Bernoulli **19**(5B), 2013, 2167–2199
DOI: 10.3150/12-BEJ448

Recurrence and transience property for a class of Markov chains

NIKOLA SANDRIĆ

Department of Mathematics, Faculty of Civil Engineering, University of Zagreb, Fra Andrije Kačića-Miošića 26, 10 000 Zagreb, Croatia. E-mail: nsandric@grad.hr

We consider the recurrence and transience problem for a time-homogeneous Markov chain on the real line with transition kernel $p(x, dy) = f_x(y - x)\,dy$, where the density functions $f_x(y)$, for large $|y|$, have a power-law decay with exponent $\alpha(x) + 1$, where $\alpha(x) \in (0, 2)$. In this paper, under a uniformity condition on the density functions $f_x(y)$ and an additional mild drift condition, we prove that when $\liminf_{|x|\longrightarrow\infty} \alpha(x) > 1$, the chain is recurrent. Similarly, under the same uniformity condition on the density functions $f_x(y)$ and some mild technical conditions, we prove that when $\limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$, the chain is transient. As a special case of these results, we give a new proof for the recurrence and transience property of a symmetric α-stable random walk on $\mathbb{R}$ with the index of stability $\alpha \in (0, 1) \cup (1, 2)$.

Keywords: Foster–Lyapunov drift criterion; Harris recurrence; petite set; recurrence; stable distribution; T-chain; transience

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$, $d \geq 1$. Let us define $X_n := \sum_{i=1}^n Z_i$ and $X_0 := 0$. The sequence $\{X_n\}_{n\geq 0}$ is called a *random walk* with jumps $\{Z_n\}_{n\in\mathbb{N}}$. The random walk $\{X_n\}_{n\geq 0}$ is said to be *recurrent* if

$$\mathbb{P}\Big(\liminf_{n\longrightarrow\infty} |X_n| = 0\Big) = 1,$$

and *transient* if

$$\mathbb{P}\Big(\lim_{n\longrightarrow\infty} |X_n| = \infty\Big) = 1.$$

It is well known that every random walk is either recurrent or transient (see [3], Theorem 4.2.1). Recall that a random walk $\{X_n\}_{n\geq 0}$ in $\mathbb{R}^d$ is called *truly d-dimensional* if $\mathbb{P}(\langle Z_1, x\rangle \neq 0) > 0$ holds for all $x \in \mathbb{R}^d \setminus \{0\}$. It is also well known that every truly d-dimensional random walk is transient if $d \geq 3$ (see [3], Theorem 4.2.13). An $\mathbb{R}^d$-valued random variable Z is said to have *stable distribution* if, for any $n \in \mathbb{N}$, there are $a_n > 0$ and $b_n \in \mathbb{R}^d$, such that

$$Z_1 + \cdots + Z_n \stackrel{\mathrm{d}}{=} a_n Z + b_n,$$

where $Z_1, \ldots, Z_n$ are independent copies of Z and $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. It turns out that $a_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$ which is called the index of stability (see [11], Definition 1.1.4

and Corollary 2.1.3). The case $\alpha = 2$ corresponds to the Gaussian random variable. A random walk $\{X_n\}_{n\geq 0}$ is said to be stable if the random variable Z_1 has stable distribution. In the class of truly two-dimensional stable random walks in $\mathbb{R}^2$, by [3], Theorem 4.2.9, the only recurrent case is the case when $\{X_n\}_{n\geq 0}$ is a truly two-dimensional random walk with zero mean Gaussian jumps. In the case $d = 1$, every stable distribution is characterized by four parameters: the stability parameter $\alpha \in (0, 2]$, the skewness parameter $\beta \in [-1, 1]$, the scale parameter $\gamma \in (0, \infty)$ and the shift parameter $\delta \in \mathbb{R}$ (see [11], Definition 1.1.6). Using the notation from [11], we denote one-dimensional stable distributions by $S_\alpha(\beta, \gamma, \delta)$. For symmetric stable distributions, that is, for $S_\alpha(0, \gamma, 0)$ (see [11], Property 1.2.5), we write SαS. A SαS random walk is recurrent if and only if $\alpha \geq 1$ (see the discussion after [3], Lemma 4.2.12). In this paper, we generalize the SαS random walk in the way that the index of stability of the jump distribution depends on the current position and study the transience and recurrence property of the generalization.

Actually, we will not need the stability property of transition jumps. All we will need is a tail behavior of transition jumps. Let us introduce the notation $f(y) \sim g(y)$, when $y \longrightarrow y_0$, for $\lim_{y\longrightarrow y_0} f(y)/g(y) = 1$, where $y_0 \in [-\infty, \infty]$. Recall that if $f(y)$ is the density function of a SαS distribution with $\alpha \in (0, 2)$ and $\gamma \in (0, \infty)$ (for the existence of densities of $S_\alpha(\beta, \gamma, \delta)$ distributions see [11], Definition 1.1.6 and [3], Theorem 3.3.5), then

$$f(y) \sim c_\alpha |y|^{-\alpha-1},$$

when $|y| \longrightarrow \infty$, where $c_1 = \frac{\gamma}{2}$ and $c_\alpha = \frac{\gamma}{\pi}\Gamma(\alpha + 1)\sin(\frac{\pi\alpha}{2})$, for $\alpha \neq 1$, see [11], Property 1.2.15. Now, let $\alpha : \mathbb{R} \longrightarrow (0, 2)$ and $c : \mathbb{R} \longrightarrow (0, \infty)$ be arbitrary functions and let $\{f_x : x \in \mathbb{R}\}$ be a family of density functions on $\mathbb{R}$ such that

(C1) $x \longmapsto f_x(y)$ is a Borel measurable function for all $y \in \mathbb{R}$ and
(C2) $f_x(y) \sim c(x)|y|^{-\alpha(x)-1}$, when $|y| \longrightarrow \infty$, for all $x \in \mathbb{R}$.

Let us define a Markov chain $\{X_n\}_{n\geq 0}$ on $\mathbb{R}$ by the following transition kernel

$$p(x, \mathrm{d}y) := f_x(y - x)\,\mathrm{d}y. \tag{1.1}$$

The chain $\{X_n\}_{n\geq 0}$ jumps from the state x with transition density $f_x(y - x)$, with the power-law decay with exponent $\alpha(x) + 1$, and this jump distribution depends only on the current state x. Transition densities $\{f_x : x \in \mathbb{R}\}$ are asymptotically equivalent to the densities of SαS distributions, and we call such chain a *stable-like chain*. The aim of this paper is to find conditions for the recurrence and transience property of the stable-like chain $\{X_n\}_{n\geq 0}$ in terms of the function $\alpha(x)$.

To the best of our knowledge, all methods used in establishing conditions for recurrence and transience in the random walk case are based on the i.i.d. property of random walk jumps, that is, laws of large numbers (Chung–Fuchs theorem), central limit theorems, characteristic functions approach (Stone–Ornstein formula) etc. (see [3], Theorems 4.2.7, 4.2.8 and 4.2.9). Although we deal with distributions similar to SαS distributions, it is not clear if these methods can be used in the case of the non-constant function $\alpha(x)$.

Special cases of this problem have been considered in [2,4–6] and [10]. In [6] and [10], the authors consider the countable state space $\mathbb{Z}$ and the function $\alpha(x)$ is a two-valued step function which takes one value on negative integers and the other one on nonnegative integers. The processes considered in [2] and [4] run in continuous time. The function $\alpha(x)$ considered in [2] is a two-valued step function which takes one value on negative reals and the other one on nonnegative reals, while in [4] the author considers the case when the function $\alpha(x)$ is periodic and continuously differentiable. The methods used in [2,6,10] and [4], actually reduce the process to random walks and Lévy processes. Also, it is not clear if these methods can be used in the general case, that is, when the function $\alpha(x)$ is an arbitrary function. In this paper, under certain assumptions on the functions $\alpha(x)$, $c(x)$ and on the family of density functions $\{f_x\colon x \in \mathbb{R}\}$, we give sufficient conditions for the recurrence and transience property of the stable-like chain $\{X_n\}_{n\geq 0}$ in terms of the function $\alpha(x)$.

Let us denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$, by λ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and for arbitrary $B \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$ we define $B - x := \{y - x\colon y \in B\}$. Assume that the family of probability densities $\{f_x\colon x \in \mathbb{R}\}$ satisfies additional three conditions:

(C3) there exists $k > 0$ such that

$$\lim_{|y|\longrightarrow\infty} \sup_{x\in[-k,k]^c} \left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| = 0;$$

(C4) $\inf_{x\in C} c(x) > 0$ for every compact set $C \subseteq [-k,k]^c$;

(C5) there exists $l > 0$ such that for every compact set $C \subseteq [-l,l]^c$ with $\lambda(C) > 0$, we have

$$\inf_{x\in[-k,k]} \int_{C-x} f_x(y)\,\mathrm{d}y > 0.$$

Condition (C3) ensures that out of some compact set all jump densities of the stable-like chain $\{X_n\}_{n\geq 0}$ can be replaced by their tail behavior uniformly. This condition is crucial in proving certain structural properties of the chain $\{X_n\}_{n\geq 0}$ and in finding sufficient conditions for the recurrence and transience. Another essential property of the chain $\{X_n\}_{n\geq 0}$ is that every compact set is a petite set. A petite set is a set which assumes a role of a singleton for Markov chains on general state space (for the exact definition of the petite set see Definition 2.2). This is the reason why compact sets are important in conditions (C3), (C4) and (C5). Besides ensuring that all compact sets are petite sets (singletons), conditions (C4) and (C5) ensure also that the chain is irreducible. Condition (C4) ensures that the scaling function $c(x)$ does not vanish on petite sets, and condition (C5) ensures that the petite set $[-k,k]$ communicates with the rest of the state space.

Remark 1.1. Note that condition (C3) implies

$$\sup_{x\in[-k,k]^c} c(x) < \infty. \tag{1.2}$$

Indeed, let $0 < \varepsilon < 1$ be arbitrary. Then there exists $y_\varepsilon \geq 1$ such that for all $|y| \geq y_\varepsilon$ we have

$$\left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon$$

for all $x \in [-k, k]^c$. Therefore, upon integrating over y we get

$$c(x) < \frac{1}{1-\varepsilon}\left(2\int_{y_\varepsilon}^{\infty} y^{-\alpha(x)-1}\,\mathrm{d}y\right)^{-1} \le \frac{1}{1-\varepsilon}\left(2\int_{y_\varepsilon}^{\infty} y^{-3}\,\mathrm{d}y\right)^{-1} = \frac{y_\varepsilon^2}{1-\varepsilon}$$

for every $x \in [-k, k]^c$.

An example of a stable-like chain which satisfies conditions (C3)–(C5) is the chain which has exactly $S_{\alpha(x)}(0, \gamma(x), \delta(x))$ jumps at each location x, where the functions $\alpha(x)$, $\gamma(x)$ and $\delta(x)$ are Borel measurable and take finitely many values (see Proposition 5.5 for details).

Before stating the main results of this paper we recall relevant definitions of recurrence and transience.

Definition 1.2. *Let $\{Y_n\}_{n\ge 0}$ be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

(i) *The chain $\{Y_n\}_{n\ge 0}$ is* φ-irreducible *if there exists a probability measure φ on $\mathcal{B}(\mathbb{R})$ such that for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $\varphi(B) > 0$ implies $\mathbb{P}(Y_n \in B | Y_0 = x) > 0$.*
(ii) *The chain $\{Y_n\}_{n\ge 0}$ is* recurrent *if it is φ-irreducible and if $\sum_{n=0}^{\infty} \mathbb{P}(Y_n \in B | Y_0 = x) = \infty$ holds for all $x \in \mathbb{R}$ and all $B \in \mathcal{B}(\mathbb{R})$, such that $\varphi(B) > 0$.*
(iii) *The chain $\{Y_n\}_{n\ge 0}$ is* transient *if it is φ-irreducible and if there exists a countable cover of $\mathbb{R}$ with sets $\{B_j\}_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R})$, such that for each $j \in \mathbb{N}$ there is a finite constant $M_j \ge 0$ such that $\sum_{n=0}^{\infty} \mathbb{P}(Y_n \in B_j | Y_0 = x) \le M_j$ holds for all $x \in \mathbb{R}$.*

The following two constants will appear in the statements of the main results: For $\alpha \in (1, 2)$, let

$$R(\alpha) := \sum_{i=1}^{\infty} \frac{1}{i(2i-\alpha)} - \frac{\ln 2}{\alpha} - \frac{1}{2\alpha}\left(\Psi\left(\frac{\alpha+1}{2}\right) - \Psi\left(\frac{\alpha}{2}\right)\right),$$

and for $\alpha \in [0, 1)$ and $\beta \in (0, 1-\alpha)$ let

$$T(\alpha, \beta) := {}_2F_1(-\alpha, \beta, 1-\alpha; 1) + \beta B(1; \alpha+\beta, 1-\alpha) - \alpha B(1; \alpha+\beta, 1-\beta),$$

where $\Psi(z)$ is the Digamma function, ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function and $B(x; z, w)$ is the incomplete Beta function (see Section 3 for the definition of these functions). The constants $R(\alpha)$ and $T(\alpha, \beta)$ are strictly positive (see proofs of Theorems 1.3 and 1.4). Furthermore, it is not hard to see that the constant $R(\alpha)$, as a function of $\alpha \in (1, 2)$, is strictly increasing, $R(1) = 0$ and $\lim_{\alpha\longrightarrow 2} R(\alpha) = \infty$. The constant $T(\alpha, \beta)$, as a function of $\beta \in (0, 1-\alpha)$ for fixed $\alpha \in (0, 1)$, is strictly positive and $T(\alpha, 0) = T(\alpha, 1-\alpha) = 0$, while considered as a function of $\alpha \in [0, 1-\beta)$ for fixed $\beta \in (0, 1)$, it is strictly decreasing, $T(0, \beta) = 2$ and $T(1-\beta, \beta) = 0$.

Theorem 1.3. *Let $\alpha : \mathbb{R} \longrightarrow (1, 2)$ be an arbitrary function such that*

$$\alpha := \liminf_{|x|\longrightarrow\infty} \alpha(x) > 1.$$

Furthermore, let $c:\mathbb{R}\longrightarrow(0,\infty)$ *be an arbitrary function and let* $\{f_x\colon x\in\mathbb{R}\}$ *be a family of density functions on* $\mathbb{R}$ *which satisfies conditions* (C1)–(C5) *and such that*

$$\limsup_{|x|\longrightarrow\infty}\operatorname{sgn}(x)\frac{|x|^{\alpha(x)-1}}{c(x)}\mathbb{E}[X_1-X_0|X_0=x]<R(\alpha) \tag{1.3}$$

when $\alpha<2$, *and the left-hand side in* (1.3) *is finite when* $\alpha=2$. *Then the stable-like Markov chain* $\{X_n\}_{n\geq 0}$ *given by the transition kernel*

$$p(x,\mathrm{d}y)=f_x(y-x)\,\mathrm{d}y,$$

is recurrent.

Theorem 1.4. *Let* $\alpha:\mathbb{R}\longrightarrow(0,1)$ *be an arbitrary function such that*

$$\alpha:=\limsup_{|x|\longrightarrow\infty}\alpha(x)<1$$

and let $\beta\in(0,1-\alpha)$ *be arbitrary. Furthermore, let* $c:\mathbb{R}\longrightarrow(0,\infty)$ *be an arbitrary function and let* $\{f_x\colon x\in\mathbb{R}\}$ *be a family of density functions which satisfies conditions* (C1)–(C5) *and there exists* $a_0>0$, *such that*

$$\liminf_{|x|\longrightarrow\infty}\frac{\alpha(x)|x|^{\alpha(x)}}{c(x)}\int_{-a}^{a}\left(1-\left(1+\operatorname{sgn}(x)\frac{y}{1+|x|}\right)^{-\beta}\right)f_x(y)\,\mathrm{d}y>-T(\alpha,\beta) \tag{1.4}$$

for all $a\geq a_0$. *Then the stable-like Markov chain* $\{X_n\}_{n\geq 0}$ *given by the transition kernel*

$$p(x,\mathrm{d}y)=f_x(y-x)\,\mathrm{d}y,$$

is transient.

Actually, instead of condition (1.3), in the proof of Theorem 1.3, we use the following more technical but equivalent condition

$$\limsup_{\delta\longrightarrow 0}\limsup_{|x|\longrightarrow\infty}\frac{(1+|x|)^{\alpha(x)}}{c(x)}\int_{-\delta(1+|x|)}^{\delta(1+|x|)}\ln\left(1+\operatorname{sgn}(x)\frac{y}{1+|x|}\right)f_x(y)\,\mathrm{d}y<R(\alpha) \tag{1.5}$$

(see Section 5 for details). Conditions (1.3) (i.e., (1.5)) and (1.4) are needed to control the behavior of the family of density functions $\{f_x\colon x\in\mathbb{R}\}$ on sets symmetric around the origin. Condition (1.3) actually says that when the chain $\{X_n\}_{n\geq 0}$ has moved far away from the origin, since $R(\alpha)>0$, it cannot have strong tendency to move further from the origin. Since $R(\alpha)>0$, it is clear that condition (1.3) is satisfied if $\alpha(x)\in(1,2)$ and if $f_x(y)=f_x(-y)$ holds for all $y\in\mathbb{R}$ and for all $|x|$ large enough. For a non-symmetric example, one can take $f_x(y)$ to be the density function of a $S_{\alpha_-}(0,\gamma_-,\delta_-)$ distribution, when $x<0$, and the density function of a $S_{\alpha_+}(0,\gamma_+,\delta_+)$ distribution, when $x\geq 0$, where $\alpha_-,\alpha_+\in(1,2)$, $\gamma_-,\gamma_+\in(0,\infty)$, $\delta_-\geq 0$ and $\delta_+\leq 0$.

Using the concavity property of the function $x \longmapsto x^{\beta}$, for $\beta \in (0, 1-\alpha)$, condition (1.4) follows from the condition

$$\limsup_{|x| \longrightarrow \infty} \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} < \frac{T(\alpha, \beta)}{a_0 \beta} \tag{1.6}$$

(see Section 5 for details). Note that condition (1.6) actually says that the function $c(x)$ cannot decrease too fast. Since $T(\alpha, \beta) > 0$ and $\alpha(x) \in (0, 1)$, a simple example which satisfies condition (1.6) is the case when $c(x) \geq d|x|^{\alpha(x)-1+\epsilon}$, for some $d > 0$ and for all $|x|$ large enough, where $0 < \epsilon < 1 - \alpha$ is arbitrary. Furthermore, one can prove that the function $\beta \longmapsto T(\alpha, \beta)/\beta$ is strictly decreasing on $(0, 1-\alpha)$. Hence, according to the condition (1.6), we choose β close to 0.

In the random walk case, that is, when the family of density functions $\{f_x \colon x \in \mathbb{R}\}$ is reduced to a single density function $f(y)$ such that $f(y) \sim c|y|^{-\alpha-1}$, when $|y| \longrightarrow \infty$, where $\alpha \in (0, 2)$ and $c \in (0, \infty)$, conditions (C1)–(C5) are trivially satisfied. Hence, by Theorem 1.3 and the condition (1.3), if $\alpha > 1$ and if

$$\int_{\mathbb{R}} y f(y) \, \mathrm{d}y = 0,$$

the random walk with the jump density $f(y)$ is recurrent, and if $\alpha < 1$, by Theorem 1.4 and the condition (1.6), the random walk with the jump density $f(y)$ is transient. This result can be strengthened. If we assume that $f(y) = f(-y)$ for all $y \in \mathbb{R}$, from the discussion in [12], page 88, the random walk with the jump density $f(y)$ is recurrent if and only if $\alpha \geq 1$. As a simple consequence of Theorems 1.3 and 1.4, we get the following well-known recurrence and transience conditions for the SαS random walk case.

Corollary 1.5. *A SαS, $1 < \alpha < 2$, random walk is recurrent. A $S_\alpha(0, \gamma, \delta)$, $0 < \alpha < 1$, random walk with arbitrary shift is transient.*

The previous corollary can be generalized. If the functions $\alpha(x)$, $\gamma(x)$ and $\delta(x)$ are Borel measurable and take finitely many values, then the stable-like chain with S$\alpha(x)$S jumps is recurrent if $\alpha(x) \in (1, 2)$ for all $x \in \mathbb{R}$. If $\alpha(x) \in (0, 1)$ for all $x \in \mathbb{R}$, then the stable-like chain with $S_{\alpha(x)}(0, \gamma(x), \delta(x))$ jumps is transient.

Remark 1.6. Conditions in Theorems 1.3 and 1.4 are only sufficient conditions for recurrence and transience of the stable-like chain $\{X_n\}_{n \geq 0}$. On the countable state space $\mathbb{Z}$, when

$$\alpha(i) = \begin{cases} \alpha, & i < 0, \\ \beta, & i \geq 0 \end{cases}$$

for $\alpha, \beta \in (0, 2)$, in [6,10] it is proved that if $\frac{\alpha+\beta}{2} > 1$, the associated chain is recurrent, and if $\frac{\alpha+\beta}{2} < 1$, the associated chain is transient. A similar result, with

$$\alpha(x) = \begin{cases} \alpha, & x < 0, \\ \beta, & x \geq 0 \end{cases}$$

for $\alpha, \beta \in (0,2)$, is proved in the continuous time case in [2], that is, a stable-like process with the symbol $|\xi|^{\alpha(x)}$ is recurrent if and only if $\frac{\alpha+\beta}{2} \geq 1$. In [4], in the case when the function $\alpha(x)$ is periodic and continuously differentiable function, it is proved that all that matters is the minimum of the function $\alpha(x)$. If $\lambda(\{x\colon \alpha(x) = \alpha_0 := \inf\{\alpha(y)\colon y \in \mathbb{R}\}\}) > 0$, then a stable-like process with the symbol $|\xi|^{\alpha(x)}$ is recurrent if and only if $\alpha_0 \geq 1$.

Now we explain our strategy of proving the main results. The proof of Theorems 1.3 and 1.4 is based on the *Foster–Lyapunov drift criterion* for recurrence and transience of Markov chains (see [9], Theorems 8.4.2 and 8.4.3). This criterion is based on finding an appropriate test function $V(x)$ (positive and unbounded in the recurrence case and positive and bounded in the transience case), and an appropriate set $C \in \mathcal{B}(\mathbb{R})$ (petite set) such that $\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \leq 0$, in the recurrence case, and $\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \geq 0$, in the transience case, for every $x \in C^c$. The idea is to find test functions $V(x)$ such that the associated level sets $C_V(r) := \{y\colon V(y) \leq r\}$ are compact sets, that is, petite sets, and that $C_V(r) \uparrow \mathbb{R}$, when $r \longrightarrow \infty$, in the case of recurrence and $C_V(r) \uparrow \mathbb{R}$, when $r \longrightarrow 1$, in the case of transience. In the recurrence case for the test function, we take $V(x) = \ln(1 + |x|)$, and in the transience case we take $V(x) = 1 - (1 + |x|)^{-\beta}$, where $0 < \beta < 1 - \alpha$ (recall that $\alpha = \limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$). Now, by proving that

$$\limsup_{|x|\longrightarrow\infty} \frac{|x|^{\alpha(x)}}{c(x)} \left(\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \right) < 0,$$

in the recurrence case, and

$$\liminf_{|x|\longrightarrow\infty} \frac{\alpha(x)|x|^{\alpha(x)+\beta}}{c(x)} \left(\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \right) > 0,$$

in the transience case, since compact sets are petite sets, the proofs of Theorems 1.3 and 1.4 are accomplished.

A similar approach, by using similar test functions $V(x)$, can be found in [7] and [8]. In [7], the author considers a Markov chain on the nonnegative real line with uniformly bounded transition jumps, while in [8] the authors generalize this result to the case of uniformly bounded $2 + \delta_0$-moments of transition jumps, for some $\delta_0 > 0$. If we allow that $\alpha(x) \in (0, \infty)$ and assume the following additional assumption: $\sup_{x \in C} \alpha(x) < \infty$, for every compact set $C \subseteq [-k, k]^c$ (recall that the constant k is defined in condition (C3)), one can prove all nice structural properties of the chain $\{X_n\}_{n\geq 0}$, given by (1.1), proved in Section 2. Hence, since the chain $\{X_n\}_{n\geq 0}$ is recurrent if and only if the chain $\{|X_n|\}_{n\geq 0}$ is recurrent, [8] covers the case when $\liminf_{|x|\longrightarrow\infty} \alpha(x) > 2$.

The paper is organized as follows. In Section 2, we give several structural properties of the stable-like chain $\{X_n\}_{n\geq 0}$ which will be crucial in finding sufficient conditions for the recurrence and transience property. In Sections 3 and 4, using Foster–Lyapunov drift criterion for recurrence and transience of Markov chains, we prove Theorems 1.3 and 1.4. In Section 5, we extend our model from the model of asymptotically symmetric transition jumps to the model of asymptotically non-symmetric transition jumps. Further, we prove that the change of the chain $\{X_n\}_{n\geq 0}$ on bounded sets will not affect the recurrence and transience property.

Throughout the paper, we use the following notation. We write $\mathbb{Z}_+$ and $\mathbb{Z}_-$ for nonnegative and nonpositive integers, respectively. For $x, y \in \mathbb{R}$ let $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

Furthermore, $\{X_n\}_{n\geq 0}$ will denote the stable-like Markov chain on $\mathbb{R}$ given by (1.1) with transition densities satisfying conditions (C1)–(C5), while $\{Y_n\}_{n\geq 0}$ will denote an arbitrary Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by the transition kernel $p(x,B)$, for $x\in\mathbb{R}$ and $B\in\mathcal{B}(\mathbb{R})$. For $x\in\mathbb{R}$, $B\in\mathcal{B}(\mathbb{R})$ and $n\in\mathbb{N}$ let $p^n(x,B) := \mathbb{P}(Y_n\in B|Y_0 = x)$ and $\tau_B := \min\{n\geq 1\colon\ Y_n\in B\}$.

2. Structural properties of the model

In this section, we discuss several structural properties of stable-like Markov chains. In Definition 1.2, we defined irreducibility of a Markov chain on the state space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. In [9], Proposition 4.2.1, it is shown that the irreducibility measure can always be maximized, that is, if $\{Y_n\}_{n\geq 0}$ is a φ-irreducible Markov chain, then there exists a probability measure ψ on $\mathcal{B}(\mathbb{R})$ such that the chain $\{Y_n\}_{n\geq 0}$ is ψ-irreducible and $\varphi'\ll\psi$, for every irreducibility measure φ' on $\mathcal{B}(\mathbb{R})$ of the chain $\{Y_n\}_{n\geq 0}$. The measure ψ is called the *maximal irreducibility measure* and from now on, when we refer to irreducibility measure we actually refer to the maximal irreducibility measure. For the ψ-irreducible Markov chain $\{Y_n\}_{n\geq 0}$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, let us set $\mathcal{B}^+(\mathbb{R}) = \{B\in\mathcal{B}(\mathbb{R})\colon\ \psi(B)>0\}$.

Proposition 2.1. *Under conditions* (C1)–(C4), *the maximal irreducibility measure for the chain* $\{X_n\}_{n\geq 0}$ *is equivalent*, *in the absolutely continuous sense*, *with the Lebesgue measure*. *Therefore*, *the chain* $\{X_n\}_{n\geq 0}$ *is* λ-*irreducible*.

Proof. First, we prove that under conditions (C1)–(C4), the chain $\{X_n\}_{n\geq 0}$ is φ-irreducible for all measures φ, such that $\varphi\ll\lambda$. We prove that for every $x\in\mathbb{R}$ and for every $B\in\mathcal{B}(\mathbb{R})$, such that $\lambda(B)>0$, there exists $n\in\mathbb{N}$, such that $p^n(x,B)>0$. It is enough to prove the claim in the case of bounded sets. Let $B\in\mathcal{B}(\mathbb{R})$, $\lambda(B)>0$, be an arbitrary bounded set. Let $x\in\mathbb{R}$ and $0<\varepsilon<1$ be arbitrary. Then, by (C2), there exists $y_{\varepsilon,x}\geq 1$ such that for all $|y|\geq y_{\varepsilon,x}$ we have

$$\left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1\right| < \varepsilon.$$

Furthermore, by (C3), there exists $k>0$ such that for given ε there exists $y_\varepsilon\geq 1$, such that for all $|y|\geq y_\varepsilon$ and all $z\in[-k,k]^c$, we have

$$\left| f_z(y)\frac{|y|^{\alpha(z)+1}}{c(z)} - 1\right| < \varepsilon.$$

Let $a := \sup B$ and $y_0 := (y_{\varepsilon,x}\vee y_\varepsilon\vee k) + |x| + |a| + 1$. Finally, by (C4) we have

$$\begin{aligned} p^2(x,B) &= \int_{\mathbb{R}} p(x,\mathrm{d}y)p(y,B) = \int_{\mathbb{R}} f_x(y-x)\int_{B-y} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \\ &\geq \int_{y_0}^{2y_0} f_x(y-x)\int_{B-y} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&> (1-\varepsilon)^2 c(x) \int_{y_0}^{2y_0} (y-x)^{-\alpha(x)-1} c(y) \int_{B-y} |z|^{-\alpha(y)-1}\,\mathrm{d}z\,\mathrm{d}y \\
&> (1-\varepsilon)^2 c(x) \Big(\inf_{y_0 \le y \le 2y_0} c(y)\Big) \int_{y_0}^{2y_0} (y-x)^{-3} \int_{B-y} |z|^{-3}\,\mathrm{d}z\,\mathrm{d}y > 0,
\end{aligned}
$$

since $B - y \subseteq (-\infty, -y_\varepsilon)$, for $y \ge y_0$.

Now, we show the maximality of the Lebesgue measure. Let ψ be the maximal irreducibility measure of the chain $\{X_n\}_{n\ge 0}$. Hence, $\lambda \ll \psi$. Let us show that $\psi \ll \lambda$. If that would not be the case, that is, if there would exist $B \in \mathcal{B}(\mathbb{R})$ such that $\lambda(B) = 0$ and $\psi(B) > 0$, then by irreducibility of the chain $\{X_n\}_{n\ge 0}$, for every $x \in \mathbb{R}$ there would exist $n \in \mathbb{N}$ such that

$$
p^n(x,B) = \int_{\mathbb{R}} p(x,\mathrm{d}x_1) \int_{\mathbb{R}} p(x_1,\mathrm{d}x_2) \cdots \int_{\mathbb{R}} p(x_{n-2},\mathrm{d}x_{n-1}) \int_{B-x_{n-1}} f_{x_{n-1}}(x_n)\,\mathrm{d}x_n > 0.
$$

But, since $\int_{B-x} f_x(y)\,\mathrm{d}y = 0$, for every $x \in \mathbb{R}$, because $\lambda(B) = 0$, we have $p^n(x,B) = 0$. $\square$

Definition 2.2. *Let $\{Y_n\}_{n\ge 0}$ be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

(i) *A set $C \in B(\mathbb{R})$ is called a ν_n-*small set *if there exist $n \in \mathbb{N}$ and a nontrivial measure ν_n on $\mathcal{B}(\mathbb{R})$ such that for every $B \in \mathcal{B}(\mathbb{R})$ and for every $x \in C$ we have*

$$
p^n(x,B) \ge \nu_n(B). \tag{2.1}
$$

(ii) *The ψ-irreducible Markov chain $\{Y_n\}_{n\ge 0}$ is called* aperiodic *if for some small set C with $\psi(C) > 0$, 1 is the greatest common divisor of all values $m \in \mathbb{N}$ for which (2.1) holds for $\nu_m = \delta_m \nu_n$, where $n \in \mathbb{N}$ is such that C is ν_n-small set with $\nu_n(C) > 0$ and $\delta_m > 0$.*

(iii) *Let $C \in \mathcal{B}(\mathbb{R})$. If there exist a probability measure $a = \{a(n)\}_{n\ge 0}$ on $\mathbb{Z}_+$ and a nontrivial measure ν_a on $\mathcal{B}(\mathbb{R})$ such that*

$$
\sum_{n=0}^{\infty} a(n) p^n(x,B) \ge \nu_a(B)
$$

*holds for every $x \in C$ and every $B \in \mathcal{B}(\mathbb{R})$, then the set C is called ν_a-*petite set.

Proposition 2.3. *Conditions* (C1)–(C4) *imply that for the chain $\{X_n\}_{n\ge 0}$ every bounded Borel set $C \subseteq [-k,k]^c$ is a ν_2-small set for some nontrivial measure ν_2.*

Proof. By (C3), there exists $k > 0$, such that for all $0 < \varepsilon < 1$ there exists $y_\varepsilon \ge k \vee 1$, such that for all $|y| \ge y_\varepsilon$ we have

$$
\left| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon
$$

for all $x \in [-k,k]^c$. Let $C \subseteq (-\infty,-k]$ be a bounded Borel set. Let $x \in C$ and $B \in \mathcal{B}(\mathbb{R})$ be arbitrary. Similarly as in Proposition 2.1, we have

$$\begin{aligned} p^2(x,B) &= \int_{\mathbb{R}} f_x(y-x)\int_{B-y} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \geq \int_{y_\varepsilon}^{2y_\varepsilon} f_x(y-x)\int_{(B-y)\cap(-\infty,-y_\varepsilon)} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \\ &> (1-\varepsilon)^2\Big(\inf_{x\in C} c(x)\Big)\Big(\inf_{y_\varepsilon\leq y\leq 2y_\varepsilon} c(y)\Big)\int_{y_\varepsilon}^{2y_\varepsilon}(y-a)^{-3}\int_{(B-y)\cap(-\infty,-y_\varepsilon)}|z|^{-3}\,\mathrm{d}z\,\mathrm{d}y, \end{aligned}$$

where $a := \inf C$. Now, by condition (C4), the measure

$$\nu_2(B) := (1-\varepsilon)^2\Big(\inf_{x\in C} c(x)\Big)\Big(\inf_{y_\varepsilon\leq y\leq 2y_\varepsilon} c(y)\Big)\int_{y_\varepsilon}^{2y_\varepsilon}(y-a)^{-3}\int_{(B-y)\cap(-\infty,-y_\varepsilon)}|z|^{-3}\,\mathrm{d}z\,\mathrm{d}y$$

is a nontrivial measure. Therefore, the set C is a ν_2-small set. Similarly, we deduce that a bounded Borel set $C \subseteq [k,\infty)$ is a ν_2-small for some nontrivial measure ν_2. □

Proposition 2.4. *Under conditions* (C1)–(C4), *the chain* $\{X_n\}_{n\geq 0}$ *is an aperiodic chain.*

Proof. From the previous proposition, we know that every bounded Borel set $C \subseteq [-k,k]^c$ is a ν_2-small set. Let us show that there exists a ν_2-small set $C \subseteq [-k,k]^c$ which is also a ν_3-small set with $\nu_3 = \delta_3\nu_2$, for some $\delta_3 > 0$. Let $C = [-4y_\varepsilon - k, -k]$, where ε and y_ε are given as in the previous proposition. The set C is a ν_2-small set. Let us show that

$$\inf_{x\in C} p(x,C) > 0.$$

Then, by [9], Proposition 5.2.4, C is a ν_3-small set, where ν_3 is a multiple of ν_2. Similarly as in Proposition 2.1, we have

$$\begin{aligned} p(x,C) &= \int_{C-x} f_x(y)\,\mathrm{d}y \geq \int_{(C-x)\cap(-\infty,-y_\varepsilon)\cup(C-x)\cap(y_\varepsilon,\infty)} f_x(y)\,\mathrm{d}y \\ &> (1-\varepsilon)\Big(\inf_{x\in C} c(x)\Big)\inf_{x\in C}\int_{(C-x)\cap(-\infty,-y_\varepsilon)\cup(C-x)\cap(y_\varepsilon,\infty)}|y|^{-3}\,\mathrm{d}y > 0. \end{aligned}$$ □

The following result is a consequence of [9], Proposition 5.5.2 and Theorem 5.5.7.

Proposition 2.5. *Conditions* (C1)–(C4) *imply that for the chain* $\{X_n\}_{n\geq 0}$, *a Borel set is a small set if and only if it is a petite set.*

Since conditions (C3), (C4) and (C5) consider compact sets, we get the following result which is essential in proving Theorems 1.3 and 1.4.

Proposition 2.6. *Conditions* (C1)–(C5) *imply that for the chain* $\{X_n\}_{n\geq 0}$, *every bounded Borel set is a small set.*

Proof. From Proposition 2.3, we know that every bounded Borel set $C \subseteq [-k,k]^c$ is a small set. By [9], Proposition 5.5.5, it is enough to show that $[-k,k]$ is a small set. Let $C \subseteq (-\infty,-k]$ be a bounded Borel set, that is, a small set. Let $0<\varepsilon<1$ be arbitrary and let $y_\varepsilon \geq (k \vee l \vee 1)$ (recall that l is defined in condition (C5)) be such that for all $|y| \geq y_\varepsilon$ we have

$$\left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon$$

for all $x \in [-k,k]^c$. Then, similarly as in Proposition 2.1, for every $x \in [-k,k]$, we have

$$\begin{aligned} p^2(x,C) &= \int_{\mathbb{R}} f_x(y-x)\int_{C-y} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \geq \int_{y_\varepsilon}^{2y_\varepsilon} f_x(y-x)\int_{(C-y)\cap(-\infty,-y_\varepsilon)} f_y(z)\,\mathrm{d}z\,\mathrm{d}y \\ &> (1-\varepsilon)\Big(\inf_{y_\varepsilon\leq y\leq 2y_\varepsilon} c(y)\Big)\inf_{x\in[-k,k]}\left(\int_{[y_\varepsilon,2y_\varepsilon]-x} f_x(y)\,\mathrm{d}y\right)\left(\int_{C-2y_\varepsilon-k} |z|^{-3}\,\mathrm{d}z\right). \end{aligned}$$

Now, using condition (C5), we have that $p^2(x,C)>0$. Therefore, by [9], Proposition 5.2.4, the set $[-k,k]$ is a small set, that is, every bounded Borel set is a small set. □

3. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Before the proof we recall several special functions we need. The Digamma function is a function defined by $\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$, for $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$, where $\Gamma(z)$ is the Gamma function.

Lemma 3.1. *Let $a>0$ be an arbitrary real number. Then*

$$\int_1^\infty \frac{\mathrm{d}y}{y^a(1+y)} = \frac{1}{2}\left(\Psi\left(\frac{a+1}{2}\right) - \Psi\left(\frac{a}{2}\right)\right).$$

Proof. From [1], formula 6.3.22, we have

$$\Psi(z) = \int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x - \gamma,$$

for $\operatorname{Re}(z)>0$, where γ is Euler's constant. Then

$$\Psi\left(\frac{a+1}{2}\right) - \Psi\left(\frac{a}{2}\right) = \int_0^1 \frac{x^{a/2-1} - x^{(a+1)/2-1}}{1-x}\,\mathrm{d}x.$$

The claim follows by change of variables $x = y^{-2}$. □

The Gauss hypergeometric function is defined by the formula

$$ {}_2F_1(a,b,c;z) := \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{3.1}$$

for $a, b, c, z \in \mathbb{C}$, $c \notin \mathbb{Z}_-$, where for $w \in \mathbb{C}$ and $n \in \mathbb{Z}_+$, $(w)_n$ is defined by

$$(w)_0 = 1 \quad \text{and} \quad (w)_n = w(w+1)\cdots(w+n-1).$$

The series (3.1) absolutely converges on $|z| < 1$, absolutely converges on $|z| \le 1$ when $\mathrm{Re}(c - a - b) > 0$, conditionally converges on $|z| \le 1$, except for $z = 1$, when $-1 < \mathrm{Re}(c - b - a) \le 0$ and diverges when $\mathrm{Re}(c - b - a) \le -1$. In the case when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, it can be analytically continued on $\mathbb{C} \setminus (1, \infty)$ by the formula

$$_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, \mathrm{d}t. \tag{3.2}$$

The incomplete Beta function is defined by the formula

$$B(x; z, w) := \int_0^x t^{z-1}(1-t)^{w-1}\, \mathrm{d}t \tag{3.3}$$

for $x \in [0, 1]$, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$. When $x = 1$, the function $B(1; z, w)$ is called the Beta function and

$$B(1; z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{3.4}$$

We need the following technical lemma.

Lemma 3.2. *Let $\alpha : \mathbb{R} \longrightarrow (1, 2)$ be an arbitrary function. Then for every $R \ge 0$ we have*

$$\lim_{|x| \longrightarrow \infty} \frac{1}{2 - \alpha(x)} \left(1 - \left(\frac{|x|}{|x| + R}\right)^{2-\alpha(x)}\right) = 0.$$

Proof. Let $0 < \varepsilon < 1$ be arbitrary. Since

$$\frac{1}{x}\big(1 - (1-\varepsilon)^x\big) \le -\ln(1-\varepsilon)$$

for all $x \in (0, 1]$, we have

$$0 \le \limsup_{|x| \longrightarrow \infty} \frac{1}{2 - \alpha(x)} \left(1 - \left(\frac{|x|}{|x| + R}\right)^{2-\alpha(x)}\right) \le \limsup_{|x| \longrightarrow \infty} \frac{1 - (1-\varepsilon)^{2-\alpha(x)}}{2 - \alpha(x)} \le -\ln(1-\varepsilon).$$

By letting $\varepsilon \longrightarrow 0$, we have the claim. □

Proof of Theorem 1.3. The proof is divided in four steps.

Step 1. In the first step, we explain our strategy of the proof. Let us define the function $V : \mathbb{R} \longrightarrow \mathbb{R}_+$ by the formula

$$V(x) := \ln\big(1 + |x|\big).$$

From Proposition 2.6, the set $C_V(r)=\{y\colon\ V(y)\le r\}$ is a petite set for all $r<\infty$. We will show that there exists $r_0>0$, big enough, such that $\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y)-V(x)\le 0$ for all $x\in C_V^c(r_0)$. Then, the desired result will follow from [9], Theorem 8.4.2. Since $C_V(r)\uparrow\mathbb{R}$, when $r\longrightarrow\infty$, it is enough to show that

$$\limsup_{|x|\longrightarrow\infty}\frac{(1+|x|)^{\alpha(x)}}{c(x)}\left(\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y)-V(x)\right)<0.$$

We have

$$\begin{aligned}\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y)&=\int_{\mathbb{R}} f_x(y-x)V(y)\,\mathrm{d}y=\int_{\mathbb{R}} f_x(y)V(y+x)\,\mathrm{d}y\\&=\int_{-x}^{\infty}\ln(1+x+y)f_x(y)\,\mathrm{d}y+\int_{-\infty}^{-x}\ln(1-x-y)f_x(y)\,\mathrm{d}y.\end{aligned}\tag{3.5}$$

Step 2. In the second step, we find an appropriate upper bound for the first summand in (3.5). For any $x>0$, we have

$$\int_{-x}^{\infty}\ln(1+x+y)f_x(y)\,\mathrm{d}y=\ln(1+x)\int_{-x}^{\infty}f_x(y)\,\mathrm{d}y+\int_{-x}^{\infty}\ln\left(1+\frac{y}{1+x}\right)f_x(y)\,\mathrm{d}y.$$

Let $0<\delta<1$ be arbitrary. By restricting $\ln(1+t)$ to intervals $(-1,-\delta)$, $[-\delta,\delta]$, $(\delta,1)$ and $[1,\infty)$, and using the Taylor expansion of the function $\ln(1+t)$, that is,

$$\ln(1+t)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}t^i$$

for $t\in(-1,1]$, we get

$$\begin{aligned}\int_{-x}^{\infty}\ln(1+x+y)f_x(y)\,\mathrm{d}y\le{}&\ln(1+x)\int_{-x}^{\infty}f_x(y)\,\mathrm{d}y\\&-\sum_{i=1}^{\infty}\frac{1}{i(1+x)^i}\int_{\{-1-x<y<-\delta(1+x)\}\cap\{y+x>0\}}|y|^i f_x(y)\,\mathrm{d}y\\&+\int_{\{-\delta(1+x)\le y\le\delta(1+x)\}\cap\{y+x>0\}}\ln\left(1+\frac{y}{1+x}\right)f_x(y)\,\mathrm{d}y\\&+\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i(1+x)^i}\int_{\{\delta(1+x)<y<1+x\}\cap\{y+x>0\}}y^i f_x(y)\,\mathrm{d}y\\&+\int_{\{y\ge 1+x\}\cap\{y+x>0\}}\ln\left(1+\frac{y}{1+x}\right)f_x(y)\,\mathrm{d}y.\end{aligned}$$

Furthermore, by taking $x > \frac{\delta}{1-\delta}$ we get

$$\begin{aligned}\int_{-x}^{\infty} \ln(1+x+y) f_x(y)\,\mathrm{d}y \le \ln(1+x)\int_{-x}^{\infty} f_x(y)\,\mathrm{d}y - \sum_{i=1}^{\infty}\frac{1}{i(1+x)^i}\int_{-x}^{-\delta(1+x)} |y|^i f_x(y)\,\mathrm{d}y\\ + \int_{-\delta(1+x)}^{\delta(1+x)} \ln\left(1+\frac{y}{1+x}\right) f_x(y)\,\mathrm{d}y\\ + \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i(1+x)^i}\int_{\delta(1+x)}^{1+x} y^i f_x(y)\,\mathrm{d}y\\ + \int_{1+x}^{\infty} \ln\left(1+\frac{y}{1+x}\right) f_x(y)\,\mathrm{d}y.\end{aligned}$$

Let us put

$$\begin{aligned}U_1^{\delta}(x) &:= -\frac{1}{1+x}\int_{\delta(1+x)}^{x} y f_x(-y)\,\mathrm{d}y + \frac{1}{1+x}\int_{\delta(1+x)}^{1+x} y f_x(y)\,\mathrm{d}y,\\ U_2^{\delta}(x) &:= -\frac{1}{2(1+x)^2}\int_{\delta(1+x)}^{x} y^2 f_x(-y)\,\mathrm{d}y - \frac{1}{2(1+x)^2}\int_{\delta(1+x)}^{1+x} y^2 f_x(y)\,\mathrm{d}y,\\ U_3^{\delta}(x) &:= -\sum_{i=3}^{\infty}\frac{1}{i(1+x)^i}\int_{\delta(1+x)}^{x} y^i f_x(-y)\,\mathrm{d}y + \sum_{i=3}^{\infty}\frac{(-1)^{i+1}}{i(1+x)^i}\int_{\delta(1+x)}^{1+x} y^i f_x(y)\,\mathrm{d}y,\\ U_4^{\delta}(x) &:= \int_{-\delta(1+x)}^{\delta(1+x)} \ln\left(1+\frac{y}{1+x}\right) f_x(y)\,\mathrm{d}y \quad \text{and}\\ U_5(x) &:= \int_{1+x}^{\infty} \ln\left(1+\frac{y}{1+x}\right) f_x(y)\,\mathrm{d}y\end{aligned}$$

for $0 < \delta < 1$ and $x > \frac{\delta}{1-\delta}$. Hence, we find

$$\begin{aligned}&\int_{-x}^{\infty} \ln(1+x+y) f_x(y)\,\mathrm{d}y\\ &\quad\le \ln(1+x)\int_{-x}^{\infty} f_x(y)\,\mathrm{d}y + U_1^{\delta}(x) + U_2^{\delta}(x) + U_3^{\delta}(x) + U_4^{\delta}(x) + U_5(x).\end{aligned}\tag{3.6}$$

Here comes the crucial step where condition (C3) is needed. In the above terms, by (C3), we can replace all the density functions $f_x(y)$ by the functions $c(x)|y|^{-\alpha(x)-1}$ and find a more operable upper bound in (3.6). Let $0 < \varepsilon < 1$ be arbitrary. Then, by (C3), there exists $y_\varepsilon \ge 1$, such that for all $|y| \ge y_\varepsilon$

$$\left| f_x(y)\frac{|y|^{\alpha(x)+1}}{c(x)} - 1\right| < \varepsilon$$

for all $x \in [-k,k]^c$. Let $x > (k \vee \frac{y_\varepsilon-\delta}{\delta} \vee \frac{\delta}{1-\delta})$. By a straightforward calculation, we have

$$\begin{aligned}
U_1^\delta(x) &< -\frac{(1-\varepsilon)c(x)}{(\alpha(x)-1)(1+x)^{\alpha(x)}}\left(\delta^{-\alpha(x)+1}-\left(\frac{x}{1+x}\right)^{-\alpha(x)+1}\right)\\
&\quad+\frac{(1+\varepsilon)c(x)}{(\alpha(x)-1)(1+x)^{\alpha(x)}}\frac{\delta-\delta^{\alpha(x)}}{\delta^{\alpha(x)}}=:U_1^{\delta,\varepsilon}(x),\\
U_2^\delta(x) &< -\frac{(1-\varepsilon)c(x)}{(1+x)^{\alpha(x)}}\frac{1}{2(2-\alpha(x))}\left(\left(\frac{x}{1+x}\right)^{2-\alpha(x)}-\delta^{2-\alpha(x)}\right)\\
&\quad-\frac{(1-\alpha)c(x)}{(1+x)^{\alpha(x)}}\frac{1}{2(2-\alpha(x))}\frac{\delta^{\alpha(x)}-\delta^2}{\delta^{\alpha(x)}}=:U_2^{\delta,\varepsilon}(x),\\
U_3^\delta(x) &< -\frac{(1-\varepsilon)c(x)}{(1+x)^{\alpha(x)}}\sum_{i=3}^\infty\frac{1}{i(i-\alpha(x))}\left(\left(\frac{x}{1+x}\right)^{i-\alpha(x)}-\delta^{i-\alpha(x)}\right)\\
&\quad+\frac{c(x)}{(1+x)^{\alpha(x)}}\sum_{i=3}^\infty\left(\frac{(-1)^{i+1}(1+(-1)^{i+1}\varepsilon)}{i(i-\alpha(x))}\frac{\delta^{\alpha(x)}-\delta^i}{\delta^{\alpha(x)}}\right)=:U_3^{\delta,\varepsilon}(x)\quad\text{and}\\
U_5(x) &< (1+\varepsilon)c(x)\int_{1+x}^\infty\ln\left(1+\frac{y}{1+x}\right)\frac{1}{y^{\alpha(x)+1}}\,\mathrm{d}y=:U_5^\varepsilon(x).
\end{aligned}$$

Hence, from (3.6), we get

$$\begin{aligned}
&\int_{-x}^\infty\ln(1+x+y)f_x(y)\,\mathrm{d}y\\
&\quad<\ln(1+x)\int_{-x}^\infty f_x(y)\,\mathrm{d}y+U_1^{\delta,\varepsilon}(x)+U_2^{\delta,\varepsilon}(x)+U_3^{\delta,\varepsilon}(x)+U_4^\delta(x)+U_5^\varepsilon(x).
\end{aligned}\tag{3.7}$$

Step 3. In the third step, we find an appropriate upper bound for the second summand in (3.5). We have

$$\int_\infty^{-x}\ln(1-x-y)f_x(y)\,\mathrm{d}y=\ln(x-1)\int_\infty^{-x}f_x(y)\,\mathrm{d}y+\int_\infty^{-x}\ln\left(-1-\frac{y}{x-1}\right)f_x(y)\,\mathrm{d}y.$$

Let $x > (k \vee \frac{y_\varepsilon-\delta}{\delta} \vee \frac{\delta}{1-\delta})$. Then, again by (C3),

$$\begin{aligned}
\int_\infty^{-x}\ln(1-x-y)f_x(y)\,\mathrm{d}y &< \ln(x-1)\int_\infty^{-x}f_x(y)\,\mathrm{d}y\\
&\quad+c(x)(1-\varepsilon)\int_x^{2x-2}\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}\\
&\quad+c(x)(1+\varepsilon)\int_{2x-2}^\infty\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \ln(x-1)\int_{\infty}^{-x} f_x(y)\,\mathrm{d}y\\
&\quad + c(x)(1-\varepsilon)\int_x^{\infty}\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}\\
&\quad + 2\varepsilon c(x)\int_{2x-2}^{\infty}\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}.
\end{aligned}
$$

Let us put

$$
\begin{aligned}
U_6^{\varepsilon}(x) := {}& c(x)(1-\varepsilon)\int_x^{\infty}\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}\\
&+ 2\varepsilon c(x)\int_{2x-2}^{\infty}\ln\left(-1+\frac{y}{x-1}\right)\frac{\mathrm{d}y}{|y|^{\alpha(x)+1}}.
\end{aligned}
$$

We have

$$
\int_{\infty}^{-x}\ln(1-x-y)f_x(y)\,\mathrm{d}y < \ln(x-1)\int_{\infty}^{-x} f_x(y)\,\mathrm{d}y + U_6^{\varepsilon}(x). \tag{3.8}
$$

Step 4. In the fourth step, we prove

$$
\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}\left(\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y) - V(x)\right) < 0.
$$

By combining (3.5), (3.7) and (3.8), we have

$$
\begin{aligned}
\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y) < {}& U_0(x) + U_1^{\delta,\varepsilon}(x) + U_2^{\delta,\varepsilon}(x) + U_3^{\delta,\varepsilon}(x)\\
&+ U_4^{\delta}(x) + U_5^{\varepsilon}(x) + U_6^{\varepsilon}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
U_0(x) &= \ln(1+x)\int_{-x}^{\infty} f_x(y)\,\mathrm{d}y + \ln(x-1)\int_{-\infty}^{-x} f_x(y)\,\mathrm{d}y\\
&= \ln(1+x) - \ln(1+x)\int_{-\infty}^{-x} f_x(y)\,\mathrm{d}y + \ln(x-1)\int_{-\infty}^{-x} f_x(y)\,\mathrm{d}y\\
&< \ln(1+x) = V(x).
\end{aligned}
$$

Hence, we have

$$
\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y) - V(x) < U_1^{\delta,\varepsilon}(x) + U_2^{\delta,\varepsilon}(x) + U_3^{\delta,\varepsilon}(x) + U_4^{\delta}(x) + U_5^{\varepsilon}(x) + U_6^{\varepsilon}(x). \tag{3.9}
$$

In the rest of the fourth step, we prove

$$
\begin{aligned}
&\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}\left(\int_{\mathbb{R}}p(x,\mathrm{d}y)V(y)-V(x)\right)\\
&\quad<\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_1^{\delta,\varepsilon}(x)+\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_2^{\delta,\varepsilon}(x)\\
&\qquad+\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_3^{\delta,\varepsilon}(x)+\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_5^{\varepsilon}(x)\\
&\qquad+\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_6^{\varepsilon}(x)+R(\alpha)\le 0.
\end{aligned}
$$

Recall that $\alpha=\liminf_{|x|\longrightarrow\infty}\alpha(x)>1$,

$$
R(\alpha)=\sum_{i=1}^{\infty}\frac{1}{i(2i-\alpha)}-\frac{\ln 2}{\alpha}-\frac{1}{2\alpha}\left(\Psi\left(\frac{\alpha+1}{2}\right)-\Psi\left(\frac{\alpha}{2}\right)\right)
$$

and

$$
\limsup_{\delta\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_4^{\delta}(x)<R(\alpha)
$$

when $\alpha<2$, and the above limit is finite when $\alpha=2$ (assumption (1.3)). We have

$$
\begin{aligned}
&\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_1^{\delta,\varepsilon}(x)\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\left[-\frac{1-\varepsilon}{\alpha(x)-1}\left(\delta^{-\alpha(x)+1}-\left(\frac{x}{1+x}\right)^{-\alpha(x)+1}\right)\right.\\
&\qquad\qquad\left.+\frac{1+\varepsilon}{\alpha(x)-1}\frac{\delta-\delta^{\alpha(x)}}{\delta^{\alpha(x)}}\right]\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\left[-\frac{1-\varepsilon}{\alpha(x)-1}\left(\frac{\delta-\delta^{\alpha(x)}}{\delta^{\alpha(x)}}+1-\left(\frac{x}{1+x}\right)^{-\alpha(x)+1}\right)\right.\\
&\qquad\qquad\left.+\frac{1+\varepsilon}{\alpha(x)-1}\frac{\delta-\delta^{\alpha(x)}}{\delta^{\alpha(x)}}\right]\qquad(3.10)\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\left[\frac{2\varepsilon}{\alpha(x)-1}\frac{\delta-\delta^{\alpha(x)}}{\delta^{\alpha(x)}}-\frac{1-\varepsilon}{\alpha(x)-1}\left(1-\left(\frac{x}{1+x}\right)^{-\alpha(x)+1}\right)\right]\\
&\quad=\limsup_{x\longrightarrow\infty}\left[\frac{1}{\alpha(x)-1}\left(\left(\frac{x}{x+1}\right)^{-\alpha(x)+1}-1\right)\right]=0.
\end{aligned}
$$

In the last two equalities, we use the assumption $\liminf_{|x|\longrightarrow\infty}\alpha(x)>1$. From Lemma 3.2, we have

$$
\begin{aligned}
&\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_2^{\delta,\varepsilon}(x)\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\Bigg[-\frac{1-\varepsilon}{2(2-\alpha(x))}\left(\left(\frac{x}{1+x}\right)^{2-\alpha(x)}-\delta^{2-\alpha(x)}\right)\\
&\qquad\qquad-\frac{1-\varepsilon}{2(2-\alpha(x))}\frac{\delta^{\alpha(x)}-\delta^2}{\delta^{\alpha(x)}}\Bigg]\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{x\longrightarrow\infty}\Bigg[-\frac{1}{2(2-\alpha(x))}\left(\left(\frac{x}{1+x}\right)^{2-\alpha(x)}+\frac{\delta^{\alpha(x)}-\delta^2}{\delta^{\alpha(x)}}-1\right)\\
&\qquad\qquad-\frac{1}{2(2-\alpha(x))}\frac{\delta^{\alpha(x)}-\delta^2}{\delta^{\alpha(x)}}\Bigg]\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{x\longrightarrow\infty}\Bigg[-\frac{1}{2-\alpha(x)}\frac{\delta^{\alpha(x)}-\delta^2}{\delta^{\alpha(x)}}\Bigg]\\
&\quad\le\begin{cases}-\dfrac{1}{2-\alpha}, & \alpha<2,\\ -\infty, & \alpha=2.\end{cases}
\end{aligned}
\tag{3.11}
$$

Using the dominated convergence theorem, we have

$$
\begin{aligned}
&\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_3^{\delta,\varepsilon}(x)\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\Bigg[-(1-\varepsilon)\sum_{i=3}^{\infty}\frac{1}{i(i-\alpha(x))}\left(\left(\frac{x}{1+x}\right)^{i-\alpha(x)}-\delta^{i-\alpha(x)}\right)\\
&\qquad\qquad+\sum_{i=3}^{\infty}\frac{(\varepsilon+(-1)^{i+1})}{i(i-\alpha(x))}\frac{\delta^{\alpha(x)}-\delta^i}{\delta^{\alpha(x)}}\Bigg]\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\Bigg[\sum_{i=3}^{\infty}\frac{-(x/(1+x))^{i-\alpha(x)}+\delta^{i-\alpha(x)}+(-1)^{i+1}-(-1)^{i+1}\delta^{i-\alpha(x)}}{i(i-\alpha(x))}\\
&\qquad\qquad+\varepsilon\sum_{i=3}^{\infty}\frac{(x/(1+x))^{i-\alpha(x)}-\delta^{i-\alpha(x)}+1-\delta^{i-\alpha(x)}}{i(i-\alpha(x))}\Bigg]\\
&\quad=\limsup_{\delta\longrightarrow 0}\limsup_{x\longrightarrow\infty}\sum_{i=3}^{\infty}\frac{-(x/(1+x))^{i-\alpha(x)}+\delta^{i-\alpha(x)}+(-1)^{i+1}-(-1)^{i+1}\delta^{i-\alpha(x)}}{i(i-\alpha(x))}
\end{aligned}
\tag{3.12}
$$

$$= \limsup_{\delta\longrightarrow 0}\limsup_{x\longrightarrow\infty}\sum_{i=3}^{\infty}\left(\frac{-(x/(1+x))^{i-\alpha(x)}+(-1)^{i+1}}{i(i-\alpha(x))}+\frac{\delta^{i-\alpha(x)}-(-1)^{i+1}\delta^{i-\alpha(x)}}{i(i-\alpha(x))}\right)$$

$$\leq -\sum_{i=2}^{\infty}\frac{2}{2i(2i-\alpha)} = -\sum_{i=2}^{\infty}\frac{1}{i(2i-\alpha)}.$$

Therefore, by combining (3.10), (3.11) and (3.12) we get

$$\begin{aligned}&\limsup_{\delta\longrightarrow 0}\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}\left(U_1^{\delta,\varepsilon}(x)+U_2^{\delta,\varepsilon}(x)+U_3^{\delta,\varepsilon}(x)\right)\\ &\quad\leq\begin{cases}-\displaystyle\sum_{i=1}^{\infty}\frac{1}{i(2i-\alpha)}, & \alpha<2,\\ -\infty, & \alpha=2.\end{cases}\end{aligned}\tag{3.13}$$

Now, let us calculate

$$\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_5^{\varepsilon}(x).$$

Using integration by parts formula, we get

$$\begin{aligned}&\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_5^{\varepsilon}(x)\\ &\quad=\limsup_{x\longrightarrow\infty}(1+x)^{\alpha(x)}\int_{1+x}^{\infty}\ln\left(1+\frac{y}{1+x}\right)\frac{1}{y^{\alpha(x)+1}}\,\mathrm{d}y\\ &\quad=\limsup_{x\longrightarrow\infty}\left(\frac{\ln 2}{\alpha(x)}+\frac{1}{\alpha(x)}\int_1^{\infty}\frac{\mathrm{d}y}{y^{\alpha(x)}(1+y)}\right).\end{aligned}$$

Furthermore, from Lemma 3.1 and the fact that the function

$$x\longmapsto\Psi\left(\frac{x+1}{2}\right)-\Psi\left(\frac{x}{2}\right)$$

is decreasing on $(0,\infty)$ (Lemma 3.1) we have

$$\begin{aligned}&\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_5^{\varepsilon}(x)\\ &\quad=\limsup_{x\longrightarrow\infty}\left(\frac{\ln 2}{\alpha(x)}+\frac{1}{2\alpha(x)}\left(\Psi\left(\frac{\alpha(x)+1}{2}\right)-\Psi\left(\frac{\alpha(x)}{2}\right)\right)\right)\\ &\quad\leq\frac{\ln 2}{\alpha}+\frac{1}{2\alpha}\left(\Psi\left(\frac{\alpha+1}{2}\right)-\Psi\left(\frac{\alpha}{2}\right)\right).\end{aligned}\tag{3.14}$$

At the end, using integration by parts formula, we have

$$\begin{aligned}
&\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_6^{\varepsilon}(x)\\
&\quad=\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}(1+x)^{\alpha(x)}\bigg[(1-\varepsilon)\int_x^{\infty}\ln\Big(-1+\frac{y}{x-1}\Big)\frac{1}{|y|^{\alpha(x)+1}}\,\mathrm{d}y\\
&\qquad\qquad+2\varepsilon\int_{2x-2}^{\infty}\ln\Big(-1+\frac{y}{x-1}\Big)\frac{1}{|y|^{\alpha(x)+1}}\,\mathrm{d}y\bigg]\\
&\quad=\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}(1+x)^{\alpha(x)}\bigg[\frac{1-\varepsilon}{\alpha(x)}\bigg(\frac{1}{x^{\alpha(x)}}\ln\Big(-1+\frac{x}{x-1}\Big)+\int_x^{\infty}\frac{\mathrm{d}y}{y^{\alpha(x)}(y-x+1)}\bigg)\\
&\qquad\qquad+\frac{2\varepsilon}{\alpha(x)}\int_{2x-2}^{\infty}\frac{\mathrm{d}y}{y^{\alpha(x)}(y-x+1)}\bigg]\\
&\quad=\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\bigg[\frac{1-\varepsilon}{\alpha(x)}\bigg(\frac{(1+x)^{\alpha(x)}}{x^{\alpha(x)}}\ln\Big(\frac{1}{x-1}\Big)+\frac{(1+x)^{\alpha(x)}}{(x-1)^{\alpha(x)}}\int_0^{(x-1)/x}\frac{y^{\alpha(x)-1}}{1-y}\,\mathrm{d}y\bigg)\\
&\qquad\qquad+\frac{2\varepsilon}{\alpha^2(x)}\frac{(1+x)^{\alpha(x)}}{(x-1)^{\alpha(x)}}{}_2F_1\big(\alpha(x),\alpha(x),\alpha(x)+1;-1\big)\bigg],
\end{aligned}$$

where in the last equality we use (3.2). From (3.2), we get

$${}_2F_1\big(\alpha(x),\alpha(x),\alpha(x)+1;-1\big)\le 2\int_0^1(1+t)^{-1}\,\mathrm{d}t=\ln 4,$$

and

$$\int_0^{(x-1)/x}\frac{y^{\alpha(x)-1}}{1-y}\,\mathrm{d}y\le\int_0^{(x-1)/x}\frac{\mathrm{d}y}{1-y}=\ln x.$$

Hence,

$$\begin{aligned}
&\limsup_{\varepsilon\longrightarrow 0}\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}U_6^{\varepsilon}(x)\\
&\quad\le\limsup_{x\longrightarrow\infty}\frac{1}{\alpha(x)}\bigg(\frac{(1+x)^{\alpha(x)}}{x^{\alpha(x)}}\ln\Big(-1+\frac{x}{x-1}\Big)+\frac{(x+1)^{\alpha(x)}}{(x-1)^{\alpha(x)}}\ln x\bigg)\qquad(3.15)\\
&\quad\le\limsup_{x\longrightarrow\infty}\frac{1}{\alpha(x)}\bigg(\ln\Big(-1+\frac{x}{x-1}\Big)+\Big(1+\frac{2}{x-1}\Big)^2\ln x\bigg)=0.
\end{aligned}$$

By combining (3.9), (3.13), (3.14) and (3.15), we have

$$\limsup_{x\longrightarrow\infty}\frac{(1+x)^{\alpha(x)}}{c(x)}\bigg(\int_{\mathbb{R}}p(x,\mathrm{d}y)V(y)-V(x)\bigg)<0.$$

The case when $x<0$ is treated in the same way. Therefore, we have proved the desired result. □

4. Proof of Theorem 1.4

Let us first list some properties of the Gauss hypergeometric function which will be needed in the proof of Theorem 1.4:

(i) for $a, b, c, z \in \mathbb{C}$, $c \notin \mathbb{Z}_-$,

$$ {}_2F_1(0, b, c; z) = {}_2F_1(a, 0, c; z) = 1; \tag{4.1} $$

(ii) for $\mathrm{Re}(c - a - b) > 0$, $c \notin \mathbb{Z}_-$,

$$ {}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}; \tag{4.2} $$

(iii) for $z \in \mathbb{C} \setminus (1, \infty)$

$$ {}_2F_1(a, b, c; z) = (1-z)^{c-b-a}\,{}_2F_1(c-a, c-b, c; z); \tag{4.3} $$

(iv) for $z \in \mathbb{C} \setminus (0, \infty)$

$$ \begin{aligned} {}_2F_1(a, b, c; z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(a, 1-c+a, 1-b+a, \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(b, 1-c+b, 1-a+b, \frac{1}{z}\right). \end{aligned} \tag{4.4} $$

For further properties of the hypergeometric functions, the incomplete Beta functions and the Beta function (see [1], Chapters 6 and 15).

Proof of Theorem 1.4. The proof is divided in three steps.

Step 1. In the first step, we explain our strategy of the proof. Let us define the function $V : \mathbb{R} \longrightarrow \mathbb{R}_+$ by the formula

$$ V(x) := 1 - (1 + |x|)^{-\beta}, $$

where $0 < \beta < 1 - \alpha$ is arbitrary (recall that $\alpha = \limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$). It is clear that $C_V(r) \in \mathcal{B}^+(\mathbb{R})$ and $C_V^c(r) \in \mathcal{B}^+(\mathbb{R})$, for every $0 < r < 1$. By [9], Theorem 8.4.3, we have to show that there exists $0 < r_0 < 1$ such that $\Delta V(x) \geq 0$, for every $x \in C_V^c(r_0)$. Since $C_V(r) \uparrow \mathbb{R}$, when $r \uparrow 1$, it is enough to show that

$$ \liminf_{|x|\longrightarrow\infty} \frac{\alpha(x)|x|^{\alpha(x)+\beta}}{c(x)} \left(\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \right) > 0. $$

We have

$$ \begin{aligned} & \int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x) \\ & \quad = \int_{\mathbb{R}} V(y + x) f_x(y)\,\mathrm{d}y - V(x) \end{aligned} \tag{4.5} $$

$$
\begin{aligned}
&= \int_{-x}^{\infty} \big(1 - (1 + y + x)^{-\beta}\big) f_x(y)\,\mathrm{d}y + \int_{-\infty}^{-x} \big(1 - (1 - y - x)^{-\beta}\big) f_x(y)\,\mathrm{d}y \\
&\quad - \big(1 - \big(1 + |x|\big)^{-\beta}\big) \int_{-x}^{\infty} f_x(y)\,\mathrm{d}y - \big(1 - \big(1 + |x|\big)^{-\beta}\big) \int_{-\infty}^{-x} f_x(y)\,\mathrm{d}y \\
&= \big(1 + |x|\big)^{-\beta} \bigg[\int_{-x}^{\infty} \bigg(1 - \bigg(\frac{1 + |x|}{1 + x + y}\bigg)^{\beta}\bigg) f_x(y)\,\mathrm{d}y \\
&\qquad + \int_{-\infty}^{-x} \bigg(1 - \bigg(\frac{1 + |x|}{1 - x - y}\bigg)^{\beta}\bigg) f_x(y)\,\mathrm{d}y \bigg].
\end{aligned}
$$

Step 2. In the second step, by use of condition (C3), we find an operable lower bound for (4.5). First, let us take a look at the case when $x > 0$. Let $0 < \varepsilon < 1$ be arbitrary. Then, by (C3), there exists $y_\varepsilon \ge a_0 \vee 1$ (the constant $a_0 > 0$ is defined in (1.3)), such that for all $|y| \ge y_\varepsilon$

$$
\bigg| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \bigg| < \varepsilon
$$

for all $x \in [-k, k]^c$. Let $x \ge k \vee y_\varepsilon$. Then we have

$$
\begin{aligned}
\int_{-x}^{\infty} \bigg(1 - \bigg(1 + \frac{y}{1 + x}\bigg)^{-\beta}\bigg) f_x(y)\,\mathrm{d}y &> c(x)(1 + \varepsilon) \int_{y_\varepsilon}^{x} \bigg(1 - \bigg(1 - \frac{y}{1 + x}\bigg)^{-\beta}\bigg) \frac{\mathrm{d}y}{y^{\alpha(x)+1}} \\
&\quad + \int_{-y_\varepsilon}^{y_\varepsilon} \bigg(1 - \bigg(1 + \frac{y}{1 + x}\bigg)^{-\beta}\bigg) f_x(y)\,\mathrm{d}y \\
&\quad + c(x)(1 - \varepsilon) \int_{y_\varepsilon}^{\infty} \bigg(1 - \bigg(1 + \frac{y}{1 + x}\bigg)^{-\beta}\bigg) \frac{\mathrm{d}y}{y^{\alpha(x)+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{-x} \bigg(1 - \bigg(\frac{1 + x}{1 - x - y}\bigg)^{\beta}\bigg) f_x(y)\,\mathrm{d}y \\
&\quad > \frac{c(x)(1 - \varepsilon)}{\alpha(x) x^{\alpha(x)}} - c(x)(1 + \varepsilon) \int_{x}^{\infty} \bigg(\frac{1 + x}{1 - x + y}\bigg)^{\beta} \frac{\mathrm{d}y}{y^{\alpha(x)+1}}.
\end{aligned}
$$

Note that this was the crucial step where we needed condition (C3). For given $0 < \varepsilon < 1$ and $x \ge k \vee y_\varepsilon$, let us put

$$
\begin{aligned}
U_1^{\varepsilon}(x) &:= c(x)(1 + \varepsilon) \int_{y_\varepsilon}^{x} \bigg(1 - \bigg(1 - \frac{y}{1 + x}\bigg)^{-\beta}\bigg) \frac{\mathrm{d}y}{y^{\alpha(x)+1}}, \\
U_2^{\varepsilon}(x) &:= \int_{-y_\varepsilon}^{y_\varepsilon} \bigg(1 - \bigg(1 + \frac{y}{1 + x}\bigg)^{-\beta}\bigg) f_x(y)\,\mathrm{d}y,
\end{aligned}
$$

$$U_3^\varepsilon(x) := c(x)(1-\varepsilon)\int_{y_\varepsilon}^\infty \left(1-\left(1+\frac{y}{1+x}\right)^{-\beta}\right)\frac{\mathrm{d}y}{y^{\alpha(x)+1}},$$

$$U_4^\varepsilon(x) := \frac{c(x)(1-\varepsilon)}{\alpha(x)x^{\alpha(x)}} \quad \text{and}$$

$$U_5^\varepsilon(x) := c(x)(1+\varepsilon)\int_{x}^\infty \left(\frac{1+x}{1-x+y}\right)^{\beta}\frac{\mathrm{d}y}{y^{\alpha(x)+1}}.$$

Hence, we have

$$\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y) - V(x) > U_1^\varepsilon(x) + U_2^\varepsilon(x) + U_3^\varepsilon(x) + U_4^\varepsilon(x) - U_5^\varepsilon(x).$$

Step 3. In the third step, we prove

$$\begin{aligned} &\liminf_{x\longrightarrow\infty} \frac{\alpha(x)x^{\alpha(x)+\beta}}{c(x)}\left(\int_{\mathbb{R}} p(x,\mathrm{d}y)V(y) - V(x)\right) \\ &\quad > \liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty}\frac{\alpha(x)x^{\alpha(x)}}{c(x)}\big(U_1^\varepsilon(x)+U_3^\varepsilon(x)+U_4^\varepsilon(x)-U_5^\varepsilon(x)\big) - T(\alpha,\beta) \ge 0. \end{aligned} \tag{4.6}$$

Recall that

$$T(\alpha,\beta) = {}_2F_1(-\alpha,\beta,1-y;1) + \beta B(1;\alpha+\beta,1-\alpha) - \alpha B(1;\alpha+\beta,1-\beta)$$

and

$$\liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty}\frac{\alpha(x)x^{\alpha(x)}}{c(x)}U_2^\varepsilon(x) > -T(\alpha,\beta)$$

(assumption (1.4)). By straightforward calculations, using (3.2), (4.3) and (3.3), we have

$$\begin{aligned} \frac{\alpha(x)x^{\alpha(x)}}{c(x)}U_1^\varepsilon(x) &= \frac{(1+\varepsilon)x^{\alpha(x)}}{y_\varepsilon^{\alpha(x)}} - \frac{(1+\varepsilon)x^{\alpha(x)}{}_2F_1(-\alpha(x),\beta,1-\alpha(x);y_\varepsilon/(1+x))}{y_\varepsilon^{\alpha(x)}} \\ &\quad -(1+\varepsilon) + (1+\varepsilon){}_2F_1\left(-\alpha(x),\beta,1-\alpha(x);\frac{x}{1+x}\right), \end{aligned}$$

$$\frac{\alpha(x)x^{\alpha(x)}}{c(x)}U_4^\varepsilon(x) = (1-\varepsilon) \quad \text{and}$$

$$\frac{\alpha(x)x^{\alpha(x)}}{c(x)}U_5^\varepsilon(x) = \frac{(1+\varepsilon)\alpha(x)x^{\alpha(x)}(1+x)^\beta B((x-1)/x;\alpha(x)+\beta,1-\beta)}{(x-1)^{\alpha(x)+\beta}}.$$

It is easy to check that

$$\frac{\partial}{\partial y}\left(-\frac{{}_2F_1(-\alpha(x),\beta,1-\alpha(x);-y/(1+x))}{\alpha(x)y^{\alpha(x)}(1+x)^\beta}\right) = \frac{1}{(1+x+y)^\beta y^{\alpha(x)+1}},$$

and from (4.4) and

$$\Gamma(z+1) = z\Gamma(z), \qquad z \in \mathbb{C} \setminus \mathbb{Z}_-, \tag{4.7}$$

we have

$$\begin{aligned}\frac{{}_2F_1(-\alpha(x), \beta, 1-\alpha(x); -y/(1+x))}{\alpha(x) y^{\alpha(x)}(1+x)^{\beta}} &= \frac{{}_2F_1(\beta, \alpha(x)+\beta, 1+\alpha(x)+\beta; -(1+x)/y)}{(\alpha(x)+\beta) y^{\alpha(x)+\beta}} \\ &\quad + \frac{\Gamma(1-\alpha(x))\Gamma(\alpha(x)+\beta)}{\alpha(x)(1+x)^{2\alpha(x)+\beta}\Gamma(\beta)}.\end{aligned}$$

Therefore,

$$\int \frac{\mathrm{d}y}{(1+x+y)^{\beta} y^{\alpha(x)+1}} = -\frac{{}_2F_1(\beta, \alpha(x)+\beta, 1+\alpha(x)+\beta; -(1+x)/y)}{(\alpha(x)+\beta) y^{\alpha(x)+\beta}},$$

that is,

$$\begin{aligned}&\frac{\alpha(x)x^{\alpha(x)}}{c(x)} U_3^{\varepsilon}(x) \\ &\quad = \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} - \frac{(1-\varepsilon)\alpha(x)x^{\alpha(x)}(1+x)^{\beta}{}_2F_1(\beta, \alpha(x)+\beta, 1+\alpha(x)+\beta, -(1+x)/y_{\varepsilon})}{y_{\varepsilon}^{\alpha(x)+\beta}(\alpha(x)+\beta)}.\end{aligned}$$

Furthermore, from (4.1), (4.4) and (4.7), we have

$$\begin{aligned}&\frac{\alpha(x)x^{\alpha(x)}}{c(x)} U_3^{\varepsilon}(x) \\ &\quad = \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} - \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} {}_2F_1\left(\beta, -\alpha(x), 1-\alpha(x); -\frac{y_{\varepsilon}}{x+1}\right) \\ &\qquad - \frac{(1-\varepsilon)\Gamma(\alpha(x)+\beta)\Gamma(-\alpha(x))\alpha(x)x^{\alpha(x)}}{\Gamma(\beta)(1+x)^{\alpha(x)}} {}_2F_1\left(\alpha(x)+\beta, 0, 1+\alpha(x); -\frac{y_{\varepsilon}}{x+1}\right) \\ &\quad = \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} - \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} {}_2F_1\left(\beta, -\alpha(x), 1-\alpha(x); -\frac{y_{\varepsilon}}{x+1}\right) \\ &\qquad - \frac{(1-\varepsilon)\Gamma(\alpha(x)+\beta)\Gamma(-\alpha(x))\alpha(x)x^{\alpha(x)}}{\Gamma(\beta)(1+x)^{\alpha(x)}}.\end{aligned}$$

Let us put

$$\begin{aligned}V_1^{\varepsilon}(x) &:= \frac{(1+\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} - \frac{(1+\varepsilon)x^{\alpha(x)}{}_2F_1(-\alpha(x), \beta, 1-\alpha(x); y_{\varepsilon}/(1+x))}{y_{\varepsilon}^{\alpha(x)}} \\ V_2^{\varepsilon}(x) &:= \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} - \frac{(1-\varepsilon)x^{\alpha(x)}}{y_{\varepsilon}^{\alpha(x)}} {}_2F_1\left(\beta, -\alpha(x), 1-\alpha(x); -\frac{y_{\varepsilon}}{x+1}\right)\end{aligned}$$

and

$$
\begin{aligned}
V_3^{\varepsilon}(x) := {} & (1+\varepsilon)\, {}_2F_1\left(-\alpha(x), \beta, 1-\alpha(x); \frac{x}{1+x}\right) \\
& - \frac{(1-\varepsilon)\Gamma(\alpha(x)+\beta)\Gamma(-\alpha(x))\alpha(x)x^{\alpha(x)}}{\Gamma(\beta)(1+x)^{\alpha(x)}} \\
& - \frac{(1+\varepsilon)\alpha(x)x^{\alpha(x)}(1+x)^{\beta}B((x-1)/x; \alpha(x)+\beta, 1-\beta)}{(x-1)^{\alpha(x)+\beta}}.
\end{aligned}
$$

Hence, (4.6) is reduced to

$$
\begin{aligned}
& \liminf_{x\longrightarrow\infty} \frac{\alpha(x)x^{\alpha(x)+\beta}}{c(x)}\left(\int_{\mathbb{R}} p(x, \mathrm{d}y)V(y) - V(x)\right) \\
& \quad > \liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty} V_1^{\varepsilon}(x) + \liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty} V_2^{\varepsilon}(x) \\
& \qquad + \liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty} V_3^{\varepsilon}(x) - T(\alpha, \beta).
\end{aligned} \tag{4.8}
$$

By (3.1) and (3.2), we have

$$
0 \le {}_2F_1\left(-\alpha(x), \beta, 1-\alpha(x), \frac{y_\varepsilon}{1+x}\right) \le 1,
$$

therefore

$$
\liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty} V_1^{\varepsilon}(x) \ge 0. \tag{4.9}
$$

Since $1-\alpha(x)-(-\alpha(x))-\beta = 1-\beta > 0$, from (3.1) and the dominated convergence theorem, we have

$$
\liminf_{\varepsilon\longrightarrow 0}\liminf_{y_\varepsilon\longrightarrow\infty}\liminf_{x\longrightarrow\infty} V_2^{\varepsilon}(x) = 0. \tag{4.10}
$$

At the end, let us calculate

$$
\liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty} V_3^{\varepsilon}(x).
$$

From (3.1), we have

$$
{}_2F_1\left(-\alpha(x), \beta, 1-\alpha(x); \frac{x}{1+x}\right) \ge {}_2F_1\big(-\alpha(x), \beta, 1-\alpha(x); 1\big),
$$

and from (3.3) we have

$$
\alpha(x)B\left(\frac{x-1}{x}; \alpha(x)+\beta, 1-\beta\right) \le \alpha(x)B\big(1; \alpha(x)+\beta, 1-\beta\big).
$$

Hence, we have

$$\begin{aligned}
&\liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty} V_3^{\varepsilon}(x)\\
&\quad\geq \liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty}\Bigg[(1+\varepsilon)\,{}_2F_1\big(-\alpha(x),\beta,1-\alpha(x);1\big)\\
&\qquad\qquad - \frac{(1-\varepsilon)\alpha(x)\Gamma(\alpha(x)+\beta)\Gamma(-\alpha(x))x^{\alpha(x)}}{\Gamma(\beta)(1+x)^{\alpha(x)}}\\
&\qquad\qquad - \frac{(1+\varepsilon)\alpha(x)B(1;\alpha(x)+\beta,1-\beta)x^{\alpha(x)}(1+x)^{\beta}}{(1-x)^{\alpha(x)+\beta}}\Bigg],
\end{aligned}$$

that is, since all terms are bounded,

$$\begin{aligned}
&\liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty} V_3^{\varepsilon}(x)\\
&\quad\geq \liminf_{x\longrightarrow\infty}\big[{}_2F_1\big(-\alpha(x),\beta,1-\alpha(x);1\big)+\beta B\big(1;\alpha(x)+\beta,1-\alpha(x)\big)\\
&\qquad\qquad -\alpha(x)B\big(1;\alpha(x)+\beta,1-\beta\big)\big].
\end{aligned}$$

One can prove that the function

$$y\longmapsto T(y,\beta):={}_2F_1(-y,\beta,1-y;1)+\beta B(1;y+\beta,1-y)-yB(1;y+\beta,1-\beta)$$

is strictly decreasing on $[0,1-\beta)$, and it easy to see that $T(1-\beta,\beta)=0$. Hence, since $0\leq\alpha<1-\beta$, we have

$$\liminf_{\varepsilon\longrightarrow 0}\liminf_{x\longrightarrow\infty} V_3^{\varepsilon}(x)\geq T(\alpha,\beta). \tag{4.11}$$

By combining (4.8), (4.9), (4.10) and (4.11), we have

$$\liminf_{x\longrightarrow\infty}\frac{\alpha(x)x^{\alpha(x)+\beta}}{c(x)}\left(\int_{\mathbb{R}}p(x,\mathrm{d}y)V(y)-V(x)\right)>0.$$

The case when $x<0$ is treated in the same way. Therefore, by [9], Theorem 8.4.3, the chain $\{X_n\}_{n\geq 0}$ is transient. □

5. Some remarks and generalizations of the model

We start this section with the proof of equivalence of conditions (1.3) and (1.5), and the proof of relaxation of condition (1.4) to condition (1.6).

(i) Recall that condition (1.5) is given by

$$\limsup_{\delta\longrightarrow 0}\limsup_{|x|\longrightarrow\infty}\frac{(1+|x|)^{\alpha(x)}}{c(x)}\int_{-\delta(1+|x|)}^{\delta(1+|x|)}\ln\left(1+\operatorname{sgn}(x)\frac{y}{1+|x|}\right)f_x(y)\,\mathrm{d}y<R(\alpha).$$

Using $\ln(1+t) \leq t$, condition (1.5) follows from the condition

$$\limsup_{\delta \longrightarrow 0} \limsup_{|x| \longrightarrow \infty} \operatorname{sgn}(x) \frac{(1+|x|)^{\alpha(x)-1}}{c(x)} \int_{-\delta(1+|x|)}^{\delta(1+|x|)} y f_x(y) \, \mathrm{d}y < R(\alpha). \tag{5.1}$$

In fact, under condition (C3), conditions (1.5) and (5.1) are equivalent, but the proof of this statement is rather elementary and technical and we omit it here. Furthermore, by (C3) and since $\alpha(x) \in (1,2)$, condition (5.1) is equivalent with

$$\limsup_{|x| \longrightarrow \infty} \operatorname{sgn}(x) \frac{(1+|x|)^{\alpha(x)-1}}{c(x)} \int_{\mathbb{R}} y f_x(y) \, \mathrm{d}y < R(\alpha),$$

that is, with condition (1.3). Indeed, let $\delta > 0$ and $0 < \varepsilon < 1$ be arbitrary. Then, by (C3), there exists $y_\varepsilon > 0$ such that for all $|y| \geq y_\varepsilon$

$$\left| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon$$

for all $x \in [-k,k]^c$. By taking $|x| \geq \frac{y_\varepsilon}{\delta} - 1$, we have (recall that $\alpha(x) \in (1,2)$)

$$\begin{aligned}
\int_{\mathbb{R}} y f_x(y) \, \mathrm{d}y &> -(1+\varepsilon) \int_{-\infty}^{-\delta(1+|x|)} \frac{c(x)}{|y|^{\alpha(x)}} \, \mathrm{d}y + \int_{-\delta(1+|x|)}^{\delta(1+|x|)} y f_x(y) \, \mathrm{d}y \\
&\quad + (1-\varepsilon) \int_{\delta(1+|x|)}^{\infty} \frac{c(x)}{|y|^{\alpha(x)}} \, \mathrm{d}y \\
&= \int_{-\delta(1+|x|)}^{\delta(1+|x|)} y f_x(y) \, \mathrm{d}y - \frac{2\varepsilon c(x)}{(\alpha(x)-1)\delta^{\alpha(x)-1}(1+|x|)^{\alpha(x)-1}}.
\end{aligned}$$

In the same way, we get

$$\int_{\mathbb{R}} y f_x(y) \, \mathrm{d}y < \int_{-\delta(1+|x|)}^{\delta(1+|x|)} y f_x(y) \, \mathrm{d}y + \frac{2\varepsilon c(x)}{(\alpha(x)-1)\delta^{\alpha(x)-1}(1+|x|)^{\alpha(x)-1}}.$$

By taking $\limsup_{|x| \longrightarrow \infty}$, $\limsup_{\varepsilon \longrightarrow 0}$ and $\limsup_{\delta \longrightarrow 0}$ we get the desired result.

(ii) From the concavity of the function $x \longmapsto x^\beta$, for $\beta \in (0, 1-\alpha)$, we have

$$\begin{aligned}
&\liminf_{|x| \longrightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)}}{c(x)} \int_{-a}^{a} \left(1 - \left(1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) \, \mathrm{d}y \\
&\quad \geq \liminf_{|x| \longrightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)}}{c(x)} \frac{(1+|x|-a)^\beta - (1+|x|)^\beta}{(1+|x|-a)^\beta} \int_{-a}^{a} f_x(y) \, \mathrm{d}y \\
&\quad \geq \liminf_{|x| \longrightarrow \infty} \left(-\frac{a\beta\alpha(x)|x|^{\alpha(x)}}{c(x)(1+|x|-a)} \right) = -a\beta \limsup_{|x| \longrightarrow \infty} \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1}.
\end{aligned}$$

In the sequel, we give several generalizations of the stable-like chain $\{X_n\}_{n\geq 0}$. Recall that a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is called *lower semicontinuous* if $\liminf_{y\longrightarrow x} f(y)\geq f(x)$ for all $x\in\mathbb{R}$.

Definition 5.1. *Let $\{Y_n\}_{n\geq 0}$ be a Markov chain on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.*

(i) *The chain $\{Y_n\}_{n\geq 0}$ is called a* T-chain *if for some probability measure $a=\{a(n)\}_{n\geq 0}$ on $\mathbb{Z}_+$ there exists a kernel $T(x,B)$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ with $T(x,\mathbb{R})>0$ for all $x\in\mathbb{R}$, such that the function $x\longmapsto T(x,B)$ is lower semicontinuous for all $B\in\mathcal{B}(\mathbb{R})$, and*

$$\sum_{n=0}^{\infty} a(n)p^n(x,B)\geq T(x,B)$$

holds for all $x\in\mathbb{R}$ and all $B\in\mathcal{B}(\mathbb{R})$.

(ii) *The chain $\{Y_n\}_{n\geq 0}$ is* Harris recurrent, *or* H-recurrent, *if it is ψ-irreducible and if $\mathbb{P}(\tau_B<\infty|Y_0=x)=1$ holds for all $x\in\mathbb{R}$ and all $B\in\mathcal{B}^+(\mathbb{R})$.*

(iii) *A state $x\in\mathbb{R}$ is called a* topologically recurrent state *if $\sum_{n=0}^{\infty}p^n(x,O_x)=\infty$ holds for all open neighborhoods O_x around x. Otherwise we call state x a* topologically transient state.

From Proposition 2.6 and [9], Theorem 6.2.5, we have the following.

Proposition 5.2. *The chain $\{X_n\}_{n\geq 0}$ is a T-chain.*

It is well known that the recurrence and H-recurrence properties of a Markov chain on the general state space are not equivalent (see [9], Section 9.1.2). Now, let us prove that these properties are equivalent for the stable-like chain $\{X_n\}_{n\geq 0}$.

Proposition 5.3. *The chain $\{X_n\}_{n\geq 0}$ is recurrent if and only if it is H-recurrent.*

Proof. We have to prove that recurrence property implies H-recurrence property, since the opposite claim is trivial. Since the Markov chain $\{X_n\}_{n\geq 0}$ is a T-chain, by [9], Theorem 9.3.6, it is enough to prove that every state is a topologically recurrent state. That follows from [9], Lemma 6.1.4 and Theorem 9.3.3. □

If we change the chain $\{X_n\}_{n\geq 0}$ on a set of Lebesgue measure zero, it can happen that its recurrence and H-recurrence properties are not equivalent anymore. Let $A\in\mathcal{B}(\mathbb{R})$ be such that $\lambda(A)=0$. Note that A can be unbounded. Let $\{\bar{X}_n\}_{n\geq 0}$ be a Markov chain on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ given by the transition kernel

$$\bar{p}(x,\mathrm{d}y)=\bar{f}_x(y-x)\,\mathrm{d}y,$$

where $\{\bar{f}_x\colon x\in\mathbb{R}\}$ is the family of density functions on $\mathbb{R}$ such that $\bar{f}_x=f_x$, for every $x\in\mathbb{R}\setminus A$. It is to easy see that the chain $(\bar{X}_n)$ is λ-irreducible and aperiodic. Therefore, a Borel set is a small set for $\{\bar{X}_n\}_{n\geq 0}$ if and only if it is a petite set for $\{\bar{X}_n\}_{n\geq 0}$. But we cannot conclude that every bounded Borel set is a petite set. The most we can get is that every bounded set $B\in\mathcal{B}(\mathbb{R}\setminus A)$ is a petite set. As a consequence of this fact, we do not know if the chain $\{\bar{X}_n\}_{n\geq 0}$ is a T-chain, so we cannot deduce equivalence between recurrence and H-recurrence property of the chain $\{\bar{X}_n\}_{n\geq 0}$. But, since the chains $\{X_n\}_{n\geq 0}$ and $\{\bar{X}_n\}_{n\geq 0}$ are λ-irreducible and since they differ on the set with zero Lebesgue measure, it is easy to see that the recurrence property of the chain $\{X_n\}_{n\geq 0}$ is equivalent with the recurrence property of the chain $\{\bar{X}_n\}_{n\geq 0}$, and the H-recurrence property of the chain $\{X_n\}_{n\geq 0}$ is equivalent with the H-recurrence property of the chain $\{\bar{X}_n\}_{n\geq 0}$. Hence, the chain $\{\bar{X}_n\}_{n\geq 0}$ is recurrent if and only if it is H-recurrent.

In Proposition 2.6, it is proved that every bounded Borel set is a petite set (singleton) for the stable-like chain $\{X_n\}_{n\geq 0}$. Therefore, it is natural to expect that a change of the chain $\{X_n\}_{n\geq 0}$ on an arbitrary bounded Borel set will not affect its recurrence and transience property. Let $B \in \mathcal{B}(\mathbb{R})$ be bounded and let $\{\tilde{X}_n\}_{n\geq 0}$ be a stable-like Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by the transition kernel

$$\tilde{p}(x, \mathrm{d}y) = \tilde{f}_x(y - x)\,\mathrm{d}y,$$

where $\{\tilde{f}_x\colon x \in \mathbb{R}\}$ is a family of density functions on $\mathbb{R}$ such that $\tilde{f}_x = f_x$ for all $x \in \mathbb{R} \setminus B$ and such that it satisfies conditions (C1)–(C5). Therefore, the chain $\{\tilde{X}_n\}_{n\geq 0}$ is either H-recurrent or transient.

Proposition 5.4. *The chain $\{X_n\}_{n\geq 0}$ is H-recurrent if and only if the chain $\{\tilde{X}_n\}_{n\geq 0}$ is H-recurrent. Hence, the chain $\{X_n\}_{n\geq 0}$ is recurrent if and only if the chain $\{\tilde{X}_n\}_{n\geq 0}$ is recurrent.*

Proof. If $\lambda(B) = 0$, the claim follows from the above discussion. Let us suppose that $\lambda(B) > 0$. By Proposition 2.6, the set B is a petite set for both chains $\{X_n\}_{n\geq 0}$ and $\{\tilde{X}_n\}_{n\geq 0}$. Let us suppose that the chain $\{X_n\}_{n\geq 0}$ is H-recurrent. Then, by [9], Theorem 9.1.4, we have $\mathbb{P}(\tau_B < \infty | X_0 = x) = 1$ for all $x \in \mathbb{R}$. Since

$$\mathbb{P}(\tau_B < \infty | X_0 = y) = \mathbb{P}(\tilde{\tau}_B < \infty | \tilde{X}_0 = y)$$

for all $y \notin B$, we have

$$\begin{aligned}\mathbb{P}(\tilde{\tau}_B < \infty | \tilde{X}_0 = x) &= \tilde{p}(x, B) + \int_{B^c} \hat{p}(x, \mathrm{d}y)\mathbb{P}(\tilde{\tau}_B < \infty | \tilde{X}_0 = y)\\ &= \tilde{p}(x, B) + \tilde{p}\bigl(x, B^c\bigr) = 1\end{aligned}$$

for all $x \in \mathbb{R}$. Therefore, by [9], Proposition 9.1.7, the chain $\{\tilde{X}_n\}_{n\geq 0}$ is H-recurrent. The proof of the opposite direction is completely the same. □

From the above discussions, we can weaken assumptions on function $\alpha(x)$ and conditions (1.3) and (1.4) in Theorems 1.3 and 1.4. In Theorem 1.3, we assumed that $\alpha : \mathbb{R} \longrightarrow (1, 2)$ and

$$\liminf_{|x|\longrightarrow\infty} \alpha(x) > 1,$$

but it is enough to request that $\alpha : \mathbb{R} \setminus (A \cup B) \longrightarrow (1, 2)$ and

$$\liminf_{x\in\mathbb{R}\setminus A, |x|\longrightarrow\infty} \alpha(x) > 1$$

for some set $A \in \mathcal{B}(\mathbb{R})$ with zero Lebesgue measure and some bounded set $B \in \mathcal{B}(\mathbb{R})$. In condition (1.3) instead of using $\limsup_{|x|\longrightarrow\infty}$, we use $\limsup_{x\in\mathbb{R}\setminus A, |x|\longrightarrow\infty}$. An analog modification can be done in Theorem 1.4.

The transition densities of the stable-like chain $\{X_n\}_{n\geq 0}$, from the current state x, have the power-law decay with exponent $\alpha(x)+1$. Let us take a look at the Markov chain with transition densities with the power-law decay with exponent $\alpha_-(x)+1$ on the left of the current state x and with the power-law decay with exponent $\alpha_+(x)+1$ on the right of the current state x. Let $\alpha_+, \alpha_- : \mathbb{R} \longrightarrow (0,2)$ and $c_+, c_- : \mathbb{R} \longrightarrow (0,\infty)$ be arbitrary functions and let (X'_n) be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by the transition kernel $p'(x, dy) = f'_x(y-x)\,dy$, where $\{f'_x : x \in \mathbb{R}\}$ is a family of density functions on $\mathbb{R}$ which satisfies:

(C1′) $x \longmapsto f'_x(y)$ is measurable, for every $y \in \mathbb{R}$;
(C2′) $f'_x(y) \sim c_+(x) y^{-\alpha_+(x)-1}$, when $y \longrightarrow \infty$, and $f'_x(y) \sim c_-(x)(-y)^{-\alpha_-(x)-1}$, when $y \longrightarrow -\infty$;
(C3′) there exists $k' > 0$ such that

$$\lim_{y\longrightarrow\infty} \sup_{x\in[-k',k']^c} \left| f'_x(y) \frac{y^{\alpha_+(x)+1}}{c_+(x)} - 1 \right| = 0$$

and

$$\lim_{y\longrightarrow-\infty} \sup_{x\in[-k',k']^c} \left| f'_x(y) \frac{(-y)^{\alpha_-(x)+1}}{c_-(x)} - 1 \right| = 0;$$

(C4′) $\inf_{x\in C}(c_+(x) \wedge c_-(x)) > 0$ for every compact set $C \subseteq [-k', k']^c$;
(C5′) there exists $l' > 0$ such that for every compact set $C \subseteq [-l', l']^c$ with $\lambda(C) > 0$, we have

$$\inf_{x\in[-k',k']} \int_{C-x} f'_x(y)\,dy > 0.$$

It is clear that the chain $\{X'_n\}_{n\geq 0}$ has the same properties, discussed in Section 2, as the chain $\{X_n\}_{n\geq 0}$. It is λ-irreducible and aperiodic and every bounded Borel set is a petite set. By assuming certain additional conditions, Theorems 1.3 and 1.4 can be generalized in terms of the chain $\{X'_n\}_{n\geq 0}$. The chain $\{X'_n\}_{n\geq 0}$ will be recurrent if $\alpha_+, \alpha_- : \mathbb{R} \longrightarrow (1,2)$ are such that

$$\lim_{|x|\longrightarrow\infty} \frac{\alpha_+(x)}{\alpha_-(x)} = 1 \quad \text{and} \quad \alpha := \liminf_{|x|\longrightarrow\infty} \alpha_+(x) \Big(= \liminf_{|x|\longrightarrow\infty} \alpha_-(x) \Big) > 1,$$

and $c_+, c_- : \mathbb{R} \longrightarrow (0,\infty)$ are such that

$$\lim_{|x|\longrightarrow\infty} \frac{c_-(x)}{c_+(x)} |x|^{\alpha_+(x)-\alpha_-(x)} = 1$$

and such that condition (1.3) is satisfied with the constant $R(\alpha)$. In this case, for the test function $V(x)$, we take $V(x) = \ln(1+|x|)$ again. Similarly, the chain (X'_n) will be transient if $\alpha_+, \alpha_- : \mathbb{R} \longrightarrow (0,1)$ are such that

$$\alpha_+ := \limsup_{|x|\longrightarrow\infty} \alpha_+(x) < 1 \quad \text{and} \quad \alpha_- := \limsup_{|x|\longrightarrow\infty} \alpha_-(x) < 1,$$

and $c_+, c_- : \mathbb{R} \longrightarrow (0, \infty)$ are such that

$$\lim_{|x|\longrightarrow\infty} \frac{\alpha_+(x)c_-(x)}{c_+(x)\alpha_-(x)}|x|^{\alpha_+(x)-\alpha_-(x)} = 1$$

and such that condition (1.4) is satisfied with the constant $T(\alpha, \beta)$, where $\alpha := \alpha_- \vee \alpha_+$ and $\beta \in (0, 1-\alpha)$. In this case, for the test function $V(x)$, we take $V(x) = 1 - (1+|x|)^{-\beta}$ again.

In the following proposition, we treat the case when the family of density functions $\{f_x\colon x \in \mathbb{R}\}$ is exactly a family of $S_{\alpha(x)}(\beta(x), \gamma(x), \delta(x))$ densities and we give sufficient conditions on functions $\alpha(x), \beta(x), \gamma(x)$ and $\delta(x)$ such that the family $\{f_x : x \in \mathbb{R}\}$ satisfies conditions (C1′)–(C5′). From [11], Properties 1.2.2, 1.2.3, 1.2.4 and 1.2.15, [3], Theorem 3.3.5, and (1.2) it follows:

Proposition 5.5. *Let* $0 < \varepsilon < 1$, $M > 0$ *and* $k' \geq 0$ *be arbitrary, and let* $F_\alpha \subseteq [1, 2)$, $F_\beta \subseteq (-1, 1)$ *and* $F_\gamma \subseteq (0, \infty)$ *be arbitrary and finite. Furthermore, let*

(i) $\bar{\alpha} : \mathbb{R} \longrightarrow (\varepsilon, 2-\varepsilon)$ *and* $\tilde{\alpha} : \mathbb{R} \longrightarrow (0, 1) \cup F_\alpha$, *such that* $\inf_{x\in C} \tilde{\alpha}(x) > 0$ *for all compact sets* $C \subseteq \mathbb{R}$,
(ii) $\bar{\beta} : \mathbb{R} \longrightarrow (-1+\varepsilon, 1-\varepsilon)$ *and* $\tilde{\beta} : \mathbb{R} \longrightarrow F_\beta$,
(iii) $\bar{\gamma} : \mathbb{R} \longrightarrow (0, M)$, $\tilde{\gamma} : \mathbb{R} \longrightarrow F_\gamma$ *and* $\hat{\gamma} : \mathbb{R} \longrightarrow (\varepsilon, M)$, *such that* $\inf_{x\in C} \bar{\gamma}(x) > 0$ *for all compact sets and* $C \subseteq \mathbb{R}$,
(iv) $\delta : \mathbb{R} \longrightarrow (-M, M)$

be arbitrary and Borel measurable. Define

$$\alpha(x) := \begin{cases} \bar{\alpha}(x), & x \in [-k', k'], \\ \tilde{\alpha}(x), & x \in [-k', k']^c, \end{cases}$$

$$\beta(x) := \begin{cases} \bar{\beta}(x), & x \in [-k', k'], \\ \bar{\beta}(x)1_{\{y:\alpha(y)<1\}}(x) + \tilde{\beta}(x)1_{\{y:\alpha(y)\geq 1\}}(x), & x \in [-k', k']^c \end{cases} \quad \textit{and}$$

$$\gamma(x) := \begin{cases} \hat{\gamma}(x), & x \in [-k', k'], \\ \bar{\gamma}(x)1_{\{y:\alpha(y)<1\}}(x) + \tilde{\gamma}(x)1_{\{y:\alpha(y)\geq 1\}}(x), & x \in [-k', k']^c. \end{cases}$$

Then, for any $l' \geq 0$, *the family of* $S_{\alpha(x)}(\beta(x), \gamma(x), \delta(x))$, $x \in \mathbb{R}$, *densities satisfies conditions* (C1′)–(C5′).

Unfortunately, Proposition 5.5 does not cover the case when the function $\alpha(x)$ takes infinitely many values in the interval $[1, 2)$ since we do not know the series representation of stable densities for $\alpha \geq 1$, as for $\alpha < 1$ (see [13], Theorems 2.4.2, 2.5.1 and 2.5.4).

At the end, note that all conclusions, methods and proofs given in this paper can also be carried out in the discrete state space $\mathbb{Z}$. Note that in this case conditions (C1)–(C5) are reduced just to conditions (C2) and (C3), since compact sets are replaced by finite sets. Therefore, we deal with a Markov chain $\{X_n^d\}_{n\geq 0}$ on $\mathbb{Z}$ given by the transition kernel

$$p_{i,j} = f_i(j-i)$$

for $i, j \in \mathbb{Z}$, where $\{f_i\colon\ i \in \mathbb{Z}\}$ is a family of probability functions which satisfies the following conditions:

(CD1) $f_i(j) \sim c(i)|j|^{-\alpha(i)-1}$, when $|j| \longrightarrow \infty$, for every $i \in \mathbb{Z}$;
(CD2) there exists $k \in \mathbb{N}$ such that

$$\lim_{|j|\longrightarrow\infty} \sup_{i\in\{-k,\dots,k\}^c} \left| f_i(j)\frac{|j|^{\alpha(i)+1}}{c(i)} - 1\right| = 0.$$

Functions $\alpha : \mathbb{Z} \longrightarrow (0,2)$ and $c : \mathbb{Z} \longrightarrow (0,\infty)$ are arbitrary given functions. Proofs and assumptions of Theorems 1.3 and 1.4 in the discrete case remain the same as in the continuous case because we can switch from sums to integrals due to the tail behavior of transition jumps.

Acknowledgements

The author would like to thank Prof. Zoran Vondraček for many discussions on the topic and for helpful comments on the presentation of the results. Many thanks to Prof. René Schilling and Dr. Björn Böttcher from TU Dresden for discussions on the topic during their visit to Zagreb and authors visit to Dresden, funded by DAAD and MZOS of the Republic of Croatia. The author also thanks the Referees and the Associate Editor for helpful and constructive criticism of an earlier version of this paper.

References

[1] Abramowitz, M. and Stegun, I.A., eds. (1984). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Wiley. MR0757537
[2] Böttcher, B. (2011). An overshoot approach to recurrence and transience of Markov processes. *Stochastic Process. Appl.* **121** 1962–1981. MR2819236
[3] Durrett, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge: Cambridge Univ. Press. MR2722836
[4] Franke, B. (2006). The scaling limit behaviour of periodic stable-like processes. *Bernoulli* **12** 551–570. MR2232732
[5] Franke, B. (2007). Correction to: "The scaling limit behaviour of periodic stable-like processes" [*Bernoulli* **12** (2006) 551–570]. *Bernoulli* **13** 600.
[6] Kemperman, J.H.B. (1974). The oscillating random walk. *Stochastic Process. Appl.* **2** 1–29. MR0362500
[7] Lamperti, J. (1960). Criteria for the recurrence or transience of stochastic process. I. *J. Math. Anal. Appl.* **1** 314–330. MR0126872
[8] Menshikov, M.V., Èĭsymont, I.M. and Yasnogorodskiĭ, R. (1995). Markov processes with asymptotically zero drift. *Probl. Inf. Transm.* **31** 60–75.
[9] Meyn, S.P. and Tweedie, R.L. (1993). *Markov Chains and Stochastic Stability*. *Communications and Control Engineering Series*. London: Springer. MR1287609
[10] Rogozin, B.A. and Foss, S.G. (1978). The recurrence of an oscillating random walk. *Theory Probab. Appl.* **23** 161–169.

[11] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*: *Stochastic Models with Infinite Variance*. *Stochastic Modeling*. New York: Chapman & Hall. MR1280932

[12] Spitzer, F. (1976). *Principles of Random Walk*, 2nd ed. *Graduate Texts in Mathematics* **34**. New York: Springer. MR0388547

[13] Zolotarev, V.M. (1986). *One-dimensional Stable Distributions*. *Translations of Mathematical Monographs* **65**. Providence, RI: Amer. Math. Soc. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver. MR0854867

Received March 2011 and revised April 2012