex functions and it is assumed that $\varphi(0) = \psi(0) = 0$, we see that $\langle \nabla_y \Psi^n(t,y), y\rangle \geq 0$, $\forall y \in H$, and therefore (2.4) becomes

$$\big\langle Y_t^n, \Phi_n\big(t, Y_t^n, Z_t^n\big)\,\mathrm{d}Q_t\big\rangle \leq \mathbb{1}_{[0,n]}(t)\big|Y_t^n\big|\big|\Phi(t,0,0)\big|\,\mathrm{d}Q_t + \big|Y_t^n\big|^2 \,\mathrm{d}\tilde{V}_t^n + \frac{1}{2a}\big|Z_t^n\big|^2 \,\mathrm{d}t.$$

Equation (3.8) can be written, for any $T \geq 0$, in the form

$$Y_t^n = Y_T^n + \int_t^T \Phi_n\big(s, Y_s^n, Z_s^n\big)\, \mathrm{d}Q_s - \int_t^T Z_s^n \,\mathrm{d}W_s, \qquad \mathbb{P}\text{-a.s., } \forall t \in [0, T].$$

Applying Proposition A.1 (see the Appendix), we deduce that, for all $q \in [2, p]$, there exists a constant $C = C(a, q) > 0$ such that such that, $\mathbb{P}$-a.s., for all $0 \leq t \leq T \leq n$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} \mathrm{e}^{q\tilde{V}_s}\big|Y_s^n\big|^q + \mathbb{E}^{\mathcal{F}_t}\bigg(\int_t^T \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n\big|^2\,\mathrm{d}s\bigg)^{q/2}\\
&\quad\leq C\mathbb{E}^{\mathcal{F}_t}\bigg[\mathrm{e}^{q\tilde{V}_T}\big|Y_T^n\big|^q + \bigg(\int_t^T \mathbb{1}_{[0,n]}(s)\mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^q\bigg]\\
&\quad\leq C\mathbb{E}^{\mathcal{F}_t}\bigg[\mathrm{e}^{q\sup_{s\in[t,T]}\tilde{V}_s}|\xi_T|^q + \bigg(\int_t^\infty \mathbb{1}_{[0,n]}(s)\mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^q\bigg]\\
&\quad\leq C\mathbb{E}^{\mathcal{F}_t}\bigg[\mathrm{e}^{q\sup_{s\in[t,T]}\tilde{V}_s}|\eta|^q + \bigg(\int_t^\infty \mathbb{1}_{[0,n]}(s)\mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^q\bigg],
\end{aligned}$$

since by Jensen's inequality we have $|\xi_T|^q = |\mathbb{E}^{\mathcal{F}_T}\eta|^q \leq \mathbb{E}^{\mathcal{F}_T}|\eta|^q$.

Using (A.2), it can be proved that the above inequality holds also for all $0 \leq t \vee n \leq T$.

Passing to limit as $T \to \infty$ we infer, using Beppo Levi's theorem, that $\mathbb{P}$-a.s.

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\geq t} \mathrm{e}^{q\tilde{V}_s}\big|Y_s^n\big|^q + \mathbb{E}^{\mathcal{F}_t}\bigg(\int_t^\infty \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n\big|^2\,\mathrm{d}s\bigg)^{q/2}\\
&\quad\leq C\mathbb{E}^{\mathcal{F}_t}\bigg[\big(\mathrm{e}^{q\sup_{s\in[t,\tau]}\tilde{V}_s}|\eta|^q\big) + \bigg(\int_t^\infty \mathbb{1}_{[0,n]}(s)\mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^q\bigg].
\end{aligned}\tag{3.10}$$

In particular, for $q = 2$, there exists another constant $C \geq 1$ such that, for all $t \geq 0$,

$$\begin{aligned}
\mathrm{e}^{2\tilde{V}_t}\big|Y_t^n\big|^2 &\leq C\mathbb{E}^{\mathcal{F}_t}\bigg[\big(\mathrm{e}^{2\sup_{s\in[t,\tau]}\tilde{V}_s}|\eta|^2\big)\\
&\qquad + \bigg(\int_t^\infty \mathbb{1}_{[0,n]}(s)\mathrm{e}^{\tilde{V}_s}\big|\Phi(s,0,0)\big|\,\mathrm{d}Q_s\bigg)^2\bigg]\\
&\leq 2CR_0^2 = R_0', \qquad \mathbb{P}\text{-a.s.,}
\end{aligned}\tag{3.11}$$

where R_0 is given by (3.2).

Remark 3.3. We emphasis that we just use, for the first time, assumption (A_9) and the obtained inequality (3.11) will be essential in order to deduce, using assumption (A_8) of locally boundedness for the generators, the subsequent step D (i.e., the boundedness of the gradient of Ψ^n) and what follows afterwards.

C. *Other boundedness results on Y^n and Z^n*. Since for all $u \in H$, $\langle u - y, \nabla_y \Psi^n(t, y)\rangle \le \Psi^n(t, u) - \Psi^n(t, y)$, we can deduce (see inequality (2.5)) that, as signed measures on $[0, n]$,

$$\begin{aligned}
&\big\langle Y_t^n - \xi_t, \Phi_n\big(s, Y_s^n, Z_s^n\big)\big\rangle \,\mathrm{d}Q_t \\
&\quad\le \big[\Psi^n(t, \xi_t) - \Psi^n\big(t, Y_t^n\big)\big]\,\mathrm{d}Q_t \\
&\qquad + \big|Y_t^n - \xi_t\big|\big|\Phi(t, \xi_t, \zeta_t)\big|\,\mathrm{d}Q_t + \big|Y_t^n\big|^2\,\mathrm{d}\tilde{V}_t^n + \frac{1}{2a}\big|Z_t^n - \zeta_t\big|^2\,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

But $0 \le \Psi^n(t, \xi_t) \le \Psi(t, \xi_t) = \mathbb{1}_{[0,\tau(\omega)]}(t)[\alpha_t(\omega)\varphi(\xi_t) + (1 - \alpha_t(\omega))\psi(\xi_t)]$, therefore (3.12) becomes

$$\begin{aligned}
&\Psi^n\big(t, Y_t^n\big)\,\mathrm{d}Q_t + \big\langle Y_t^n - \xi_t, \Phi_n\big(s, Y_s^n, Z_s^n\big)\big\rangle\,\mathrm{d}Q_t \\
&\quad\le \Psi(t, \xi_t)\,\mathrm{d}Q_t + \big|Y_t^n - \xi_t\big|\big|\Phi(t, \xi_t, \zeta_t)\big|\,\mathrm{d}Q_t + \big|Y_t^n\big|^2\,\mathrm{d}\tilde{V}_t^n + \frac{1}{2a}\big|Z_t^n - \zeta_t\big|^2\,\mathrm{d}t.
\end{aligned}$$

From (3.9), we see that (Y^n, Z^n) satisfies the equation

$$Y_t^n - \xi_t = \int_t^n \Phi_n\big(s, Y_s^n, Z_s^n\big)\,\mathrm{d}Q_s - \int_t^n \big(Z_s^n - \zeta_s\big)\,\mathrm{d}W_s, \qquad \forall t \in [0, n],$$

since $\xi_t = \xi_n - \int_t^n \zeta_s\,\mathrm{d}W_s$, $\forall t \in [0, n]$.

Applying again Proposition A.1, there exists a constant $C = C(a, p) > 0$ such that, $\mathbb{P}$-a.s., for all $t \in [0, n]$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,n]} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n - \xi_s\big|^p + \mathbb{E}^{\mathcal{F}_t}\bigg(\int_t^n \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n - \zeta_s\big|^2\,\mathrm{d}s\bigg)^{p/2} \\
&\qquad + \mathbb{E}\bigg(\int_t^n \mathrm{e}^{2\tilde{V}_s}\Psi^n\big(s, Y_s^n\big)\,\mathrm{d}Q_s\bigg)^{p/2} \\
&\quad\le C\mathbb{E}^{\mathcal{F}_t}\bigg[\bigg(\int_t^n \mathrm{e}^{2\tilde{V}_s}\Psi(s, \xi_s)\,\mathrm{d}Q_s\bigg)^{p/2} + \bigg(\int_t^n \mathrm{e}^{\tilde{V}_s}\big|\Phi(s, \xi_s, \zeta_s)\big|\,\mathrm{d}Q_s\bigg)^{p}\bigg].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\ge t} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n - \xi_s\big|^p + \mathbb{E}^{\mathcal{F}_t}\bigg(\int_t^\infty \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n - \zeta_s\big|^2\,\mathrm{d}s\bigg)^{p/2} \\
&\quad\le C\mathbb{E}^{\mathcal{F}_t}\bigg[\bigg(\int_t^\infty \mathrm{e}^{2\tilde{V}_s}\Psi(s, \xi_s)\,\mathrm{d}Q_s\bigg)^{p/2} + \bigg(\int_t^\infty \mathrm{e}^{\tilde{V}_s}\big|\Phi(s, \xi_s, \zeta_s)\big|\,\mathrm{d}Q_s\bigg)^{p}\bigg]
\end{aligned} \tag{3.13}$$

since $(Y_s^n, Z_s^n) = (\xi_s, \zeta_s)$ for $s > n$.

D. *Boundedness of $\nabla\varphi_{1/n}(Y_t^n)$ and $\nabla\psi_{1/n}(Y_t^n)$*. In order to obtain the boundedness for $|\nabla\varphi_{1/n}(Y_s^n)|^2$ it is essential to use the following stochastic subdifferential inequality (see Proposition 11 in Maticiuc and Răşcanu [11]), written first for $\varphi_{1/n}$: for all $0 \le t \le s \le n$

$$e^{2\tilde{V}_s}\varphi_{1/n}\big(Y_s^n\big) \ge e^{2\tilde{V}_t}\varphi_{1/n}\big(Y_t^n\big) + \int_t^s \varphi_{1/n}\big(Y_r^n\big)\,d\big(e^{2\tilde{V}_r}\big) + \int_t^s e^{2\tilde{V}_r}\nabla\varphi_{1/n}\big(Y_r^n\big)\,dY_r^n.$$

Hence,

$$e^{2\tilde{V}_s}\varphi_{1/n}\big(Y_s^n\big) \ge e^{2\tilde{V}_t}\varphi_{1/n}\big(Y_t^n\big) + 2\int_t^s e^{2\tilde{V}_r}\varphi_{1/n}\big(Y_r^n\big)\,d\tilde{V}_r + \int_t^s e^{2\tilde{V}_r}\nabla\varphi_{1/n}\big(Y_r^n\big)\,dY_r^n.$$

It follows that, $\mathbb{P}$-a.s. for all $0 \le t \le s \le n$,

$$\begin{aligned}
&e^{2\tilde{V}_t}\varphi_{1/n}\big(Y_t^n\big) + \int_t^s e^{2\tilde{V}_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big), \mathbb{1}_{[0,n]}(r)\nabla_y\Psi^n\big(r, Y_r^n\big)\big\rangle\,dQ_r \\
&\quad\le e^{2\tilde{V}_s}\varphi_{1/n}\big(Y_s^n\big) + \int_t^s e^{2\tilde{V}_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big), \mathbb{1}_{[0,n]}(r)\Phi\big(r, Y_r^n, Z_r^n\big)\,dQ_r\big\rangle \\
&\qquad - \int_t^s e^{2\tilde{V}_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big), Z_r^n\,dW_r\big\rangle - 2\int_t^s e^{2\tilde{V}_r}\varphi_{1/n}\big(Y_r^n\big)\,d\tilde{V}_r
\end{aligned}$$

(and a similar inequality for $\psi_{1/n}$).

Since

$$\varphi_{1/n}(0) + \psi_{1/n}(0) = 0 \le \varphi_{1/n}(y) + \psi_{1/n}(y) \le \varphi(y) + \psi(y), \qquad \forall y \in H, \tag{3.14}$$

we infer that

$$\begin{aligned}
&e^{2\tilde{V}_t}\big[\varphi_{1/n}\big(Y_t^n\big) + \psi_{1/n}\big(Y_t^n\big)\big] + \int_t^s \mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde{V}_r}\big[\alpha_r\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2 \\
&\qquad + \big\langle\nabla\varphi_{1/n}\big(Y_r^n\big), \nabla\psi_{1/n}\big(Y_r^n\big)\big\rangle + (1-\alpha_r)\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2\big]\,dQ_r \\
&\quad\le e^{2\tilde{V}_s}\big[\varphi\big(Y_s^n\big) + \psi\big(Y_s^n\big)\big] - 2\int_t^s e^{2\tilde{V}_r}\big[\varphi_{1/n}\big(Y_r^n\big) + \psi_{1/n}\big(Y_r^n\big)\big]\,d\tilde{V}_r \\
&\qquad - \int_t^s e^{2\tilde{V}_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big) + \nabla\psi_{1/n}\big(Y_r^n\big), Z_r^n\,dW_r\big\rangle \\
&\qquad + \int_t^s e^{2\tilde{V}_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big) + \nabla\psi_{1/n}\big(Y_r^n\big), \mathbb{1}_{[0,n]}(r)\Phi\big(r, Y_r^n, Z_r^n\big)\big\rangle\,dQ_r.
\end{aligned} \tag{3.15}$$

Using definition of Φ, compatibility assumptions (2.9) gives us

$$\begin{aligned}
&\big\langle\nabla\varphi_\varepsilon(y), \Phi(t, y, z)\big\rangle \\
&\quad= \mathbb{1}_{[0,\tau]}(t)\big\langle\nabla\varphi_\varepsilon(y), \alpha_t F(t, y, z) + (1-\alpha_t)G(t, y)\big\rangle \\
&\quad\le \mathbb{1}_{[0,\tau]}(t)\big(\alpha_t\big|F(t, y, z)\big|\big|\nabla\varphi_\varepsilon(y)\big| + (1-\alpha_t)\big|G(t, y)\big|\big|\nabla\psi_\varepsilon(y)\big| \\
&\qquad + (1-\alpha_t)\nu_t^-|y|\big|\nabla\psi_\varepsilon(y)\big| - (1-\alpha_t)\nu_t^-\big\langle\nabla\varphi_\varepsilon(y), y\big\rangle\big)
\end{aligned} \tag{3.16}$$

and respectively,

$$\begin{aligned}&\big\langle\nabla\psi_\varepsilon(y),\Phi(t,y,z)\big\rangle\\&\quad=\mathbb{1}_{[0,\tau]}(t)\big\langle\nabla\psi_\varepsilon(y),\alpha_tF(t,y,z)+(1-\alpha_t)G(t,y)\big\rangle\\&\quad\leq\mathbb{1}_{[0,\tau]}(t)\big(\alpha_t\big|F(t,y,z)\big|\big|\nabla\varphi_\varepsilon(y)\big|+(1-\alpha_t)\big|G(t,y)\big|\big|\nabla\psi_\varepsilon(y)\big|\\&\qquad+\alpha_t\mu_t^-|y|\big|\nabla\varphi_\varepsilon(y)\big|-\alpha_t\mu_t^-\big\langle\nabla\psi_\varepsilon(y),y\big\rangle\big).\end{aligned}\tag{3.17}$$

From (2.9)(i), (3.11), (3.15)–(3.17) and inequality $2ab\leq\frac{1}{\alpha}a^2+\alpha b^2$ with $\alpha\in\{2,4\}$, we obtain

$$\begin{aligned}&e^{2\tilde V_t}\big[\varphi_{1/n}\big(Y_t^n\big)+\psi_{1/n}\big(Y_t^n\big)\big]+\frac12\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\big[\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2dr+\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2dA_r\big]\\&\quad\leq e^{2\tilde V_s}\big[\varphi\big(Y_s^n\big)+\psi\big(Y_s^n\big)\big]+\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\big|Y_r^n\big|^2\big[\big|\mu_r^-\big|^2dr+\big|\nu_r^-\big|^2dA_r\big]\\&\qquad+\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)4e^{2\tilde V_r}\big[\big|F\big(r,Y_r^n,Z_r^n\big)\big|^2dr+\big|G\big(r,Y_r^n\big)\big|^2dA_r\big]\\&\qquad-\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\big[\mu_r^-\big\langle\nabla\psi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dr+\nu_r^-\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dA_r\big]\\&\qquad-\int_t^s e^{2\tilde V_r}\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big)+\nabla\psi_{1/n}\big(Y_r^n\big),Z_r^n\,dW_r\big\rangle-2\int_t^s e^{2\tilde V_r}\big[\varphi_{1/n}\big(Y_r^n\big)+\psi_{1/n}\big(Y_r^n\big)\big]d\tilde V_r.\end{aligned}\tag{3.18}$$

Using (3.14), the definition of $\tilde V$ and inequality $0\leq\varphi_{1/n}(y)\leq\langle\nabla\varphi_{1/n}(y),y\rangle$, $\forall y\in H$, we have

$$\begin{aligned}&-\mu_r^-\big\langle\nabla\psi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dr-\nu_r^-\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dA_r-2\big[\varphi_{1/n}\big(Y_r^n\big)+\psi_{1/n}\big(Y_r^n\big)\big]d\tilde V_r\\&\quad\leq-\mu_r^-\big\langle\nabla\psi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dr-\nu_r^-\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dA_r\\&\qquad+\big[\varphi_{1/n}\big(Y_r^n\big)+\psi_{1/n}\big(Y_r^n\big)\big]\big[\mu_r^-\,dr+\nu_r^-\,dA_r\big]\\&\quad\leq\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle\mu_r^-\,dr+\big\langle\nabla\psi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle\nu_r^-\,dA_r,\end{aligned}\tag{3.19}$$

since $2d\tilde V_r\geq dV_r\geq-\mu_r^-\,dr-\nu_r^-\,dA_r$ on $[0,\tau]$.

Hence,

$$\begin{aligned}&-\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\big[\mu_r^-\big\langle\nabla\psi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dr+\nu_r^-\big\langle\nabla\varphi_{1/n}\big(Y_r^n\big),Y_r^n\big\rangle dA_r\big]\\&\qquad-2\int_t^s e^{2\tilde V_r}\big[\varphi_{1/n}\big(Y_r^n\big)+\psi_{1/n}\big(Y_r^n\big)\big]d\tilde V_r\\&\quad\leq\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\bigg[\frac14\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2dr+\frac14\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2dA_r\bigg]\\&\qquad+\int_t^s\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde V_r}\big|Y_r^n\big|^2\big[\big|\mu_r^-\big|^2dr+\big|\nu_r^-\big|^2dA_r\big].\end{aligned}\tag{3.20}$$

Moreover, we see that $\mathbb{E}^{\mathcal{F}_t}\int_t^s e^{2\tilde{V}_r}\langle\nabla\varphi_{1/n}(Y_r^n)+\nabla\psi_{1/n}(Y_r^n), Z_r^n\,\mathrm{d}W_r\rangle=0$, because

$$\begin{aligned}
&\mathbb{E}\bigg(\int_t^s e^{4\tilde{V}_r}\big(\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|+\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|\big)^2\big|Z_r^n\big|^2\,\mathrm{d}r\bigg)^{1/2}\\
&\quad\le\mathbb{E}\bigg[\sup_{r\in[t,s]}\big(2ne^{\tilde{V}_r}\big|Y_r^n\big|\big)\bigg(\int_t^s e^{2\tilde{V}_r}\big|Z_r^n\big|^2\,\mathrm{d}r\bigg)^{1/2}\bigg]\\
&\quad\le 2\sqrt{2}n\sqrt{R_0'}\bigg[\mathbb{E}\bigg(\int_t^s e^{2\tilde{V}_r}\big|Z_r^n-\tilde{\zeta}_r\big|^2\,\mathrm{d}r\bigg)^{1/2}+\mathbb{E}\bigg(\int_t^s e^{2\tilde{V}_r}|\tilde{\zeta}_r|^2\,\mathrm{d}r\bigg)^{1/2}\bigg]\\
&\quad<\infty.
\end{aligned}$$

For $s=n$, Jensen's inequality yields

$$\mathbb{E}\big[e^{2\tilde{V}_s}\varphi\big(Y_s^n\big)+\psi\big(Y_s^n\big)\big]=\mathbb{E}\big[e^{2\tilde{V}_n}\big(\varphi(\xi_n)+\psi(\xi_n)\big)\big]\le\mathbb{E}\big[e^{2\tilde{V}_n}\big(\varphi(\eta)+\psi(\eta)\big)\big],$$

and using (3.20), inequality (3.18) becomes

$$\begin{aligned}
&\mathbb{E}\big[e^{2\tilde{V}_t}\big(\varphi_{1/n}\big(Y_t^n\big)+\psi_{1/n}\big(Y_t^n\big)\big)\big]\\
&\qquad+\frac{1}{4}\mathbb{E}\int_t^n\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde{V}_r}\big[\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2\,\mathrm{d}r+\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2\,\mathrm{d}A_r\big]\\
&\quad\le\mathbb{E}\big[e^{2\tilde{V}_n}\big(\varphi(\eta)+\psi(\eta)\big)\big]+4\mathbb{E}\int_t^n\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde{V}_r}\big(\big|F\big(r,Y_r^n,Z_r^n\big)\big|^2\,\mathrm{d}r+\big|G\big(r,Y_r^n\big)\big|^2\,\mathrm{d}A_r\big)\\
&\qquad+2\mathbb{E}\int_t^n\mathbb{1}_{[0,n\wedge\tau]}(r)e^{2\tilde{V}_r}\big|Y_r^n\big|^2\big[\big|\mu_r^-\big|^2\,\mathrm{d}r+\big|\nu_r^-\big|^2\,\mathrm{d}A_r\big].
\end{aligned}$$

The right hand side in the above inequality is bounded since

$$\begin{aligned}
e^{2\tilde{V}_r}\big|G\big(r,Y_r^n\big)\big|^2&\le\sup_{|y|\le\sqrt{R_0'}}e^{2\tilde{V}_r}\big|G\big(r,e^{-\tilde{V}_r}y\big)\big|^2,\\
e^{2\tilde{V}_r}\big|F\big(r,Y_r^n,Z_r^n\big)\big|^2&\le 3\sup_{|y|\le\sqrt{R_0'}}e^{2\tilde{V}_r}\big|F\big(r,e^{-\tilde{V}_r}y,0\big)\big|^2+3\ell^2e^{2\tilde{V}_r}\big|Z_r^n-\zeta_r\big|^2+3\ell^2e^{2\tilde{V}_r}|\zeta_r|^2.
\end{aligned}$$

Therefore

$$\mathbb{E}\big[e^{2\tilde{V}_t}\big(\varphi_{1/n}\big(Y_t^n\big)+\psi_{1/n}\big(Y_t^n\big)\big)\big]\le C,\qquad\text{for all } t\ge 0\tag{3.21}$$

and

$$\mathbb{E}\int_0^\infty\mathbb{1}_{[0,n\wedge\tau]}(r)\big[e^{2\tilde{V}_r}\big|\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2\,\mathrm{d}r+e^{2\tilde{V}_r}\big|\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2\,\mathrm{d}A_r\big]\le C.\tag{3.22}$$

From (3.21) and (2.8)(a) we see that, for all $t\ge 0$,

$$\mathbb{E}\big[e^{2\tilde{V}_t}\big(\big|1/n\nabla\varphi_{1/n}\big(Y_r^n\big)\big|^2+\big|1/n\nabla\psi_{1/n}\big(Y_r^n\big)\big|^2\big)\big]\le 2C/n\tag{3.23}$$

and

$$\mathbb{E}\big[\mathrm{e}^{2\tilde{V}_t}\big(\varphi\big(Y_t^n - 1/n\nabla\varphi_{1/n}\big(Y_r^n\big)\big) + \psi\big(Y_t^n - 1/n\nabla\psi_{1/n}\big(Y_r^n\big)\big)\big)\big] \le C. \tag{3.24}$$

E. *Cauchy sequences and convergence.* From (3.13), we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\ge n}\mathrm{e}^{p\tilde{V}_s}\big|Y_s^{n+l} - \xi_s\big|^p + \mathbb{E}\bigg(\int_n^\infty \mathrm{e}^{2\tilde{V}_s}\big|Z_s^{n+l} - \zeta_s\big|^2\,\mathrm{d}s\bigg)^{p/2}\\
&\quad\le C\mathbb{E}\bigg[\bigg(\int_n^\infty \mathrm{e}^{2\tilde{V}_s}\Psi(s,\xi_s)\,\mathrm{d}Q_s\bigg)^{p/2}\\
&\qquad+\bigg(\int_n^\infty \mathrm{e}^{\tilde{V}_s}\big|\Phi(s,\xi_s,\zeta_s)\big|\,\mathrm{d}Q_s\bigg)^p\bigg] \to 0, \qquad n\to\infty.
\end{aligned}\tag{3.25}$$

By uniqueness it follows that, for all $t\in[0,n]$,

$$Y_t^{n+l} - Y_t^n = Y_n^{n+l} - \xi_n + \int_t^n \mathrm{d}K_s^{n,l} - \int_t^n \big(Z_s^{n+l} - Z_s^n\big)\,\mathrm{d}W_s, \qquad \text{a.s.},$$

where $\mathrm{d}K_s^{n,l} = (\Phi(s,Y_s^{n+l},Z_s^{n+l}) - \Phi(s,Y_s^n,Z_s^n) - [\nabla_y\Psi^{n+l}(Y_s^{n+l}) - \nabla_y\Psi^n(Y_s^n)])\,\mathrm{d}Q_s$.

By (2.8)(d) with $\varepsilon = 1/(n+l)$ and $\delta = 1/n$

$$\begin{aligned}
&-\big\langle Y_s^{n+l} - Y_s^n, \big(\nabla_y\Psi^{n+l}\big(s,Y_s^{n+l}\big) - \nabla_y\Psi^n\big(s,Y_s^n\big)\big)\,\mathrm{d}Q_s\big\rangle\\
&\quad\le(\varepsilon+\delta)\mathbb{1}_{[0,\tau]}(s)\big(\big\langle\nabla\varphi_\varepsilon\big(Y_s^{n+l}\big),\nabla\varphi_\delta\big(Y_s^n\big)\big\rangle\,\mathrm{d}s + \big\langle\nabla\psi_\varepsilon\big(Y_s^{n+l}\big),\nabla\psi_\delta\big(Y_s^n\big)\big\rangle\,\mathrm{d}A_s\big),
\end{aligned}$$

and using (2.5) we have on $[0,n]$

$$\begin{aligned}
&\big\langle Y_s^{n+l} - Y_s^n, \mathrm{d}K_s^{n,l}\big\rangle\\
&\quad\le\frac{\varepsilon+\delta}{2}\mathbb{1}_{[0,\tau]}(s)\big[\big(\big|\nabla\varphi_\varepsilon\big(Y_s^n\big)\big|^2 + \big|\nabla\varphi_\delta\big(Y_s^{n+l}\big)\big|^2\big)\,\mathrm{d}s\\
&\qquad+\big(\big|\nabla\psi_\varepsilon\big(Y_s^n\big)\big|^2 + \big|\nabla\psi_\delta\big(Y_s^{n+l}\big)\big|^2\big)\,\mathrm{d}A_s\big] + \big|Y_s^{n+l} - Y_s^n\big|^2\,\mathrm{d}\tilde{V}_s^n + \frac{1}{2a}\big|Z_s^{n+l} - Z_s^n\big|^2\,\mathrm{d}s.
\end{aligned}$$

Proposition A.1 yields once again

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,n]}\mathrm{e}^{2\tilde{V}_s}\big|Y_s^{n+l} - Y_s^n\big|^2 + \mathbb{E}\int_0^n \mathrm{e}^{2\tilde{V}_s}\big|Z_s^{n+l} - Z_s^n\big|^2\,\mathrm{d}s\\
&\quad\le C\mathbb{E}\mathrm{e}^{2\tilde{V}_n}\big|Y_n^{n+l} - \xi_n\big|^2 + (\varepsilon+\delta)C\mathbb{E}\int_0^{n\wedge\tau}\mathrm{e}^{2\tilde{V}_s}\big(\big|\nabla\varphi_\varepsilon\big(Y_s^n\big)\big|^2 + \big|\nabla\varphi_\delta\big(Y_s^{n+l}\big)\big|^2\big)\,\mathrm{d}s\\
&\qquad+(\varepsilon+\delta)C\mathbb{E}\int_0^{n\wedge\tau}\mathrm{e}^{2\tilde{V}_s}\big(\big|\nabla\psi_\varepsilon\big(Y_s^n\big)\big|^2 + \big|\nabla\psi_\delta\big(Y_s^{n+l}\big)\big|^2\big)\,\mathrm{d}A_s.
\end{aligned}$$

The estimates (3.22) and (3.25) give us, for $n \to \infty$,

$$\mathbb{E} \sup_{s\in[0,n]} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^{n+l} - Y_s^n\right|^2 + \mathbb{E}\int_0^n \mathrm{e}^{2\tilde{V}_s} \left|Z_s^{n+l} - Z_s^n\right|^2 \mathrm{d}s$$
$$\leq \mathbb{E}\sup_{s\geq n} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^{n+l} - \xi_s\right|^2 + \frac{C}{n} \to 0.$$

Hence, for $n \to \infty$,

$$\mathbb{E}\sup_{s\geq 0} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^{n+l} - Y_s^n\right|^2 \leq \mathbb{E} \sup_{s\in[0,n]} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^{n+l} - Y_s^n\right|^2 + \mathbb{E}\sup_{s\geq n} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^{n+l} - \xi_s\right|^2 \to 0$$

and

$$\mathbb{E}\int_0^\infty \mathrm{e}^{2\tilde{V}_s} \left|Z_s^{n+l} - Z_s^n\right|^2 \mathrm{d}s \leq \mathbb{E}\int_0^n \mathrm{e}^{2\tilde{V}_s} \left|Z_s^{n+l} - Z_s^n\right|^2 \mathrm{d}s + \mathbb{E}\int_n^\infty \mathrm{e}^{2\tilde{V}_s} \left|Z_s^{n+l} - \zeta_s\right|^2 \mathrm{d}s \to 0.$$

F. *Passage to the limit.* Consequently there exists $(Y, Z) \in \mathcal{S}^2 \times \Lambda^2$ such that

$$\mathbb{E}\sup_{s\geq 0} \mathrm{e}^{2\tilde{V}_s} \left|Y_s^n - Y_s\right|^2 + \mathbb{E}\int_0^\infty \mathrm{e}^{2\tilde{V}_s} \left|Z_s^n - Z_s\right|^2 \mathrm{d}s \to 0, \qquad \text{as } n \to \infty.$$

We have $(Y_t, Z_t) = (\eta, 0)$ for $t > \tau$, since $Y_t^n = \xi_t = \eta$ and $Z_t^n = \zeta_t = 0$ for $t > \tau$.

Applying Fatou's lemma to (3.10) and (3.13), we obtain (3.5)(a), (b) and taking the limit along a subsequence in (3.11), we deduce that $\mathrm{e}^{2\tilde{V}_t}|Y_t|^2 \leq R_0'$, $\mathbb{P}$-a.s., for all $t \geq 0$.

From (3.22), there exist two p.m.s.p. U^1 and U^2, such that along a subsequence still indexed by n, $\mathbb{1}_{[0,\tau\wedge n]}\mathrm{e}^{2\tilde{V}}\nabla\varphi_{1/n}(Y^n) \rightharpoonup \mathbb{1}_{[0,\tau]}U^1$, weakly in $L^2(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; H)$ and $\mathbb{1}_{[0,\tau\wedge n]}\mathrm{e}^{2\tilde{V}}\nabla\psi_{1/n}(Y^n) \rightharpoonup \mathbb{1}_{[0,\tau]}U^2$, weakly in $L^2(\Omega \times \mathbb{R}_+, \mathrm{d}\mathbb{P} \otimes \mathrm{d}A_t; H)$.

Applying Fatou's lemma to (3.22), we obtain (3.6) and from (3.23) we deduce that for all $t \geq 0$ fixed, there exists a subsequence indexed also by n, such that

$$\frac{1}{n}\nabla\varphi_{1/n}\left(Y_t^n\right) \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad \frac{1}{n}\nabla\psi_{1/n}\left(Y_t^n\right) \xrightarrow{\text{a.s.}} 0.$$

We now apply Fatou's lemma to (3.24) and we conclude (3.5)(d).

From (3.8), we have for all $0 \leq t \leq T \leq n$, $\mathbb{P}$-a.s.

$$Y_t^n + \int_t^T \nabla_y \Psi^n\left(s, Y_s^n\right) \mathrm{d}Q_s = Y_T^n + \int_t^T \Phi\left(s, Y_s^n, Z_s^n\right) \mathrm{d}Q_s - \int_t^T Z_s^n \,\mathrm{d}W_s,$$

and passing to the limit we conclude that

$$Y_t + \int_t^T U_s \,\mathrm{d}Q_s = Y_T + \int_t^T \Phi(s, Y_s, Z_s)\,\mathrm{d}Q_s - \int_t^T Z_s \,\mathrm{d}W_s, \qquad \text{a.s.}$$

with $U_s = \mathbb{1}_{[0,\tau]}(s)[\alpha_s U_s^1 + (1 - \alpha_s)U_s^2]$, for $s \geq 0$.

Since (2.8)(b), we see that, for all $E \in \mathcal{F}$, $0 \le s \le t$ and $X \in \mathcal{S}^2[0,T]$,

$$\mathbb{E}\int_s^t \mathbb{1}_E\big(\big\langle e^{2\tilde{V}_r}\nabla\varphi_{1/n}\big(Y_r^n\big), X_r - Y_r^n\big\rangle + e^{2\tilde{V}_r}\varphi\big(Y_r^n - 1/n\nabla\varphi_{1/n}\big(Y_s^n\big)\big)\big)\,dr$$
$$\le \mathbb{E}\int_s^t \mathbb{1}_E e^{2\tilde{V}_r}\varphi(X_r)\,dr.$$

Passing to lim inf for $n \to \infty$ in the above inequality we obtain $U_s^1 \in \partial\varphi(Y_s)$, $d\mathbb{P}\otimes ds$-a.e. and, with similar arguments, $U_s^2 \in \partial\psi(Y_s)$, $d\mathbb{P}\otimes dA_s$-a.e.

Summarizing the above conclusions we see that (Y, Z, U) is solution of the BSVI (2.7) under assumptions (A_1)–(A_9). □

4. Variational weak formulation

In this section, we generalize the notion of solution for (1.1), or (2.7), in order to give up to the assumption (A_9). The existence and the uniqueness of a weak solution (Y, Z) will be given. We mention that without (A_9) we cannot prove the existence of a process K such that $dK_t = U_t\,dQ_t \in \partial_y\Psi(t, Y_t)\,dQ_t$ (see Remarks 3.2 and 3.3); more precisely we cannot obtain the boundedness in L^2 of the gradients, see (3.22), and respectively, the existence of a process U such that $K_t = \int_0^t U_s\,dQ_s$. Therefore, we shall give the definition of a weak solution of the BSVI (2.7).

Let us define the space $\mathcal{M}$ of the semimartingales $M \in \mathcal{S}^1$ of the form

$$M_t = \gamma - \int_0^t N_r\,dQ_r + \int_0^t R_r\,dW_r,$$

where N and R are two p.m.s.p. such that

$$\int_0^T |N_r|\,dQ_r + \int_0^T |R_r|^2\,dr < \infty \qquad \text{a.s., } \forall T > 0 \text{ and } \gamma \in L^0(\Omega; \mathcal{F}_0, \mathbb{P}; H).$$

For a intuitive introduction, let $M \in \mathcal{M}$ and (Y, Z) be strong a solution of (2.7), in the sense of Definition 3.1. By Itô's formula, we deduce inequality

$$\begin{aligned}
&\frac{1}{2}|M_t - Y_t|^2 + \frac{1}{2}\int_t^T |R_r - Z_r|^2\,dr + \int_t^T \Psi(r, Y_r)\,dQ_r\\
&\quad\le \frac{1}{2}|M_T - Y_T|^2 + \int_t^T \Psi(r, M_r)\,dQ_r\\
&\qquad + \int_t^T \big\langle M_r - Y_r, N_r - \Phi(r, Y_r, Z_r)\big\rangle\,dQ_r - \int_t^T \big\langle M_r - Y_r, (R_r - Z_r)\,dW_r\big\rangle,
\end{aligned} \tag{4.1}$$

since $dK_t = U_t\,dQ_t \in \partial_y\Psi(t, Y_t)\,dQ_t$ (see inequality (2.6)).

Following the approach for the forward stochastic variational inequalities from article Răşcanu [20], we propose the next weak formulation of Definition 3.1:

Definition 4.1. *We call $(Y_t, Z_t)_{t\geq 0}$ a variational weak solution of* (2.7) *if* $(Y, Z) \in \mathcal{S}^2 \times \Lambda^2$, $(Y_t, Z_t) = (\xi_t, \zeta_t) = (\eta, 0)$ *for* $t > \tau$ *and*

$$\text{(i)} \quad \int_0^T \big(\big|\Phi(s, Y_s, Z_s)\big| + \big|\Psi(s, Y_s)\big|\big)\,\mathrm{d}Q_s < \infty, \qquad \mathbb{P}\text{-}a.s., \textit{for all } T \geq 0,$$

$$\begin{aligned}\text{(ii)} \quad & \frac{1}{2}|M_t - Y_t|^2 + \frac{1}{2}\int_t^s |R_r - Z_r|^2\,\mathrm{d}r + \int_t^s \Psi(r, Y_r)\,\mathrm{d}Q_r \\ & \leq \frac{1}{2}|M_s - Y_s|^2 \\ & \quad + \int_t^s \Psi(r, M_r)\,\mathrm{d}Q_r + \int_t^s \big\langle M_r - Y_r, N_r - \Phi(r, Y_r, Z_r)\big\rangle\,\mathrm{d}Q_r \\ & \quad - \int_t^s \big\langle M_r - Y_r, (R_r - Z_r)\,\mathrm{d}W_r\big\rangle, \\ & \quad \forall 0 \leq t \leq s, \forall (N, R) \in L^2\big(\Omega \times [0, \infty); H\big) \times \Lambda^2, \forall M \in \mathcal{M},\end{aligned} \tag{4.2}$$

$$\text{(iii)} \quad \mathbb{E}\mathrm{e}^{2V_T}|Y_T - \xi_T|^2 + \mathbb{E}\int_T^\infty \mathrm{e}^{2V_s}|Z_s - \zeta_s|^2\,\mathrm{d}s \to 0, \qquad \textit{as } T \to \infty.$$

Remark 4.1. It is obviously that a strong solution for (2.7) is also a weak solution (see the intuitive introduction for inequality (4.1)).

Remark 4.2. We highlight the connection between this definition and the Fitzpatrick function approach for the multivalued BSDEs driven by a maximal monotone operator or, in particular, by a subdifferential operator (see Răşcanu and Rotenstein [21]).

Theorem 4.2. *Let assumptions* (A_1)–(A_8) *be satisfied. Then the backward stochastic variational inequality* (2.7) *has a unique solution* (Y, Z) *in the sense of Definition* 4.1 *such that* $\mathbb{E}\sup_{s\in[0,T]} \mathrm{e}^{p\tilde{V}_s}|Y_s|^p < \infty$, *for all* $T \geq 0$. *Moreover, inequalities* (3.5) *hold.*

Proof. First, we shall approximate the data η and Φ by η^n, respectively Φ^n which satisfy (3.2).

Let

$$\eta^n(\omega) = \eta(\omega)\mathbb{1}_{[0,n]}\Big(\big|\eta(\omega)\big| + \Big|\sup_{s\in[0,\tau]} \tilde{V}_s\Big|\Big),$$

$$\Phi^n(t, y, z) = \Phi(t, y, z) - \Phi(t, 0, 0) + \Phi(t, 0, 0)\mathbb{1}_{[0,n]}\big(t + \big|\Phi(t, 0, 0)\big| + |\tilde{V}_t|\big).$$

Obviously, as $n \to \infty$,

$$\mathbb{E}\big(\mathrm{e}^{p\sup_{s\in[0,\tau]}\tilde{V}_s}\big|\eta^n - \eta\big|^p\big) + \mathbb{E}\bigg(\int_0^\tau \mathrm{e}^{\tilde{V}_s}\big|\Phi^n(s, 0, 0) - \Phi(s, 0, 0)\big|\,\mathrm{d}Q_s\bigg)^p \to 0.$$

Theorem 3.2 shows that there exists a unique solution (Y^n, Z^n, U^n) of the BSVI (2.7) corresponding to η^n and Φ^n:

$$\begin{cases} Y_t^n + \int_t^\infty U_s^n \, \mathrm{d}Q_s = \eta^n + \int_t^\infty \Phi^n\big(s, Y_s^n, Z_s^n\big) \, \mathrm{d}Q_s - \int_t^\infty Z_s^n \, \mathrm{d}W_s, \qquad \text{a.s.,} \\ U_t^n \in \partial_y \Psi\big(t, Y_t^n\big), \qquad \forall t \ge 0. \end{cases}$$

This solution satisfies inequalities (3.5) and (3.6) with $Y, Z, U, \Phi, \eta, \xi, \zeta$ replaced respectively, with $Y^n, Z^n, U^n, \Phi^n, \eta^n, \xi^n, \zeta^n$.

Since $|\eta^n| \le |\eta|$ and $|\Phi^n(t, 0, 0)| \le |\Phi(t, 0, 0)|$,

$$\mathrm{e}^{2\tilde{V}_t} \big|Y_t^n\big|^2 \le C\mathbb{E}^{\mathcal{F}_t}\bigg[\big(\mathrm{e}^{2\sup_{s\in[t,\tau]} \tilde{V}_s}|\eta|^2\big) + \bigg(\int_t^\infty \mathrm{e}^{\tilde{V}_s}\big|\Phi(s, 0, 0)\big| \, \mathrm{d}Q_s\bigg)^2\bigg], \qquad \forall t \ge 0, \ \mathbb{P}\text{-a.s.}$$

Using (2.5), we see that

$$\begin{aligned} &\big\langle Y_s^n - Y_s^m, \Phi^n\big(s, Y_s^n, Z_s^n\big) - \Phi^m\big(s, Y_s^m, Z_s^m\big) - \big(U_s^n - U_s^m\big)\big\rangle \mathrm{d}Q_s \\ &\quad \le \big\langle Y_s^n - Y_s^m, \Phi\big(s, Y_s^n, Z_s^n\big) - \Phi\big(s, Y_s^m, Z_s^m\big)\big\rangle \mathrm{d}Q_s \\ &\qquad + \big\langle Y_s^n - Y_s^m, \Phi(s, 0, 0)\big\rangle (\mathbb{1}_{[0,n]} - \mathbb{1}_{[0,m]})\big(t + \big|\Phi(t, 0, 0)\big| + |\tilde{V}_t|\big) \, \mathrm{d}Q_s \\ &\quad \le \big|Y_s^n - Y_s^m\big|\big|\Phi(s, 0, 0)\big|\big|(\mathbb{1}_{[0,n]} - \mathbb{1}_{[0,m]})\big(t + \big|\Phi(t, 0, 0)\big| + |\tilde{V}_t|\big)\big| \, \mathrm{d}Q_s \\ &\qquad + \big|Y_s^n - Y_s^m\big|^2 \, \mathrm{d}\tilde{V}_s + \frac{1}{2a}\big|Z_s^n - Z_s^m\big|^2 \, \mathrm{d}s, \end{aligned}$$

since $\langle Y_s^n - Y_s^m, U_s^n - U_s^m\rangle \ge 0$, for $U_s^n \in \partial_y \Psi(s, Y_s^n)$ and $U_s^m \in \partial_y \Psi(s, Y_s^m)$, and $\mathrm{d}V_s \le \mathrm{d}\tilde{V}_s$ on $[0, \tau]$.

Applying Proposition A.1 (see the Appendix) for the equation satisfied by $Y^n - Y^m$ on $[0, T]$, it follows that

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,T]} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n - Y_s^m\big|^p + \mathbb{E}\bigg(\int_0^T \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n - Z_s^m\big|^2 \, \mathrm{d}s\bigg)^{p/2} \\ &\quad \le C\mathbb{E}\bigg(\int_0^T \mathrm{e}^{\tilde{V}_s}\big|\Phi(s, 0, 0)\big|\big|(\mathbb{1}_{[0,n]} - \mathbb{1}_{[0,m]})\big(t + \big|\Phi(t, 0, 0)\big| + |\tilde{V}_t|\big)\big| \, \mathrm{d}Q_s\bigg)^p \\ &\qquad + C\mathbb{E}\mathrm{e}^{p\tilde{V}_T}\big(\big|Y_T^n - \xi_T^n\big|^p + \big|\xi_T^n - \xi_T^m\big|^p + \big|\xi_T^m - Y_T^m\big|^p\big). \end{aligned}$$

Therefore, passing to the limit for $T \to \infty$,

$$\begin{aligned} &\mathbb{E} \sup_{s\ge 0} \mathrm{e}^{p\tilde{V}_s}\big|Y_s^n - Y_s^m\big|^p + \mathbb{E}\bigg(\int_0^\infty \mathrm{e}^{2\tilde{V}_s}\big|Z_s^n - Z_s^m\big|^2 \, \mathrm{d}s\bigg)^{p/2} \\ &\quad \le C\mathbb{E}\bigg(\int_0^\infty \mathrm{e}^{\tilde{V}_s}\big|\Phi(s, 0, 0)\big|\big|(\mathbb{1}_{[0,n]} - \mathbb{1}_{[0,m]})\big(t + \big|\Phi(t, 0, 0)\big| + |\tilde{V}_t|\big)\big| \, \mathrm{d}Q_s\bigg)^p \\ &\qquad + C\mathbb{E}\mathrm{e}^{p\sup_{s\ge 0} \tilde{V}_s}\big|\eta^n - \eta^m\big|^p \xrightarrow[n,m\to\infty]{} 0. \end{aligned}$$

Consequently there exists $(Y, Z) \in \mathcal{S}^0 \times \Lambda^0$ a solution of the BSVI (2.7) such that

$$\mathbb{E} \sup_{s\geq 0} e^{2\tilde{V}_s} \left|Y_s^n - Y_s\right|^2 + \mathbb{E} \int_0^\infty e^{2\tilde{V}_s} \left|Z_s^n - Z_s\right|^2 \mathrm{d}s \to 0, \qquad \text{as } n \to \infty$$

and $(Y_t, Z_t) = (\eta, 0)$ for $t > \tau$, since $Y_t^n = \xi_t^n = \eta^n$ and $Z_t^n = \zeta_t^n = 0$ for $t > \tau$.

Let $M \in \mathcal{M}$ given by $M_t = \gamma - \int_0^t N_r \,\mathrm{d}Q_r + \int_0^t R_r \,\mathrm{d}W_r$. From the Itô's formula applying to $|M_t - Y_t^n|^2$, we deduce that, for all $0 \leq t \leq s \leq \tau$,

$$\begin{aligned}
&\frac{1}{2}\left|M_t - Y_t^n\right|^2 + \frac{1}{2}\int_t^s \left|R_r - Z_r^n\right|^2 \mathrm{d}r + \int_t^s \Psi\left(r, Y_r^n\right) \mathrm{d}Q_r \\
&\quad\leq \frac{1}{2}\left|M_s - Y_s^n\right|^2 \\
&\qquad + \int_t^s \Psi(r, M_r)\,\mathrm{d}Q_r + \int_t^s \left\langle M_r - Y_r^n, N_r - \Phi\left(r, Y_r^n, Z_r^n\right)\right\rangle \mathrm{d}Q_r \\
&\qquad - \int_t^s \left\langle M_r - Y_r^n, \left(R_r - Z_r^n\right) \mathrm{d}W_r\right\rangle.
\end{aligned}$$

Since on a subsequence (still denoted by n)

$$\sup_{s\in[0,T]} \left|Y_s^n - Y_s\right|^2 + \int_0^T \left|Z_s^n - Z_s\right|^2 \mathrm{d}s \to 0, \qquad \text{a.s.,}$$

it follows easily, passing to the lim inf, that the couple (Y, Z) satisfies inequality (4.2)(ii).

In the same manner, inequalities (3.5) follow now from the similar properties satisfied by the approximate solution (Y^n, Z^n).

In order to prove the uniqueness of the solution let (Y^1, Z^1) and (Y^2, Z^2) be two solutions of (2.7) corresponding to η^1 and η^2, respectively. Hence,

$$\begin{aligned}
&\frac{1}{2}\left(\left|M_t - Y_t^1\right|^2 + \left|M_t - Y_t^2\right|^2\right) + \frac{1}{2}\int_t^s \left(\left|R_r - Z_r^1\right|^2 + \left|R_r - Z_r^2\right|^2\right) \mathrm{d}r \\
&\qquad + \int_t^s \left(\Psi\left(r, Y_r^1\right) + \Psi\left(r, Y_r^2\right)\right) \mathrm{d}Q_r \\
&\quad\leq \frac{1}{2}\left(\left|M_s - Y_s^1\right|^2 + \left|M_s - Y_s^2\right|^2\right) \\
&\qquad + \int_t^s \left(\left\langle M_r - Y_r^1, N_r - \Phi\left(r, Y_r^1, Z_r^1\right)\right\rangle + \left\langle M_r - Y_r^2, N_r - \Phi\left(r, Y_r^2, Z_r^2\right)\right\rangle\right) \mathrm{d}Q_r \\
&\qquad + 2\int_t^s \Psi(r, M_r)\,\mathrm{d}Q_r - \int_t^s \left\langle M_r - Y_r^1, \left(R_r - Z_r^1\right) \mathrm{d}W_r\right\rangle \\
&\qquad - \int_t^s \left\langle M_r - Y_r^2, \left(R_r - Z_r^2\right) \mathrm{d}W_r\right\rangle, \qquad \forall 0 \leq t \leq s, \forall M \in \mathcal{M}.
\end{aligned}$$

Let $Y=\frac{Y^1+Y^2}{2}$, $Z=\frac{Z^1+Z^2}{2}$ and $\Phi(r)=\frac{\Phi(r,Y_r^1,Z_r^1)+\Phi(r,Y_r^2,Z_r^2)}{2}$. From the convexity of Ψ we see that $2\Psi(r,Y_r)\le\Psi(r,Y_r^1)+\Psi(r,Y_r^2)$ and, using the identity

$$2\left\langle\frac{y^1+y^2}{2},\frac{f^1+f^2}{2}\right\rangle+\frac{1}{2}\langle y^1-y^2,f^1-f^2\rangle=\langle y^1,f^1\rangle+\langle y^2,f^2\rangle,$$

we obtain

$$\begin{aligned}&\langle M_r-Y_r^1,N_r-\Phi(r,Y_r^1,Z_r^1)\rangle+\langle M_r-Y_r^2,N_r-\Phi(r,Y_r^2,Z_r^2)\rangle\\&\quad-2\langle M_r-Y_r,N_r-\Phi(r)\rangle-\frac{1}{2}\langle Y_r^1-Y_r^2,\Phi(r,Y_r^1,Z_r^1)-\Phi(r,Y_r^2,Z_r^2)\rangle=0\end{aligned}$$

and

$$\langle M_r-Y_r^1,R_r-Z_r^1\rangle+\langle M_r-Y_r^2,R_r-Z_r^2\rangle-2\langle M_r-Y_r,R_r-Z_r\rangle-\frac{1}{2}\langle Y_r^1-Y_r^2,Z_r^1-Z_r^2\rangle=0.$$

Therefore, since $\frac{1}{2}(|m-y^1|^2+|m-y^2|^2)=|m-\frac{y^1+y^2}{2}|^2+\frac{1}{4}|y^1-y^2|^2$, we have

$$\begin{aligned}&|Y_t^1-Y_t^2|^2+\int_t^s|Z_r^1-Z_r^2|^2\,\mathrm{d}r\\&\quad\le 8B_{t,s}(M)+|Y_s^1-Y_s^2|^2\\&\qquad+2\int_t^s\langle Y_r^1-Y_r^2,\Phi(r,Y_r^1,Z_r^1)-\Phi(r,Y_r^2,Z_r^2)\rangle\,\mathrm{d}Q_r\\&\qquad-2\int_t^s\langle Y_r^1-Y_r^2,(Z_r^1-Z_r^2)\,\mathrm{d}W_r\rangle,\qquad\forall 0\le t\le s,\forall M\in\mathcal{M},\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}B_{t,s}(M)&=\frac{1}{2}|M_s-Y_s|^2+\int_t^s\Psi(r,M_r)\,\mathrm{d}Q_r-\int_t^s\Psi(r,Y_r)\,\mathrm{d}Q_r-\frac{1}{2}|M_t-Y_t|^2\\&\quad-\frac{1}{2}\int_t^s|R_r-Z_r|^2\,\mathrm{d}r+\int_t^s\langle M_r-Y_r,N_r-\Phi(r)\rangle\,\mathrm{d}Q_r\\&\quad-\int_t^s\langle M_r-Y_r,(R_r-Z_r)\,\mathrm{d}W_r\rangle.\end{aligned}$$

Our next goal will be to prove that

$$\text{there exists } M^\varepsilon\in\mathcal{M} \text{ such that } \lim_{\varepsilon\to 0}B_{t,s}(M^\varepsilon)=0,\qquad\text{a.s., }\forall 0\le t\le s.\tag{4.4}$$

Let $M_t^\varepsilon=\mathrm{e}^{-Q_t/Q_\varepsilon}[Y_0+\frac{1}{Q_\varepsilon}\int_0^t\mathrm{e}^{Q_r/Q_\varepsilon}Y_r\,\mathrm{d}Q_r]$. Clearly, $M^\varepsilon\in\mathcal{M}$, since $M_t^\varepsilon=M_0^\varepsilon+\int_0^t\mathrm{d}M_r^\varepsilon$.

The next result it is necessary in order to obtain the limit in the Stieltjes type integrals:

Lemma 4.3. *Let $a:[0,T]\to\mathbb{R}$ be a strictly increasing continuous function such that $a(0)=0$ and $f:[0,T]\to H$ be a measurable function such that $|f(t)|\le C$ a.e. $t\in[0,T]$. Define, for $\varepsilon>0$,*

$$f_\varepsilon(t)=f(0)\mathrm{e}^{-a(t)/a(\varepsilon)}+\frac{1}{a(\varepsilon)}\int_0^t \mathrm{e}^{(a(r)-a(t))/a(\varepsilon)}f(r)\,\mathrm{d}a(r).$$

Then as $\varepsilon\to 0$, $f_\varepsilon(t)\to f(t)$, a.e. $t\in[0,T]$ and, if f is continuous, then $\sup_{s\in[0,T]}|f_\varepsilon(t)-f(t)|\to 0$.

Remark 4.3. The same conclusions are true if we consider $f_\varepsilon(t)$ replaced by

$$g_\varepsilon(t)=f(T)\mathrm{e}^{(a(t)-a(T))/a(\varepsilon)}+\frac{1}{a(\varepsilon)}\int_t^T \mathrm{e}^{(a(t)-a(r))/a(\varepsilon)}f(r)\,\mathrm{d}a(r),\qquad t\in[0,T].$$

Proof of Lemma 4.3. Obviously, we have

$$\begin{aligned}
&\int_0^t \frac{1}{a(\varepsilon)}\mathrm{e}^{(a(r)-a(t))/a(\varepsilon)}f(r)\,\mathrm{d}a(r)\\
&\quad=\int_{-a(t)/a(\varepsilon)}^0 \mathrm{e}^u f\bigl(\bigl(a^{-1}\bigl(ua(\varepsilon)+a(t)\bigr)\bigr)\bigr)\,\mathrm{d}u\\
&\quad=\int_{-a(t)/a(\varepsilon)}^0 \mathrm{e}^u\bigl[f\bigl(a^{-1}\bigl(ua(\varepsilon)+a(t)\bigr)\bigr)-f\bigl(a^{-1}\bigl(a(t)\bigr)\bigr)\bigr]\,\mathrm{d}u+f(t)\int_{-a(t)/a(\varepsilon)}^0 \mathrm{e}^u\,\mathrm{d}u.
\end{aligned}$$

But

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int_{-a(t)/a(\varepsilon)}^0 \mathrm{e}^u\bigl[f\bigl(a^{-1}\bigl(ua(\varepsilon)+a(t)\bigr)\bigr)-f\bigl(a^{-1}\bigl(a(t)\bigr)\bigr)\bigr]\,\mathrm{d}u\right|\\
&\quad\le\limsup_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{e}^u\bigl|f\bigl(a^{-1}\bigl(\bigl(ua(\varepsilon)+a(t)\bigr)\vee 0\bigr)\bigr)-f\bigl(a^{-1}\bigl(a(t)\bigr)\bigr)\bigr|\,\mathrm{d}u\\
&\quad\le 2C\int_{-\infty}^{-n}\mathrm{e}^u\,\mathrm{d}u+\int_{-n}^0 \mathrm{e}^u\bigl|f\bigl(a^{-1}\bigl(\bigl(ua(\varepsilon)+a(t)\bigr)\vee 0\bigr)\bigr)-f\bigl(a^{-1}\bigl(a(t)\bigr)\bigr)\bigr|\,\mathrm{d}u\\
&\quad\le 2C\mathrm{e}^{-n}+\limsup_{\varepsilon\to 0}\int_{-n}^0 \bigl|f\bigl(a^{-1}\bigl(\bigl(ua(\varepsilon)+a(t)\bigr)\vee 0\bigr)\bigr)-f\bigl(a^{-1}\bigl(a(t)\bigr)\bigr)\bigr|\,\mathrm{d}u\le 2C\mathrm{e}^{-n},
\end{aligned}$$

for all n, since $\lim_{\delta\to 0}\int_\alpha^\beta|f(a^{-1}(s+\delta u))-f(a^{-1}(s))|\,\mathrm{d}u=0$ a.e.

Therefore, there exists

$$\lim_{\varepsilon\to 0}\left|\int_{-a(t)/a(\varepsilon)}^0 \mathrm{e}^u\bigl[f\bigl(a^{-1}\bigl(ua(\varepsilon)+a(t)\bigr)\bigr)-f(t)\bigr]\,\mathrm{d}u\right|=0,$$

and the first conclusion follows.

In the case of continuity for f it is sufficient to write

$$\begin{aligned} f_\varepsilon(t) &= f(0)\mathrm{e}^{-a(t)/a(\varepsilon)} + \frac{1}{a(\varepsilon)}\int_0^t \mathrm{e}^{(a(r)-a(t))/a(\varepsilon)} f(r)\,\mathrm{d}a(r) \\ &= f(0)\mathrm{e}^{-a(t)/a(\varepsilon)} + \frac{1}{a(\varepsilon)}\int_0^{t_\varepsilon} \mathrm{e}^{(a(r)-a(t))/a(\varepsilon)} f(r)\,\mathrm{d}a(r) \\ &\quad + \frac{1}{a(\varepsilon)}\int_{t_\varepsilon}^t \mathrm{e}^{(a(r)-a(t))/a(\varepsilon)} f(r)\,\mathrm{d}a(r), \end{aligned}$$

where $t_\varepsilon := a^{-1}(a(t) - \sqrt{a(\varepsilon)}) \to t$, as $\varepsilon \to 0$, and $t_\varepsilon < t$. □

Applying the above lemma, we can conclude that

$$M_t^\varepsilon \to Y_t, \qquad \forall t \in [0,T]. \tag{4.5}$$

Next, we shall prove that, for all $t \le s$,

$$\int_t^s \Psi\big(r, M_r^\varepsilon\big)\,\mathrm{d}Q_r \to \int_t^s \Psi(r, Y_r)\,\mathrm{d}Q_r.$$

Using definition of M^ε and the convexity of the functions φ and ψ we deduce that

$$\begin{aligned} \int_t^s \varphi\big(M_r^\varepsilon\big)\alpha_r\,\mathrm{d}Q_r &\le \int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\varphi(Y_0)\,\mathrm{d}r + \int_t^s \left(\int_0^r \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\varphi(Y_u)\,\mathrm{d}Q_u\right)\mathrm{d}r \\ &= \varphi(Y_0)\int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\,\mathrm{d}r + \int_0^s \left(\int_0^s \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\varphi(Y_u)\mathbb{1}_{[0,r]}(u)\,\mathrm{d}Q_u\right)\mathrm{d}r \\ &\quad - \int_0^t \left(\int_0^t \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\varphi(Y_u)\mathbb{1}_{[0,r]}(u)\,\mathrm{d}Q_u\right)\mathrm{d}r \qquad (4.6) \\ &= \varphi(Y_0)\int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\,\mathrm{d}r + \int_0^s \left(\varphi(Y_u)\int_0^s \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\mathbb{1}_{[u,s]}(r)\,\mathrm{d}r\right)\mathrm{d}Q_u \\ &\quad - \int_0^t \left(\varphi(Y_u)\int_0^t \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\mathbb{1}_{[u,t]}(r)\,\mathrm{d}r\right)\mathrm{d}Q_u \end{aligned}$$

and

$$\begin{aligned} &\int_t^s \psi\big(M_r^\varepsilon\big)(1-\alpha_r)\,\mathrm{d}Q_r \\ &\quad\le \int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\psi(Y_0)\,\mathrm{d}A_r + \int_t^s \left(\int_0^r \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\psi(Y_u)\,\mathrm{d}Q_u\right)\mathrm{d}A_r \\ &\quad= \psi(Y_0)\int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\,\mathrm{d}A_r + \int_0^s \left(\int_0^r \frac{1}{Q_\varepsilon}\mathrm{e}^{(Q_u-Q_r)/Q_\varepsilon}\psi(Y_u)\,\mathrm{d}Q_u\right)\mathrm{d}A_r \qquad (4.7) \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^t \left(\int_0^r \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} \psi(Y_u)\,\mathrm{d}Q_u \right) \mathrm{d}A_r \\
&= \psi(Y_0) \int_t^s \mathrm{e}^{-Q_r/Q_\varepsilon}\,\mathrm{d}A_r + \int_0^s \left(\psi(Y_u) \int_0^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} \mathbb{1}_{[u,s]}(r)\,\mathrm{d}A_r \right) \mathrm{d}Q_u \\
&\quad - \int_0^t \left(\psi(Y_u) \int_0^t \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} \mathbb{1}_{[u,t]}(r)\,\mathrm{d}A_r \right) \mathrm{d}Q_u.
\end{aligned}
$$

On the other hand, using Remark 4.3 and Lebesgue's dominated convergence theorem, we conclude that

$$
\lim_{\varepsilon \to 0} \int_u^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} \alpha_r\,\mathrm{d}Q_r = \lim_{\varepsilon \to 0} \int_u^t \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} \alpha_r\,\mathrm{d}Q_r = \alpha_u, \qquad \text{a.e.} \tag{4.8}
$$

and respectively,

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \int_u^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} (1 - \alpha_r)\,\mathrm{d}Q_r &= \lim_{\varepsilon \to 0} \int_u^t \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon} (1 - \alpha_r)\,\mathrm{d}Q_r \\
&= \alpha_u, \qquad \text{a.e.}
\end{aligned} \tag{4.9}
$$

From inequalities (4.6) and (4.7), we obtain

$$
\begin{aligned}
&\int_t^s \Psi(r, Y_r)\,\mathrm{d}Q_r \\
&\quad \le \int_t^s \Psi\bigl(r, M_r^\varepsilon\bigr)\,\mathrm{d}Q_r \\
&\quad \le \varphi(Y_0) Q_\varepsilon \mathrm{e}^{-Q_t/Q_\varepsilon} \int_t^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_t - Q_r)/Q_\varepsilon}\,\mathrm{d}r + \psi(Y_0) Q_\varepsilon \mathrm{e}^{-Q_t/Q_\varepsilon} \int_t^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_t - Q_r)/Q_\varepsilon}\,\mathrm{d}A_r \\
&\qquad + \int_0^s \left(\varphi(Y_u) \int_u^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon}\,\mathrm{d}r + \psi(Y_u) \int_u^s \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon}\,\mathrm{d}A_r \right) \mathrm{d}Q_u \\
&\qquad - \int_0^t \left(\varphi(Y_u) \int_u^t \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon}\,\mathrm{d}r + \psi(Y_u) \int_u^t \frac{1}{Q_\varepsilon} \mathrm{e}^{(Q_u - Q_r)/Q_\varepsilon}\,\mathrm{d}A_r \right) \mathrm{d}Q_u,
\end{aligned}
$$

and applying the limits (4.8) and (4.9), we deduce

$$
\begin{aligned}
\int_t^s \Psi\bigl(r, M_r^\varepsilon\bigr)\,\mathrm{d}Q_r &\xrightarrow{\varepsilon \to 0} \int_t^s \bigl(\varphi(Y_u)\alpha_u + \psi(Y_u)(1 - \alpha_u)\bigr)\,\mathrm{d}Q_u \\
&= \quad \int_t^s \Psi(u, Y_u)\,\mathrm{d}Q_u.
\end{aligned} \tag{4.10}
$$

Therefore (4.4) follows immediately, since we have (4.5) and (4.10).

Now returning to inequality (4.3), for all $0 \le t \le s$,

$$\begin{aligned} &\left|Y_t^1 - Y_t^2\right|^2 + \int_t^s \left|Z_r^1 - Z_r^2\right|^2 \mathrm{d}r \\ &\quad \le \left|Y_s^1 - Y_s^2\right|^2 + 2\int_t^s \left\langle Y_r^1 - Y_r^2, \Phi\left(r, Y_r^1, Z_r^1\right) - \Phi\left(r, Y_r^2, Z_r^2\right)\right\rangle \mathrm{d}Q_r \\ &\qquad - 2\int_t^s \left\langle Y_r^1 - Y_r^2, \left(Z_r^1 - Z_r^2\right) \mathrm{d}W_r\right\rangle. \end{aligned} \tag{4.11}$$

From (2.5),

$$\left\langle Y_r^1 - Y_r^2, \Phi\left(r, Y_r^1, Z_r^1\right) - \Phi\left(r, Y_r^2, Z_r^2\right)\right\rangle \mathrm{d}Q_r \le \left|Y_r^1 - Y_r^2\right|^2 \mathrm{d}\tilde{V}_r + \frac{1}{2a}\left|Z_r - Z_r^2\right|^2 \mathrm{d}r,$$

and therefore inequality (4.11) becomes

$$\begin{aligned} &\left|Y_t^1 - Y_t^2\right|^2 + (1 - 1/a)\int_t^s \left|Z_r^1 - Z_r^2\right|^2 \mathrm{d}r \\ &\quad \le \left|Y_s^1 - Y_s^2\right|^2 + 2\int_t^s \left|Y_r^1 - Y_r^2\right|^2 \mathrm{d}\tilde{V}_r - 2\int_t^s \left\langle Y_r^1 - Y_r^2, \left(Z_r^1 - Z_r^2\right) \mathrm{d}W_r\right\rangle. \end{aligned}$$

Applying a Gronwall's type stochastic inequality (see Lemma 12 from the Appendix of Maticiuc and Răşcanu [10]) we conclude that, for all $0 \le t \le s$, $\mathbb{P}$-a.s.

$$\mathrm{e}^{2\tilde{V}_t}\left|Y_t^1 - Y_t^2\right|^2 \le \mathrm{e}^{2\tilde{V}_s}\left|Y_s^1 - Y_s^2\right|^2 - 2\int_t^s \mathrm{e}^{2\tilde{V}_r}\left\langle Y_r^1 - Y_r^2, \left(Z_r^1 - Z_r^2\right) \mathrm{d}W_r\right\rangle.$$

Therefore, using also the condition (4.2)(iii) form the definition of weak variational solution, the uniqueness follows. □

5. Examples

Let $\mathcal{D} \subset \mathbb{R}^d$ be an open bounded subset with boundary $\mathrm{Bd}(\mathcal{D})$ sufficiently smooth. In what follows $H^m(\mathcal{D})$ and $H_0^m(\mathcal{D})$ stand for the usual Sobolev spaces. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ be a complete probability space, $\{W_s : 0 \le s \le t\}$ a real Wiener process and set $H = H_1 := L^2(\mathcal{D})$. We notice that the space of Hilbert–Schmidt operators from $L^2(\mathcal{D})$ to $L^2(\mathcal{D})$ can be identified with $L^2(\mathcal{D} \times \mathcal{D})$.

Let $j : \mathbb{R} \to (\infty, \infty]$ be a proper convex l.s.c. function, for which we assume that $j(u) \ge j(0) = 0$, $\forall u \in \mathbb{R}$.

Our aim is to obtain, via Theorem 3.2, the existence and uniqueness of the solution for some backward stochastic partial differential equations (SPDE) suggested in Pardoux and Răşcanu [18]. We recall assumptions (A_1)–(A_5), (A_8)(2.12), condition $\mathbb{E}(Q_T^p) < \infty$, $\forall T > 0$, and definitions of Φ and V from Section 2.2.

Example 5.1. First we consider the backward SPDE with Dirichlet boundary condition

$$\begin{cases} -dY(t,x) + \partial j\big(Y(t,x)\big)\,\mathrm{d}Q_t \ni \Delta Y(t,x)\,\mathrm{d}Q_t + \Phi\big(t, Y(t,x), Z(t,x)\big)\,\mathrm{d}Q_t \\ \qquad\qquad\qquad\qquad\qquad - Z(t,x)\,\mathrm{d}W_t, \qquad \text{in } \Omega\times[0,\tau]\times\mathcal{D}, \\ Y(\omega,t,x) = 0 \qquad \text{on } \Omega\times[0,\tau]\times\mathrm{Bd}(\mathcal{D}), \\ \mathrm{e}^{2V_T}\left\|Y(T)-\xi_T\right\|^2 + \displaystyle\int_T^\infty \mathrm{e}^{2V_s}\left\|Z(s)-\zeta_s\right\|^2\mathrm{d}s \xrightarrow[T\to\infty]{\text{prob.}} 0, \end{cases} \tag{5.1}$$

where $\|f\|^2 := \int_{\mathcal{D}}|f(x)|^2\,\mathrm{d}x$.

Let us apply Theorem 3.2, with $\Psi=\varphi=\psi$ (in which case the compatibility assumptions (2.9) are satisfied), where $\varphi: L^2(\mathcal{D})\to(-\infty,\infty]$ is given by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{D}}\left|\nabla u(x)\right|^2\mathrm{d}x + \int_{\mathcal{D}} j\big(u(x)\big)\,\mathrm{d}x, & \text{if } u\in H_0^1(\mathcal{D}),\, j(u)\in L^1(\mathcal{D}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 2.8 from Barbu [1], Chapter II, shows that the following properties hold:

(a) function φ is proper, convex and l.s.c.,
(b) $\partial\varphi(u) = \{u^*\in L^2(\mathcal{D}): u^*(x)\in\partial j(u(x)) - \Delta u(x) \text{ a.e. on } \mathcal{D}\}$, $\forall u\in\mathrm{Dom}(\partial\varphi)$,
(c) $\mathrm{Dom}(\partial\varphi) = \{u\in H_0^1(\mathcal{D})\cap H^2(\mathcal{D}): u(x)\in\mathrm{Dom}(\partial j) \text{ a.e. on } \mathcal{D}\}$.

Moreover, there exists a positive constant C such that

(d) $\|u\|_{H_0^1(\mathcal{D})\cap H^2(\mathcal{D})} \le C\|u^*\|_{L^2(\mathcal{D})}$, $\forall(u,u^*)\in\partial\varphi$.

Let η be a $H_0^1(\mathcal{D})$-valued random variable, $\mathcal{F}_\tau$-measurable such that (A_9) is satisfied and

$$\mathbb{E}\big[\mathrm{e}^{p\sup_{s\in[0,\tau]}\tilde{V}_s}|\eta|^p\big] < \infty, \qquad \mathrm{e}^{2\sup_{s\in[0,\tau]}\tilde{V}_s} j(\eta)\in L^1(\Omega\times\mathcal{D}),$$

and the stochastic processes ξ,ζ, associated to η by the martingale representation theorem, such that

$$\mathbb{E}\bigg(\int_0^\tau \mathrm{e}^{2\tilde{V}_s}\varphi(\xi_s)\,\mathrm{d}Q_s\bigg)^{p/2} + \mathbb{E}\bigg(\int_0^\tau \mathrm{e}^{\tilde{V}_s}\big|\Phi(s,\xi_s,\zeta_s)\big|\,\mathrm{d}Q_s\bigg)^p < \infty, \tag{5.2}$$

where $\tilde{V}$ is defined by (2.10).

Applying now Theorem 3.2, we deduce that, under the above assumptions, backward SPDE (5.1) has a unique solution $(Y,Z,U)\in\mathcal{S}^0_{L^2(\mathcal{D})}\times\Lambda^0_{L^2(\mathcal{D}\times\mathcal{D})}\times\Lambda^0_{L^2(\mathcal{D})}$ such that $(Y(t),Z(t)) = (\xi_t,\zeta_t) = (\eta,0)$, for $t\ge\tau$, and

(i) $Y(t,x)+\int_t^T U(s,x)\,\mathrm{d}Q_s = Y(T,x)+\int_t^T\Delta Y(s,x)\,\mathrm{d}Q_s+\int_t^T\Phi(s,Y(s,x),Z(s,x))\,\mathrm{d}Q_s - \int_t^T Z(s,x)\,\mathrm{d}W_s$, in $[0,T]\times\mathcal{D}$, a.s.,
(ii) $Y(t)\in H_0^1(\mathcal{D})\cap H^2(\mathcal{D})$, $\mathrm{d}\mathbb{P}\times\mathrm{d}t$ a.e.,
(iii) $Y(t,x)\in\mathrm{Dom}(\partial j)$, $\mathrm{d}\mathbb{P}\times\mathrm{d}Q_t\times\mathrm{d}x$ a.e.,
(iv) $U(t,x)\in\partial j(Y(t,x))$, $\mathrm{d}\mathbb{P}\times\mathrm{d}Q_t\times\mathrm{d}x$ a.e.,

(v) $e^{2\tilde{V}} Y \in L^{\infty}(0,T;L^2(\Omega;H_0^1(\mathcal{D})))$ and $e^{2\tilde{V}} j(Y) \in L^{\infty}(0,T;L^1(\Omega\times\mathcal{D}))$, $\forall T>0$,

(vi) $\mathbb{E}\int_0^{\tau} e^{2\tilde{V}_s}\|Y(s)\|^2_{H_0^1(\mathcal{D})\cap H^2(\mathcal{D})}\,dQ_s<\infty$.

Remark 5.1. If we renounce at assumption (A_9), then it follows that backward SPDE (5.1) admits a variational weak solution. More precisely, Theorem 4.2 shows that there exists a unique solution $(Y,Z)\in \mathcal{S}^0_{L^2(\mathcal{D})}\times \Lambda^0_{L^2(\mathcal{D}\times\mathcal{D})}$ such that $(Y(t),Z(t))=(\xi_t,\zeta_t)=(\eta,0)$, for $t\ge\tau$, and for all $0\le t\le s$

(i) $\frac{1}{2}\|M(t)-Y(t)\|^2+\frac{1}{2}\int_t^s|R(r)-Z(r)|^2\,dr+\int_t^s\int_{\mathcal{D}} j(Y(r,x))\,dx\,dQ_r\le \frac{1}{2}\|M(s)-Y(s)\|^2+\int_t^s\int_{\mathcal{D}} j(M(r,x))\,dx\,dQ_r+\int_t^s\langle\langle M(r)-Y(r),N(r)-\Delta Y(r)-\Phi(r,Y(r),Z(r))\rangle\rangle\,dQ_r-\int_t^s\langle\langle M(r)-Y(r),(R(r)-Z(r))\,dW_r\rangle\rangle$, $\forall (N,R)\in L^2(\Omega\times[0,\infty);L^2(\mathcal{D}))\times\Lambda^2_{L^2(\mathcal{D}\times\mathcal{D})}$, $\forall M\in\mathcal{M}$,

(ii) $Y(t)\in H_0^1(\mathcal{D})$, $d\mathbb{P}\times dt$ a.e.,

(iii) $Y(t,x)\in \mathrm{Dom}(j)$, $d\mathbb{P}\times dQ_t\times dx$ a.e.,

where $\mathcal{M}$ is defined at the beginning of the Section 4 and

$$\|f\|^2:=\int_{\mathcal{D}}\big|f(x)\big|^2\,dx \quad\text{and}\quad \big\langle\langle f,g\rangle\big\rangle:=\int_{\mathcal{D}} f(x)g(x)\,dx.$$

Example 5.2. As a second example we consider the backward SPDE with Neumann boundary condition

$$\begin{cases} -dY(t,x)=\Delta Y(t,x)\,dQ_t \\ \qquad\qquad +\Phi\big(t,Y(t,x),Z(t,x)\big)\,dQ_t-Z_t\,dW_t, & \text{in } \Omega\times[0,\tau]\times\mathcal{D},\\ -\dfrac{\partial Y(\omega,t,x)}{\partial n}\in\partial j\big(Y(\omega,t,x)\big), & \text{on } \Omega\times[0,\tau]\times \mathrm{Bd}(\mathcal{D}),\\ e^{2V_T}\big\|Y(T)-\xi_T\big\|^2+\displaystyle\int_T^{\infty} e^{2V_s}\big\|Z(s)-\zeta_s\big\|^2\,ds\xrightarrow[T\to\infty]{\text{prob.}}0. \end{cases}\tag{5.3}$$

We apply again Theorem 3.2, with $\Psi=\varphi=\psi$, where $\varphi:L^2(\mathcal{D})\to(-\infty,\infty]$ is given by

$$\varphi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_{\mathcal{D}}\big|\nabla u(x)\big|^2\,dx+\int_{\mathrm{Bd}(\mathcal{D})} j\big(u(x)\big)\,dx, & \text{if } u\in H^1(\mathcal{D}) \text{ and } k(u)\in L^1\big(\mathrm{Bd}(\mathcal{D})\big),\\ +\infty, & \text{otherwise.}\end{cases}$$

Proposition 2.9 from Barbu [1], Chapter II, shows that:

(a) function φ is proper, convex and l.s.c.,

(b) $\partial\varphi(u)=-\Delta u(x)$, $\forall u\in\mathrm{Dom}(\partial\varphi)$,

(c) $\mathrm{Dom}(\partial\varphi)=\{u\in H^2(\mathcal{D}):-\frac{\partial u(x)}{\partial n}\in\partial j(u(x))$ a.e. on $\mathrm{Bd}(\mathcal{D})\}$,

where $\frac{\partial u}{\partial n}$ is the outward normal derivative to the boundary.

Moreover, there are some positive constants C_1, C_2 such that

(d) $\|u\|_{H^2(\mathcal{D})}\le C_1\|u-\Delta u\|_{L^2(\mathcal{D})}+C_2$, $\forall u\in\mathrm{Dom}(\partial\varphi)$.

Let η be a $H^1(\mathcal{D})$-valued random variable, $\mathcal{F}_\tau$-measurable such that (A₉) is satisfied and

$$\mathbb{E}\big[\mathrm{e}^{p\sup_{s\in[0,\tau]}\tilde{V}_s}|\eta|^p\big]<\infty,\qquad \mathrm{e}^{2\sup_{s\in[0,\tau]}\tilde{V}_s}j(\eta)\in L^1\big(\Omega\times\mathrm{Bd}(\mathcal{D})\big),$$

and the stochastic processes ξ and ζ (from the martingale representation theorem) be such that (5.2) holds.

Applying Theorem 3.2 we conclude that, under the above assumptions, backward SPDE (5.3) has a unique solution $(Y,Z)\in\mathcal{S}^0_{L^2(\mathcal{D})}\times\Lambda^0_{L^2(\mathcal{D}\times\mathcal{D})}$ such that $(Y(t),Z(t))=(\xi_t,\zeta_t)=(\eta,0)$, for $t\geq\tau$, and

(i) $Y(t,x)=Y(T,x)+\int_t^T\Delta Y(s,x)\,\mathrm{d}Q_s+\int_t^T\Phi(s,Y(s,x),Z(s,x))\,\mathrm{d}Q_s-\int_t^T Z(s,x)\,\mathrm{d}W_s$, in $[0,T]\times\mathcal{D}$, a.s.,
(ii) $Y(t)\in H^2(\mathcal{D})$, $\mathrm{d}\mathbb{P}\times\mathrm{d}t$ a.e.,
(iii) $-\frac{\partial Y(t,x)}{\partial n}\in\mathrm{Dom}(\partial j)$, $\mathrm{d}\mathbb{P}\times\mathrm{d}Q_t\times\mathrm{d}x$ a.e.,
(iv) $\mathrm{e}^{2\tilde{V}}Y\in L^\infty(0,T;L^2(\Omega;H^1(\mathcal{D})))$ and $\mathrm{e}^{2\tilde{V}}j(Y)\in L^\infty(0,T;L^1(\Omega\times\mathrm{Bd}(\mathcal{D})))$, $\forall T>0$,
(v) $\mathbb{E}\int_0^\tau\mathrm{e}^{2\tilde{V}_s}\|Y(s)\|^2_{H^2(\mathcal{D})}\,\mathrm{d}Q_s<\infty$.

Example 5.3. The third example is the backward stochastic porous media equation

$$\begin{cases}-\mathrm{d}Y(t,x)=\Delta(\partial j)\big(Y(t,x)\big)\,\mathrm{d}Q_t+\Phi\big(t,Y(t,x),Z(t,x)\big)\,\mathrm{d}Q_t\\ \qquad\qquad -Z(t,x)\,\mathrm{d}W_t,\qquad \text{in }\Omega\times[0,\tau]\times\mathcal{D},\\ \partial j\big(Y(\omega,t,x)\big)\ni 0\qquad \text{on }\Omega\times[0,\tau]\times\mathrm{Bd}(\mathcal{D}),\\ \mathrm{e}^{2V_T}\big\|Y(T)-\xi_T\big\|^2+\displaystyle\int_T^\infty\mathrm{e}^{2V_s}\big\|Z(s)-\zeta_s\big\|^2\,\mathrm{d}s\xrightarrow[T\to\infty]{\text{prob.}}0.\end{cases}\tag{5.4}$$

In Theorem 3.2, let $H=H^{-1}(\mathcal{D})$ (the dual of $H_0^1(\mathcal{D})$), $H_1=\mathbb{R}^d$ and $\Psi=\varphi=\psi$, where $\varphi:H^{-1}(\mathcal{D})\to(-\infty,\infty]$ is given by

$$\varphi(u)=\begin{cases}\displaystyle\int_{\mathcal{D}}j\big(u(x)\big)\,\mathrm{d}x, & \text{if } u\in L^1(\mathcal{D}),\ j(u)\in L^1(\mathcal{D}),\\ +\infty, & \text{otherwise},\end{cases}$$

and $j:\mathbb{R}\to\mathbb{R}_+$ is suppose, moreover, to be continuous with $\lim_{r\to\infty}j(r)/r=\infty$.

Proposition 2.10 from Barbu [1], Chapter II, shows that:

(a) function φ is proper, convex and l.s.c.,
(b) $\partial\varphi(u)=\{u^*\in H^{-1}(\mathcal{D}):u^*(x)=-\Delta v(u(x)),\ v\in H_0^1(\mathcal{D}),\ v(x)\in\partial j(u(x))\text{ a.e. on }\mathcal{D}\}$, $\forall u\in\mathrm{Dom}(\partial\varphi)$,
(c) $\mathrm{Dom}(\partial\varphi)=\{u\in H^{-1}(\mathcal{D})\cap L^1(\mathcal{D}):u(x)\in\mathrm{Dom}(\partial j)\text{ a.e. on }\mathcal{D}\}$.

Let η be a $H^{-1}(\mathcal{D})$-valued random variable, $\mathcal{F}_\tau$-measurable such that (A₉) is satisfied and

$$\mathbb{E}\big[\mathrm{e}^{p\sup_{s\in[0,\tau]}\tilde{V}_s}|\eta|^p\big]<\infty,\qquad \eta\in L^1(\Omega\times\mathcal{D}),\qquad \mathrm{e}^{2\sup_{s\in[0,\tau]}\tilde{V}_s}j(\eta)\in L^1(\Omega\times\mathcal{D}),$$

and the stochastic processes ξ and ζ (from the martingale representation theorem) be such that (5.2) holds.

From Theorem 3.2 it follows that, under the above assumptions, backward SPDE (5.4) has a unique solution $(Y, Z) \in \mathcal{S}^0_{H^{-1}(\mathcal{D})} \times \Lambda^0_{(H^{-1}(\mathcal{D}))^d}$ such that $(Y(t), Z(t)) = (\xi_t, \zeta_t) = (\eta, 0)$, for $t \geq \tau$, and

(i) $Y(t,x) + \int_t^T \Delta U(s,x)\,\mathrm{d}Q_s = Y(T,x) + \int_t^T \Phi(s, Y(s,x), Z(s,x))\,\mathrm{d}Q_s - \int_t^T Z(s,x)\,\mathrm{d}W_s$, in $[0,T] \times \mathcal{D}$, a.s.,
(ii) $Y(t,x) \in \mathrm{Dom}(\partial j)$, $\mathrm{d}\mathbb{P} \times \mathrm{d}t \times \mathrm{d}x$ a.e.,
(iii) $U(t,x) \in \partial j(Y(t,x))$, $\mathrm{d}\mathbb{P} \times \mathrm{d}t \times \mathrm{d}x$ a.e.,
(iv) $\mathrm{e}^{2\tilde{V}} j(Y) \in L^\infty(0,T; L^1(\Omega \times \mathcal{D}))$, $\forall T > 0$,
(v) $\mathbb{E}\int_0^\tau \mathrm{e}^{2\tilde{V}_s} \|U(s)\|^2_{H^1_0(\mathcal{D})}\,\mathrm{d}Q_s < \infty$.

Appendix

In this section, we first present some useful and general estimates on $(Y, Z) \in \mathcal{S}^0[0,T] \times \Lambda^0(0,T)$ satisfying an identity of type

$$Y_t = Y_T + \int_t^T \mathrm{d}K_s - \int_t^T Z_s\,\mathrm{d}W_s, \qquad t \in [0,T],\ \mathbb{P}\text{-a.s.},$$

where $K \in \mathcal{S}^0[0,T]$ and $t \longmapsto K_t(\omega)$ is a bounded variation function, $\mathbb{P}$-a.s.

The following results are proved in monograph Pardoux and Răşcanu [19], Annex C.

Assume there exist: three progressively measurable increasing continuous stochastic processes D, R, N such that $D_0 = R_0 = N_0 = 0$, a progressively measurable bounded variation continuous stochastic process V with $V_0 = 0$, some constants $a, p > 1$ such that, as signed measures on $[0,T]$:

$$\mathrm{d}D_t + \langle Y_t, \mathrm{d}K_t\rangle \leq \left(\mathbb{1}_{p\geq 2}\,\mathrm{d}R_t + |Y_t|\,\mathrm{d}N_t + |Y_t|^2\,\mathrm{d}V_t\right) + \frac{n_p}{2a}\|Z_t\|^2\,\mathrm{d}t, \tag{A.1}$$

where $n_p = (p-1) \wedge 1$.

Let $\|\mathrm{e}^V Y\|_{[t,T]} := \sup_{s\in[t,T]} |\mathrm{e}^{V_s} Y_s|$.

Proposition A.1. *Assume* (A.1) *and that*

$$\mathbb{E}\|Y\mathrm{e}^V\|^p_{[0,T]} + \mathbb{E}\left(\int_0^T \mathrm{e}^{2V_s}\mathbb{1}_{p\geq 2}\,\mathrm{d}R_s\right)^{p/2} + \mathbb{E}\left(\int_0^T \mathrm{e}^{V_s}\,\mathrm{d}N_s\right)^p < \infty.$$

Then there exists a positive constant $C = C(a,p)$ such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left[\|\mathrm{e}^V Y\|^p_{[t,T]} + \left(\int_t^T \mathrm{e}^{2V_s}\,\mathrm{d}D_s\right)^{p/2} + \left(\int_t^T \mathrm{e}^{2V_s}\|Z_s\|^2\,\mathrm{d}s\right)^{p/2}\right] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \mathrm{e}^{pV_s}|Y_s|^{p-2}\mathbb{1}_{Y_s\neq 0}\,\mathrm{d}D_s + \int_t^T \mathrm{e}^{pV_s}|Y_s|^{p-2}\mathbb{1}_{Y_s\neq 0}\|Z_s\|^2\,\mathrm{d}s\right] \\
&\leq C\mathbb{E}^{\mathcal{F}_t}\left[\left|\mathrm{e}^{V_T}Y_T\right|^p + \left(\int_t^T \mathrm{e}^{2V_s}\mathbb{1}_{p\geq 2}\,\mathrm{d}R_s\right)^{p/2} + \left(\int_t^T \mathrm{e}^{V_s}\,\mathrm{d}N_s\right)^p\right].
\end{aligned}$$

In particular for all $t \in [0, T]$:

$$|Y_t|^p \le C\mathbb{E}^{\mathcal{F}_t}\big[\big(|Y_T|^p + \mathbb{1}_{p\ge 2}R_T^p + N_T^p\big)\mathrm{e}^{p\|(V_\cdot - V_t)^+\|_{[t,T]}}\big], \qquad \mathbb{P}\text{-}a.s.$$

As a simple consequence we can deduce, from the above proposition, an estimate for the stochastic processes (ξ, ζ) associated to η as in Proposition 2.2:

Corollary A.1. *Let* $(V_t)_{t\ge 0}$ *be a bounded variation and continuous p.m.s.p. with* $V_0 = 0$, $\eta : \Omega \to H$ *a random variable such that* $\mathbb{E}(\mathrm{e}^{p\sup_{s\in[0,T]} V_s}|\eta|^p) < \infty$ *and* $(\xi, \zeta) \in \mathcal{S}^0 \times \Lambda^0(0, \infty)$ *the unique solution of the following equation* (*see the martingale representation formula* (2.1)): $\xi_s = \mathbb{E}^{\mathcal{F}_T}\eta - \int_s^T \zeta_r\,\mathrm{d}W_r$, $s \in [0, T]$, *a.s. Therefore, there exists* $C = C(p) > 0$ *such that for all* $t \in [0, T]$,

$$\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\mathrm{e}^{pV_s}|\xi_s|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T \mathrm{e}^{2V_s}|\zeta_s|^2\,\mathrm{d}s\right)^{p/2} \le C\mathbb{E}^{\mathcal{F}_t}\big(\mathrm{e}^{p\sup_{s\in[t,T]} V_s}|\eta|^p\big). \tag{A.2}$$

Proof. We see at once that the stochastic pair (ξ, ζ) satisfy equation $\xi_s = \xi_T - \int_s^T \zeta_r\,\mathrm{d}W_r$, $s \in [0, T]$ a.s. For any fixed $t \in [0, T]$ let $(\bar{V}_s^t)_{s\in[0,T]}$ be the increasing continuous p.m.s.p. defined by $\bar{V}_s^t = V_t$, $s < t$, and $\bar{V}_s^t = \sup_{r\in[t,s]} V_r$, $s \ge t$. Applying Jensen's inequality and Proposition A.1 for (ξ, ζ) (which satisfies an inequality of type (A.1), with $K = 0$ and $R = N = 0$), we deduce that for all $p > 1$, there exists $C = C(p) > 0$ such that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\mathrm{e}^{pV_s}|\xi_s|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T \mathrm{e}^{2V_s}|\zeta_s|^2\,\mathrm{d}s\right)^{p/2} \\
&\quad\le \mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\mathrm{e}^{p\bar{V}_s^t}|\xi_s|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T \mathrm{e}^{2\bar{V}_s^t}|\zeta_s|^2\,\mathrm{d}s\right)^{p/2} \\
&\quad\le C\mathbb{E}^{\mathcal{F}_t}\big(\mathrm{e}^{p\bar{V}_T^t}|\xi_T|^p\big) \le C\mathbb{E}^{\mathcal{F}_t}\big(\mathrm{e}^{p\sup_{s\in[t,T]}|V_s|}|\eta|^p\big).
\end{aligned}$$ □

Let us now discuss the existence and uniqueness of a solution for the backward stochastic equation of the form

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\,\mathrm{d}Q_s - \int_t^T Z_s\,\mathrm{d}B_s, \qquad \text{a.s., } \forall t \in [0, T]. \tag{A.3}$$

We will need the following basic assumptions:

(A_3') *the process* $\{Q_t : t \ge 0\}$ *is a progressively measurable increasing continuous stochastic process such that* $Q_0 = 0$, *and* $\{\alpha_t : t \ge 0\}$ *is a real positive p.m.s.p. such that* $\alpha \in [0, 1]$ *and* $\mathrm{d}t = \alpha_t\,\mathrm{d}Q_t$;

(A_4') *the function* $\Phi : \Omega \times [0, \infty) \times H \times L_2(H_1, H) \to H$ *is such that*

$$\begin{cases} \Phi(\cdot, \cdot, y, z) \text{ is p.m.s.p.}, & \forall (y, z) \in H \times L_2(H_1, H), \\ \Phi(\omega, t, \cdot, \cdot) \text{ is continuous function}, & \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.e.}, \end{cases}$$

and $\mathbb{P}$*-a.s.*, $\int_0^T \Phi_\rho^\#(s)\,\mathrm{d}Q_s < \infty$, $\forall \rho \geq 0$, *where* $\Phi_\rho^\#(\omega, s) := \sup_{|u|\leq\rho} |\Phi(\omega, s, u, 0)|$;

(A_5') *there exist a p.m.s.p.* $\mu : \Omega \times [0,\infty) \to \mathbb{R}$ *and a function* $\ell : [0,\infty) \to [0,\infty)$ *such that* $\int_0^T |\mu_t|\,\mathrm{d}Q_t + \int_0^T \ell^2(t)\,\mathrm{d}t < \infty$, $\mathbb{P}$*-a.s. and, for all* $y, y' \in H$, $z, z' \in L_2(H_1, H)$,

$$\langle y' - y, \Phi(t, y', z) - \Phi(t, y, z)\rangle \leq \mu_t |y' - y|^2,$$
$$|\Phi(t, y, z') - \Phi(t, y, z)| \leq \ell(t)\alpha_t |z' - z|.$$

Let $a > 1$ and $V_t = \int_0^t (\mu_s + \frac{a}{2n_p}\ell^2(s)\alpha_s)\,\mathrm{d}Q_s$.

Proposition A.2. *Let* $p > 1$ *and* $\eta : \Omega \to H$ *be a random variable measurable with respect to* $\sigma(\{\mathcal{F}_t : t \geq 0\})$. *Under the hypotheses* (A_3')–(A_5'), *if moreover,*

$$\mathbb{E}\big(e^{pV_T}|\eta|^p\big) + \mathbb{E}\bigg(\int_0^T \sup_{|y|\leq\rho} \big|e^{V_t}\Phi(t, e^{-V_t}y, 0) - \mu_s y\big|\,\mathrm{d}Q_s\bigg)^p < \infty, \qquad \forall \rho \geq 0, \tag{A.4}$$

there exists a unique pair $(Y_t, Z_t)_{t\geq 0} \in \mathcal{S}^0 \times \Lambda^0$ *solution of the BSDE* (A.3) *in the sense that*

(j) $Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\,\mathrm{d}Q_s - \int_t^T Z_s\,\mathrm{d}B_s$, *a.s.*, $\forall t \in [0, T]$,

(jj) $\mathbb{E}\|e^V Y\|_{[0,T]}^p + \mathbb{E}(\int_0^T e^{2V_s}|Z_s|^2\,\mathrm{d}s)^{p/2} < \infty$.

Remark A.1. If $(V_t)_{t\geq 0}$ is a deterministic process, then assumption (A.4) is equivalent to

$$\mathbb{E}\big(|\eta|^p\big) + \mathbb{E}\bigg(\int_0^T \Phi_\rho^\#(s)\,\mathrm{d}Q_s\bigg)^p < \infty.$$

Acknowledgements

The authors would like to thank the referees for the attention in reading this paper and for their helpful suggestions and comments that have led to the improvement of the paper.

The work was supported by grant POSDRU/89/1.5/S/49944 and respectively by IDEAS project, "Deterministic and stochastic systems with state constraints", code 241/05.10.2011.

References

[1] Barbu, V. (2010). *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. *Springer Monographs in Mathematics*. New York: Springer. MR2582280

[2] Brézis, H. (1973). *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. Amsterdam: North-Holland. MR0348562

[3] Briand, Ph., Delyon, B., Hu, Y., Pardoux, E. and Stoica, L. (2003). L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** 109–129. MR2008603

[4] Cvitanić, J. and Karatzas, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab.* **24** 2024–2056. MR1415239

[5] Darling, R.W.R. and Pardoux, E. (1997). Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *Ann. Probab.* **25** 1135–1159. MR1457614

[6] Da Prato, G. and Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge: Cambridge Univ. Press. MR1207136

[7] Diomande, B. and Maticiuc, L. (2014). Multivalued backward stochastic differential equations with time delayed generators. *Cent. Eur. J. Math.* To appear.

[8] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M.C. (1997). Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.* **25** 702–737. MR1434123

[9] Maticiuc, L. and Nie, T. (2014). Fractional backward stochastic differential equations and fractional backward variational inequalities. *J. Theoret. Probab.* To appear.

[10] Maticiuc, L. and Răşcanu, A. (2007). Viability of moving sets for a nonlinear Neumann problem. *Nonlinear Anal.* **66** 1587–1599. MR2301340

[11] Maticiuc, L. and Răşcanu, A. (2010). A stochastic approach to a multivalued Dirichlet–Neumann problem. *Stochastic Process. Appl.* **120** 777–800. MR2610326

[12] Maticiuc, L., Răşcanu, A. and Zălinescu, A. (2014). Backward stochastic variational inequalities with locally bounded generators. *An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.)*. To appear. DOI:10.2478/aicu-2013-0023.

[13] Maticiuc, L. and Rotenstein, E. (2012). Numerical schemes for multivalued backward stochastic differential systems. *Cent. Eur. J. Math.* **10** 693–702. MR2886566

[14] Pardoux, É. (1999). BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear Analysis*, *Differential Equations and Control* (*Montreal*, *QC*, 1998). *NATO Sci. Ser. C Math. Phys. Sci.* **528** 503–549. Dordrecht: Kluwer Academic. MR1695013

[15] Pardoux, É. and Peng, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications* (*Charlotte*, *NC*, 1991) (B.L. Rozovskii and R.B. Sowers, eds.). *Lecture Notes in Control and Inform. Sci.* **176** 200–217. Berlin: Springer. MR1176785

[16] Pardoux, É. and Peng, S.G. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55–61. MR1037747

[17] Pardoux, E. and Răşcanu, A. (1998). Backward stochastic differential equations with subdifferential operator and related variational inequalities. *Stochastic Process. Appl.* **76** 191–215. MR1642656

[18] Pardoux, E. and Răşcanu, A. (1999). Backward stochastic variational inequalities. *Stochastics* **67** 159–167. MR1729473

[19] Pardoux, E. and Răşcanu, A. (2014). *Stochastic Differential Equations*, *Backward SDEs*, *Partial Differential Equations*. *Springer Series*: *Stochastic Modelling and Applied Probability* **69**. Berlin: Springer.

[20] Răşcanu, A. (1981). Existence for a class of stochastic parabolic variational inequalities. *Stochastics* **5** 201–239. MR0631996

[21] Răşcanu, A. and Rotenstein, E. (2011). The Fitzpatrick function – A bridge between convex analysis and multivalued stochastic differential equations. *J. Convex Anal.* **18** 105–138. MR2777600

Received June 2012 and revised January 2014