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Estimation of absolutely continuous distributions for censored variables in two-sample nonparametric and semi-parametric models

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This paper considers the estimation of the density of an absolutely continuous distribution with respect to an unknown baseline distribution F, and the estimation of F, from censored observations. For parametric and nonparametric densities, an $n^{1/2}$-consistent estimator of F is defined from the two samples and the asymptotic distribution of the estimators is studied. The efficient score functions and the minimal variances of the estimators are established.

Keywords: absolutely continuous distributions; censoring; efficiency; nonparametric estimation

1. Introduction

Semi-parametric stratified models for censored variables are usually chosen when studying populations stratified into homogeneous groups, as in case–control studies when the assumption of a proportional hazards model is not satisfied. Here the model is a set of absolutely continuous distributions. In the simplest case, there are two samples of positive random variables in $\mathbb{R}_+$, $X_1, \ldots, X_{n_1}$ with distribution function F_0 in a set $\mathcal{F}$, and $X_{n_1+1}, \ldots, X_n$ with distribution function F_{φ_0} absolutely continuous with respect to F_0, with a density φ_0. The function φ_0 belongs either to a parametric family $\{\varphi_\theta;\ \theta \in \Theta\}$, where the parameter space Θ is a subset of $\mathbb{R}^p$, or to a nonparametric space of continuous functions Φ. The model extends to K homogeneous groups, each having a distribution absolutely continuous with respect to the unknown nonparametric distribution of one of them. The general distribution is a mixture of several distributions $\sum_k \alpha_k F_{\varphi_k}$ where the distribution functions F_{φ_k} are absolutely continuous with respect to F, the distribution of the first sample, and the other ones have mixture distributions with parametric or nonparametric derivatives $\alpha_k \varphi_k$, with $\varphi_k = \mathrm{d}F_{\varphi_k}/\mathrm{d}F$, $1 \leqslant k \leqslant K$. In the following, $K = 2$ but all the results obviously extend in an obvious way to larger values.

The variables $X_1, \ldots, X_n$ may be right-censored by independent and identically distributed variables $C_1, \ldots, C_n$, respectively, with distribution G, independent of $X_1, \ldots, X_n$ and non-informative for α, φ and F. The observed variables are the censored variables $T_i = X_i \wedge C_i$, the censoring indicator $\delta_i = 1_{\{X_i \leqslant C_i\}}$ and a group indicator. The problem is to estimate the parameter θ_0 or φ_0 and the distribution function F_0 from the censored variables in the semi-parametric model $\mathcal{P}_{\mathcal{F},\Theta} = \{(F, \theta); F \in \mathcal{F}, \theta \in \Theta, \int_0^\infty \varphi_\theta \, \mathrm{d}F = 1\}$ or in the nonparametric model $\mathcal{P}_{\mathcal{F},\Phi} = \{(F, \varphi); F \in \mathcal{F}, \varphi \in \Phi, \int_0^\infty \varphi \, \mathrm{d}F = 1\}$. The distribution function G and the probabilities α and $1 - \alpha$ are nuisance parameters which are estimated independently of the parameters of interest since C is independent of T and each individual belongs to a known subpopulation.

Estimation of the distributions of stratified populations, without censoring, has already been studied, in particular by Anderson (1979) with a specific parametric form for φ_θ, by Gill *et al.* (1988) in biased sampling models with group distributions $\int_0^\cdot w_k \, \mathrm{d}F$, where the weight functions are known, by Gilbert (2000) in biased sampling models with parametric weight functions, and by Cheng and Chu (2004) with the Lebesgue measure and kernel density estimators. Here, the uncensored problem may also be viewed as a special case of the semi-parametric biased sampling model with $W(\theta_k, F) = \int \varphi_{\theta_k} \, \mathrm{d}F = 1$. This integral may also be considered as a real parameter without link to θ and F, and all these models are different parametrizations of similar problems.

Section 2 introduces notation and conditions. In Section 3, $n^{1/2}$-consistent estimators of θ and F are defined by an iterative procedure where the distribution function F is estimated from the two samples, and their asymptotic distribution is studied. Similar results are established in Section 4 for the nonparametric model $\mathcal{P}_{\mathcal{F},\Phi}$, with the same approach as for the nonparametric regression kernel estimators. Efficient estimators are defined in Section 5.

2. Notation and conditions

Let $\varphi_0 = \varphi_{\theta_0}$ be the true value of the density in models $\mathcal{P}_{\mathcal{F},\Theta}$ or $\mathcal{P}_{\mathcal{F},\Phi}$. The observations are the counting processes $N(t) = \delta 1_{\{T \leqslant t\}}$, indicator of an uncensored variable X before t, $Y(t) = 1_{\{T \geqslant t\}}$, risk indicator at t, for $t \geqslant 0$, and ρ the sample indicator defined by $\rho = 1$ for individuals of the first sample and $\rho = 0$ for individuals of the second sample. The density of a distribution function F is denoted by f, and the related survival function is $\bar{F} = 1 - F$. The endpoints of the supports of the distributions F, G and $H = 1 - \bar{F}\bar{G}$ are denoted by τ_F, τ_G and τ_H respectively, and $\int_0^\cdot \varphi \, \mathrm{d}F$ has the same support as F. Let $n_1 = n_1(n)$ and $n_2 = n_2(n) = n - n_1$ increasing with n, such that $\lim_n n^{-1} n_1(n) = \alpha = \Pr(\rho = 1) < 1$ in $]0, 1[$. For each sample, the maximum observed variable is $\tau_n^{(1)} = \max_{1 \leqslant i \leqslant n} \rho_i \delta_i X_i$ and $\tau_n^{(2)} = \max_{1 \leqslant i \leqslant n} (1 - \rho_i) \delta_i X_i$, and $\tau_n = \tau_n^{(1)} \wedge \tau_n^{(2)}$.

The log-likelihood of the variables under both models is defined by

$$l(\varphi, F)(T, \delta, \rho) \equiv \rho\{\delta \log f(T) + (1 - \delta) \log \bar{F}(T)\} + \{\delta \log \bar{G}(T) + (1 - \delta) \log g(T)\}$$
$$+ (1 - \rho)[\delta\{\log \varphi(T) + \log f(T)\} + (1 - \delta) \log \bar{F}_\varphi(T)], \qquad (2.1)$$

with φ in Φ or $\varphi = \varphi_\theta$, θ in Θ.

Under censoring, the likelihood of the sample is conveniently and equivalently written using the hazard functions related to the distribution functions F and F_θ,

$$\lambda(t) = \frac{f(t)}{\overline{F}(t)}, \qquad \lambda_\theta(t) = \frac{\varphi_\theta(t) f(t)}{\overline{F}_\theta(t)}.$$

Denoting by $\Lambda(t) = \int_0^t \lambda(y)\mathrm{d}y$ the cumulative hazard function for F is equivalent to defining λ from F or F from λ by the relationship $\overline{F}(t) = \exp\{-\Lambda(t)\}$ and $\overline{F}_\theta(t) = \exp\{-\Lambda_\theta(t)\}$ with $\Lambda_\theta(t) = \int_0^t \lambda_\theta(y)\,\mathrm{d}y$.

Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration defined by the σ-algebras $\mathcal{F}_t = \sigma\{\delta 1_{\{T\leqslant s\}}, 1_{\{T\leqslant s\}}, 0 \leqslant s \leqslant t\}$. Then the processes $M(t) = N(t) - \int_0^t Y\,\mathrm{d}\Lambda$ and $M_\theta(t) = N(t) - \int_0^t Y\,\mathrm{d}\Lambda_\theta$ are local square-integrable martingales adapted to $(\mathcal{F}_t)_{t\geqslant 0}$ under the probabilities associated with the distribution functions F and F_θ, respectively. Let Λ_0 and Λ_{θ_0} be the cumulative hazard functions related to the true distribution functions F_0 and F_{θ_0}.

From the observations write, for the first sample,

$$N_n^{(1)}(t) = \sum_{i=1}^n \rho_i \delta_i 1_{\{T_i\leqslant t\}} \quad \text{and} \quad Y_n^{(1)}(t) = \sum_{i=1}^n \rho_i 1_{\{T_i\geqslant t\}}$$

the number of uncensored variables observed before t and the number of individuals at risk at time t respectively, and, for the second sample,

$$N_n^{(2)}(t) = \sum_{i=1}^n (1-\rho_i)\delta_i 1_{\{T_i\leqslant t\}},$$

$$Y_n^{(2)}(t) = \sum_{i=1}^n (1-\rho_i) 1_{\{T_i\geqslant t\}},$$

$$N_n^{(3)}(t) = \sum_{i=1}^n (1-\rho_i)(1-\delta_i) 1_{\{T_i\leqslant t\}}.$$

Lemma 1.

$$\left\| n^{-1}N_n^{(1)} - \alpha\int_0^{\cdot} \overline{G}\,\mathrm{d}F_0 \right\| \xrightarrow{\mathcal{P}}_n 0, \ \|n^{-1}Y_n^{(1)} - \alpha \overline{F}_0\overline{G}\| \xrightarrow{\mathcal{P}}_n 0,$$

$$\left\| n^{-1}N_n^{(2)} - (1-\alpha)\int_0^{\cdot} \varphi_0\overline{G}\,\mathrm{d}F_0 \right\| \xrightarrow{\mathcal{P}}_n 0, \ \|n^{-1}Y_n^{(2)} - (1-\alpha)\overline{F}_{\varphi_0}\,\overline{G}\| \xrightarrow{\mathcal{P}}_n 0,$$

$$\left\| n^{-1}N_n^{(3)} - (1-\alpha)\int_0^{\cdot} \overline{F}_{\theta_0}\,\mathrm{d}G \right\| \xrightarrow{\mathcal{P}}_n 0.$$

In the semi-parametric model, densities with respect to F satisfy the condition $\int \varphi_\theta \, \mathrm{d}F = 1$ and multiplicative constants of φ_θ are identified; conversely, if a multiplicative constant is omitted from the expression of the density, it is deduced from the dominating distribution F by this condition. Let

$$v(\theta, F)(T, \delta, \rho) = (1-\rho)\left[\delta \log \varphi_\theta(T) + (1-\delta)\log \int_T^\infty \varphi_\theta \, \mathrm{d}F_0\right]. \tag{2.2}$$

The following conditions are assumed to be satisfied:

(C_1) The parameter θ_0 belongs to the interior of a compact and convex set Θ and its closure does not contain the constants.

(C_2) For every F in $\mathcal{F}$, the integrals $\int \overline{F}_{\theta_0}^{-2} \overline{G}^{-1} \, \mathrm{d}F_0$, $\int (\dot{\varphi}_{\theta_0} \varphi_{\theta_0}^{-1})^{\otimes 2} \overline{G} \, \mathrm{d}F_{\theta_0}$, $\int (\dot{\overline{F}}_{\theta_0} \overline{F}_{\theta_0}^{-1})^{\otimes 2} \overline{F}_{\theta_0} \, \mathrm{d}G$ and $\int \overline{F}_0^{-2} \overline{F}_\theta^2 \overline{G}^{-1} \, \mathrm{d}F_0$ are finite, for every θ in Θ.

(C_3) The function φ_θ is twice continuously differentiable with respect to θ, with derivatives $\dot{\varphi}_\theta$ and $\ddot{\varphi}_\theta$, which satisfy

$$\int_0^\infty \sup_{\theta \in \Theta} \{\log(\varphi_\theta) \overline{G} \, \mathrm{d}F_{\theta_0} + \log(\overline{F}_\theta) \, \overline{F}_{\theta_0} \, \mathrm{d}G\}^2 < \infty,$$

and the matrix

$$\Sigma(\theta) = (1-\alpha)\left[\int \left\{ \left(\frac{\dot{\varphi}_\theta}{\varphi_\theta}\right)^{\otimes 2} - \frac{\ddot{\varphi}_\theta}{\varphi_\theta} \right\} \overline{G} \, \mathrm{d}F_{\theta_0} + \int_0^\infty \left\{ \left(\frac{\int_\cdot^\infty \dot{\varphi}_\theta \, \mathrm{d}F_0}{\int_\cdot^\infty \varphi_\theta \, \mathrm{d}F_0} \right)^{\otimes 2} - \frac{\int_\cdot^\infty \ddot{\varphi}_\theta \, \mathrm{d}F_0}{\int_\cdot^\infty \varphi_\theta \, \mathrm{d}F_0} \right\} \overline{F}_{\theta_0} \, \mathrm{d}G \right]$$

is finite and positive definite.

Remark. Let $u_\theta = \log \varphi_\theta$, $\dot{\varphi}_\theta = \dot{u}_\theta \varphi_\theta$ and $\ddot{\varphi}_\theta = \ddot{u}_\theta \varphi_\theta + \dot{u}_\theta^2 \varphi_\theta$ and the conditions are easily written in exponential models. In the model where the log-likelihood is written $\theta_1 + \theta_2 r$ with r a known function and $\exp\{-\theta_1\} = \int \exp\{\theta_2 r\} \mathrm{d}F_0$, the derivatives of $u_\theta = \theta_2 r - \log \int_\cdot^\infty \mathrm{e}^{\theta_2 r} \, \mathrm{d}F_0$ with respect to θ_2 are

$$\dot{u}_\theta = r - \mathrm{E}_\theta r(X), \qquad \ddot{u}_\theta = -\mathrm{var}_\theta r(X).$$

The matrix

$$\Sigma(\theta) = (1-\alpha)\left[-\int_0^\infty \ddot{u}_\theta \bar{G}\,\mathrm{d}F_{\theta_0} + \int_0^\infty \left\{\left(\frac{\int_\cdot^\infty \dot{u}_\theta\,\mathrm{d}F_\theta}{\bar{F}_\theta}\right)^{\otimes 2} - \frac{\int_\cdot^\infty (\ddot{u}_\theta + \dot{u}_\theta^2)\,\mathrm{d}F_\theta}{\bar{F}_\theta}\right\}\bar{F}_{\theta_0}\,\mathrm{d}G\right]$$

$$= (1-\alpha)\left[\mathrm{var}_\theta r(X) P_{\theta_0}(\delta = 1) + \int_0^\infty \{\mathrm{E}_\theta(r(X)|X > x) - \mathrm{E}_\theta r(X)\}^2 \bar{F}_{\theta_0}(x)\,\mathrm{d}G(x)\right.$$

$$\left. - \int_0^\infty \{\mathrm{E}_\theta(r^2(X)|X > x) - 2\mathrm{E}_\theta(r(X)|X > x)\mathrm{E}_\theta r(X) + \mathrm{E}_\theta r^2(X)\}\bar{F}_{\theta_0}(x)\,\mathrm{d}G(x)\right]$$

$$= (1-\alpha)\mathrm{E}_{\theta_0}[\delta\,\mathrm{var}_\theta r(X) - (1-\delta)\mathrm{var}_\theta(r(X)|X > C)]$$

is finite if $\mathrm{E}_{\theta_0}\{r(T)\}^2 < \infty$ and positive definite under the condition that $\mathrm{E}_{\theta_0}\delta\,\mathrm{var}_\theta r(X) > \mathrm{E}_{\theta_0}\{(1-\delta)\mathrm{var}_\theta(r(X)|X > C)\}$. The other condition of (C_3) is satisfied if $\mathrm{E}_{\theta_0}\sup_{\theta\in\Theta}\{r(X)\}^2 < \infty$.

In the nonparametric model, we assume that the following conditions are satisfied instead of (C_1):

- (C_1') The space of functions Φ does not contain 1.
- (C_2') The kernel K is a positive, continuous and symmetric function on $[-1, 1]$ such that $\int K(t)\,\mathrm{d}t = 1$ and $\int |\mathrm{d}K(t)| < \infty$, $\int K^r(t)\,\mathrm{d}t < \infty$ for $r \geqslant 0$, the bandwidth h_n tends to 0 with $nh_n^3 \to \infty$ and $nh_n^5 \to 0$.
- (C_3') For every (F, φ) in $\mathcal{P}_{\mathcal{F},\Phi}$, F and φ have continuous derivatives of order 3 and 2, respectively.

Let $\kappa_1 = \int K^2(t)\mathrm{d}t$ and $\kappa_2 = \int t^2 K(t)\mathrm{d}t$.

3. Estimation for the semi-parametric model

3.1. Definition of the estimators

With censored variables, the likelihood cannot be factorized as in models studied in van der Vaart (1988), and the parameters θ and F must be estimated jointly. Due to the integral in the last term $(1-\rho)(1-\delta)\log\int_T^\infty \varphi_\theta\,\mathrm{d}F$ of the log-likelihood, there is no simple and explicit expression for their maximum likelihood estimator. Though the observations of the first sample are not sufficient for an efficient estimation of F, a first estimator of the unknown distribution function is defined from this sample by the product-limit estimator $\hat{F}_n^{(1)}(t) = 1 - \prod_{T_i \leqslant t}\{1 - \delta_i(Y_n^{(1)}(T_i))^{-1}\}$ and is introduced in (2.2) for the estimation of θ. Let

$$K_n(\theta) = \sum_{i=1}^n v(\theta, \hat{F}_n^{(1)})(T_i, \delta_i, \rho_i). \tag{3.1}$$

An estimator $\hat{\theta}_n$ of θ_0 is obtained by maximization of K_n, and the normalizing constant $(\int \varphi_\theta \,\mathrm{d}F)^{-1}$ of the density is estimated by $(\int \varphi_{\hat{\theta}_n} \,\mathrm{d}\hat{F}_n^{(1)})^{-1}$.

If θ were known, F_θ could be estimated from the first sample by

$$\hat{F}_{n,\theta}(t) = \int_0^t \varphi_\theta \,\mathrm{d}\hat{F}_n^{(1)}. \tag{3.2}$$

Let $\hat{\bar{F}}_n^{(1)} = 1 - \hat{F}_n^{(1)}$, $\hat{\bar{F}}_{n,\theta} = 1 - \hat{F}_{n,\theta}$ and

$$\hat{Y}_{n,\theta}^{(2)}(t) = Y_n^{(2)}(t)\{\hat{\bar{F}}_{n,\theta}(t)\}^{-1}\hat{\bar{F}}_n^{(1)}(t), \qquad 0 \leqslant t \leqslant \tau_n; \tag{3.3}$$

from the consistency of the Kaplan–Meier estimator $\hat{\bar{F}}_n^{(1)}$ on $[0, \tau_n^{(1)}]$, $n^{-1}\hat{Y}_{n,\theta_0}^{(2)}$ converges to $(1-\alpha)\bar{F}_0\bar{G}$ on this interval. A global estimator of Λ_0 based on the two samples is deduced as

$$\hat{\Lambda}_n(t) = \int_0^t \frac{\mathrm{d}N_n^{(1)} + \varphi_{\hat{\theta}_n}^{-1}\,\mathrm{d}N_n^{(2)}}{Y_n^{(1)} + \hat{Y}_{n,\hat{\theta}_n}^{(2)}}. \tag{3.4}$$

Finally, a product-limit estimator of F_0 is based on $\hat{\Lambda}_n$ given in (3.4),

$$\hat{\bar{F}}_n(t) = \prod_{T_i \leqslant t}\{1 - \mathrm{d}\hat{\Lambda}_n(T_i)\} \tag{3.5}$$

and the probability α is estimated by $\hat{\alpha}_n = n^{-1}\sum_i \rho_i$.

Remark. The estimators may be numerically improved by iterations of this procedure in the following way. In step 1, let $\bar{F}_{n,0} = \hat{\bar{F}}_n^{(1)}$ the estimator of $\bar{F}_0$ from the first sample and $\theta_{n,0} = \hat{\theta}_n$ that maximizes $K_n(\theta, \bar{F}_{n,0}) = K_n(\theta)$ given by (3.1); then a second estimator of $\bar{F}_0$ is $\bar{F}_{n,1} = \hat{\bar{F}}_{n,\theta_{n,0}}$ defined from the n-sample by $\Lambda_{n,\theta_{n,0},\bar{F}_{n,0}} = \hat{\Lambda}_n$ in (3.4) and by $\bar{F}_{n,1} = \hat{\bar{F}}_{n,\Lambda_{n,0}}$ with (3.5). In step k, $\theta_{n,k}$ is the estimator that maximizes $K_n(\theta, \bar{F}_{n,k-1})$ given by (3.1), $\Lambda_{n,k} = \Lambda_{n,\theta_{n,k-1},\bar{F}_{n,k-1}}$ is defined by (3.4) and $\bar{F}_{n,k} = \hat{\bar{F}}_{n,\Lambda_{n,k}}$ is defined (3.5).

3.2. Weak convergence of the estimators

The convergence of $\hat{\theta}_n$, $\hat{\Lambda}_n$ and $\hat{\bar{F}}_n$ (equations (3.1), (3.4) and (3.5) is related to the behaviour of the estimators $\hat{Y}_{n,\theta}^{(2)}$ and $\hat{F}_{n,\theta}$ given by (3.2) and (3.3). The proof of the consistency of $\hat{\theta}_n$ relies on the uniform convergence of $n^{-1}K_n(\theta)$ to a concave function in a subset of Θ containing θ_0. These convergences are used to study the asymptotic properties of the Kaplan–Meier estimator. For a function u on $\mathbb{R}_+$, let $\|u\|_t = \sup_{s\in[0,t]}|u(s)|$ if $t < \infty$ and $\|u\| = \sup_{t\in\mathbb{R}_+}|u(s)|$. The derivatives of $K_n(\theta)$ are

$$\dot{K}_n(\theta) = \sum_{i=1}^{n}(1-\rho_i)\left\{\delta_i \frac{\dot{\varphi}_\theta}{\varphi_\theta}(T_i) + (1-\delta_i)\frac{\int_{T_i}^{\infty} \dot{\varphi}_\theta \, \mathrm{d}\hat{F}_n^{(1)}}{\int_{T_i}^{\infty} \varphi_\theta \, \mathrm{d}\hat{F}_n^{(1)}}\right\}, \tag{3.6}$$

$$\ddot{K}_n(\theta) = \sum_{i=1}^{n}(1-\rho_i)\left[\delta_i\left\{\frac{\ddot{\varphi}_\theta}{\varphi_\theta}(T_i) - \left(\frac{\dot{\varphi}_\theta}{\varphi_\theta}\right)^{\otimes 2}(T_i)\right\}\right. \tag{3.7}$$

$$\left. + (1-\delta_i)\left\{\frac{\int_{T_i}^{\infty} \ddot{\varphi}_\theta \, \mathrm{d}\hat{F}_n^{(1)}}{\int_{T_i}^{\infty} \varphi_\theta \, \mathrm{d}\hat{F}_n^{(1)}} - \left(\frac{\int_{T_i}^{\infty} \dot{\varphi}_\theta \, \mathrm{d}\hat{F}_n^{(1)}}{\int_{T_i}^{\infty} \varphi_\theta \, \mathrm{d}\hat{F}_n^{(1)}}\right)^{\otimes 2}\right\}\right],$$

and the limits of $n^{-1}\dot{K}_n$ and $n^{-1}\ddot{K}_n$ are expressed as functions of the limits of $n^{-1}\hat{Y}_{n,\theta}^{(2)}$ and $n^{-1}K_n(\theta)$,

$$y_\theta(t) = (1-\alpha)\frac{\overline{F}_{\theta_0}(t)}{\overline{F}_\theta(t)}\overline{F}_0(t)\overline{G}(t),$$

$$K(\theta) = (1-\alpha)\left\{\int_0^\infty \log\left(\int_\cdot^\infty \varphi_\theta \, \mathrm{d}F_0\right)\overline{G}\, \mathrm{d}F_{\theta_0} + \int_0^\infty \log \overline{F}_\theta \, \overline{F}_{\theta_0} \, \mathrm{d}G\right\}.$$

Lemma 2. *Under* (C_3)*, the variables* $\sup_{\theta\in\Theta}\|\hat{F}_{n,\theta} - F_\theta\|_{\tau_n^{(1)}}$ *and* $\sup_{\theta\in\Theta}\|n^{-1}\hat{Y}_{n,\theta}^{(2)} - y_\theta\|_{\tau_n}$ *converge in probability to zero as* $n \to \infty$*. Moreover,* $\sup_{\theta\in\Theta}\|\hat{\Lambda}_{n,\theta} - \Lambda_0\|_\tau$ *converges in probability to zero for every* $0 < \tau < \tau_H$*.*

Proof. From Gill (1983), the Kaplan–Meier estimator of $\overline{F}_0$ is such that $\|\hat{\overline{F}}_n^{(1)}(t) - \overline{F}_0(t)\|_{\tau_n^{(1)}}$ converge in probability to zero. Let $0 < \tau < \tau_H$, $\eta > 0$; and $\varepsilon > 0$; by the continuity of the function $\theta \mapsto \varphi_\theta$ uniformly on $[0, \tau_H)$, the compact closure of Θ may be covered by a finite number K of balls with radius ε', Θ_k with centres θ_k, $k = 1, \ldots, K$, such that $\sup_{\theta\in\Theta_k}\|\varphi_\theta - \varphi_{\theta_k}\|_{\tau_H} \leqslant (8K)^{-1}\eta\varepsilon$ and

$$\Pr\left(\sup_{\theta\in\Theta}\|\hat{F}_{n,\theta} - F_\theta\|_\tau > \eta\right)$$

$$\leqslant \sum_{k\leqslant K}\left\{\Pr\left(\sup_{\theta\in\Theta_k}\|\hat{F}_{n,\theta} - F_\theta - (\hat{F}_{n,\theta_k} - F_{\theta_k})\|_\tau > \frac{\eta}{2}\right) + \Pr\left(\|\hat{F}_{n,\theta_k} - F_{\theta_k}\|_\tau > \frac{\eta}{2}\right)\right\}$$

$$\leqslant \frac{4K}{\eta}\left\{\sup_{\theta\in\Theta_k}\|\varphi_\theta - \varphi_{\theta_k}\|_{\tau_H} + \sum_{k\leqslant K}\Pr\left(\|\hat{F}_{n,\theta_k} - F_{\theta_k}\|_\tau > \frac{\eta}{2}\right)\right\},$$

where the first sum on the right-hand side is smaller than $\varepsilon/2$. For the second term, the representation $\hat{F}_n^{(1)} - \mathrm{F}_0 = \overline{F}_0 \int_0^{\cdot} \hat{\overline{F}}_n^{(1)-} (\overline{F}_0 Y_n^{(1)})^{-1} \, \mathrm{d}M_n^{(1)}$, where $M_n^{(1)} = N_n^{(1)} - \int_0^{\cdot} Y_n^{(1)} \, \mathrm{d}\Lambda_0$, implies

$$\hat{F}_{n,\theta_k}(t) - F_{\theta_k}(t) = \overline{F}_0(t) \int_0^t \varphi_{\theta_k} \frac{\hat{\overline{F}}_n^{(1)-}}{\overline{F}_0} \frac{\mathrm{d}M_n^{(1)}}{Y_n^{(1)}} - F_{\theta_k}(t) \int_0^t \frac{\hat{\overline{F}}_n^{(1)-}}{\overline{F}_0} \frac{\mathrm{d}M_n^{(1)}}{Y_n^{(1)}},$$

and $\sup_{\theta \in \Theta} \| \hat{\overline{F}}_{n,\theta} - \overline{F}_\theta \|_\tau$ converges in probability to zero, for every $\tau < \tau_H$. As $\overline{F}_\theta$ is bounded, this uniform convergence extends to the convergence of $\sup_{\theta \in \Theta} \| \hat{\overline{F}}_{n,\theta} - \overline{F}_\theta \|_{\tau_n^{(1)}}$. The convergence to zero of $\sup_{\theta \in \Theta} \| n^{-1} \hat{Y}_{n,\theta}^{(2)} - y_\theta \|_{\tau_n}$ is then a consequence of the uniform convergence of $n^{-1} Y_n^{(2)}$ to $(1-\alpha) \overline{F}_{\theta_0} \overline{G}$. The result for $\hat{\Lambda}_{n,\theta}$ is deduced by the δ-method.

□

Proposition 1. *Under* (C_1)–(C_3)*, the estimator* $\hat{\theta}_n$ *converges in probability to* θ_0.

Proof. By Lemma 2, $n^{-1} K_n$ converges uniformly on Θ to the function K with derivatives

$$\dot{K}(\theta) = (1-\alpha) \left\{ \int_0^\infty \frac{\dot{\varphi}_\theta}{\varphi_\theta} \varphi_{\theta_0} \overline{G} \, \mathrm{d}F_0 + \int_0^\infty \frac{\int_\cdot^\infty \dot{\varphi}_\theta \, \mathrm{d}F_0}{\int_\cdot^\infty \varphi_\theta \, \mathrm{d}F_0} \overline{F}_{\theta_0} \, \mathrm{d}G \right\},$$

$$\ddot{K}(\theta) = -\Sigma(\theta).$$

$\dot{K}(\theta_0) = (1-\alpha) \{ \int_0^\infty \dot{\varphi}_{\theta_0} \overline{G} \, \mathrm{d}F_0 + \int_0^\infty \dot{\overline{F}}_{\theta_0} \mathrm{d}G \} = 0$ since $\overline{F}_\theta G$ is maximal at the true parameter value θ_0. Under the integrability conditions, the empirical matrix $\Sigma_n = -n^{-1} \ddot{K}_n$ converges in probability to Σ (defined in (C_3)), uniformly on Θ, by the uniform convergence of the Kaplan–Meier estimator and of $\hat{\overline{F}}_{n,\theta}$ to $\overline{F}_0$ and $\overline{F}_{\theta_0}$ (Lemma 2), and by the uniform convergence of the processes $N_n^{(2)}$ and $N_n^{(3)}$ (Lemma 1). Since $\ddot{K}(\theta)$ is a positive definite matrix, the function K is strictly concave in a subset of Θ containing θ_0, with a maximum at θ_0. These results imply the consistency of $\hat{\theta}_n$. □

Proposition 2. $\| \hat{F}_n - F_0 \|_{\tau_n}$ *and* $\| \hat{\Lambda}_n - \Lambda_0 \|_\tau$*, for every* $\tau < \tau_n$*, converge in probability to 0, and* $\int_0^{\tau_n} \varphi_{\hat{\theta}_n} \, \mathrm{d}\hat{F}_n$ *converges in probability to 1.*

Proof. By Proposition 1 and the continuity of φ_θ with respect to θ, $\| \varphi_{\hat{\theta}_n} - \varphi_{\theta_0} \|_{\tau_H}$ converges in probability to 0 and, by Lemma 2, $\| \hat{Y}_{n,\hat{\theta}_n}^{(2)} - (1-\alpha) \overline{F}_0 \overline{\mathrm{G}} \|_{\tau_\mathrm{n}}$ converges in probability to 0. The consistency of $\hat{\Lambda}_n$ then follows from the weak convergence to zero of $\sup_t | n^{-1} (N_n^{(1)}(t) + \int_0^t \varphi_{\theta_0}^{-1} \, \mathrm{d}N_n^{(2)}) - \int_0^t \overline{G} \, \mathrm{d}F_0 |$. Since τ_n tends to τ, the last result is a consequence of the weak convergence to zero of $\hat{\overline{F}}_n(\tau_n)$ and $\| \varphi_{\hat{\theta}_n} - \varphi_{\theta_0} \|_{\tau_n}$. □

Let $K_0(s, t) = \alpha \overline{F}_0(s) \overline{F}_0(t) \int_0^{s \wedge t} F_0^{-2} \overline{G}^{-1} \, \mathrm{d}F_0$ be the covariance of the process W_{F_0}, the limit of W_{n,F_0} is $n^{1/2}(\hat{F}_n^{(1)} - F_0)$, and

$$V_0=(1-\alpha)\left[\int\left\{\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\right\}^{\otimes 2}\overline{G}\,\mathrm{d}F_{\theta_0}+\int\left\{\frac{\int_\cdot^\infty \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_\cdot^\infty \varphi_{\theta_0}\,\mathrm{d}F_0}\right\}^{\otimes 2}\overline{F}_{\theta_0}\,\mathrm{d}G+\frac{(1-\alpha)^2}{(\int\varphi_{\theta_0}\,\mathrm{d}F_0)^{\otimes 2}}\right.$$

$$\left.\times\int\left\{\dot\varphi_{\theta_0}(s)\overline{G}(s)+\int_s^\infty\frac{\int_\cdot^\infty \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_\cdot^\infty \varphi_{\theta_0}\,\mathrm{d}F_0}\,\mathrm{d}G\right\}\left\{\dot\varphi_{\theta_0}(t)\overline{G}(t)+\varphi_{\theta_0}(t)\int_t^\infty\frac{\int_\cdot^\infty \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_\cdot^\infty \varphi_{\theta_0}\,\mathrm{d}F_0}\,\mathrm{d}G\right\}\mathrm{d}K_0(s,\,t)\right].$$

Proposition 3. *$n^{1/2}(\hat\theta_n-\theta_0)$ converges weakly to a centred Gaussian variable $\mathcal{N}_0$ with variance $\Omega=\{\Sigma(\theta_0)\}^{-1}V_0\{\Sigma(\theta_0)\}^{-1}$.*

Proof. Let $U_n=n^{-1/2}\dot K_n(\theta_0)$ and $\Sigma_n(\theta)=-n^{-1}\ddot K_n$ defined in (C_3). By the mean value theorem, the asymptotic normality of $n^{1/2}(\hat\theta_n-\theta_0)$ is obtained from the weak convergence of U_n and from the convergence in probability of $\Sigma_n(\theta_n)$ to $\Sigma(\theta_0)$ for a sequence θ_n between θ_0 and $\hat\theta_n$, converging to θ_0.

Let $W_{n,F_0}=n^{1/2}(\hat F_n^{(1)}-F_0)$, $\nu_{2n}(t)=n^{-1/2}(N_n^{(2)}(t)-(1-\alpha)\int_0^t\overline{G}\,\mathrm{d}F_{\theta_0})$ and $\nu_{3n}(t)=n^{-1/2}(N_n^{(3)}(t)-(1-\alpha)\int_0^t\overline{F}_{\theta_0}\,\mathrm{d}G)$. From (3.6), $\int\dot\varphi_{\theta_0}\overline{G}\,\mathrm{d}F_0+\int(\int_\cdot^\infty\dot\varphi_\theta\,\mathrm{d}F_0)\,\mathrm{d}G=0$, therefore

$$U_n=n^{-1/2}\int\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}N_n^{(2)}+n^{-1/2}\int\frac{\int_\cdot^\infty\dot\varphi_\theta\,\mathrm{d}\hat F_n^{(1)}}{\int_\cdot^\infty\varphi_\theta\,\mathrm{d}\hat F_n^{(1)}}\,\mathrm{d}N_n^{(3)}=\int\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}\nu_{2n}+\int\frac{\int_\cdot^\infty\dot\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}{\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}\,\mathrm{d}\nu_{3n}$$

$$-\frac{(1-\alpha)}{\int_0^\infty\varphi_{\theta_0}\,\mathrm{d}F_0}n^{1/2}\int\left\{\int_\cdot^\infty\dot\varphi_{\theta_0}\,\mathrm{d}F_0-\frac{\int_\cdot^\infty\dot\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}{\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}F_0\right\}\mathrm{d}G$$

$$=\int\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}\nu_{2n}+\int\frac{\int_\cdot^\infty\dot\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}{\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}}\,\mathrm{d}\nu_{3n}$$

$$+(1-\alpha)\int\left\{\dot W_{n,F_{\theta_0}}\left(\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}F_0\right)-W_{n,F_{\theta_0}}\left(\int_\cdot^\infty\dot\varphi_{\theta_0}\,\mathrm{d}F_0\right)\right\}\frac{\mathrm{d}G}{\int_\cdot^\infty\varphi_{\theta_0}\,\mathrm{d}\hat F_n^{(1)}},$$

with $W_{n,F_{\theta_0}} = n^{1/2}(\hat F_{n,\theta_0} - F_{\theta_0}) = \int_0^{\cdot} \varphi_{\theta_0}\, \mathrm{d}W_{n,F_0}$ and $\dot W_{n,F_{\theta_0}} = \{\int_0^{\cdot} \dot\varphi_{\theta_0}\, \mathrm{d}W_{n,F_0}\} \times \{\int_0^{\infty} \varphi_{\theta_0}\, \mathrm{d}F_0\}^{-1}$. The variable U_n converges weakly to a Gaussian variable by the weak convergence of the independent processes (ν_{2n}, ν_{3n}) based on the second sample and $\{\int_0^{\cdot}(\varphi_{\theta_0}, \dot\varphi_{\theta_0})\, \mathrm{d}W_{n,F_0}\}\{\int_0^{\infty} \varphi_{\theta_0}\, \mathrm{d}F_0\}^{-1}$ based on the first sample. Therefore it may be written

$$U_n = \int \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\, \mathrm{d}\nu_{2n} + \int \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\, \mathrm{d}\hat F_n^{(1)}}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\, \mathrm{d}\hat F_n^{(1)}}\, \mathrm{d}\nu_{3n}$$

$$+ (1-\alpha) \int \left\{ \dot W_{n,F_{\theta_0}} - W_{n,F_{\theta_0}} \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\, \mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\, \mathrm{d}F_0} \right\} \mathrm{d}G + o_p(1)$$

$$= \int \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\, \mathrm{d}\nu_{2n} + \int \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\, \mathrm{d}\hat F_n^{(1)}}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\, \mathrm{d}\hat F_n^{(1)}}\, \mathrm{d}\nu_{3n}$$

$$+ \frac{(1-\alpha)}{\int_0^{\infty} \varphi_{\theta_0}\, \mathrm{d}F_0} \int \left\{ \dot\varphi_{\theta_0} \overline{G} - \int_{\cdot}^{\infty} \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\, \mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\, \mathrm{d}F_0}\, \mathrm{d}G \right\} \mathrm{d}W_{n,F_0} + o_p(1).$$

The covariances of (ν_{2n}, ν_{3n}) are given by

$$C_2(s,\ t) = \mathrm{cov}(\nu_{2n}(s),\ \nu_{2n}(t)) = (1-\alpha)\int_0^{s\wedge t} \overline{G}\, \mathrm{d}F_{\theta_0} - (1-\alpha)^2 \int_0^s \overline{G}\, \mathrm{d}F_{\theta_0} \int_0^t \overline{G}\, \mathrm{d}F_{\theta_0},$$

$$C_3(s,\ t) = \mathrm{cov}(\nu_{3n}(s),\ \nu_{3n}(t)) = (1-\alpha)\int_0^{s\wedge t} \overline{F}_{\theta_0}\, \mathrm{d}G - (1-\alpha)^2 \int_0^s \overline{F}_{\theta_0}\, \mathrm{d}G \int_0^t \overline{F}_{\theta_0}\, \mathrm{d}G,$$

$$C_{2,3}(s,\ t) = \mathrm{cov}(\nu_{2n}(s),\ \nu_{3n}(t)) = -(1-\alpha)^2 \int_0^s \overline{G}\, \mathrm{d}F_{\theta_0} \int_0^t \overline{F}_{\theta_0}\, \mathrm{d}G,$$

therefore

$$\operatorname{cov}\left(\int \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}\nu_{2n}, \int \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0}\,\mathrm{d}\nu_{3n}\right) = -(1-\alpha)^2\left\{\int_0^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0\right\}^{-2}\left\{\int \dot\varphi_{\theta_0}\,\overline{G}\,\mathrm{d}F_0\right\}^{\otimes 2},$$

$$\operatorname{var}\left(\int \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}\nu_{2n}\right) = (1-\alpha)\int\left\{\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\right\}^{\otimes 2}\overline{G}\,\mathrm{d}F_{\theta_0} - (1-\alpha)^2\,\frac{\left\{\int \dot\varphi_{\theta_0}\,\overline{G}\,\mathrm{d}F_0\right\}^{\otimes 2}}{\left\{\int \varphi_{\theta_0}\,\mathrm{d}F_0\right\}^{2}},$$

$$\operatorname{var}\left(\int \frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0}\,d\nu_{3n}\right) = (1-\alpha)\int\left\{\frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0}\right\}^{\otimes 2}\overline{F}_{\theta_0}\,\mathrm{d}G - (1-\alpha)^2\,\frac{\left\{\int \dot\varphi_{\theta_0}\,\overline{G}\,\mathrm{d}F_0\right\}^{\otimes 2}}{\left\{\int \varphi_{\theta_0}\,\mathrm{d}F_0\right\}^{2}},$$

$$\operatorname{var}\left(\int \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\,\mathrm{d}\nu_{2n} - \int \frac{\dot F_{\theta_0}}{\overline{F}_{\theta_0}}\,\mathrm{d}\nu_{3n}\right) = (1-\alpha)\left[\int\left\{\frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\right\}^{\otimes 2}\overline{G}\,\mathrm{d}F_{\theta_0} + \int\left\{\frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0}\right\}^{\otimes 2}\overline{F}_{\theta_0}\,\mathrm{d}G\right]$$

and W_{n,F_0} converges weakly to a Gaussian process W_0 with mean zero and covariance K_0. □

Proposition 4. *On every finite interval* $[0,\tau]$ *with* $\tau < \tau_H$, *the process* $n^{1/2}(\hat\Lambda_n - \Lambda_0)$ *converges weakly to a centred Gaussian process given by*

$$W_\Lambda(t) = \int_0^t \frac{1}{\overline{F}_0\overline{G}}\{\mathrm{d}M^{(1)} + \varphi_{\theta_0}^{-1}\mathrm{d}M^{(2)}\} + (1-\alpha)\mathcal{N}_0\int_0^t \left\{\frac{\int_{\cdot}^{\infty} \dot\varphi_{\theta_0}\,\mathrm{d}F_0}{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}F_0} - \frac{\dot\varphi_{\theta_0}}{\varphi_{\theta_0}}\right\}\mathrm{d}\Lambda_0(s)$$

$$+ (1-\alpha)\int_0^t \left\{\frac{W_{F_0}}{\overline{F}_0} - \frac{\int_{\cdot}^{\infty} \varphi_{\theta_0}\,\mathrm{d}W_{F_0}}{\overline{F}_{\theta_0}}\right\}\mathrm{d}\Lambda_0,$$

where $M^{(1)}$ *and* $M^{(2)}$ *are independent centred Gaussian processes with independent increments and variances* $\alpha\int_0^{\cdot} \overline{F}_0\overline{G}\,\mathrm{d}\Lambda_0$ *and* $(1-\alpha)\int_0^{\cdot} \overline{F}_{\theta_0}\overline{G}\,\mathrm{d}\Lambda_{\theta_0}$ *and they are independent of* W_{F_0}, *the limit of* W_{n,F_0}.

Proof. Let $M_n^{(1)} = N_n^{(1)} - \int_0^t Y_n^{(1)}\,\mathrm{d}\Lambda_0$ and $M_n^{(2)} = N_n^{(2)} - \int_0^t Y_n^{(2)}\,\mathrm{d}\Lambda_{\theta_0}$. Then

$$W_{n,\Lambda}(t) = n^{1/2}(\hat{\Lambda}_n - \Lambda_0)(t)$$
$$= n^{1/2}\int_0^t \frac{1}{Y_n^{(1)} + \hat{Y}^{(2)}_{n,\hat{\theta}}}\{\mathrm{d}M_n^{(1)} + \varphi_{\hat{\theta}_n}^{-1}\mathrm{d}M_n^{(2)} + \varphi_{\hat{\theta}_n}^{-1} Y_n^{(2)}\,\mathrm{d}\Lambda_{\theta_0} - \hat{Y}^{(2)}_{n,\hat{\theta}}\,\mathrm{d}\Lambda_0\},$$

where

$$\varphi_{\hat{\theta}_n}^{-1}\frac{\mathrm{d}\Lambda_{\theta_0}}{\mathrm{d}\Lambda_0} - \frac{\hat{Y}^{(2)}_{n,\hat{\theta}}}{Y_n^{(2)}} = \left(\frac{\varphi_{\theta_0}}{\varphi_{\hat{\theta}_n}} - 1\right)\frac{\overline{F}_0}{\overline{F}_{\theta_0}} + \frac{\overline{F}_0}{\overline{F}_{\theta_0}} - \frac{\hat{\overline{F}}{}^{(1)}_n}{\hat{\overline{F}}_{n,\hat{\theta}_n}}$$
$$= -\left\{\int_\cdot^\infty (\varphi_{\hat{\theta}_n} - \varphi_{\theta_0})\,\mathrm{d}F_0 + \int_\cdot^\infty \varphi_{\hat{\theta}_n} n^{-1/2}\,\mathrm{d}W_{n,F_0}\right\}\frac{\overline{F}_0}{\overline{F}_{\theta_0}\hat{\overline{F}}_{n,\hat{\theta}_n}}$$
$$+ \frac{n^{-1/2}W_{n,F_0}}{\hat{\overline{F}}_{n,\hat{\theta}_n}} + \left(\frac{\varphi_{\theta_0}}{\varphi_{\hat{\theta}_n}} - 1\right)\frac{\overline{F}_0}{\overline{F}_{\theta_0}}$$
$$= (\hat{\theta}_n - \theta_0)\left\{\left(\int_\cdot^\infty \dot{\varphi}_{\theta_n}\,\mathrm{d}F_0\right)\frac{\overline{F}_0}{\overline{F}_{\theta_0}\hat{\overline{F}}_{n,\hat{\theta}_n}} - \frac{\dot{\varphi}_{\theta_n}}{\varphi_{\hat{\theta}_n}}\frac{\overline{F}_0}{\overline{F}_{\theta_0}}\right\}$$
$$- \frac{\overline{F}_0}{\overline{F}_{\theta_0}\hat{\overline{F}}_{n,\hat{\theta}_n}}\int_\cdot^\infty \varphi_{\hat{\theta}_n} n^{-1/2}\,\mathrm{d}W_{n,F_0} + \frac{n^{-1/2}W_{n,F_0}}{\hat{\overline{F}}_{n,\hat{\theta}_n}}$$

with θ_n between $\hat{\theta}_n$ and θ_0 and with

$$W_{n,F_0}(t) = n^{1/2}\int_0^t \frac{\hat{\overline{F}}{}^{(1)}_n(s^-)}{\overline{F}_0(s)}\frac{\mathrm{d}M_n^{(1)}(s)}{Y_n^{(1)}(s)}.$$

The processes $n^{-1}(Y_n^{(1)} + \hat{Y}^{(2)}_{n,\hat{\theta}_n})$ and $n^{-1}Y_n^{(2)}$ converge uniformly, in probability, to $\overline{F}_0\overline{G}$ and $(1-\alpha)\overline{F}_{\theta_0}\overline{G}$, respectively; $n^{-1/2}(M_n^{(1)}, M_n^{(2)})$ converges weakly to a centred Gaussian process $(M^{(1)}, M^{(2)})$ with independent components and variance $(\alpha\int_0^\cdot \overline{F}_0\overline{G}\,\mathrm{d}\Lambda_0,$ $(1-\alpha)\int_0^\cdot \overline{F}_{\theta_0}\overline{G}\,\mathrm{d}\Lambda_{\theta_0})$. The expression of the score variable U_n in the proof of the previous proposition ensures the joint convergence in distribution of $n^{1/2}(\hat{\theta}_n - \theta_0)$, W_{n,F_0}, $n^{-1}(Y_n^{(1)} + \hat{Y}^{(2)}_{n,\hat{\theta}_n})$ and $n^{-1}Y_n^{(2)}$. □

The weak convergence of $n^{1/2}(\hat{F}_n - F_0)$ on $[0, \tau_n^{(1)}]$ follows by the representation of the product-limit estimator under integrability conditions (Gill 1983).

4. Estimators for the nonparametric model

Let $f^{(1)}$ and $f^{(2)}$ denote the densities of F and F_φ such that $f^{(j)}(t) > 0$ for t in $]0, \tau_F[$, $j = 1, 2$. In the nonparametric model, the distribution function F_φ of the second sample may be estimated from the product-limit estimator by $\hat{F}_n^{(2)} = 1 - \hat{\overline{F}}{}^{(2)}_n$, and by

$$\hat{F}_{n,\varphi}(t) = \int_0^t \varphi \,\mathrm{d}\hat{F}_n^{(1)} = -\int_0^t \varphi \,\mathrm{d}\hat{\overline{F}}_n^{(1)}. \tag{4.1}$$

For the estimation of the function φ, let K is a symmetric kernel with support $[-1, 1]$, h_n a bandwidth converging to 0 as $n \to \infty$ and $K_h(t) = h^{-1}K(h^{-1}t)$. For t in $]h_n, \tau_n - h_n[$, denote

$$\hat{f}^{(j)}_{n,h_n}(t) = \int K_{h_n}(t-s)\mathrm{d}\hat{F}_n^{(j)}(s),$$

$$\tilde{f}^{(j)}_{n,h_n}(t) = \int K_{h_n}(t-s)\mathrm{d}F^{(j)}(s),$$

$$\hat{\varphi}_{n,h_n}(t) = \hat{f}^{(2)}_{n,h_n}(t)\{\hat{f}^{(1)}_{n,h_n}(t)\}^{-1},$$

$$\hat{Y}^{(2)}_{n,\varphi}(t) = Y_n^{(2)}(t)\{\hat{\overline{F}}_{n,\varphi}(t)\}^{-1}\hat{\overline{F}}_n^{(1)}(t)$$

and $\hat{Y}^{(2)}_{n,\hat{\varphi}_n}$. As $h_n \to 0$, $\Lambda(h_n) \to 0$ and $\overline{F}(h_n) \to 1$, hence Λ and $\overline{F}$ may be estimated for $h_n < t < \tau_n - h_n$ by

$$\hat{\Lambda}_n(t) = \hat{\Lambda}_{n,\hat{\varphi}_n}(t) = \int_{h_n}^t \frac{\mathrm{d}N_n^{(1)} + \hat{\varphi}_n^{-1}\,\mathrm{d}N_n^{(2)}}{Y_n^{(1)} + \hat{Y}^{(2)}_{n,\hat{\varphi}_n}}, \tag{4.2}$$

$$\hat{\overline{F}}_n(t) = \prod_{h_n < T_i \leqslant t} \{1 - \mathrm{d}\hat{\Lambda}_n(T_i)\}. \tag{4.3}$$

Proposition 5. *For every $0 < \tau_1 < \tau_2 < \tau_F$ with τ_1 such that $0 < f^{(1)}(\tau_1)$, $\|\hat{\varphi}_{n,h_n} - \varphi_0\|_{[\tau_1,\tau_2]}$, $\|\hat{F}_{n,\varphi} - F_\varphi\|_{[\tau_1,\tau_2]}$, $\|\hat{Y}^{(2)}_{n,\varphi} - (1-\alpha)\overline{F}_{\varphi_0}\overline{G}_0\|_{[\tau_1,\tau_2]}$, $\|\hat{\Lambda}_n - \Lambda_0\|_{[\tau_1,\tau_2]}$, and $\|\hat{\overline{F}}_n - \overline{F}_0\|_{[\tau_1,\tau_2]}$ converge in probability to 0.*

Proof. The consistency of $\hat{\varphi}_{n,h_n}$ is a consequence of the uniform convergence of the product-limit estimator and of the weak convergence of $n^{1/2}(\hat{F}_n^{(j)} - F^{(j)})$ on $[\tau_1, \tau_2]$, writing

$$\begin{aligned}\|\hat{f}^{(j)}_{n,h_n} - f_0^{(j)}\|_{[\tau_1,\tau_2]} &= \left\| \int_0^\infty K_{h_n}(t-s)\,\{\mathrm{d}\hat{F}_n^{(j)}(s) - \mathrm{d}F^{(j)}(s)\} \right\|_{[\tau_1,\tau_2]} \\ &\leqslant \sup_{t\in[\tau_1,\tau_2]} \int_0^\infty |\hat{F}_n^{(j)}(s) - F^{(j)}(s)|\,|\mathrm{d}K_{h_n}(t-s)| + \sup|K_{h_n}|\,\|\hat{F}_n^{(j)} - F^{(j)}\|_{[\tau_1,\tau_2]} \\ &\leqslant (n_j^{1/2}h_n)^{-1}\left(\sup|K| + \int|\mathrm{d}K|\right) n_j^{1/2}\|\hat{F}_n^{(j)} - F^{(j)}\|_{[\tau_1,\tau_2]}.\end{aligned}$$

The other consistency results are proved as in Section 3. □

Proposition 6. *On intervals $[\tau_1, \tau_2]$ such that $0 < \tau_1 < \tau_2 < \tau_F$ and $0 < f^{(1)}(\tau_1)$, the bias of $\hat{\varphi}_{n,h_n}(t)$ is*

$$b_{n,h_n}(t) = (\hat{\varphi}_{n,h_n} - \varphi_{n,h_n})(t) + O((nh_n)^{-1})$$
$$= \frac{h^2}{2f^{(1)}(t)}[\varphi(t)\{f^{(1)}(t)\}'' + 2\varphi'(t)\{f^{(1)}(t)\}']\kappa_2 + o(h_n^2),$$

its variance is

$$v_{n,h_n}(t) = (nh_n)^{-1}\{f^{(1)}(t)\}^{-2}\Big[\alpha^{-1}\operatorname{var}\{\hat{f}^{(2)}_{n,h_n}(t)\} + (1-\alpha)^{-1}\varphi^2(t)\operatorname{var}\{\hat{f}^{(1)}_{n,h_n}(t)\}\Big] + o((nh_n)^{-1})$$

with

$$\operatorname{var}\{\hat{f}^{(j)}_{n,h_n}(t)\} = (n_j h_n)^{-1}\kappa_1 \frac{f^{(j)}}{\bar{G}}(t)(1+o(1)),$$

and its higher order moments are $o(n^{-1}h_n^{-1})$ *and*

$$(nh_n)^{1/2}(\hat{\varphi}_{n,h_n} - \varphi_{n,h_n})(t) = \{f^{(1)}(t)\}^{-1}(nh_n)^{1/2}\{(\hat{f}^{(2)}_{n,h_n} - f^{(2)}_{n,h_n})(t) - \varphi(t)(\hat{f}^{(1)}_{n,h_n} - f^{(1)}_{n,h_n})(t)\}$$
$$+ o_{L_2}(1). \tag{4.4}$$

Proof. The mean $\varphi_{n,h_n}(t)$ of $\hat{\varphi}_{n,h_n}(t)$ may be approximated as

$$\varphi_{n,h_n}(t) = \frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)} - \frac{\mathrm{E}\Big\{\hat{f}^{(2)}_{n,h_n}(t)[\hat{f}^{(1)}_{n,h_n}(t) - f^{(1)}_{n,h_n}(t)]\Big\}}{\{f^{(1)}_{n,h_n}(t)\}^2} + \frac{\mathrm{E}\Big\{\hat{\varphi}_{n,h_n}(t)[\hat{f}^{(1)}_{n,h_n}(t) - f^{(1)}_{n,h_n}(t)]^2\Big\}}{\{f^{(1)}_{n,h_n}(t)\}^2},$$

therefore

$$\left|\varphi_{n,h_n}(t) - \frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)}\right| \leqslant \frac{\Big[\operatorname{var}\{\hat{f}^{(1)}_{n,h_n}(t)\}\operatorname{var}\{\hat{f}^{(2)}_{n,h_n}(t)\Big]^{1/2}}{\{f^{(1)}_{n,h_n}(t)\}^2} + \sup|\hat{\varphi}_{n,h_n}(t)|\frac{\operatorname{var}\hat{f}^{(1)}_{n,h_n}(t)}{\{f^{(1)}_{n,h_n}(t)\}^2}.$$

For an approximation of $\varphi_{n,h_n}(t)$, the bias and variance of the estimators $\hat{f}^{(j)}_n(t)$ must be calculated. Let

$$\hat{\Lambda}^{(j)}_n(t) = \int_0^t \{Y^{(j)}_n(s)\}^{-1}\,\mathrm{d}N^{(j)}_n(s),$$

$$\Lambda^{(j)}(t) = \int_0^t \{\bar{F}^{(j)}(s)\}^{-1}\,\mathrm{d}F^{(j)}(s),$$

$$\hat{f}^{(j)}_{n,h_n}(t) = \int K_{h_n}(t-s)\,\mathrm{d}\hat{F}^{(j)}_n(s) = \int \hat{F}^{(j)}_n(s^-)K_{h_n}(t-s)\mathrm{d}\hat{\Lambda}^{(j)}_n(s), \qquad t \in]h_n, \tau_n - h_n[.$$

The processes $(n_j h_n)^{1/2}(\hat{f}^{(j)}_n - f^{(j)})$, $j = 1, 2$, are independent and split into a deterministic term $b^{(j)}_{n,h_n} = (n_j h_n)^{1/2}(\tilde{f}^{(j)}_n - f^{(j)})$ and a random term $(n_j h_n)^{1/2}(\hat{f}^{(j)}_n - \tilde{f}^{(j)}_n)$. The deterministic terms satisfy

$$(n_j h_n)^{-1/2} b^{(j)}_{n,h_n}(t) = \frac{h^2}{2}\ddot{f}^{(j)}(t)\kappa_2 + o(h^2).$$

From the representation

$$\hat{F}_n^{(j)}(t) - F^{(j)}(t) = \overline{F}^{(j)}(t)\int_0^t \frac{\hat{\overline{F}}_n^{(j)}(s^-)}{\overline{F}^{(j)}(s)} \frac{\mathrm{d}M_n^{(j)}(s)}{Y_n^{(j)}(s)},$$

we have

$$\hat{f}_{n,h_n}^{(j)}(t) - \tilde{f}_{n,h_n}^{(j)}(t) = \int K_{h_n}(t-s)\left\{\int_0^s \frac{\hat{\overline{F}}_n^{(j)}(u^-)}{\overline{F}^{(j)}(u)} \frac{\mathrm{d}M_n^{(j)}(u)}{Y_n^{(j)}(u)}\right\}\mathrm{d}\overline{F}^{(j)}(s)$$
$$+ \int K_{h_n}(t-s)\hat{\overline{F}}_n^{(j)}(s^-)\frac{\mathrm{d}M_n^{(j)}(s)}{Y_n^{(j)}(s)}$$

and

$$\begin{aligned}
\mathrm{E}\{\hat{f}_{n,h_n}^{(j)}(t) - \tilde{f}_{n,h_n}^{(j)}(t)\}^2 = \mathrm{E}\Bigg[&\int K_{h_n}^2(t-s)\{\hat{\overline{F}}_n^{(j)}(s^-)\}^2 \frac{\mathrm{d}\Lambda^{(j)}(s)}{Y_n^{(j)}(s)}.\\
&+ \int\int K_{h_n}(t-s)K_{h_n}(t-v)\left\{\int_0^s \frac{\hat{\overline{F}}_n^{(j)}(y^-)}{\overline{F}^{(j)}(y)} \frac{\mathrm{d}M_n^{(j)}(y)}{Y_n^{(j)}(y)}\right\}\\
&\times \left\{\int_0^v \frac{\hat{\overline{F}}_n^{(j)}(u^-)}{\overline{F}^{(j)}(u)} \frac{\mathrm{d}M_n^{(j)}(u)}{Y_n^{(j)}(u)}\right\}\mathrm{d}\overline{F}^{(j)}(v)\mathrm{d}\overline{F}^{(j)}(s)\\
&+ 2\int K_{h_n}(t-s)\left\{\int_0^s \frac{\hat{\overline{F}}_n^{(j)}(y^-)}{\overline{F}^{(j)}(y)} \frac{\mathrm{d}M_n^{(j)}(y)}{Y_n^{(j)}(y)}\right\}\\
&\times \left\{\int K_{h_n}(t-v)\hat{\overline{F}}_n^{(j)}(v^-)\frac{\mathrm{d}M_n^{(j)}(v)}{Y_n^{(j)}(v)}\right\}\mathrm{d}\overline{F}^{(j)}(s)\Bigg]\\
= \mathrm{E}\Bigg[&\int K_{h_n}^2(t-s)\{\hat{\overline{F}}_n^{(j)}(s^-)\}^2 \frac{\mathrm{d}\Lambda^{(j)}(s)}{Y_n^{(j)}(s)}.\\
&+ \int\int K_{h_n}(t-s)K_{h_n}(t-v)\left\{\int_0^{s\wedge v} \left\{\frac{\hat{\overline{F}}_n^{(j)}(y^-)}{\overline{F}^{(j)}(y)}\right\}^2 \frac{\mathrm{d}\Lambda^{(j)}(y)}{Y_n^{(j)}(y)}\right\}\\
&\times \mathrm{d}\overline{F}^{(j)}(s)\,\mathrm{d}\overline{F}^{(j)}(v)\\
&+ 2\int K_{h_n}(t-s)\left\{\int_0^s \frac{\hat{\overline{F}}_n^{(j)2}(y^-)}{\overline{F}^{(j)}(y)} K_{h_n}(t-y)\frac{\mathrm{d}\Lambda^{(j)}(y)}{Y_n^{(j)}(y)}\right\}\mathrm{d}\overline{F}^{(j)}(s)\Bigg].
\end{aligned}$$

On every interval $[\tau_1, \tau_2]$ such that $c_H < \overline{H}^{(j)}(\tau_k) = \overline{F}^{(j)}(\tau_k)\overline{G}(\tau_k) < C_H$, $k = 1, 2$, $c_H < n_j^{-1}Y_n^{(j)}(\tau_k) < C_H$ for n sufficiently large then

$$\mathrm{E}\{(n_j^{-1}Y_n^{(j)}(t))^{-1}\} = (\overline{H}^{(j)}(t))^{-1} + \mathrm{E}\left\{\frac{(\overline{H}^{(j)}(t) - n_j^{-1}Y_n^{(j)}(t))^2}{(\overline{H}^{(j)2}n_j^{-1}Y_n^{(j)}(t)}\right\} = (\overline{H}^{(j)}(t))^{-1} + O(n^{-1}).$$

Moreover

$$\hat{\overline{F}}_n^{(j)}(t) = \overline{F}^{(j)}(t)\left\{1 + n_j^{-1}\sum_{i=1}^{n_j}\zeta_{ij}(t)\right\} + r_{n,j}(t), \qquad \|r_{n,j}\| = o(1), \qquad \text{almost surely,}$$

where the processes $\zeta_{ij}(t)$ are independent and identically distributed with finite covariances. Hence

$$\begin{aligned} n_j h_n \mathrm{E}\{\hat{f}^{(j)}_{n,h_n}(t) - \tilde{f}^{(j)}_{n,h_n}(t)\}^2 &= h_n\left[\int K^2_{h_n}(t-s)\,\frac{f^{(j)}}{\overline{G}}(s)\,\mathrm{d}s\right.\\ &+ \int\int K_{h_n}(t-s)K_{h_n}(t-v)\left\{\int_0^{s\wedge v}\frac{\mathrm{d}\Lambda^{(j)}}{\overline{F}^{(j)}\overline{G}}\right\}\mathrm{d}\overline{F}^{(j)}(s)\,\mathrm{d}\overline{F}^{(j)}(v)\\ &\left.+ 2\int K_{h_n}(t-s)\left\{\int_0^s K_{h_n}(t-y)\,\frac{\mathrm{d}\Lambda^{(j)}(y)}{\overline{G}(y)}\right\}\mathrm{d}\overline{F}^{(j)}(s)\right](1+o(1))\\ &= \kappa_1\frac{f^{(j)}}{\overline{G}}(t)\,(1+o(1)), \end{aligned}$$

and higher-order moments of $\hat{f}^{(j)}_{n,h_n}(t)$ are $o(h_n^2)$. Therefore, for any $t \in [\tau_1, \tau_2]$

$$\varphi_{n,h_n}(t) = \frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)} + O(n^{-1}h_n^{-1}), \tag{4.5}$$

the expression of the bias of $\hat{\varphi}_{n,h_n}(t)$ follows and its variance $v_{n,h_n}(t)$ is

$$v_{n,h_n}(t) = \mathrm{E}\left\{\hat{\varphi}_{n,h_n}(t) - \frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)}\right\}^2 + o(h_n^2).$$

By similar expansions,

$$f^{(1)}_{n,h_n}(t)\left\{\hat\varphi_{n,h_n}(t)-\frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)}\right\}=\hat f^{(2)}_{n,h_n}(t)-f^{(2)}_{n,h_n}(t)-\varphi_{n,h_n}(t)\left\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\right\}$$
$$-\frac{\left\{\hat f^{(2)}_{n,h_n}(t)-f^{(2)}_{n,h_n}(t)\right\}\left\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\right\}}{f^{(1)}_{n,h_n}(t)}$$
$$+\frac{\hat\varphi_{n,h_n}(t)\left\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\right\}^2}{f^{(1)}_{n,h_n}(t)}$$
$$+\left\{\varphi_{n,h_n}(t)-\frac{f^{(2)}_{n,h_n}(t)}{f^{(1)}_{n,h_n}(t)}\right\}\left\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\right\}$$

and the approximation of $v_{n,h_n}(t)$ follows from the independence of $\hat f^{(2)}_{n,h_n}(t)$ and $\hat f^{(1)}_{n,h_n}(t)$ and from the order of their moments of order $k>2$.

From formula (4.5) and the expression for the variance of $\hat\varphi_{n,h_n}$,

$$(nh_n)^{1/2}\{\hat\varphi_{n,h_n}(t)-\varphi_{n,h_n}(t)\}$$
$$=(nh_n)^{1/2}\{f^{(1)}_{n,h_n}(t)\}^{-1}[\{\hat f^{(2)}_{n,h_n}(t)-f^{(2)}_{n,h_n}(t)\}-\varphi_{n,h_n}(t)\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\}$$
$$-\{f^{(1)}_{n,h_n}(t)\}^{-1}\{\hat f^{(2)}_{n,h_n}(t)-f^{(2)}_{n,h_n}(t)\}\{\hat f^{(1)}_{n,h_n}(t)-f^{(1)}_{n,h_n}(t)\}]+o_{L_2}(1)$$

and (4.4) follows. □

The next result is a consequence of (4.4) and of the weak convergence of the independent processes $(n_jh_n)^{1/2}(\hat f^{(j)}_{n,h_n}-f^{(j)}_{n,h_n})$.

Proposition 7. *On every interval* $[\tau_1,\tau_2]$ *such that* $0<\tau_1<\tau_2<\tau_F$ *and* $0<f^{(1)}(\tau_1)$*, the process* $W_{n,\varphi}=(h_n)^{1/2}(\hat\varphi_{n,h_n}-\varphi_0)$ *converges weakly on* $[\tau_1,\tau_2]$ *to a centred Gaussian process* W_φ *with null covariances and with variance*

$$\sigma(t)=\lim_{n\to\infty}v_{n,h_n}(t)=\{f^{(1)}(t)\}^{-2}\left[\alpha^{-1}\sigma^{(2)}(t)\}+(1-\alpha)^{-1}\varphi^2(t)\sigma^{(1)}(t)\}\right],$$

with $\sigma^{(j)}(t)=\kappa_1\overline{G}^{-1}(t)f^{(j)}$.

Proposition 8. *On every interval* $[\tau_1,\tau_2]$ *such that* $0<\tau_1<\tau_2<\tau_F$*, the process* $n^{1/2}(\hat\Lambda_n-\Lambda_0)$ *defined by* (4.2) *converges weakly to a centred Gaussian process* W_Λ *such that*

$$W_\Lambda(t) = \int_0^t \frac{1}{\overline{F}_0 \overline{G}} \left\{ \mathrm{d}M^{(1)} + \varphi_0^{-1} \mathrm{d}M^{(2)} \right\} + (1-\alpha) W_\varphi(t) \int_0^t \left\{ \frac{\dot{\overline{F}}_{\varphi_0}}{\overline{F}_{\varphi_0}} - \frac{\dot{\varphi}}{\varphi_0} \right\} \mathrm{d}\Lambda_0(s)$$

$$+ (1-\alpha) \int_0^t \left\{ \frac{W_{F_0}}{\overline{F}_0} - \frac{\int_\cdot^\infty \varphi_0 \mathrm{d}W_{F_0}}{\overline{F}_{\varphi_0}} \right\} \mathrm{d}\Lambda_0,$$

where $(M^{(1)}, M^{(2)})$ *and* W_{F_0} *are defined in Proposition 4.*

Proof. $W_{n,\Lambda}(t)$ is now written

$$n^{1/2} \int_{h_n}^t \frac{1}{Y_n^{(1)} + \hat{Y}_{n,\hat{\varphi}_n}^{(2)}} \{ \mathrm{d}M_n^{(1)} + \varphi_{\hat{\varphi}_n}^{-1} \mathrm{d}M_n^{(2)} + \varphi_{\hat{\varphi}_n}^{-1} Y_n^{(2)} \mathrm{d}\Lambda_{\varphi_0} - \hat{Y}_{n,\hat{\varphi}_n}^{(2)} \mathrm{d}\Lambda_0 \}$$

where $t \in]h_n, \tau_n - h_n[$,

$$\hat{\varphi}_n^{-1} \frac{\mathrm{d}\Lambda_{\varphi_0}}{\mathrm{d}\Lambda_0} - \frac{\hat{Y}_{n,\hat{\varphi}_n}^{(2)}}{Y_n^{(2)}} = \frac{\varphi_0}{\hat{\varphi}_n} \frac{\overline{F}_0}{\overline{F}_{\varphi_0}} - \frac{\hat{\overline{F}}_n^{(1)}}{\hat{\overline{F}}_{n,\hat{\varphi}_n}}$$

$$= -\left\{ \int_0^\cdot (\varphi_n - \varphi_0) \mathrm{d}F_0 + \int_0^\cdot \hat{\varphi}_n n^{-1/2} \mathrm{d}W_{n,F_0} \right\} \frac{\overline{F}_0}{\overline{F}_{\varphi_0} \hat{\overline{F}}_{n,\hat{\varphi}_n}} + \frac{n^{-1/2} W_{n,F_0}}{\hat{\overline{F}}_{n,\hat{\varphi}_n}} + \left(\frac{\varphi_0}{\hat{\varphi}_n} - 1 \right) \frac{\overline{F}_0}{\overline{F}_{\varphi_0}}$$

and the weak convergence of $W_{n,\Lambda}$ is a consequence of the joint convergence of $(nh_n)^{1/2}(\hat{f}_{n,h_n}^{(1)} - f_{n,h_n}^{(1)})$, $(nh_n)^{1/2}(\hat{f}_{n,h_n}^{(2)} - f_{n,h_n}^{(2)})$, $n^{-1}(Y_n^{(1)} + \hat{Y}_{n,\hat{\varphi}_n}^{(2)})$ and $n^{-1} Y_n^{(2)}$. The limit distribution of the nonparametric density is caracterized in Pinçon and Pons (2006). □

As in Section 3, the weak convergence of $n^{1/2}(\hat{F}_n - F_0)$ defined with (4.3) follows by the representation of the product-limit estimator under integrability conditions.

5. Efficient estimators

Efficient estimators of F and θ or φ are obviously independent of those of α and G which are only estimated from $(\rho_i)_{i \leqslant n}$ and from $(\delta_i \log \overline{G}(T_i) + (1-\delta_i) \log g(T_i))_{i \leqslant n}$ in (2.1).

For $(\theta, F) \in \Theta \times \mathcal{F}$, let $L_2(F) = \{a : \mathbb{R}_+ \to \mathbb{R}^p;\ \int \|a\|^2 \mathrm{d}F < \infty\}$, $L_2^0(F) = \{a \in L_2(F);\ \int a \,\mathrm{d}F = 0\}$ and let $L_2(F_\theta)$ and $L_2^0(F_\theta)$ be similarly defined for F_θ. Following the approach presented in Bickel *et al.* (1993), we consider the linear operators R on $L_2^0(F)$ and R_θ on $L_2^0(F_\theta)$ defined by

$$Ra(t) = a(t) - \overline{F}^{-1}(t) \int_t^\infty a \,\mathrm{d}F, \qquad R_\theta a(t) = a(t) - \overline{F}_\theta^{-1}(t) \int_t^\infty a \varphi_\theta \,\mathrm{d}F.$$

Their adjoint operators are

$$R^* b(t) = b(t) - \int_0^t b\, \mathrm{d}\Lambda, \qquad R_\theta^* b(t) = b(t) - \int_0^t b\, \mathrm{d}\Lambda_\theta,$$

which satisfy $R^* = R^{-1}$ and $R_\theta^* = R_\theta^{-1}$. The scores $\dot{l}_\theta$ for θ and $\dot{l}_F$ for the distribution function F satisfy

$$\dot{l}_\theta(T, \delta, \rho) = (1-\rho)\int R_\theta \frac{\dot{\varphi}_\theta}{\varphi_\theta}\, \mathrm{d}M_\theta = (1-\rho) L_\theta R_\theta \frac{\dot{\varphi}_\theta}{\varphi_\theta},$$

$$\dot{l}_F a(T, \delta, \rho) = \rho \int Ra\, \mathrm{d}M + (1-\rho)\int R_\theta a\, \mathrm{d}M_\theta = \rho LR \oplus (1-\rho) L_\theta R_\theta$$

with

$$Lb(T, \delta) = \int b\, \mathrm{d}M, \qquad L_\theta b(T, \delta) = \int b\, \mathrm{d}M_\theta.$$

The linear space $R(\dot{l}_F)$ generated by $\dot{l}_F$ is closed since $K = \rho LR$ and $K_\theta = (1-\rho) L_\theta R_\theta$ generate orthogonal and closed spaces. The adjoint operators of K, K_θ and $\dot{l}_F$ are given by

$$K^* b(t) = \alpha R^{-1} L^* b(t), \qquad K_\theta^* b(t) = (1-\alpha) R_\theta^{-1} L_\theta^* b(t),$$

$$\dot{l}_F^* b(t) = (K^* + K_\theta^*) b(t).$$

Let $Da(t) = L^* La(t) = \overline{G}(t) a(t) = L_\theta^* L_\theta a(t)$. Then for every a in $L_2^0(F_\theta)$,

$$\begin{aligned} \dot{l}_F^* \dot{l}_F a(t) &= \alpha R^{-1} L^* LRa(t) + (1-\alpha) R_\theta^{-1} L_\theta^* L_\theta R_\theta a(t), \\ &= \alpha R^{-1} DRa(t) + (1-\alpha)\varphi_\theta R_\theta^{-1} DR_\theta a(t). \end{aligned}$$

The efficient score function l_θ^* for θ satisfies $l_\theta^* = \dot{l}_\theta - \dot{l}_F a_\theta$, for some a_θ in $L_2^0(F) \cap L_2^0(F_\theta)$, and $\mathrm{E}\{l_\theta^*(T, \delta, \rho)\}\{\dot{l}_F b(T, \delta, \rho)\}^{\mathrm{T}} = 0$ for every $b \in L_2^0(F) \cap L_2^0(F_\theta)$, so $l_\theta^* = (1-\rho) L_\theta R_\theta(\dot{\varphi}_\theta/\varphi_\theta - a_\theta) - \rho LRa_\theta$,

$$\begin{aligned} 0 = \langle l_\theta^*, \dot{l}_F b\rangle &= \left\langle K_\theta^* K_\theta\left(\frac{\dot{\varphi}_\theta}{\varphi_\theta} - a_\theta\right), b\right\rangle_{F_\theta} - \langle K^* K a_\theta, b\rangle_F, \qquad (5.1) \\ &= (1-\alpha)\left\langle R_\theta^{-1} DR_\theta\left(\frac{\dot{\varphi}_\theta}{\varphi_\theta} - a_\theta\right), b\right\rangle_{F_\theta} - \alpha\langle R^{-1} DRa_\theta, b\rangle_F, \qquad \text{for all } b, \end{aligned}$$

therefore

$$\alpha R^{-1} DRa_\theta = (1-\alpha)\varphi_\theta\, R_\theta^{-1} DR_\theta\left(\frac{\dot{\varphi}_\theta}{\varphi_\theta} - a_\theta\right),$$

$$\dot{l}_F^* \dot{l}_F a_\theta(t) = (1-\alpha)\varphi_\theta(t) R_\theta^{-1} DR_\theta \frac{\dot{\varphi}_\theta}{\varphi_\theta}(t). \qquad (5.2)$$

The operator $\dot{l}_F^* \dot{l}_F$ may be inverted since

$$\|\dot{l}_F^* \dot{l}_F\|^2 = \sup_{a \in L_2^0(F) \cap L_2^0(F_\theta): \|a\|_{L_2(F)} = \|a\|_{L_2(F_\theta)} = 1} \left\{ \alpha \int (Ra)^2 \overline{G}^2 \, \mathrm{d}F + (1-\alpha) \int (R_\theta a)^2 \overline{G}^2 \, \mathrm{d}F_\theta \right\};$$

without censoring it equals 1, and it is strictly positive because $\alpha(Ra)^2 + (1-\alpha)\varphi_\theta(R_\theta a)^2 \not\equiv 0$ for any $\varphi_\theta \neq 0$. A solution of (5.2) is deduced to be

$$a_\theta(t) = (1-\alpha)\{\dot{l}_F^* \dot{l}_F\}^{-1} \varphi_\theta(t) \left\{ R_\theta^{-1} D R_\theta \frac{\dot{\varphi}_\theta}{\varphi_\theta}(t) \right\}. \tag{5.3}$$

Example 1. Let $\mathcal{F}$ a parametric family $\mathcal{F} = \{F_\eta, \eta \in \mathcal{H}\}$ on $\mathbb{R}_+$, with densities f_η with respect to a measure μ, and $\mathcal{F}_{\Theta,\mathcal{H}}$ a set of distributions with densities $\varphi_{\theta,\eta}$ on a finite interval $[0, \tau]$ with respect to F_η. Let $a_\eta = c f_\eta^{-1} \dot{f}_\eta 1_{\{f_\eta > 0\}}$ in $L_2^0(F_\eta)$ and $L_2^0(F_{\theta,\eta})$ for every (θ, η) in $(\Theta \times \mathcal{H}\}$, where the constant c and τ are such that the unit norms of a_η equal 1, therefore $\int_0^\tau a_\eta^2 \varphi_{\theta,\eta} \, \mathrm{d}\mu = \int_0^\infty a_\eta^2 \, \mathrm{d}\mu$. Then $\|\dot{l}_F^* \dot{l}_F\| > 0$ if $\alpha\{\dot{f}_\eta \overline{F}_\eta - f_\eta \dot{\overline{F}}_\eta\}^2 + (1-\alpha)\varphi_{\theta,\eta}\{\dot{f}_\eta \int_\cdot^\tau \varphi_{\theta,\eta} \, \mathrm{d}F_\eta - f_\eta \int_\cdot^\tau \varphi_{\theta,\eta} \, \mathrm{d}\dot{F}_\eta\}^2 \neq 0$. For exponential distributions F_η with parameter η and $\varphi_\theta(x) = \lambda_{\theta,\tau} \mathrm{e}^{\theta x} 1_{[0,\tau]}(x)$ such that $0 < \theta < \eta$ and $1 - \theta\eta^{-1} = \lambda_{\theta,\tau}(1 - \mathrm{e}^{-(\eta-\theta)\tau})$, this condition is satisfied with $a_\eta(x) = c(1 - \eta x)$.

Example 2. Let $\mathcal{F}$ a nonparametric family of distributions with densities f with respect to a measure μ and L_2-derivative $\dot{f}$, and let φ_θ a non-uniform parametric density on a finite interval $[0, \tau]$ with respect to F in $\mathcal{F}$. Let $a = c f^{-1} \dot{f} 1_{\{f > 0\}}$ in $L_2^0(F)$ and $L_2^0(F_\theta)$, where τ and c are chosen such that a has unit norms. Then $\alpha\{\dot{f}\overline{F} - f\dot{\overline{F}}\}^2 + (1 - \alpha)\varphi_\theta\{\dot{f} \int_\cdot^\tau \varphi_\theta \, \mathrm{d}F - f \int_\cdot^\tau \varphi_\theta \, \mathrm{d}\dot{F}\}^2 \equiv 0$ and $\|\dot{l}_F^* \dot{l}_F\| > 0$.

The efficient influence function for θ is

$$\tilde{l}_\theta(T, \delta, \rho; \alpha, \theta, F, G) = (I^*(\alpha, \theta, F, G))^{-1} l_\theta^*(T, \delta, \rho; \alpha, \theta, F, G)$$

with the efficient score function defined by (5.1) and the efficient information matrix

$$\begin{aligned} I^*(\alpha, \theta, F, G) &= \langle l_\theta^*(T, \delta, \rho; \alpha, \theta, F, G), l_\theta^*(T, \delta, \rho; \alpha, \theta, F, G) \rangle, \\ &= (1-\alpha) \int \left\{ R_\theta \left(\frac{\dot{\varphi}_\theta}{\varphi_\theta} - a_\theta \right) \right\}^{\otimes 2} \varphi_\theta \overline{G} \, \mathrm{d}F + \alpha \int \{R a_\theta\}^{\otimes 2} \overline{G} \, \mathrm{d}F. \end{aligned} \tag{5.4}$$

A solution a_θ of (5.2) and the functions l_θ^* and I_θ^* depend on all the parameters α, θ, F and G, they may be estimated from $(T_i, \delta_i, \rho_i)_{i=1,\dots,n}$. In the uncensored case, $R^{-1} D R = id = R_\theta^{-1} D R_\theta$ and

$$a_\theta = \frac{(1-\alpha)\dot{\varphi}_\theta}{\alpha + (1-\alpha)\varphi_\theta}$$

which does not depend on F and G. Moreover $LR = L_\theta R_\theta = id$, therefore

$$l_\theta^*(T,\,\delta,\,\rho;\,\alpha,\,\theta)=\alpha(1-\rho)\frac{\dot\varphi_\theta}{\varphi_\theta\{\alpha+(1-\alpha)\varphi_\theta\}}-\rho(1-\alpha)\frac{\dot\varphi_\theta}{\alpha+(1-\alpha)\varphi_\theta},$$

$$\begin{aligned}I_\theta^*(T,\,\delta,\,\rho;\,\alpha,\,\theta)&=\alpha(1-\alpha)\left[\int\alpha\left\{\frac{\dot\varphi_\theta}{\varphi_\theta\{\alpha+(1-\alpha)\varphi_\theta\}}\right\}^{\otimes 2}\mathrm{d}F_\theta\right.\\&\qquad\left.+(1-\alpha)\int\left\{\frac{\dot\varphi_\theta}{\alpha+(1-\alpha)\varphi_\theta}\right\}^{\otimes 2}\mathrm{d}F\right]\\&=\alpha(1-\alpha)\int\frac{\dot\varphi_\theta^{\otimes 2}}{\varphi_\theta^2\{\alpha+(1-\alpha)\varphi_\theta\}}\,\mathrm{d}F.\end{aligned}$$

In the nonparametric model $M_{\mathcal{F},\Phi}$, the efficient score function for φ is defined on $L_2^0(F)\cap L_2^0(F_\varphi)$, where $L_2^0(F_\varphi)=\{a:\mathbb{R}_+\to\mathbb{R}^p;\,\int a\varphi\,\mathrm{d}F=0,\,\int\|a\|^2\varphi\,\mathrm{d}F<\infty\}$, by

$$\begin{aligned}l_\varphi^*a&=(1-\rho)L_\varphi R_\varphi(a-b_a)-\rho LRb_a,\\b_a(t)&=(1-\alpha)\{\dot l_F^*\dot l_F\}^{-1}\varphi(t)\{R_\varphi^{-1}DR_\varphi a(t)\},\\\dot l_F^*\dot l_F a(t)&=\alpha R^{-1}DRa(t)+(1-\alpha)\varphi(t)\{R_\varphi^{-1}DR_\varphi a(t)\},\end{aligned}$$

with $R_\varphi a(t)=a(t)-\overline{F}_\varphi^{-1}(t)\int_t^\infty a\varphi\,\mathrm{d}F$, $L_\varphi a(T,\,\delta)=\delta a(T)-\int_0^T b\overline{F}_\varphi^{-1}\varphi\,\mathrm{d}F$. In the uncensored case, the solution is

$$\begin{aligned}b_a&=\frac{(1-\alpha)a\varphi}{\alpha+(1-\alpha)\varphi},\\l_\varphi^*(T,\,\delta,\,\rho;\,\alpha,\,\varphi)a&=\{\alpha(1-\rho)-\rho(1-\alpha)\varphi\}\frac{a}{\alpha+(1-\alpha)\varphi},\\I_\varphi^*(T,\,\delta,\,\rho;\,\alpha,\,\varphi)a&=\alpha(1-\alpha)\int\frac{a^{\otimes 2}}{\alpha+(1-\alpha)\varphi}\,\mathrm{d}F_\varphi.\end{aligned}$$

More generally, it is defined by (5.4).

As for efficient estimation in parametric models, an estimator of θ is deduced by a Newton iterative procedure of the form $\theta_n=\hat\theta_n+n^{1/2}(\mathrm{I}_{n,\theta}^*)^{-1}l_{n,\theta}^*$ where $I_{n,\theta}^*$ and $l_{n,\theta}^*$ are the estimators of I_θ^* and l_θ^* obtained by replacing the unknown parameters α, θ, F and G by the above estimators (Bickel *et al.* 1993). In the uncensored case, $l_{n,\theta}^*$ and $I_{n,\theta}^*$ are simply given by

$$
\begin{aligned}
l^*_{n,\theta} &= n^{-1}\sum_{i=1}^{n} l^*_{\theta}(T_i,\, \rho_i;\, \hat{\alpha}_n,\, \hat{\theta}_n) \\
&= n^{-1}\sum_{i=1}^{n}\left\{(1-\rho_i)\left\{\frac{\dot{\varphi}_{\hat{\theta}_n}}{\varphi_{\hat{\theta}_n}}(T_i) - a_\theta(T_i;\, \hat{\alpha}_n,\, \hat{\theta}_n)\right\} - \rho_i a_\theta(T_i;\, \hat{\alpha}_n,\, \hat{\theta}_n)\right\}, \\
I^*_{n,\theta} &= \hat{\alpha}_n(1-\hat{\alpha}_n)n^{-1}\sum_{i=1}^{n}\left\{\frac{\dot{\varphi}_{\hat{\theta}_n}}{\hat{\alpha}_n + (1-\hat{\alpha}_n)\varphi_{\hat{\theta}_n}}(T_i)\right\}^{\otimes 2}.
\end{aligned}
$$

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