E. CIECHANOWICZ AND I. I. MARCHENKO
KODAI MATH. J.
40 (2017), 379–404

ON DEVIATIONS, SMALL FUNCTIONS AND STRONG ASYMPTOTIC FUNCTIONS OF MEROMORPHIC FUNCTIONS

Ewa Ciechanowicz and Ivan Ivanovich Marchenko

Abstract

The paper addresses two long-standing problems: of extending the second main theorem of Nevanlinna to the case of small functions, and of finding an upper limit for the number of asymptotic functions of a function of finite lower order. Upper estimates of the sum of deviations and the numbers of strong asymptotic functions and strong functional asymptotic spots of meromorphic functions of finite lower order are presented. The structure of the set of Petrenko's deviations from small functions for meromorphic functions of finite lower order is examined. An analogue of Denjoy's question for strong asymptotic small functions of meromorphic functions of finite lower order is also considered.

1. Introduction

Throughout the paper we apply the standard notations of value distribution theory of meromorphic functions: $N(r,a,f)$, $\overline{N}(r,a,f)$, $N(r,f)$, $m(r,a,f)$, $m(r,f)$, $T(r,f)$, $\delta(a,f)$ and $\theta(a,f)$ ([18]). Since 1920's the problem of generalizing Nevanlinna's second main theorem has been approached a number of times, beginning with Nevanlinna's own theorem on three small functions. Following earlier results of Chuang, Yang Le and Osgood, in 1986 Frank and Weissenborn proved the theorem for rational defective functions ([8]).

Theorem 1.1. *Let f be a transcendental meromorphic function. Then for distinct rational functions $q_1, \dots, q_k$ and every $\varepsilon > 0$ we have*

$$m(r,f) + \sum_{v=1}^{k} m(r,q_v,f) \le (2+\varepsilon)T(r,f)$$

for $r \to \infty$, possibly except for r in a set of finite linear measure.

2010 *Mathematics Subject Classification*. 30D35 (primary); 30D30 (secondary).

Key words and phrases. Meromorphic function, deviation, asymptotic function, small function.

The first author was supported by the National Science Center under grant number 2011/01/D/ST1/05873.

Received July 12, 2016.

A meromorphic function a is called *a small function with respect to* f if

$$T(r,a) = o(T(r,f)) \quad (r \to \infty).$$

The set of all small functions of f is denoted by $\mathscr{S}(f)$. In 1986 Steinmetz proved a result which was more general than the theorem obtained by Frank and Weissenborn ([25]).

THEOREM 1.2. *Let f be a nonconstant meromorphic function and let $\{a_\nu\}_{\nu=1}^{k}$ be a set of pairwise distinct meromorphic small functions of f. Then for every $\varepsilon > 0$,*

$$m(r,f) + \sum_{\nu=1}^{k} m(r,a_\nu,f) \le (2+\varepsilon)T(r,f)$$

for $r \to \infty$, possibly except for r in a set of finite linear measure.

The analogue of the second main theorem including ramification factor was finally obtained by Yamanoi in 2004 ([26]).

THEOREM 1.3. *Let f be a nonconstant meromorphic function on $\mathbf{C}$ and let $a_1, \dots, a_k$ be distinct meromorphic functions on $\mathbf{C}$. Assume that for $\nu = 1, \dots, k$,*

$$T(r,a_\nu) = o(T(r,f)) \quad (r \to \infty).$$

Then for every $\varepsilon > 0$ we have

$$(k-2-\varepsilon)T(r,f) \le \sum_{\nu=1}^{k} \overline{N}(r,a_\nu,f)$$

for $r \to \infty$, $r \notin E$, $\int_E d\log r < \infty$. We also have the defect relation,

$$\sum_{a \in \mathscr{S}(f)} (\delta(a,f) + \theta(a,f)) \le 2.$$

In 1969 Petrenko set up the question: *how will Nevanlinna's theory change if we measure the proximity of a meromorphic function f to a value a applying a different metric*? He introduced the following *function of deviation*:

$$\mathscr{L}(r,a,f) = \begin{cases} \max\limits_{|z|=r} \log^+|f(z)| & \text{for } a = \infty, \\ \max\limits_{|z|=r} \log^+\left|\dfrac{1}{f(z)-a}\right| & \text{for } a \ne \infty. \end{cases}$$

The quantity

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathscr{L}(r,a,f)}{T(r,f)}$$

is called *the magnitude of deviation of* f *with respect to* a, and

$$\Omega(f) := \{a \in \overline{\mathbf{C}} : \beta(a, f) > 0\},$$

the set of positive deviations of f. Let us remind that the values

$$\varrho := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda := \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

are called *order* and *lower order* of a meromorphic function f, respectively. In case of meromorphic functions of finite lower order the properties of $\beta(a, f)$ and $\delta(a, f)$ are similar. Petrenko himself obtained the sharp upper estimate of $\beta(a, f)$ and also an estimate for the sum $\sum_{a\in\overline{\mathbf{C}}} \beta(a, f)$ ([20]).

Theorem 1.4. *If* f *is a meromorphic function of finite lower order* λ, *then for all* $a \in \overline{\mathbf{C}}$ *we have*

$$\beta(a, f) \le B(\lambda) := \begin{cases} \dfrac{\pi\lambda}{\sin \pi\lambda} & \text{if } \lambda \le 0.5, \\ \pi\lambda & \text{if } \lambda > 0.5. \end{cases}$$

$$\sum_{a\in\overline{\mathbf{C}}} \beta(a, f) \le 816\pi(\lambda + 1)^2.$$

The conjecture asserting that the inequality $\beta(\infty, f) \le \pi\varrho$ is true for entire functions of order ϱ, $0.5 \le \varrho < \infty$, was stated by Paley in 1932 and proved in 1969 by Govorov in [11]. In [17] Marchenko and Shcherba presented the following exact estimate of the sum of deviations for functions of finite lower order and solved the problem posed by Petrenko in [21].

Theorem 1.5. *If* f *is a meromorphic function of finite lower order* λ, *then*

$$\sum_{a\in\overline{\mathbf{C}}} \beta(a, f) \le 2B(\lambda).$$

The value

$$\Delta(a, f) := \limsup_{r\to\infty} \frac{m(r, a, f)}{T(r, f)}$$

is called *Valiron's defect of* f *at* a. If $\Delta(a, f) > 0$ we say that *a is a defective value of* f *in the sense of Valiron* and denote $V(f) := \{a \in \overline{\mathbf{C}} : \Delta(a, f) > 0\}$. There is an interesting relationship between the set of positive deviations and the set of Valiron's defective values. The result belongs to Shea and was presented by Fuchs in [9] (see also [21]).

For $\gamma \geq 0$ we put

$$B(\gamma, \Delta) := \begin{cases} \pi\gamma\sqrt{\Delta(2-\Delta)} & \text{if } \gamma > 0.5 \text{ or } \sin\frac{\pi\gamma}{2} > \sqrt{\frac{\Delta}{2}} \\ \frac{\pi\gamma}{\sin\pi\gamma}(1-(1-\Delta)\cos\pi\gamma) & \text{if } 0 \leq \gamma \leq 0.5 \text{ and } \sin\frac{\pi\gamma}{2} \leq \sqrt{\frac{\Delta}{2}} \end{cases}.$$

THEOREM 1.6. *Let f be a meromorphic function of finite lower order λ. Then for each $a \in \mathbf{C}$ we have*

$$\beta(a, f) \leq B(\lambda, \Delta(a, f)).$$

COROLLARY 1.6.1. *For meromorphic functions of finite lower order $\Omega(f) \subset V(f)$.*

The estimate in Theorem 1.6 is sharp, which was shown by Ryshkov in [23]. In 1973 the result of Shea was extended to *n*-valued algebroid functions by Niino in [19].

If $\beta(a, f) > 0$ and $a \in \mathbf{C}$ then a meromorphic function f approaches the value a fast in appropriate components. It could be expected that in those components the derivative f' approaches 0. Hence a natural question is if the sum $\sum_{a\neq\infty}\beta(a, f)$ can be estimated by $\Delta(0, f')$. This problem was solved by Marchenko in 1999 ([14]).

THEOREM 1.7. *For a meromorphic function of finite lower order λ the following inequality holds*

$$\sum_{a\neq\infty}\beta(a, f) \leq 2B(\lambda, \Delta(0, f')).$$

Let $E \subset (0, \infty)$ be a measurable set. The quantities

$$\overline{logdens}\, E = \limsup_{R\to\infty}\frac{1}{\log R}\int_{E\cap[1,R]}\frac{dt}{t},$$

$$\underline{logdens}\, E = \liminf_{R\to\infty}\frac{1}{\log R}\int_{E\cap[1,R]}\frac{dt}{t}$$

are called, respectively, *upper* and *lower logarithmic density* of the set E. In [13] we can find the following analogue of the second main theorem for Petrenko's theory.

THEOREM 1.8. *Let f be a meromorphic function of finite lower order λ and order ρ. Let $\{a_\nu\}_{\nu=1}^{k}$ be a finite set of distinct values, $a_\nu \in \overline{\mathbf{C}}$, $1 \leq \nu \leq k$. For*

$0 < \gamma < \infty$, *put*

$$E_1(\gamma) = \left\{ r : \sum_{\nu=1}^{k} \mathscr{L}(r, a_\nu, f) < 2B(\gamma)T(r, f) \right\}.$$

If g is an entire function, put

$$E_2(\gamma) = \left\{ r : \sum_{\nu=1}^{k} \mathscr{L}(r, a_\nu, g) < B(\gamma)T(r, g) \right\}.$$

We have

$$\overline{\mathit{logdens}}\, E_j(\gamma) \geq 1 - \frac{\lambda}{\gamma} \quad \mathit{and} \quad \underline{\mathit{logdens}}\, E_j(\gamma) \geq 1 - \frac{\rho}{\gamma} \quad j = 1, 2.$$

The following extension of Theorem 1.8 appears in [15].

Theorem 1.9. *Let f be a meromorphic function of finite lower order λ and order ρ. Let $\{a_\nu\}_{\nu=1}^{k}$ be a finite set of distinct complex numbers, and let ε be a fixed positive number. For $0 < \gamma < \infty$, we put*

$$E(\gamma) := \left\{ r : \sum_{\nu=1}^{k} \mathscr{L}(r, a_\nu, f) < 2B(\gamma, \Delta(0, f'))T(r, f) \right\}$$

if $\Delta(0, f') > 0$, and

$$E(\gamma) := \left\{ r : \sum_{1 \leq \nu \leq k} \mathscr{L}(r, a_\nu, f) < \varepsilon T(r, f) \right\}$$

if $\Delta(0, f') = 0$. Then, in both cases,

$$\overline{\mathit{logdens}}\, E(\gamma) \geq 1 - \frac{\lambda}{\gamma} \quad \mathit{and} \quad \underline{\mathit{logdens}}\, E(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

Let now f be a meromorphic function and a—a meromorphic small function of f. We put

$$\beta(a, f) = \liminf_{r \to \infty} \frac{\mathscr{L}(r, a, f)}{T(r, f)},$$

where $\mathscr{L}(r, a, f) = \log^+ \max_{|z|=r} \dfrac{1}{|f(z) - a(z)|}$. If $\beta(a, f) > 0$ we say that a is *a defective function of f in the sense of Petrenko*. Let us remind that for a meromorphic function f and $h : \mathbf{R}_+ \to \mathbf{R}$ we write $h(r) = S(r, f)$ if $h(r) = o(T(r, f))$ for $r \to \infty$, $r \notin E$, $\operatorname{mes} E < \infty$. In 2011 the authors obtained the following theorem, which is a generalization of Theorem 1.8 ([3]).

Theorem 1.10. *Let f be a transcendental meromorphic function of finite lower order λ and order ρ, such that $N(r,f) = S(r,f)$ and let $0 < \gamma < \infty$. Let also $\{q_\nu\}_{\nu=1}^{k}$ be distinct rational functions. Put*

$$E(\gamma) = \left\{ r : \sum_{\nu=1}^{k} \mathscr{L}(r, q_\nu, f) < B(\gamma) T(r,f) \right\}.$$

We have

$$\overline{logdens}\, E(\gamma) \geq 1 - \frac{\lambda}{\gamma} \quad \textit{and} \quad \underline{logdens}\, E(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

Theorem 1.10 implies, that the set of rational functions q with positive deviation from a function f with $N(r,f) = S(r,f)$ is at most countable and

$$\sum_{(q)} \beta(q,f) \leq B(\lambda).$$

A standard definition (see, for instance, [4], or [10] p. 233) says that $a \in \overline{\mathbf{C}}$ is *an asymptotic value* of a meromorphic function f if there exists a continuous curve $\Gamma \subset \mathbf{C}$,

$$\Gamma : z = z(t) \quad (0 \leq t < \infty), \qquad z(t) \to \infty \quad \text{for } t \to \infty,$$

such that

$$\lim_{z \to \infty, z \in \Gamma} f(z) = \lim_{t \to \infty} f(z(t)) = a.$$

We call a pair $\{a, \Gamma\}$, defined as above, *an asymptotic spot of f*. Two asymptotic spots $\{a_1, \Gamma_1\}$ and $\{a_2, \Gamma_2\}$ are considered equal if $a_1 = a_2 = a$ and there exists a sequence of continuous curves γ_k with one end of each γ_k belonging to Γ_1 and the other to Γ_2, and

$$\lim_{k \to \infty} \min_{z \in \gamma_k} |z| = \infty, \qquad \lim_{z \to \infty, z \in \bigcup_k \gamma_k} f(z) = a.$$

A classical theorem of Denjoy-Carleman-Ahlfors gives the sharp upper estimate of the number of asymptotic spots for entire functions of finite lower order (see: [10]).

Theorem 1.11. *An entire function of finite lower order λ cannot have more than $[2\lambda]$ different asymptotic spots connected with finite values. Here $[x]$ denotes the integer part of x.*

The number of asymptotic spots of an entire function of infinite lower order may be infinite (it is indeed for $f(z) = \exp(\exp z)$). The number of asymptotic values of a meromorphic function may be infinite even for functions of finite order. In 1986 Eremenko proved that for every value ϱ, $0 \leq \varrho \leq \infty$ there exists a meromorphic function of order ϱ with the set of asymptotic values equal to $\overline{\mathbf{C}}$

([6]). It shows that it is not possible to put an upper bound for the number of classically defined asymptotic values for mermorphic functions in general. In 2004 Marchenko introduced the following definition.

Definition 1.1. We say that $a \in \overline{\mathbf{C}}$ is *an α_0-strong asymptotic value* of a meromorphic function f, if there exists a continuous curve

$$\Gamma : z = z(t), \quad 0 \le t < \infty, \, z(t) \to \infty \text{ as } t \to \infty,$$

such that

$$\liminf_{t \to \infty} \frac{\log|f(z(t)) - a|^{-1}}{T(|z(t)|, f)} = \alpha(a) \ge \alpha_0 > 0, \quad \text{if } a \ne \infty;$$

$$\liminf_{t \to \infty} \frac{\log|f(z(t))|}{T(|z(t)|, f)} = \alpha(\infty) \ge \alpha_0 > 0, \qquad \text{if } a = \infty.$$

If a is an α_0-strong asymptotic value of f, then an asymptotic spot $\{a, \Gamma\}$ is called *an α_0-strong asymptotic spot*.

In other words, a is a strong asymptotic value of a meromorphic function f if on an asymptotic curve Γ the function tends to a with the speed comparable with characteristic $T(r, f)$. It is easy to notice that, if a is an α_0-strong asymptotic value of f, then the magnitude of Petrenko's deviation $\beta(a, f) \ge \alpha_0$. It means that a is also a defective value in the sense of Petrenko. In the same paper from 2004 Marchenko proved an estimate of the number of strong asymptotic spots ([16]).

Theorem 1.12. *Let f be a meromorphic function of finite lower order λ and let $\{a_v, \Gamma_v\}$, $v = 1, 2, \dots, k$, be distinct α_0-strong asymptotic spots of f. Then* $k \le \left[\dfrac{2B(\lambda)}{\alpha_0}\right]$.

The example of $f(z) = \exp(\exp z)$ again shows that no similar upper estimate exists for functions of infinite order.

In 1956 Denjoy made the following conjecture concerning the set of asymptotic functions of an entire function ([5]).

If f is an entire function of finite lower order λ and $a_1, a_2, \dots, a_k$ are entire functions of order less than $1/2$ such that $f(z) - a_v(z) \to 0$ for z tending to infinity along the path Γ_v, $1 \le v \le k$, then $k \le [2\lambda]$.

The problem is still open, although there have been some results, for instance of Denjoy himself ([5]), Somorjai ([24]) or Fenton, who proved in [7] that the conjecture holds for asymptotic functions of order less than $1/4$. The assumption that the asymptotic functions should be of order less than $1/2$ is essential. If we take $f(z) = \sin\sqrt{z}/\sqrt{z}$ $(f(0) = 1)$, then all $a_c(z) = c \sin\sqrt{z}/\sqrt{z}$, $c \in \mathbf{C}$ are its asymptotic functions as $f(x) - a_c(x) = (1 - c)\dfrac{\sin\sqrt{x}}{\sqrt{x}} \to 0$ for $x \to +\infty$.

Even the number of rational asymptotic functions of a meromorphic function can be infinite, which the example of $f(z) = \exp z$ and its asymptotic functions $b_c(z) = c/z$ $(c \in \mathbf{C})$ shows. It is interesting to learn, however, if for meromorphic functions the number of strong asymptotic functions can be estimated.

Definition 1.2. Let f be a transcendental meromorphic function. We say that a meromorphic function a is an α_0-strong asymptotic function of f, if there exists a continuous curve $\Gamma : z = z(t)$, $0 \le t < \infty$, $z(t) \to \infty$ as $t \to \infty$, such that

$$\liminf_{t\to\infty} \frac{\log|f(z(t)) - a(z(t))|^{-1}}{T(|z(t)|, f)} \ge \alpha_0 > 0.$$

A pair $\{a, \Gamma\}$ is called an α_0-strong functional asymptotic spot of f.

In [3] we have given two estimates of the number of strong rational asymptotic functions.

Theorem 1.13. *Let f be a meromorphic function of finite lower order λ with $N(r, f) = S(r, f)$ and let m denote the number of distinct α_0-strong rational asymptotic spots of f. Then $m \le \left[\frac{B(\lambda)}{\alpha_0}\right]$.*

Theorem 1.14. *Let f be a meromorphic function of finite lower order λ,*

$$\{p_1, \Gamma_1^1\}, \dots, \{p_1, \Gamma_{i_1}^1\}, \dots, \{p_k, \Gamma_1^k\}, \dots, \{p_k, \Gamma_{i_k}^k\}, \quad i_1 + i_2 + \cdots + i_k = m,$$

—m distinct α_0-strong polynomial asymptotic spots of f and $d := \max_{1\le v\le k} \deg(p_v)$. Then $m \le \left[\frac{(d+2)B(\lambda)}{\alpha_0}\right]$.

2. Main results

Applying a method introduced by Petrenko it is possible to examine the structure of the set of positive deviations from small functions for meromorphic functions of finite lower order.

Theorem 2.1. *Let f be a meromorphic function of finite lower order. Then the set $\{a \in \mathcal{S}(f) : \beta(a, f) > 0\}$ of meromorphic small functions of f, which are defective in the sense of Petrenko is at most countable and*

$$\beta^2(\infty, f) + \sum_{a\in\mathcal{S}(f)} \beta^2(a, f) \le \begin{cases} 2(1+\sqrt{2})(\pi\lambda)^2 & \text{if } \lambda \ge 0.5, \\ 2[2 + (1+\sqrt{2})\sin\pi\lambda]\left(\dfrac{\pi\lambda}{\sin\pi\lambda}\right)^2 & \text{if } 0 \le \lambda < 0.5. \end{cases}$$

In case when functions a are constant the statement follows from Petrenko's result for entire curves (Theorem 3.3 in [22]). Applying Theorem 2.1 we arrive at an interesting conclusion concerning asymptotic functions.

Corollary 2.1.1. *Let f be a meromorphic function of finite lower order λ. The set of α_0-strong small asymptotic functions of f is finite and the number k of such functions fulfills the inequality*

$$k \leq \begin{cases} \left[2(1+\sqrt{2})\left(\dfrac{\pi\lambda}{\alpha_0}\right)^2\right] & \text{if } \lambda \geq 0.5, \\ \left[2(2+(1+\sqrt{2})\sin \pi\lambda)\left(\dfrac{\pi\lambda}{\alpha_0 \sin \pi\lambda}\right)^2\right] & \text{if } 0 \leq \lambda < 0.5. \end{cases}$$

As can be observed, there is no direct analogue of Denjoy's hypothesis for strong asymptotic functions. Although the number of strong asymptotic small functions depends on the order of f, there is no strict upper bound for the order of a strong asymptotic function other that being a small function of f. A simple example of a function $\exp z$ and its strong asymptotic functions $a_c(z) = c \exp z$ ($c \in \mathbf{C}$) shows that the condition that a should be a small function of f is essential.

The following extension of Theorem 1.9 concerns meromorphic functions with $N(r, f) = S(r, f)$.

Theorem 2.2. *Let f be a meromorphic function of finite lower order λ and order ρ, with $N(r, f) = S(r, f)$. Let $0 < \gamma < \infty$ and let $\{p_\nu\}_{\nu=1}^k$ be distinct polynomials with $d = \max_{1 \leq \nu \leq k} \deg p_\nu$. We put*

$$E(\gamma) := \left\{ r : \sum_{1 \leq \nu \leq k} \mathscr{L}(r, p_\nu, f) < B(\gamma, \Delta(0, f^{(d+1)}))T(r, f) \right\}$$

if $\Delta(0, f^{(d+1)}) > 0$, and for a fixed positive number ε,

$$E(\gamma) := \left\{ r : \sum_{1 \leq \nu \leq k} \mathscr{L}(r, p_\nu, f) < \varepsilon T(r, f) \right\}$$

if $\Delta(0, f^{(d+1)}) = 0$. Then we have

$$\overline{\mathit{logdens}}\, E(\gamma) \geq 1 - \frac{\lambda}{\gamma} \quad \text{and} \quad \underline{\mathit{logdens}}\, E(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

As a result, we obtain an upper estimate of the sum of deviations with respect to polynomials of limited degree.

Corollary 2.2.1. *Let f fulfill the conditions of Theorem 2.2 and let $\mathfrak{P}_d$ denote the set of all polynomials of a degree less or equal to d. Then*

$$\sum_{p \in \mathfrak{P}_d} \beta(p, f) \leq B(\lambda, \Delta(0, f^{(d+1)})).$$

Also the following estimate of the number of asymptotic polynomials holds.

Corollary 2.2.2. *If f fulfills the conditions of Theorem 2.2 and $\{p_\nu\}_{\nu=1}^{k}$ are distinct α_0-strong asymptotic polynomials of f with $d = \max_{1\le\nu\le k} \deg p_\nu$, then* $k \le \left[\dfrac{B(\lambda, \Delta(0, f^{(d+1)}))}{\alpha_0}\right]$, *where $[x]$ is the integer part of the number x.*

For meromorphic functions with $N(r, f) = S(r, f)$ we can put an upper limit not only on the number of strong asymptotic polynomials, but also on the number of strong polynomial asymptotic spots.

Theorem 2.3. *Let f be a meromorphic function of finite lower order λ with $N(r, f) = S(r, f)$ and*

$$\{p_1, \Gamma_1^1\}, \dots, \{p_1, \Gamma_{i_1}^1\}, \dots, \{p_k, \Gamma_1^k\}, \dots, \{p_k, \Gamma_{i_k}^k\}, \quad i_1 + i_2 + \cdots + i_k = m,$$

—α_0-strong polynomial asymptotic spots of f with $d = \max_{1\le\nu\le k} \deg p_\nu$. Then $m \le \left[\dfrac{B(\lambda, \Delta(0, f^{(d+1)}))}{\alpha_0}\right]$.

The following estimates hold for transcendental meromorphic functions of finite lower order without restrictions on the quantity of their poles.

Theorem 2.4. *Let f be a transcendental meromorphic function of finite lower order λ and order ρ, and let $0 < \gamma < \infty$. Let also $\{p_\nu\}_{\nu=1}^{k}$ be distinct polynomials with $d = \max_{1\le\nu\le k} \deg p_\nu$. We put*

$$\hat{E}(\gamma) := \left\{ r : \sum_{1\le\nu\le k} \mathscr{L}(r, p_\nu, f) < (d+2)B(\gamma, \Delta(0, f^{(d+1)}))T(r, f) \right\}$$

if $\Delta(0, f^{(d+1)}) > 0$, and for a fixed positive number ε,

$$\hat{E}(\gamma) := \left\{ r : \sum_{1\le\nu\le k} \mathscr{L}(r, p_\nu, f) < \varepsilon T(r, f) \right\}$$

if $\Delta(0, f^{(d+1)}) = 0$. Then we have

$$\overline{logdens}\, \hat{E}(\gamma) \ge 1 - \frac{\lambda}{\gamma} \quad \text{and} \quad \underline{logdens}\, \hat{E}(\gamma) \ge 1 - \frac{\rho}{\gamma}.$$

Corollary 2.4.1. *Let f fulfill the conditions of Theorem 2.4 and let $\mathfrak{P}_d$ again denote the set of all polynomials of a degree less or equal to d. Then*

$$\sum_{p\in\mathfrak{P}_d} \beta(p, f) \le (d+2)B(\lambda, \Delta(0, f^{(d+1)})).$$

We also have the following upper bound for the number of polynomial asymptotic spots.

THEOREM 2.5. *Let f be a meromorphic function of finite lower order λ,*

$$\{p_1, \Gamma_1^1\}, \dots, \{p_1, \Gamma_{i_1}^1\}, \dots, \{p_k, \Gamma_1^k\}, \dots, \{p_k, \Gamma_{i_k}^k\}, \quad i_1 + i_2 + \cdots + i_k = m,$$

—m distinct α_0-strong polynomial asymptotic spots of f and $d := \max_{1 \le v \le k} \deg(p_v)$. Then $m \le \left[\dfrac{(d+2)B(\lambda, \Delta(0, f^{(d+1)}))}{\alpha_0}\right]$.

3. Auxiliary results

In order to prove Theorem 2.1 we apply the following formula by Petrenko concerning the intervals where the value of $\mathscr{L}(r, \infty, f)$ is not too big in comparison with $T(r, f)$ ([21]).

LEMMA 3.1. *Let f be a meromorphic function of finite lower order $\lambda, x > \max(\lambda, 0.5)$. Then for any numbers S, R such that $2S < 0.5R$ we have*

$$\int_{2S}^{0.5R} \frac{\mathscr{L}(r, \infty, f)}{r^{\lambda+1}}\, dr \le \frac{\pi\lambda}{\sin \dfrac{\pi\lambda}{2x}} \int_{2S}^{0.5R} \frac{m(r, \infty, f)}{r^{\lambda+1}}\, dr + \pi\lambda \tan \frac{\pi\lambda}{4x} \int_{2S}^{0.5R} \frac{N(r, \infty, f)}{r^{\lambda+1}}\, dr$$
$$+ \frac{c}{x - \lambda} \left\{ \frac{T(2S, f)}{S^{\lambda}} + \frac{T(2R, f)}{R^{\lambda}} \right\}.$$

We also need a modification of the lemma on the logarithmic derivative, which follows from Lemma 4 in [13].

LEMMA 3.2. *Let f be a meromorphic function. Then, possibly except for r in a set of finite linear measure, for $k = 1, 2, \dots$ we have*

$$\mathscr{L}\left(r, \infty, \frac{f^{(k)}}{f}\right) = O(\log(rT(r, f))), \quad (r \to \infty),$$

where $f^{(k)}$ means the k-th derivative of f.

Let f be a transcendental meromorphic function of finite lower order λ and for $1 \le v \le k$, let $\{p_v\}_{v=1}^{k}$ be a set of distinct polynomials, such that $\deg(p_v) \le d$ for $1 \le v \le k$, and $d \ge 1$. Let $S_0 > 0$ be chosen in such a way that if $|z| \ge S_0$, then for all $1 \le v, \eta \le k, v \ne \eta$ we have $p_v(z) \ne p_\eta(z)$. We put for $v \ne \eta$

$$c_{v,\eta} = \min_{|z| \ge S_0} |p_v(z) - p_\eta(z)| > 0, \tag{3.1}$$

$$\min_{1 \le v, \eta \le k} c_{v,\eta} = c > 0.$$

Let $\{R_n\}$ be a sequence of positive numbers, $R_n \to \infty$ $(n \to \infty)$, fulfilling the condition

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} = \lim_{n\to\infty} \frac{\log T(3R_n,f)}{\log R_n}. \tag{3.2}$$

For $n \geq n_0$ we consider the set

$$G_n = \{z : S_0 < |z| < R_n, |f^{(d+1)}(z)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}}\}, \tag{3.3}$$

where n_0 is chosen in such a way that for $n \geq n_0$ we have

$$T(3R_n,f) < R_n^{\lambda+1} \quad \text{and} \quad \frac{\{(2d+2)^{d+1} + (2d+2)^d\}\pi^{d+1}}{R_n} < \frac{c}{4}. \tag{3.4}$$

Now for $1 \leq v \leq k$ we put $G_{n,v}$ for the union of those connected components of G_n which contain a point z_1 such that

$$|f(z_1) - p_v(z_1)| < \frac{c}{4}$$

and points $z_2, z_3, \ldots, z_{d+1}$ such that for $j = 2, \ldots, d+1$

$$|f^{(j-1)}(z_j) - p_v^{(j-1)}(z_j)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}}.$$

Applying the method introduced by Weitsman in [27] and following the same lines as in [2], we may show that for $n \geq n_0$ the sets $G_{n,v}$ and $G_{n,\eta}$ are disjoint for $v \neq \eta$. In particular, for all $z \in G_{n,v}$, $n \geq n_0$ we get

$$|f(z) - p_v(z)| < \frac{[(2d+2)^{d+1} + (2d+2)^d]\pi^{d+1}}{R_n} + \frac{c}{4} < \frac{c}{2}.$$

Thus for $1 \leq v \leq k$ and $n \geq n_0$ we may consider the functions

$$u_{n,v}(z) := \begin{cases} \log \dfrac{1}{|f^{(d+1)}(z)|} & z \in G_{n,v}, \\ \left\{\dfrac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n & z \notin G_{n,v}. \end{cases} \tag{3.5}$$

By definition, the value of each $u_{n,v}$ is relatively high in some of the components of G_n, that is in places where $f^{(d+1)}$ is close to zero. Moreover, for $v \neq \eta$ the sets where the values of $u_{n,v}$ or $u_{n,\eta}$ are relatively high do not overlap. We should note here that each $u_{n,v}(z)$ is a δ-subharmonic function in $S_0 < |z| < R_n$, which can be shown in a similar way as Lemma 6 in [17]. We now conduct a symmetrization of functions $u_{n,v}$. For a complex number $z = re^{i\theta}$ we put:

$$m^*(z,u_{n,v}) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_{n,v}(re^{i\varphi})\, d\varphi,$$

$$T^*(z,u_{n,v}) = m^*(z,u_{n,v}) + N(r,u_{n,v}),$$

where $r \in (S_0, R_n)$, $\theta \in [0, \pi]$, $|E|$ is Lebesgue's measure of the set E and

$$N(r, u_{n,v}) = \int_1^r \frac{\mu_{n,v}(t)}{t}\, dt,$$

where for $1 \le v \le k$, $\mu_{n,v}(r)$ is the number of the zeros of $f^{(d+1)}(z)$ in $G_{n,v} \cap \{z : |z| < r\}$. For δ-subharmonic functions u, the star function $T^*(z, u)$ was introduced by Baernstein in [1]. For a number t, $0 < t \le +\infty$, let us consider the set

$$F_t = \{re^{i\theta} : u_{n,v}(re^{i\theta}) > t\}$$

and let

$$\tilde{u}_{n,v}(re^{i\theta}) = \sup\{t : re^{i\theta} \in F_t^*\},$$

where F_t^* is the symmetric rearrangement of F_t through the circular symmetrization with respect to the ray $\arg z = 0$ ([12]). The function $\tilde{u}_{n,v}(re^{i\theta})$ is non-negative an non-increasing with respect to θ for $\theta \in [0, \pi]$, even in θ and, for a fixed r, equimeasureable with $u_{n,v}(re^{i\theta})$. Moreover, for $1 \le v \le k$,

$$\tilde{u}_{n,v}(r) = \max\left(\log \max_{|z|=r} \frac{1}{|f^{(d+1)}(z)|}, \left\{\frac{d+1}{2}(\lambda + 1) + d + 2\right\} \log R_n\right),$$

$$\tilde{u}_{n,v}(-r) = \max\left(\log \min_{|z|=r} \frac{1}{|f^{(d+1)}(z)|}, \left\{\frac{d+1}{2}(\lambda + 1) + d + 2\right\} \log R_n\right).$$

Let us also notice that

$$m^*(z, u_{n,v}) = \frac{1}{\pi} \int_0^\theta \tilde{u}_{n,v}(re^{i\varphi})\, d\varphi.$$

The function $T^*(z, u_{n,v})$ is subharmonic in $D = \{re^{i\theta} : S_0 < r < R_n, 0 < \theta < \pi\}$, continuous on $D \cup (-R_n, S_0) \cup (S_0, R_n)$ and also logarithmically convex in $r \in (S_0, R_n)$ for each fixed $\theta \in [0, \pi]$ ([1]). What is more, for $r \in (S_0, R_n)$,

$$T^*(r, u_{n,v}) = N(r, u_{n,v}), \quad T^*(re^{i\pi}, u_{n,v}) = T(r, u_{n,v}),$$

$$\frac{\partial}{\partial \theta} T^*(re^{i\theta}, u_{n,v}) = \frac{\tilde{u}_{n,v}(re^{i\theta})}{\pi} \quad \text{for } 0 < \theta < \pi,$$

where $T(r, u_{n,v})$ is the Nevanlinna characteristic of $u_{n,v}(z)$. The following lemma points to the existence of intervals where for a fixed θ the star function $T^*(re^{i\theta}, u_{n,v})$ is increasing with r ([3]).

LEMMA 3.3. *Let $S_0 > 0$, as before, be such that if $|z| \ge S_0$ then $p_v(z) \ne p_\eta(z)$ $(1 \le v, \eta \le k, v \ne \eta)$. Let also $\{R_n\}$ be defined as in (3.2), n_0 as in (3.4). For each number $S_1 > S_0$, there exists $n_1 \ge n_0$ such that for all $n \ge n_1$ and $\theta \in [0, \pi]$ the function $T^*(re^{i\theta}, u_{n,v})$ is monotonically increasing in r on the interval $[S_1, R_n)$.*

For $\alpha(r)$—a real-valued function of a real variable r we consider the operator

$$L\alpha(r) = \liminf_{h\to 0} \frac{\alpha(re^h) + \alpha(re^{-h}) - 2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable in r, then

$$L\alpha(r) = r\frac{d}{dr}\left(r\frac{d}{dr}\alpha(r)\right).$$

As $T^*(re^{i\theta}, u_{n,v})$ is a convex function of $\log r$, for $S_0 < r < R_n$, $\theta \in [0, \pi]$ we have

$$LT^*(re^{i\theta}, u_{n,v}) \geq 0.$$

The following lemma gives a stronger convexity condition.

Lemma 3.4. *Let S_0, $\{R_n\}$ be as in Lemma 3.3. For almost all $\theta \in [0, \pi]$ and for almost all $r \in (S_0, R_n)$ we have*

$$LT^*(re^{i\theta}, u_{n,v}) \geq -\frac{1}{\pi}\frac{\partial \tilde{u}_{n,v}(re^{i\theta})}{\partial\theta}.$$

The proof of Lemma 3.4 can be conducted similarly as the proof of Lemma 1 in [13]. We now put

$$T_0^*(z, f) := \sum_{v=1}^{k} T^*(z, u_{n,v}).$$

It follows from the definition of operator L and from logarithmic convexity of each $T^*(z, u_{n,v})$ that

$$LT_0^*(z, f) \geq \sum_{v=1}^{k} LT^*(z, u_{n,v}) \geq 0.$$

Moreover, Lemma 3.4 implies that

$$LT_0^*(z, f) \geq -\frac{1}{\pi}\sum_{v=1}^{k}\frac{\partial \tilde{u}_{n,v}(re^{i\theta})}{\partial\theta}.$$

For $\tau > 0$ we choose the numbers α and ψ such that

$$0 < \alpha \leq \min\left(\pi, \frac{\pi}{2\tau}\right), \quad -\frac{\pi}{2\tau} \leq \psi \leq \frac{\pi}{2\tau} - \alpha$$

and let

$$\begin{aligned} h_n(r, \tau) :={}& \frac{1}{\pi}\sum_{v=1}^{k}\tilde{u}_{n,v}(r)\cos\tau\psi - \frac{1}{\pi}\sum_{v=1}^{k}\tilde{u}_{n,v}(re^{i\alpha})\cos\tau(\alpha+\psi) \\ &- \tau\sin\tau(\alpha+\psi)T_0^*(re^{i\alpha}, f) + \tau\sin\tau\psi N(r, 0, f^{(d+1)}). \end{aligned}$$

The following result shows that the set of real numbers r where $h_n(r,\tau)$ is positive cannot be relatively big ([3]).

Lemma 3.5. *Let S_0, $\{R_n\}$, n_1 be defined as in Lemma 3.3. Let $S_1 = S_0 + 1$, $n \geq n_1$. Put $A = \{r : S_1 < r < R_n, h_n(r,\tau) > 0\}$. We have*

$$\tau \int_A \frac{dt}{t} \leq \log T(3R_n, f) + \log\log R_n + O(1) \quad (n \to \infty).$$

4. Proof of Theorem 2.1

Let $a_1, \ldots, a_k$ be meromorphic functions small with respect to f. We shall apply Lemma 3.1 to $\dfrac{1}{f(z) - a_v(z)}$ $(1 \leq v \leq k)$:

$$\int_{2S}^{0.5R} \frac{\mathscr{L}(r, a_v, f)}{r^{\lambda+1}}\,dr \leq \frac{\pi\lambda}{\sin\dfrac{\pi\lambda}{2x}} \int_{2S}^{0.5R} \frac{m(r, a_v, f)}{r^{\lambda+1}}\,dr + \pi\lambda \tan\frac{\pi\lambda}{4x} \int_{2S}^{0.5R} \frac{N(r, 0, f - a_v)}{r^{\lambda+1}}\,dr + \frac{c}{x-\lambda}\left\{\frac{T(2S, f)}{S^{\lambda}} + \frac{T(2R, f)}{R^{\lambda}}\right\}. \tag{4.1}$$

Moreover,

$$N(r, 0, f - a_v) \leq T(r, f - a_v) = (1 + o(1))T(r, f) \quad (r \to \infty).$$

By [21] (Lemma 1.9.2, p. 40), we can find two sequences $\{2S_n\}$, $\{2R_n\}$ such that $\dfrac{S_n}{R_n} \to 0$ and

$$\frac{T(2S_n, f)}{S_n^{\lambda}} + \frac{T(2R_n, f)}{R_n^{\lambda}} = o\left(\int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\,dr\right).$$

It follows from (4.1) that

$$\int_{2S_n}^{0.5R_n} \frac{\mathscr{L}(r, a_v, f)}{r^{\lambda+1}}\,dr \leq \frac{\pi\lambda}{\sin\dfrac{\pi\lambda}{2x}} \int_{2S_n}^{0.5R_n} \frac{m(r, a_v, f)}{r^{\lambda+1}}\,dr + \pi\lambda \tan\frac{\pi\lambda}{4x}(1 + o(1)) \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\,dr. \tag{4.2}$$

Moreover, $\mathscr{L}(r, a_v, f) \geq (\beta(a_v, f) - \varepsilon)T(r, f)$ for all $r \geq r_0$. Thus from (4.2) we get the inequality

$$\left(\beta(a_v, f) - \varepsilon - \pi\lambda \tan\frac{\pi\lambda}{4x}\right) \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\,dr \leq \frac{\pi\lambda}{\sin\dfrac{\pi\lambda}{2x}} \int_{2S_n}^{0.5R_n} \frac{m(r, a_v, f)}{r^{\lambda+1}}\,dr. \tag{4.3}$$

For the sake of simplicity we put $\beta = \beta(a_\nu, f) - \varepsilon$, $\sigma(x, \lambda) = \sigma = 0.5\pi\lambda x^{-1}$. If $\lambda \geq \frac{1}{2}$ then σ may take up any value from the interval $\left(0, \dfrac{\pi}{2}\right)$. From (4.3) we get

$$\left(\frac{\beta \sin \sigma}{\pi\lambda} - 1 + \cos \sigma\right) \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\, dr \leq \int_{2S_n}^{R_n} \frac{m(r, a_\nu, f)}{r^{\lambda+1}}\, dr.$$

Let $F(\sigma) = \beta(\pi\lambda)^{-1} \sin \sigma - 1 + \cos \sigma$. This function attains its maximum on the interval $\left(0, \dfrac{\pi}{2}\right)$ at the point $\sigma_0 = \dfrac{\arctan \beta}{\pi\lambda}$ and

$$F(\sigma_0) = \frac{\beta^2}{\pi\lambda(\sqrt{\beta^2 + (\pi\lambda)^2} + \pi\lambda)} \geq \beta^2(\pi\lambda)^{-2}(1 + \sqrt{2})^{-1}.$$

Then for $n \geq n_0$ we have,

$$\left\{\frac{\beta^2}{(\pi\lambda)^2(1 + \sqrt{2})} - \varepsilon\right\} \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\, dr \leq \int_{2S_n}^{0.5R_n} \frac{m(r, a_\nu, f)}{r^{\lambda+1}}\, dr.$$

Applying this estimate to k distinct functions a_ν we get

$$\sum_{\nu=1}^{k} \left\{\frac{\beta^2(a_\nu)}{(\pi\lambda)^2(1 + \sqrt{2})} - \varepsilon\right\} \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\, dr \leq \int_{2S_n}^{0.5R_n} \frac{\sum_{\nu=1}^{k} m(r, a_\nu, f)}{r^{\lambda+1}}\, dr.$$

For $r \geq r_0$ we have ([25]),

$$\sum_{\nu=1}^{k+1} m(r, a_\nu, f) \leq 2T(r, f) + C \log T(2r, f), \quad a_{k+1} = \infty.$$

Thus for $n \geq n_0$,

$$\int_{2S_n}^{0.5R_n} \frac{\sum_{\nu=1}^{k+1} m(r, a_\nu, f)}{r^{\lambda+1}}\, dr \leq (2 + \varepsilon) \int_{2S_n}^{0.5R_n} \frac{T(r, f)}{r^{\lambda+1}}\, dr.$$

It follows that

$$\sum_{\nu=1}^{k+1} \beta^2(a_\nu, f) \leq 2(\pi\lambda)^2(1 + \sqrt{2}).$$

This completes the proof of Theorem 2.1 in case $\lambda \geq 0.5$.

If $\lambda < 0.5$ then the function $\sigma = 0.5\pi\lambda x^{-1}$ for $x > 0.5$ takes up any value from the interval $(0, \pi\lambda)$. If $0 < \beta(a_\nu, f) \leq \pi\lambda \tan \pi\lambda$, then again the maximum value of $F(\sigma)$ on $(0, \pi\lambda)$ is attained at $\sigma_0 = \arctan \beta(\pi\lambda)^{-1}$. Similarly as in the case $\lambda \geq 0.5$, we get

$$F(\sigma_0) = \frac{\beta^2}{\pi\lambda(\sqrt{\beta^2 + (\pi\lambda)^2} + \pi\lambda)} \geq \frac{\beta^2 \sin \pi\lambda}{(\pi\lambda)^2(\sin \pi\lambda + \sqrt{1 + \sin^2 \pi\lambda})} \geq \frac{\beta^2 \sin \pi\lambda}{(\pi\lambda)^2(1 + \sqrt{2})}.$$

Repeating the reasoning of the case $\lambda \geq 0.5$, we arrive at the estimate

$$\sum_{v=1}^{k+1} \beta^2(a_v, f) \leq 2\frac{(\pi\lambda)^2}{\sin \pi\lambda}(1+\sqrt{2}),$$

where the sum is taken over those functions a_v for which $\beta(a_v, f) \leq \pi\lambda \tan \pi\lambda$.

Let now $0 \leq \lambda < 0.5$ and $\beta(a_v, f) \geq \pi\lambda \tan \pi\lambda$. In this case for $\sigma \in [0, \pi\lambda)$ we have

$$\max F(\sigma) = F(\pi\lambda) = \beta\frac{\sin \pi\lambda}{\pi\lambda} - 2\sin^2 0.5\pi\lambda = \frac{\sin \pi\lambda}{\pi\lambda}\{\beta - \pi\lambda \tan 0.5\pi\lambda\}.$$

For $\lambda \leq 0.5$ we have $\pi\lambda \tan 0.5\pi\lambda \leq 0.5\pi\lambda \tan \pi\lambda < 0.5\beta$. Thus

$$\max_{0 \leq \sigma < \pi\lambda} F(\sigma) \geq 0.5\beta\frac{\sin \pi\lambda}{\pi\lambda}.$$

Similarly as before we get that for k distinct functions $a_n u$,

$$\sum_{v=1}^{k+1} \beta(a_n, f) \leq \frac{4\pi\lambda}{\sin \pi\lambda}$$

and from that, since in this case $\beta(a_v, f) \leq \dfrac{\pi\lambda}{\sin \pi\lambda}$, we have

$$\sum_{v=1}^{k+1} \beta^2(a_v, f) \leq \frac{4(\pi\lambda)^2}{\sin^2 \pi\lambda}.$$

Put $A = \{a_v : \beta(a_v, f) > \pi\lambda \tan \pi\lambda\}$, $B = \{a_v : \beta(a_v, f) \leq \pi\lambda \tan \pi\lambda\}$. Then

$$\sum_{v=1}^{k+1} \beta^2(a_v, f) \leq \sum_{a_v \in A} \beta^2(a_v, f) + \sum_{a_v \in B} \beta^2(a_v, f) \leq \frac{2(\pi\lambda)^2}{\sin^2 \pi\lambda}\{2 + (1+\sqrt{2}) \sin \pi\lambda\}.$$

The statement in this case follows directly from this estimate. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

We show the estimate for the upper logarithmic density of $E(\gamma)$. The proof for the estimate of the lower logarithmic density can be conducted in a similar way, only instead of R_n we take any positive number R, and we replace the lower order λ with the order ρ.

We start with the following sum

$$\sum_{v=1}^{k} \mathscr{L}(r, p_v, f) = \sum_{v=1}^{k} \log^+ \max_{|z|=r} \frac{1}{|f(z) - p_v(z)|} = \sum_{v=1}^{k} \log^+ \frac{1}{|f(re^{i\theta_v}) - p_v(re^{i\theta_v})|},$$

where $r \in (S_1, R_n)$, p_v are polynomials such that for $1 \leq v \leq k$, $\deg(p_v) \leq d$.

If $|f(re^{i\theta_v}) - p_v(re^{i\theta_v})| \geq \frac{c}{4}$ then

$$\log^+ \frac{1}{|f(re^{i\theta_v}) - p_v(re^{i\theta_v})|} \leq \log^+ \frac{4}{c}.$$

Let $|f(re^{i\theta_v}) - p_v(re^{i\theta_v})| < \frac{c}{4}$. Then we have

$$\log^+ \frac{1}{|(f-p_v)(re^{i\theta_v})|} \leq \log^+ \left|\frac{(f-p_v)^{(d+1)}(re^{i\theta_v})}{(f-p_v)(re^{i\theta_v})}\right| + \log^+ \left|\frac{1}{f^{(d+1)}(re^{i\theta_v})}\right|.$$

As $u_{n,v}(re^{i\theta_v}) \leq \tilde{u}_{n,v}(r)$, it follows from the inequalities above that in general for $r \in (S_1, R_n)$, $1 \leq v \leq k$ we have

$$\log^+ \frac{1}{|(f-p_v)(re^{i\theta_v})|} \leq \tilde{u}_{n,v}(r) + \log^+ M\left(r, \frac{(f-p_v)^{(d+1)}}{f-p_v}\right) + \log^+ \frac{4}{c}.$$

This way we obtain

$$\sum_{v=1}^{k} \mathscr{L}(r, p_v, f) \leq \sum_{v=1}^{k} \left\{ \tilde{u}_{n,v}(r) + \log^+ M\left(r, \frac{(f-p_v)^{(d+1)}}{f-p_v}\right) \right\} + k \log^+ \frac{4}{c}. \tag{5.1}$$

Notice that if $\gamma \leq \lambda$ the theorem is obvious. Let then $\gamma > \lambda$. We take $\lambda < \tau < \gamma$. We choose $\psi = \frac{\pi}{2\tau} - \alpha$. Thus

$$h_n(r,\tau) = \frac{\sin \tau\alpha}{\pi} \sum_{v=1}^{k} \tilde{u}_{n,v}(r) - \tau T_0^*(re^{i\alpha}) + \tau \cos \tau\alpha N(r, 0, f^{(d+1)}).$$

From the definition of Valiron's defect,

$$\Delta(0, f^{(d+1)}) = 1 - \liminf_{r\to\infty} \frac{N(r, 0, f^{(d+1)})}{T(r, f^{(d+1)})},$$

so if $\varepsilon > 0$ is a fixed number, for $r > r_0(\varepsilon)$ we have

$$N(r, 0, f^{(d+1)}) > (1 - \Delta(0, f^{(d+1)}) - \varepsilon) T(r, f^{(d+1)}). \tag{5.2}$$

Notice that when $\Delta(0, f^{(d+1)}) = 1$ we have $B(\gamma, \Delta(0, f^{(d+1)})) = B(\gamma)$ and the statement follows from Theorem 1.7.

Let us take $0 < \Delta(0, f^{(d+1)}) < 1$ and $0 < \varepsilon < 1 - \Delta(0, f^{(d+1)})$.

Applying the first theorem of Nevanlinna we obtain

$$\begin{aligned} T_0^*(z, f) = T_0^*(re^{i\theta}, f) &= \sum_{v=1}^{k} T^*(z, u_{n,v}) \\ &= \sum_{v=1}^{k} (m^*(z, u_{n,v}) + N(r, u_{n,v})) \end{aligned} \tag{5.3}$$

$$\le m\left(r, \frac{1}{f^{(d+1)}}\right) + N\left(r, \frac{1}{f^{(d+1)}}\right) + k\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n$$
$$= T(r, f^{(d+1)}) + C_f + k\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n,$$

where C_f is a constant appearing in the first theorem of Nevanlinna. For $r \in (r_0(\varepsilon), R_n)$, from (5.2) and (5.3), we get

$$\begin{aligned} h_n(r,\tau) &> \frac{\sin \tau\alpha}{\pi} \sum_{v=1}^{k} \tilde{u}_{n,v}(r) - \tau T(r, f^{(d+1)}) - \tau k\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n \\ &\quad - \tau C_f + \tau \cos \tau\alpha (1 - \Delta(0, f^{(d+1)}) - \varepsilon) T(r, f^{(d+1)}) \\ &= \frac{\sin \tau\alpha}{\pi} \left\{ \sum_{v=1}^{k} \tilde{u}_{n,v}(r) - \frac{\pi\tau}{\sin \tau\alpha} \left[T(r, f^{(d+1)}) \right.\right. \\ &\qquad \left. + k\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n + C_f \right] \\ &\qquad \left. + \pi\tau \operatorname{ctg} \tau\alpha (1 - \Delta(0, f^{(d+1)}) - \varepsilon) T(r, f^{(d+1)}) \right\} := \frac{\sin \tau\alpha}{\pi} H_n(r,\tau) \end{aligned}$$

We now consider the set

$$A_1 = \{r \in (S_1, R_n) : H_n(r,\tau) > 0\}.$$

If $r \in A_1$ then $h_n(r,\tau) > 0$ and by Lemma 3.5,

$$\tau \int_{A_1 \cap [S_1, R_n]} \frac{dt}{t} \le \log T(3R_n, f) + \log \log R_n + O(1). \tag{5.4}$$

If $r \notin A_1$, then

$$\begin{aligned} \sum_{v=1}^{k} \tilde{u}_{n,v}(r) \le \frac{\pi\tau}{\sin \tau\alpha} &\left[(1 - \cos \tau\alpha (1 - \Delta(0, f^{(d+1)}) - \varepsilon)) T(r, f^{(d+1)}) \right. \\ &\left. + k\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n + C_f \right]. \end{aligned}$$

Thus for $r \in [r_0(\varepsilon), R_n] \backslash A_1$ from (5.1) we get

$$\begin{aligned} \sum_{v=1}^{k} \mathscr{L}(r, p_v, f) \le &\frac{\pi\tau}{\sin \tau\alpha} (1 - \cos \tau\alpha (1 - \Delta(0, f^{(d+1)}) - \varepsilon)) T(r, f^{(d+1)}) \\ &+ \frac{\pi\tau k}{\sin \tau\alpha} \left[\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n + C_f \right] \\ &+ \sum_{v=1}^{k} \log^+ M\left(r, \frac{(f - p_v)^{(d+1)}}{f - p_v}\right) + k \log^+ \frac{4}{c}. \end{aligned}$$

Applying Lemma 3.2, lemma on the logarithmic derivative, and the fact that $N(r,f) = S(r,f)$, we obtain that, possibly except for r in a set of finite linear measure, for $r \in [r_0(\varepsilon), R_n] \backslash A_1$ and for all α such that $0 < \alpha \le \min\left(\pi, \frac{\pi}{2\tau}\right)$, for $n \to \infty$,

$$\sum_{\nu=1}^{k} \mathscr{L}(r, p_\nu, f) \le \frac{\pi\tau}{\sin \tau\alpha}(1 - \cos \tau\alpha(1 - \Delta(0, f^{(d+1)}) - \varepsilon))T(r,f) + o(T(r,f)) + O(\log R_n).$$

We now calculate the minimum of $\frac{\pi\tau}{\sin \tau\alpha}(1 - \cos \tau\alpha(1 - \Delta(0, f^{(d+1)}) - \varepsilon))$ over all the numbers α with $0 < \alpha \le \min\left(\pi, \frac{\pi}{2\tau}\right)$. Thus for $n \to \infty$ we get

$$\sum_{\nu=1}^{k} \mathscr{L}(r, p_\nu, f) \le (B(\tau, \Delta(0, f^{(d+1)})) + \varepsilon)T(r,f) + o(T(r,f)) + O(\log R_n).$$

Let $\delta(R) \to 0$ be chosen in such a way that $\frac{T(R^{\delta(R)}, f)}{\log R} \to \infty$ for $R \to \infty$. We put $S_n = R_n^{\delta(R_n)}$, where $\{R_n\}$ is a sequence from (3.2). Let also $r \in [S_n, R_n]$, $S_n > S_1$. It follows from the definition of S_n that

$$T(r,f) \ge T(S_n, f) = \log R_n \frac{T(R_n^{\delta(R_n)}, f)}{\log R_n},$$

for $r \in [S_n, R_n]$, which implies that $\log R_n = o(T(S_n, f))$ $(n \to \infty)$. Therefore, as $\gamma > \tau$, for $r \in [S_n, R_n] \backslash A_1$, possibly except for $r \in E_0$, where E_0 is a set of finite linear measure we have

$$\sum_{\nu=1}^{k} \mathscr{L}(r, p_\nu, f) \le (B(\tau, \Delta(0, f^{(d+1)})) + o(1))T(r,f) < B(\gamma, \Delta(0, f^{(d+1)})T(r,f) \quad (n \to \infty),$$

so $[S_n, R_n] \backslash A_1 \subset E(\gamma) \cup E_0$. This, together with Lemma 3.5 leads to the estimate

$$\tau \int_{E(\gamma)\cap[1,R_n]} \frac{dt}{t} \ge \tau \int_{E(\gamma)\cap[S_n,R_n]} \frac{dt}{t} \ge \tau \int_{[S_n,R_n]\backslash A_1} \frac{dt}{t} + O(1) \ge \tau(1 - \delta(R_n)) \log R_n - \log T(3R_n, f) - \log\log R_n + O(1),$$

for $n \to \infty$. We divide this inequality by $\tau \log R_n$

$$\frac{1}{\log R_n} \int_{E(\gamma)\cap[1,R_n]} \frac{dt}{t} \ge (1 - \delta(R_n)) - \frac{\log T(3R_n, f)}{\tau \log R_n} - \frac{\log\log R_n + O(1)}{\tau \log R_n}.$$

From the definition of $\{R_n\}$ we obtain for all $\tau < \gamma$:

$$\overline{logdens}\, E(\gamma) \geq 1 - \frac{\lambda}{\tau}.$$

Passing to the limit with $\tau \to \gamma$ we get

$$\overline{logdens}\, E(\gamma) \geq 1 - \frac{\lambda}{\gamma}.$$

Thus we obtain the statement in this case.

Finally, let us observe that if $\Delta(0, f^{(d+1)}) = 0$ we may conduct the proof similarly, only taking a fixed positive number instead of $\Delta(0, f^{(d+1)})$.

6. Proof of Theorem 2.3

In the beginning let us notice that we need to consider only the case when

$$0 < \Delta(0, f^{(d+1)}) < 1.$$

If $\Delta(0, f^{(d+1)}) = 0$ we have no α_0-strong asymptotic polynomials (see: Corollary 2.2.2) and if $\Delta(0, f^{(d+1)}) = 1$, the statement follows from Theorem 1.10. Let f be a transcendental meromorphic function of finite lower order λ with $N(r, f) = S(r, f)$, $p_1, \dots, p_k$—distinct polynomials of $\deg(p_v) \leq d$ and let

$$\{p_1, \Gamma_1^1\}, \dots, \{p_1, \Gamma_{i_1}^1\}, \dots, \{p_k, \Gamma_1^k\}, \dots, \{p_k, \Gamma_{i_k}^k\}, \quad i_1 + i_2 + \cdots + i_k = m,$$

be m distinct α_0-strong polynomial asymptotic spots of f. Let G_n be the set defined in (3.3) and let $\tilde{G}_{n,v,j} \subset G_n$, $1 \leq j \leq j_v$, $v = 1, \dots, k$, be the components of G_n, each of which contains points $z_1, z_2, \dots, z_{d+1}$ such that

$$|f^{(s-1)}(z_s) - p_v^{(s-1)}(z_s)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}},$$

$s = 1, 2, \dots, d+1$ $(f^{(0)}(z) = f(z))$. Applying the method introduced by Weitsman in [27] and following the same lines as in [2], we may show that for $n \geq n_0$ the components $\tilde{G}_{n,v,j}$ are pairwise disjoint. In particular,

$$|f(z) - p_v(z)| < \frac{c(d)}{R_n} \quad \forall z \in \tilde{G}_{n,v},$$

where $\tilde{G}_{n,v} = \bigcup_{1 \leq j \leq j_v} \tilde{G}_{n,v,j}$.

Let (p_v, Γ_1^v) and (p_v, Γ_2^v) be two distinct asymptotic spots and let $\tilde{G}_{n,v,1}$ contain a part of the asymptotic curve Γ_1^v. It is easy to show that for $n \geq n_0, \Gamma_2^v$ does not intersect with $\tilde{G}_{n,v,1}$ ([3]). For $n \geq n_0$, $v = 1, \dots, k$, $1 \leq j \leq j_v$ we put

$$u_{n,v,j}(z) := \begin{cases} \log \dfrac{1}{|f^{(d+1)}(z)|} & z \in \tilde{G}_{n,v,j} \\ \left\{\dfrac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n & z \notin \tilde{G}_{n,v,j} \end{cases}.$$

For $\tau > 0$ we also put

$$\tilde{H}_n(r,\tau) := \sum_{v=0}^{k} \tilde{u}_{n,v}(r) - B(\tau, \Delta(0, f^{(d+1)}) \times \left[T(r, f^{(d+1)}) + \tilde{m}\left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n + C_f \right].$$

$$\tilde{A}_1 := \{r : \tilde{H}_n(r,\tau) > 0\},$$

where $\tilde{m} = j_1 + \cdots + j_k$. Following the same lines as in the proof of (5.4), we can show, that

$$\tau \int_{\tilde{A}_1 \cap [S_1, R_n]} \frac{dt}{t} \leq \log T(3R_n, f) + \log\log R_n + O(1) \quad (n \to \infty), \tag{6.1}$$

where τ is any fixed positive number such that $\tau > \lambda$. From Lemma 3.2 we get

$$\begin{aligned} \log^+ \frac{1}{|f(z) - p_v(z)|} &\leq \log^+ \frac{|f^{(s)}(z) - p_v^{(s)}(z)|}{|f(z) - p_v(z)|} + \log^+ \frac{1}{|f^{(s)}(z) - p_v^{(s)}(z)|} \\ &\leq \log^+ \frac{1}{|f^{(s)}(z) - p_v^{(s)}(z)|} + O(\log(rT(r,f))) \end{aligned} \tag{6.2}$$

$$|z| = r \notin E_{v,s}, \quad \mathrm{mes}(E_{v,s}) < \infty \quad \text{and} \quad s = 1, \ldots, d+1.$$

Put $E_v = \bigcup_{1 \leq s \leq d+1} E_{v,s}$ and notice, that $\mathrm{mes}(E_v) < \infty$. As $\{p_v, \Gamma_1^v\}, \ldots, \{p_v, \Gamma_{i_v}^v\}$, $v = 1, \ldots, k$ are α_0-strong polynomial asymptotic spots of f,

$$\liminf_{z \to \infty, z \in \Gamma_l^v} \frac{\log^+ \dfrac{1}{|f(z) - p_v(z)|}}{T(|z|, f)} \geq \alpha_0 > 0, \quad l = 1, \ldots, i_v. \tag{6.3}$$

Thus, on asymptotic curves Γ_l^v, for $|z| \in [S_n, R_n]$, we have

$$\log^+ \frac{1}{|f(z) - p_v(z)|} > \frac{\alpha_0}{2} T(|z|, f).$$

As $\log R_n = o(T(S_n, f))$ $(n \to \infty)$, for $|z| \in [S_n, R_n]$ we get

$$\frac{\alpha_0}{2} T(|z|, f) > \left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n \quad (n \to \infty),$$

which means that

$$|f(z) - p_v(z)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}} \quad (n \to \infty).$$

Moreover, it follows from (5.4) that for $s = 1, \ldots, d+1$ we have

$$\liminf_{z \to \infty, z \in \Gamma_l^v, |z| \notin E_{v,s}} \frac{\log^+ \dfrac{1}{|f^{(s)}(z) - p_v^{(s)}(z)|}}{T(|z|, f)} \geq \alpha_0 > 0. \tag{6.4}$$

Therefore for $z \to \infty$ on asymptotic curves Γ_l^ν, for $s = 1, \dots, d+1$ we have

$$\log^+ \frac{1}{|f^{(s)}(z) - p_\nu^{(s)}(z)|} > \frac{\alpha_0}{2} T(|z|, f) > \left\{\frac{d+1}{2}(\lambda+1) + d + 2\right\} \log R_n,$$

so

$$|f^{(s)}(z) - p_\nu^{(s)}(z)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}}$$

for $|z| \in [S_n, R_n]$, $|z| \notin E_\nu$ $(n \to \infty)$. Put $\tilde{E} := \bigcup_{1 \le \nu \le k} E_\nu$ and notice that mes $\tilde{E} < \infty$. If

$$z \in \Gamma_l^\nu, \quad |z| \in [S_n, R_n] \backslash \tilde{E},$$

for $1 \le l \le i_\nu$, $\nu = 1, \dots, k$ we get

$$|f(z) - p_\nu(z)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}} \quad (n \to \infty),$$

and for $1 \le s \le d+1$

$$|f^{(s)}(z) - p_\nu^{(s)}(z)| < R_n^{-\{((d+1)/2)(\lambda+1)+d+2\}} \quad (n \to \infty).$$

This way we get that for each asymptotic curve Γ_l^ν $(1 \le \nu \le k, 1 \le l \le i_\nu)$, there exists a component $\tilde{G}_{n,\nu,j_0}$ of G_n such that for $n \ge n_0$ we have the inclusion

$$\Gamma_l^\nu \cap \{z : |z| \in [S_n, R_n] \backslash \tilde{E}\} \subset \tilde{G}_{n,\nu,j_0} \quad (j_0 = j_0(l)).$$

From Lemma 3.2 we conclude that for $r \to \infty$, $r \notin E_{\nu,d+1}$, mes $E_{\nu,d+1} < \infty$, $\nu = 1, \dots k$,

$$\log^+ M\left(r, \frac{f^{(d+1)} - p_\nu^{(d+1)}}{f - p_\nu}\right) = \log^+ M\left(r, \frac{f^{(d+1)}}{f - p_\nu}\right) = O(\log(rT(r, f))).$$

Put $\hat{E} = \bigcup_{1 \le \nu \le k} E_{\nu,d+1}$. For $|z| = r$, $r \notin \hat{E}$, for each $1 \le \nu \le k$ we have

$$\log^+ \frac{1}{|f(z) - p_\nu(z)|} = \log^+ \frac{1}{|f^{(d+1)}(z)|} + O(\log(rT(r, f))).$$

Notice that $\hat{E} \subset \tilde{E}$.

From lemma on the logarithmic derivative and from the fact that $N(r, f) = S(r, f)$ it follows that

$$T(r, f^{(d+1)}) \le T(r, f) + o(T(r, f)), \quad r \to \infty, r \notin E, \text{mes}(E) < \infty.$$

Thus for $r \in [S_n, R_n] \backslash (\tilde{A}_1 \cup E \cup \tilde{E})$, we have

$$\sum_{\nu=1}^{k} \sum_{j=1}^{j_\nu} \max_{|z|=r} u_{n,\nu,j}(z) < (B(\tau, \Delta(0, f^{(d+1)})) + o(1))T(r, f), \quad n \to \infty, \tag{6.5}$$

and from (6.1),

$$\tau \int_{[S_n, R_n] \setminus (\tilde{A}_1 \cup E \cup \tilde{E})} \frac{dt}{t} \geq \tau \left(1 - \frac{\log S_n}{\log R_n} \right) \log R_n - \log T(3R_n, f) - \log \log R_n + O(1).$$

From this and from the definition of the sequence R_n (see: (3.2)),

$$\limsup_{n \to \infty} \frac{1}{\log R_n} \int_{[S_n, R_n] \setminus (\tilde{A}_1 \cup E \cup \tilde{E})} \frac{dt}{t} \geq 1 - \frac{\lambda}{\tau} > 0.$$

Therefore, from (6.3), (6.4) and (6.5), there exists a sequence $r_n \in [S_n, R_n] \setminus (\tilde{A}_1 \cup E \cup \tilde{E})$ such, that for each Γ_l^v $(1 \leq v \leq k,\ 1 \leq l \leq i_v)$ we have

$$\begin{aligned} \max_{|z|=r_n} u_{n,v,j_0(l)}(z) &= \max_{|z|=r_n,\, z \in \tilde{G}_{n,v,j_0(l)}} \log \frac{1}{|f^{(d+1)}(z)|} \\ &\geq \max_{|z|=r_n,\, z \in \Gamma_l^v} \log \frac{1}{|f^{(d+1)}(z)|} \geq (\alpha_0 + o(1)) T(r_n, f), \quad n \to \infty, \end{aligned}$$

and

$$\begin{aligned} m(\alpha_0 + o(1)) T(r_n, f) &\leq \sum_{v=1}^{k} \sum_{l=1}^{i_v} \max_{|z|=r_n} u_{n,v,j_0(l)}(z) \\ &\leq \sum_{v=1}^{k} \sum_{j=1}^{j_v} \max_{|z|=r_n} u_{n,v,j}(z) \\ &< (B(\tau, \Delta(0, f^{(d+1)})) + o(1)) T(r_n, f), \quad n \to \infty. \end{aligned}$$

It follows, that

$$m(\alpha_0 + o(1)) \leq B(\tau, \Delta(0, f^{(d+1)}) + o(1), \quad n \to \infty.$$

Passing with $n \to \infty$, we get $m\alpha_0 \leq B(\tau, \Delta(0, f^{(d+1)}))$. As we can choose any $\tau > \lambda$, this way we get the statement. $\square$

Let us remark, that the proofs of Theorem 2.4 and Theorem 2.5 can be obtained by similar reasoning as in the proofs of Theorem 2.2 and Theorem 2.3, bearing in mind that this time the restriction $N(r, f) = S(r, f)$ does not hold. It means that we can apply only the estimate $T(r, f^{(d+1)}) \leq (d+2) T(r, f)$ instead of $T(r, f^{(d+1)}) \leq T(r, f) + o(T(r, f))$ (for r outside a set of finite linear measure).

References

[1] A. Baernstein, Integral means, univalent functions and circular symmetrization, Acta Math. **133** (1974), 139–169.

[2] E. Ciechanowicz and I. I. Marchenko, On deviations from rational functions of entire functions of finite lower order, Ann. Pol. Math. **91** (2007), 161–177.

[3] E. Ciechanowicz and I. I. Marchenko, On deviations and strong asymptotic functions of meromorphic functions of finite lower order, J. Math. Anal. Appl. **382** (2011), 383–398.

[4] A. Denjoy, Sur les fonctions entieres de genre fini, C.R. Acad. Sci. Paris **145** (1907), 106–108.

[5] A. Denjoy, L'allure asymptotique des fonctions entieres d'ordre fini, C.R. Acad. Sci. Paris **242** (1956), 213–218.

[6] A. E. Eremenko, The inverse problem of the theory of value distribution of finite lower order meromorphic functions, Sib. Math. Sb. **27** (1986), 87–102. Engl. transl.: Sib. Math. J. **27** (1986), 377–390.

[7] P. C. Fenton, Entire functions having asymptotic functions, Bull. Austral. Math. Soc. **27** (1983), 321–328.

[8] G. Frank and G. Weissenborn, Rational deficient functions of meromorphic functions, Bull. London Math. Soc. **18** (1986), 29–33.

[9] W. H. J. Fuchs, Topics in Nevanlinna theory, Proc. NRL Conf. on Classical Function Theory, Naval Res. Lab., Washington D.C., 1970, 1–32.

[10] A. A. Gol'dberg and I. V. Ostrowskii, Distribution of values of meromorphic functions, Nauka, Moscow 1970. Engl. transl.: Translations of Mathematical Monographs **236**, Value Distribution of Meromorphic Functions, AMS, Providence, 2008.

[11] N. V. Govorov, On Paley's problem, Funct. Anal. and Its Applications **3** (1969), 38–43.

[12] W. K. Hayman, Multivalent functions, Cambridge Univ. Press, Cambridge, 1958.

[13] I. I. Marchenko, An analogue of the second main theorem for the uniform metric, Mat. fizika, analiz, geometria **5** (1998), 212–227.

[14] I. I. Marchenko, On the deviations and defects of meromorphic functions of finite lower order, Ukr. Mat. Journ. **51** (1999), 784–791. Engl. transl.: Ukr. Math. Journ. **51** (1999), 796–803.

[15] I. I. Marchenko, On the estimates of the upper and lower logarithmic density in the theory of meromorphic functions, Izv. Vysch. Uchebn. Zaved., Mat. **10** (2002), 64–70.

[16] I. I. Marchenko, On the strong asymptotic spots of meromorphic functions of finite lower order, Mat. fizika, analiz, geometriya **11** (2004), 484–491.

[17] I. I. Marchenko and A. I. Shcherba, On the magnitudes of deviations of meromorphic functions, Mat. Sb. **181** (1990), 3–24. Engl. transl.: Math. USSR Sbornik **69** (1991), 1–24.

[18] R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936.

[19] K. Niino, On the growth of algebroid functions of finite lower order, Kodai Math. Sem. Rep. **25** (1973), 385–391.

[20] V. P. Petrenko, The growth of meromorphic functions of finite lower order, Izv. Akad. Nauk SSSR **33** (1969), 414–454.

[21] V. P. Petrenko, Growth of meromorphic functions, Vyshcha Shkola, Kharkov, 1978.

[22] V. P. Petrenko, Entire curves, Vyshcha Shkola, Kharkov, 1984.

[23] M. A. Ryshkov, On the exactness of the estimate of the value of deviation for a meromorphic function, Resp. Sb. Teor. Funkts., Funkts. Anal., Prilozh. **37** (1982), 114–115.

[24] G. Somorjai, On asymptotic functions, J. London Math. Soc. **21** (1980), 297–303.

[25] N. Steinmetz, Eine Verallgemainerung des zweiten Nevanlinnaschen Hauptsatzes, J. Reine Angew. Math. **368** (1986), 134–141.

[26] K. Yamanoi, The second main theorem for small functions and related problems, Acta Math. **192** (2004), 225–294.

[27] A. Weitsman, A theorem on Nevanlinna deficiencies, Acta Math. **128** (1972), 41–52.

Ewa Ciechanowicz
Faculty of Mathematics and Physics
University of Szczecin
ul. Wielkopolska 15
70-451 Szczecin
Poland
E-mail: ewa.ciechanowicz@usz.edu.pl

Ivan Ivanovich Marchenko
Faculty of Mathematics and Physics
University of Szczecin
ul. Wielkopolska 15
70-451 Szczecin
Poland
E-mail: iwan.marczenko@usz.edu.pl