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THE STRUCTURE JACOBI OPERATOR OF THREE-DIMENSIONAL REAL HYPERSURFACES IN NON-FLAT COMPLEX SPACE FORMS

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Abstract

In this paper new results concerning three dimensional real hypersurfaces in non-flat complex space forms in terms of their stucture Jacobi operator are presented. More precisely, the conditions of 1) the structure Jacobi operator being of Codazzi type with respect to the generalized Tanaka-Webster connection and commuting with the shape operator and 2) η-invariance of the structure Jacobi operator and commutativity of it with the shape operator are studied. Furthermore, results concerning Hopf hypersurfaces and ruled hypersurfaces of dimension greater than three satisfying the previous conditions are also included.

1. Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbf{C}P^n$ if $c > 0$, or to a complex Euclidean space $\mathbf{C}^n$ if $c = 0$, or to a complex hyperbolic space $\mathbf{C}H^n$ if $c < 0$. The complex projective and complex hyperbolic spaces are called non-flat complex space forms, since $c \neq 0$ and the symbol $M_n(c)$ is used to denote them when it is not necessary to distinguish them.

A real hypersurface M is an immersed submanifold with real co-dimension one in $M_n(c)$. The Kähler structure (J, G), where J is the complex structure and G is the Kähler metric of $M_n(c)$, induces on M an almost contact metric structure (φ, ξ, η, g). The vector field ξ is called *structure vector field* and when it is an eigenvector of the shape operator A of M the real hypersurface is called *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$.

The study of real hypersurfaces M in $M_n(c)$ was initiated by Takagi, who classified homogeneous real hypersurfaces in $\mathbf{C}P^n$ and divided them into six types, namely (A_1), (A_2), (B), (C), (D) and (E) in [16]. These real hypersurfaces are

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Hopf ones with constant principal curvatures. In case of $\mathbf{C}H^n$ the study of real hypersurfaces with constant principal curvatures was started by Montiel in [9] and completed by Berndt in [1]. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures. The real hypersurfaces found by them are homogeneous and Hopf ones.

The last years many geometers have studied real hypersurfaces in $M_n(c)$ when they satisfy certain geometric conditions. More precisely, the *structure Jacobi operator* of them plays an important role in their study. Generally, the Jacobi operator with respect to a vector field X on a manifold is defined by $R(\cdot, X)X$, where R is the Riemmanian curvature of the manifold. In case of real hypersurfaces for $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R_\xi = R(\cdot, \xi)\xi$.

One of the geometric conditions concerning the structure Jacobi operator that has been studied is that of Codazzi type. Generally, a tensor field T of type (1,1) on M is of Codazzi type when it satisfies

$$(\nabla_X T)Y = (\nabla_Y T)X, \quad \text{where } X, Y \in TM.$$

In [14] the non-existence of real hypersurfaces in complex projective space whose structure Jacobi operator is of Codazzi type is proved. In [18] and [19] the previous result is extended for the case of three dimensional real hypersurfaces in non-flat complex space forms and for real hypersurfaces in complex hyperbolic space. In these cases it is also proved the non-existence of real hypersurfaces satisfying the Codazzi-type condition for the structure Jacobi operator.

Another topic that has been of great importance is the study of real hypersurfaces in $M_n(c)$ in terms of their generalized Tanaka-Webster connection. The notion of generalized Tanaka-Webster connection was first introduced by Tanno in [17] in case of contact metric manifolds in the following way

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y.$$

In [2], [3] Cho extended Tanno's work by defining the notion of generalized Tanaka-Webster connection for real hypersurfaces M in $M_n(c)$ in the following way

$$\hat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y,$$

where X, Y are tangent to M and k is a non-zero real number. Denote by $F_X^{(k)} Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$ which is called the *k-th Cho operator* corresponding to a vector field X. The above relation becomes

$$\hat{\nabla}_X^{(k)} Y = \nabla_X Y + F_X^{(k)} Y. \tag{1.1}$$

The importance of the above connection lies in the fact that studying real hypersurfaces in $M_n(c)$ which satisfy geometric conditions such as parallelness or Codazzi type with respect to the generalized Tanaka-Webster connection leads to different results to those obtained with respect to the Levi-Civita connection.

More precisely, in [15] real hypersurfaces in $M_n(c)$, $n \geq 3$, whose shape operator is of Codazzi type with respect to the generalized Tanaka-Webster connection are classified in contrast to the fact that there are no real hypersurfaces in $M_n(c)$ whose shape operator is of Codazzi type with respect to the Levi-Civita connection.

Motivated by all the above the following question raises naturally.

QUESTION. Are there real hypersurfaces in $M_n(c)$ whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection?

First of all, the structure Jacobi operator is called of *Codazzi type with respect to the generalized Tanaka-Webster connection*, when the following relation is satisfied

$$(\hat{\nabla}_X^{(k)} l)Y = (\hat{\nabla}_Y^{(k)} l)X, \tag{1.2}$$

for any X, Y tangent to M.

In this paper we study three dimensional real hypersurfaces in $M_2(c)$ when the structure Jacobi operator satisfies relation (1.2) and also commutes with the shape operator, i.e.

$$AlX = lAX \tag{1.3}$$

for any X tangent to M.

More precisely, the following Theorem is proved.

THEOREM 1.1. *Every real hypersurface M in $M_2(c)$, whose structure Jacobi operator satisfies relations* (1.2) *and* (1.3), *is a Hopf hypersurface. Furthermore, if $\alpha \neq 2k$ then M is locally congruent*

i) *to a real hypersurface of type* (A)

ii) *or to a Hopf hypersurface with $A\xi = 0$, which in case of $\mathbf{C}P^2$ it is a non-homogeneous real hypersurface, considered as a tube of radius $r = \dfrac{\pi}{4}$ over a holomorphic curve.*

Furthermore, in this paper three dimensional real hypersurfaces in $M_2(c)$ whose structure Jacobi operator is *η-invariant* are also studied. The condition of η-invariance implies that the structure Jacobi operator satisfies the following

$$g((\mathscr{L}_X l)Y, Z) = 0, \quad X, Y, Z \in \mathbf{D}, \tag{1.4}$$

where $\mathscr{L}$ denotes the Lie derivative on M. More precisely, the following Theorem is proved

THEOREM 1.2. *Every real hypersurface M in $M_2(c)$, whose structure Jacobi operator satisfies relations* (1.3) *and* (1.4), *is a Hopf hypersurface. Furthermore, M is locally congruent*

i) *to a real hypersurface of type* (A)

ii) *or to a Hopf hypersurface with* $A\xi = 0$, *which in case of* $\mathbf{C}P^2$ *it is a non-homogeneous real hypersurface, considered as a tube of radius* $r = \frac{\pi}{4}$ *over a holomorphic curve,*

iii) *or to a real hypersurface of type* (B). *In case of* $\mathbf{C}P^2$ *it is a tube of radius* $r \in \left(0, \frac{\pi}{4}\right)$ *around the complex quadric* Q^1 *and in case of* $\mathbf{C}H^2$ *it is a tube of some radius* r *around the canonically (totally geodesic) embedded* 2*-dimensional real hyperbolic space.*

This paper is organized as follows: In Section 2 basic relations and results about real hypersurfaces in $M_n(c)$, $n \geq 2$, are given. In Section 3 the proof of Theorem 1.1 is provided. Furthermore, in this Section some Propositions for Hopf and ruled hypersurfaces, whose structure Jacobi operator is only of Codazzi type with respect to the generalized Tanaka-Webster connection, are proved. In Section 4 the proof of Theorem 1.2 is given. Finally, in this Section Propositions for Hopf and ruled hypersurfaces, whose structure Jacobi operator satisfies only the condition of η-invariance, are included.

2. Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c.

Let N be a locally defined unit normal vector field on M and $\xi = -JN$ be the structure vector field of M. For a vector field X tangent to M relation

$$JX = \varphi X + \eta(X)N$$

holds, where φX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.

The shape operator A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from J of $M_n(c)$, where φ is the *structure tensor* which is a tensor field of type (1,1) and η is an 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).$$

The fact that J is parallel implies $\overline{\nabla} J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi A X, \quad (\nabla_X \varphi) Y = \eta(Y) A X - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \\ -g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi], \end{aligned} \tag{2.2}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

Relation (2.2) implies that the structure Jacobi operator l is given by

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{2.3}$$

for any X tangent to M and $\alpha = \eta(A\xi) = g(A\xi, \xi)$.

The tangent space $T_P M$, for every point $P \in M$, can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbf{D},$$

where $\mathbf{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$). Due to the above decomposition, the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\varphi\nabla_\xi \xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi \xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

The following Theorem is necessary in the proof of our Theorems. It was proved by Okumura in case of $\mathbf{C}P^n$ ([12]) and by Montiel and Romero in case of $\mathbf{C}H^n$ ([10]) and it provides the classification of real hypersurfaces in $M_n(c)$ whose shape operator commutes with the structure tensor field φ.

THEOREM 2.1. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\varphi = \varphi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely*:

In case of $\mathbf{C}P^n$

(A_1) *a geodesic hypersphere of radius r, where* $0 < r < \frac{\pi}{2}$,

(A_2) *a tube of radius r over a totally geodesic* $\mathbf{C}P^k$, $(1 \leq k \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$.

In case of $\mathbf{C}H^n$

(A_0) *a horosphere in* $\mathbf{C}H^n$, *i.e a Montiel tube*,

(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane* $\mathbf{C}H^{n-1}$,

(A_2) *a tube over a totally geodesic* $\mathbf{C}H^k$ $(1 \leq k \leq n-2)$.

The above real hypersurfaces are called real hypersurfaces of type (A).

Finally, we mention the following Theorem which in case of $\mathbf{C}P^n$ is owed to Maeda [8] and in case of $\mathbf{C}H^n$ is owed to Montiel [9] (also Corollary 2.3 in [11]).

THEOREM 2.2. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*

i) *α is constant.*

ii) *If W is a vector field which belongs to $\mathbf{D}$ such that $AW = \lambda W$, then*

$$\left(\lambda - \frac{\alpha}{2}\right) A\varphi W = \left(\frac{\lambda\alpha}{2} + \frac{c}{4}\right)\varphi W. \tag{2.4}$$

iii) *If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.5}$$

Remark 2.3. In case of real hypersurfaces of dimension greater than or equal to three the third case of Theorem 2.2 occurs when $\alpha^2 + c \neq 0$, since in this case relation $\lambda \neq \frac{\alpha}{2}$ holds. Furthermore, the first of (2.1) and the structure Jacobi operator for $X = W$ and $X = \varphi W$ becomes

$$\nabla_W \xi = \lambda\varphi W \quad \text{and} \quad \nabla_{\varphi W}\xi = -\nu W, \tag{2.6}$$

$$lW = \left(\frac{c}{4} + \alpha\lambda\right) W \quad \text{and} \quad l\varphi W = \left(\frac{c}{4} + \alpha\nu\right)\varphi W. \tag{2.7}$$

2.1. Auxiliary facts about three dimensional real hypersurfaces in non-flat complex space forms

Let M be a non-Hopf real hypersurface in $M_2(c)$ and $\{U, \varphi U, \xi\}$ be a local orthonormal basis at some point P of M. Then the following Lemma holds

Lemma 2.4. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$AU = \gamma U + \delta\varphi U + \beta\xi, \quad A\varphi U = \delta U + \mu\varphi U, \quad A\xi = \alpha\xi + \beta U \tag{2.8}$$
$$\nabla_U\xi = -\delta U + \gamma\varphi U, \quad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \quad \nabla_\xi\xi = \beta\varphi U,$$
$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \quad \nabla_{\varphi U} U = \kappa_2\varphi U + \mu\xi, \quad \nabla_\xi U = \kappa_3\varphi U,$$
$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \quad \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi,$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.5. The proof of Lemma 2.4 is included in [13].

The structure Jacobi operator for $X = U$, $X = \varphi U$ and $X = \xi$ due to (2.8) is given by

$$lU = \left(\frac{c}{4} + \alpha\gamma - \beta^2\right)U + \alpha\delta\varphi U, \tag{2.9}$$
$$l\varphi U = \alpha\delta U + \left(\frac{c}{4} + \alpha\mu\right)\varphi U \quad \text{and} \quad l\xi = 0.$$

The Codazzi equation for $X \in \{U, \varphi U\}$ and $Y = \xi$ because of Lemma 2.4 implies the following relations

$$U\beta - \xi\gamma = \alpha\delta - 2\delta\kappa_3 \tag{2.10}$$
$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 \tag{2.11}$$
$$U\alpha - \xi\beta = -3\beta\delta \tag{2.12}$$
$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3 \tag{2.13}$$
$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu \tag{2.14}$$
$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu \tag{2.15}$$

and for $X = U$ and $Y = \varphi U$

$$U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu \tag{2.16}$$
$$U\mu - (\varphi U)\delta = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1 \tag{2.17}$$

Furthermore, combination of the Gauss equation (2.2) with the formula of Riemannian curvature $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$, taking into account relations of Lemma 2.4, implies

$$U\kappa_2 - (\varphi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c, \tag{2.18}$$
$$(\varphi U)\kappa_3 - \xi\kappa_2 = 2\beta\mu - \mu\kappa_1 + \delta\kappa_2 + \kappa_3\kappa_1 + \beta\kappa_3. \tag{2.19}$$

Remark 2.6. In case of Hopf three dimensional real hypersurfaces we consider a local orthonormal basis $\{W, \varphi W, \xi\}$ at some point $P \in M$ such that $AW = \lambda W$ and $A\varphi W = \nu \varphi W$. So relation (2.5) holds. Moreover, relations (2.6) and (2.7) hold.

3. Proof of Theorem 1.1

Let M be a three-dimensional real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relations (1.2), (1.3) and $\alpha \neq 2k$. More analytically, relation (1.2) due to (1.1) is written

$$\begin{aligned} \nabla_X(lY) + F_X^{(k)}(lY) - l\nabla_X Y - lF_X^{(k)} Y \\ = \nabla_Y(lX) + F_Y^{(k)}(lX) - l\nabla_Y X - lF_Y^{(k)} X, \end{aligned} \tag{3.1}$$

where X, Y are tangent to M.

We consider the open subset $\mathcal{N}$ of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ relation (2.8) holds. Moreover, relation (1.3) for $X = \xi$ implies that $lU = 0$ and due to the first of (2.9) we have $\alpha\delta = 0$ and $\alpha\gamma + \frac{c}{4} = \beta^2$. Suppose that $\alpha \neq 0$. Then $\delta = 0$ and relation (2.8) becomes

$$AU = \gamma U + \beta\xi \quad \text{and} \quad A\varphi U = \mu\varphi U.$$

Relation (3.1) for $X = U$ and $Y = \xi$ and $X = U$ and $Y = \varphi U$ due to the above relation, relation (2.9) and Lemma 2.4 implies respectively

$$(\kappa_3 - k)\left(\frac{c}{4} + \alpha\mu\right) = 0 \quad \text{and} \quad \kappa_1\left(\frac{c}{4} + \alpha\mu\right) = 0.$$

Suppose that $\kappa_3 \neq k$ then the first relation implies that $\frac{c}{4} + \alpha\mu = 0$ and the second of (2.9) yields $l\varphi U = 0$. So we have that $l = 0$ which due to Proposition 8 in [4] is impossible. For the same reason as in the previous case the second relation implies that $\kappa_1 = 0$. So the following relations hold

$$\kappa_3 = k, \quad \kappa_1 = 0 \quad \text{and} \quad \frac{c}{4} + \alpha\mu \neq 0, \tag{3.2}$$

since $lU = 0$ and Proposition 8 in [4]. Differentiation of $\frac{c}{4} + \alpha\gamma = \beta^2$ with respect to φU taking into account relations (2.14), (2.15), (2.16) and (3.2) implies

$\gamma k + \gamma\mu = c$. On the other hand, relation (2.11) because of the first two relations of (3.2) and $\alpha\gamma + \frac{c}{4} = \beta^2$ implies that

$$\mu k = \gamma\mu + \gamma k. \tag{3.3}$$

Combination of the last two relations yields $\mu = \frac{c}{k}$ and this results in $U\mu = 0$. The last one due to (2.17) implies $(\gamma - \mu)\kappa_2 = 0$. If $\kappa_2 \neq 0$ then $\gamma = \mu$ and substitution of the latter in (3.3) leads to $\mu = 0$. Then since $\mu = \frac{c}{k}$ this results in $c = 0$, which is impossible. So relation $\kappa_2 = 0$ holds.

Relation (2.18) due to $\kappa_1 = \kappa_2 = 0$, $\gamma k + \gamma\mu = c$ and $\mu = \frac{c}{k}$ implies that $\gamma = -3k$. Differentiation of the latter with respect to φU due to (2.16), (3.2) and the relations for γ and μ yields $c = \frac{3\kappa^2}{2}$. On the other hand, relation (2.19) because of (3.2), $\kappa_2 = 0$ and $\mu = \frac{c}{k}$ leads to $c = -\frac{\kappa^2}{2}$. Combination of the two relations for c leads to a contradiction.

So on $\mathcal{N}$ relation $\alpha = 0$ holds and this results in $\frac{c}{4} = \beta^2$ and $l\varphi U = \frac{c}{4}\varphi U$.

Relation (3.1) for $X = U$ and $Y = \xi$ and for $X = U$ and $Y = \varphi U$ due to the above relations and Lemma 2.4 implies

$$\kappa_3 = k \quad \text{and} \quad \kappa_1 = \kappa_2 = 0.$$

Relation (2.14) because of the first of the above relations implies $\mu = \frac{k}{3}$. Furthermore, relation (2.13) yields $\delta = 0$, thus relation (2.11) due to the previous relations results in $\gamma = \frac{k}{4}$. Moreover, differentiation of $\frac{c}{4} = \beta^2$ taking into account the relations for γ, μ and relation (2.15) implies $c = \frac{k^2}{3}$. Substitution of the previous relations for γ, μ, κ_3 and c in (2.18) results in $k = 0$, which is impossible.

Thus, $\mathcal{N}$ is empty and the following Proposition is proved

Proposition 3.1. *Every real hypersurface in* $M_2(c)$ *whose structure Jacobi operator satisfies relations* (1.2) *and* (1.3) *is Hopf.*

Due to the above Proposition, relations of Theorem 2.2 and remark 2.3 hold. The inner product of relation (3.1) for $X = W$ and $Y = \xi$ with φW due to (2.6) and (2.7) implies

$$\alpha(\lambda - \nu) g(\nabla_\xi W, \varphi W) = \alpha\kappa(\lambda - \nu).$$

Suppose that $\alpha(\lambda - \nu) \neq 0$ then since $g(\nabla_\xi W, W) = g(\nabla_\xi W, \xi) = 0$ we have $\nabla_\xi W = \kappa\varphi W$ and taking into account the second of (2.1) we obtain $\nabla_\xi \varphi W = -\kappa W$.

The inner product of Codazzi equation for $X = \xi$ and $Y = W$ with φW and for $X = \xi$ and $Y = \varphi W$ with W due to the above relations and (2.6) implies

$$\lambda\kappa - \nu\kappa - \alpha\lambda + \lambda\nu = \frac{c}{4},$$
$$\lambda\kappa - \nu\kappa + \alpha\nu - \lambda\nu = -\frac{c}{4}.$$

Combination of the above relations due to $\lambda \neq \nu$ results in $\alpha = 2\kappa$ which is a contradiction.

So on M we have that $\alpha(\lambda - \nu) = 0$. If $\alpha = 0$ in case of $\mathbf{C}P^2$ we have two cases: 1) if $\lambda \neq \nu$ then M is locally congruent to a non-homegeneous real hypersurface considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve, 2) if $\lambda = \nu$ then M is locally congruent to a geodesic hypersphere of radius $r = \frac{\pi}{4}$. In case of $\mathbf{C}H^2$ M is a Hopf hypersurface with $A\xi = 0$ (for the construction of such real hypersurfaces see [5]).

If $\alpha \neq 0$ then $\lambda = \nu$. The latter implies that

$$(A\varphi - \varphi A)X = 0$$

for any X tangent to M. So due to Theorem 2.1 M is locally congruent to a real hypersurface of type (A).

Conversely, it will be proved that real hypersurfaces of type (A) and Hopf hypersurfaces with $A\xi = 0$ in $M_2(c)$ satisfy relation (1.2), since it is known that every Hopf hypersurface satisfies relation (1.3).

Let M be a real hypersurface of type (A) in $M_2(c)$, then the shape operator of M is given by

$$A\xi = \alpha\xi \quad \text{and} \quad AZ = \rho Z, \quad \text{for any } Z \in \mathbf{D},$$

where α and ρ are constants. Then relation (2.3) for $X = \xi$ and $X = Z$, $Z \in \mathbf{D}$ due to the above relation of the shape operator implies

$$l\xi = 0 \quad \text{and} \quad lZ = \left(\frac{c}{4} + \alpha\rho\right)Z, \quad \text{for any } Z \in \mathbf{D}.$$

Relation (1.2) yields relation (3.1). The last one for any $X = Y \in TM$ is identity, for any $X \in \mathbf{D}$ and $Y = \xi$ because of the above relations for the structure Jacobi operator is satisfied and for any $X, Y \in \mathbf{D}$ is satisfied. So real hypersurfaces of type (A) satisfy relation (1.2).

Let M be a Hopf hypersurface with $A\xi = 0$ in $M_2(c)$, then the shape operator of M at some point P is given by

$$A\xi = 0, \quad AZ = tZ \quad \text{and} \quad AZ_1 = t_1 Z_1,$$

where t, t_1 are non-constant functions. Then relation (2.3) for $X = \xi$ and $Y \in \{Z, Z_1\}$ because of the above relation yields

$$l\xi = 0, \quad lZ = \frac{c}{4}Z \quad \text{and} \quad lZ_1 = \frac{c}{4}Z_1.$$

Then, following the same combinations as in the case of real hypersurfaces of type (A), it is proved that Hopf hypersurfaces with $A\xi = 0$ satisfy relation (1.2).

Remark 3.2. The structure Jacobi operator is parallel with respect to the generalized Tanaka-Webster connection when it satisfies

$$(\nabla_X^{(k)} l)Y = 0,$$

for any X, Y tangent to M. The above relation implies that the structure Jacobi operator satisfies relation (1.2). Thus M is locally congruent to one of the above mentioned real hypersurfaces.

Conversely, it will be proved that real hypersurfaces of type (A) and Hopf hypersurfaces with $A\xi = 0$ in $M_2(c)$ has parallel structure Jacobi operator with respect to the generalized Tanaka-Webster connection, i.e. $(\nabla_X^{(k)} l)Y = 0$, for any X, Y tangent to M. The latter relation because of (1.1) is written more analytically as

$$\nabla_X(lY) + F_X^{(\kappa)}(lY) = l\nabla_X Y + lF_X^{(\kappa)} Y. \tag{3.4}$$

Let M be a real hypersurface of type (A) in $M_2(c)$, then the shape operator of M is given by

$$A\xi = \alpha\xi \quad \text{and} \quad AZ = \rho Z, \quad \text{for any } Z \in \mathbf{D},$$

where α and ρ are constants. Then relation (2.3) for $X = \xi$ and $X = Z$, $Z \in \mathbf{D}$ due to the above relation of the shape operator implies

$$l\xi = 0 \quad \text{and} \quad lZ = \left(\frac{c}{4} + \alpha\rho\right)Z, \quad \text{for any } Z \in \mathbf{D}.$$

Relation (3.4) for $X = Y \in TM$, for $X = \xi$ and $Y \in \mathbf{D}$, for any $X \in \mathbf{D}$ and $Y = \xi$ and for any $X, Y \in \mathbf{D}$ is satisfied. So the structure Jacobi operator of real hypersurfaces of type (A) is parallel with respect to the generalized Tanaka-Webster connection.

Let M be a Hopf hypersurface with $A\xi = 0$ in $M_2(c)$, then the shape operator of M at some point P is given by

$$A\xi = 0, \quad AZ = tZ \quad \text{and} \quad AZ_1 = t_1 Z_1,$$

where t, t_1 are non-constant functions. Then relation (2.3) for $X = \xi$ and $Y \in \{Z, Z_1\}$ because of the above relation yields

$$l\xi = 0, \quad lZ = \frac{c}{4}Z \quad \text{and} \quad lZ_1 = \frac{c}{4}Z_1.$$

Then, following the same combinations as in the case of real hypersurfaces of type (A), it is proved that Hopf hypersurfaces with $A\xi = 0$ have parallel structure Jacobi operator with respect to the generalized Tanaka-Webster connection.

Corollary. *Let M be a real hypersurface in $M_2(c)$ with $\alpha \neq 2k$ whose structure Jacobi operator is parallel with respect to the generalized Tanaka-Webster connection and also satisfies* (1.3). *Then M is locally congruent to*

i) *a real hypersurface of type* (A)

ii) *or to a Hopf hypersurface with $A\xi = 0$, which in case of $\mathbf{C}P^2$ is a non-homegeneous real hypersurface considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve.*

3.1. Real hypersurfaces in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator satisfies relation (1.2)

3.1.1. Hopf hypersurfaces in non-flat complex space forms

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator satisfies relation (1.2) and $\alpha \neq 2k$. Relation (3.1) also holds. We consider two cases

Case I: $\alpha^2 + c \neq 0$.

In this case relations of Theorem 2.2 and remark 2.3 hold. Thus, following similar steps to those as in the proof of three dimensional Hopf hypersurfaces we obtain

$$\alpha(\lambda - \nu) = 0. \tag{3.5}$$

Case II: $\alpha^2 + c = 0$.

In this case the ambient space is $\mathbf{C}H^n$ and $\alpha \neq 0$. Suppose that $\lambda \neq \frac{\alpha}{2}$ then $A\varphi W = \nu\varphi W$ and (2.5) results in $\nu = \frac{\alpha}{2}$. The same steps as in the previous case lead to $\alpha = 2k$, which is a contradiction.

Therefore, $\lambda = \frac{\alpha}{2}$ is the only eigenvalue for all vector fields in $\mathbf{D}$ and M is locally congruent to a horosphere. Therefore, due to Theorem 1.1, relation (3.5) and Theorem 2.1 we obtain

Proposition 3.3. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection and $\alpha \neq 2k$. Then M is locally congruent*

i) *to a real hypersurface of type* (A)

ii) *or to a real hypersurface with $A\xi = 0$.*

Due to remark 3.2 we also obtain the following Proposition

Proposition 3.4. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is parallel with respect to the generalized Tanaka-Webster connection and $\alpha \neq 2k$. Then M is locally congruent*

i) *to a real hypersurface of type (A)*

ii) *or to a real hypersurface with $A\xi = 0$.*

3.1.2. Ruled hypersurfaces in non-flat complex space forms

A ruled real hypersurface in $M_n(c)$, $n \geq 2$, is a real hypersurface such that $\mathbf{D}$ is integrable and its integral manifold is $M_{n-1}(c)$ (see [6] and [7]). Thus, the shape operator of a ruled real hypersurface satisfies the following relations

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi \quad \text{and} \quad AZ = 0 \quad \text{for any } Z \text{ orthogonal to } \mathrm{span}\{\xi, U\},$$

where $\beta \neq 0$. The inner product of Codazzi equation for $X = \xi$ and $Y \in \{U, \varphi U\}$ with U, φU and $Z \in \mathbf{D}_U$, for $n \geq 3$, which is the orthogonal complement of $\mathrm{span}\{U, \varphi U, \xi\}$ yields

$$\beta^2 = \beta\kappa_1 + \frac{c}{4}, \quad \text{where } \kappa_1 = g(\nabla_U U, \varphi U), \tag{3.6}$$

$$g(\nabla_U U, Z) = 0, \quad \text{for any } Z \in \mathbf{D}_U, \tag{3.7}$$

$$g(\nabla_{\varphi U} U, \varphi U) = 0, \tag{3.8}$$

$$g(\nabla_{\varphi U} U, Z) = 0, \quad \text{for any } Z \in \mathbf{D}_U, \tag{3.9}$$

$$(\varphi U)\beta = \beta^2 + \frac{c}{4}. \tag{3.10}$$

The following relations, taking into account the second of (2.1) and relations (3.7), (3.8), (3.9), $g(\nabla_U U, U) = g(\nabla_U U, \xi) = 0$ and $g(\nabla_{\varphi U} U, U) = g(\nabla_{\varphi U} U, \xi) = 0$, hold

$$\nabla_U U = \kappa_1 \varphi U, \quad \nabla_U \varphi U = -\kappa_1 U, \quad \nabla_{\varphi U} U = \nabla_{\varphi U} \varphi U = 0. \tag{3.11}$$

Moreover, the structure Jacobi operator (2.3) becomes

$$l\xi = 0, \quad lU = \left(\frac{c}{4} - \beta^2\right) U \quad \text{and} \tag{3.12}$$
$$lZ = \frac{c}{4} Z, \quad \text{for any } Z \text{ orthogonal to } U \text{ and } \xi.$$

Let M be a ruled real hypersurface whose structure Jacobi operator satisfies relation (1.2). Relation (3.1) also holds. The inner product of relation (3.1) for $X = U$ and $Y = \xi$ due to the first of (2.1) and (3.12) with φU and $Z \in \mathbf{D}_U$ implies respectively

$$\kappa_3 = g(\nabla_\xi U, \varphi U) = k \quad \text{and} \quad g(\nabla_\xi U, Z) = 0.$$

So because of the above and the second of (2.1) we have

$$\nabla_\xi U = k\varphi U \quad \text{and} \quad \nabla_\xi \varphi U = -kU - \beta\xi. \tag{3.13}$$

Moreover, relation (3.1) for $X = \varphi U$ and $Y = U$ yields $(\varphi U)\beta = \frac{\kappa_1 \beta}{2}$. So relation (3.10) due to the latter and (3.6) implies $\beta^2 = -\frac{3c}{4}$. Differentiation of the latter with respect to φU implies $(\varphi U)\beta = 0$. So relation (3.10) implies $\beta^2 + \frac{c}{4} = 0$. Combination of the latter with $\beta^2 = -\frac{3c}{4}$ results in $c = 0$ which is a contradiction. So the following Proposition is proved

PROPOSITION 3.5. *There are no ruled real hypersurfaces in* $M_n(c)$, $n \geq 2$, *whose structure Jacobi operator satisfies relation* (1.2).

4. Proof of Theorem 1.2

Let M be a three-dimensional real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relations (1.3) and (1.4). The last one implies

$$g(\nabla_X lY - \nabla_{lY} X - l\nabla_X Y + l\nabla_Y X, Z) = 0, \quad X, Y, Z \in \mathbf{D}. \tag{4.1}$$

We consider the open subset $\mathcal{N}$ of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ relation (2.8) holds. Moreover, relation (1.3) for $X = \xi$ implies that $lU = 0$.

Relation (4.1) for $X = \varphi U$ and $Y = U$ due to the latter and relations of Lemma 2.4 implies

$$\kappa_2 g(l\varphi U, Z) = 0, \quad Z \in \mathbf{D}.$$

Suppose that $\kappa_2 \neq 0$ then $l\varphi U = 0$ and this results in $l = 0$, which due to Lemma 9 in [4] is impossible.

So on $\mathcal{N}$ $\kappa_2 = 0$ and since $lU = 0$ the first of (2.9) implies

$$\alpha\delta = 0 \quad \text{and} \quad \frac{c}{4} + \alpha\gamma = \beta^2. \tag{4.2}$$

Suppose that $\alpha \neq 0$ then $\delta = 0$. Relation (4.1) for $X = U$ and $Y = \varphi U$ due to (4.2), the second of (2.9), relations of Lemma 2.4, $\kappa_2 = 0$ and $lU = l\xi = 0$ yields

$$U(\alpha\mu)g(\varphi U, Z) - \kappa_1\left(\frac{c}{4} + \alpha\mu\right)g(U, Z) = 0.$$

For $Z = U$ the above implies that $\kappa_1\left(\frac{c}{4} + \alpha\mu\right) = 0$. If $\kappa_1 \neq 0$ then $\left(\frac{c}{4} + \alpha\mu\right) = 0$ and the second of (2.9) results in $l\varphi U = 0$. The latter implies that $l = 0$ and due to Lemma 9 in [4] we have a contradiction.

So $\kappa_1 = 0$. Differentiation of the second one of (4.2) with respect to φU taking into account relations (2.14), (2.15) and (2.16) implies

$$\gamma\mu + \gamma\kappa_3 = c. \tag{4.3}$$

Since $\kappa_2 = 0$ relations (2.13) and (2.17) implies

$$\xi\mu = U\mu = 0.$$

The Lie bracket $[U, \xi]\mu$ is given by $[U, \xi]\mu = U(\xi\mu) - \xi(U\mu) = 0$. On the other hand, because of Lemma 2.4 we obtain $[U, \xi]\mu = (\nabla_U \xi - \nabla_\xi U)\mu = (\gamma - \kappa_3)(\varphi U)\mu$. Combination of the above relations results in $(\gamma - \kappa_3)(\varphi U)\mu = 0$.

Suppose that $(\varphi U)\mu \neq 0$ then $\gamma = \kappa_3$ and relation (2.18) taking into account (4.3) implies $\gamma\mu = -c$. Substitution of the last one in (4.3) implies $\gamma^2 = 2c$. Moreover, relation (2.11) yields $\gamma = 0$. So we conclude that $c = 0$, which is impossible. Thus, relation $(\varphi U)\mu = 0$ holds.

Relation (4.1) for $X = Y = Z = \varphi U$ because of the latter implies $\mu(\varphi U)\alpha = 0$. Suppose that $(\varphi U)\alpha \neq 0$ then $\mu = 0$ and relation (2.11) results in $\gamma\kappa_3 = 0$. So (4.3) leads to $c = 0$, which is a contradiction. Thus $(\varphi U)\alpha = 0$ and because of (2.14) we obtain $\kappa_3 = 3\mu - \alpha$. Substitution of the last one in (4.3) implies $(4\mu - \alpha)\gamma = c$. Differentiation of the last one with respect to φU because of $(\varphi U)\mu = (\varphi U)\alpha = 0$ yields $(4\mu - \alpha)(\varphi U)\gamma = 0$. If $(\varphi U)\gamma \neq 0$ then $4\mu = \alpha$ and $\kappa_3 = -\mu$. On the other hand, relation (2.19) since $(\varphi U)\kappa_3 = 0$ implies $\kappa_3 = \mu = 0$ and relation (4.3) leads to $c = 0$, which is impossible.

So we have $(\varphi U)\gamma = 0$ and (2.16) implies $\gamma = -2\mu$. Relation (2.11) because of the relations for γ and κ_3 and (4.3) implies $\mu(11\mu - 3\alpha) = 0$. If $11\mu - 3\alpha \neq 0$ then $\mu = 0$ and this results in $\gamma = 0$. So $\beta^2 = \frac{c}{4}$ and differentiating the latter with respect to φU taking into account (2.15) we obtain $c = 0$, which is a contradiction.

So $\alpha = \frac{11\mu}{3}$ and $\kappa_3 = -\frac{2\mu}{3}$. So (2.18) implies $c = \frac{10\mu^2}{3}$. Substitution of the relations for γ, κ_3 and c in (4.3) yields $\mu = 0$ and this results in $c = 0$, which is a contradiction.

Therefore, on $\mathcal{N}$ we have $\alpha = 0$ and the second of (4.2) implies $\beta^2 = \frac{c}{4}$. Moreover, relation (2.14) implies $\kappa_3 = 3\mu$. Relation (4.1) for $X = U$, $Y = \varphi U$ and $Z = U$ yields $\kappa_1 = 0$. So relations (1.3) and (2.11) taking into account the previous results yield $\mu(4\gamma - 3\mu) = 0$. If $\mu \neq 0$ then $\gamma = \frac{3\mu}{4}$ and relation (2.15) implies $c = 3\mu^2$. Thus, relation (2.18) yields $\mu = 0$, which is a contradiction.

So $\mu = 0$ and this results in $\kappa_3 = 0$. Therefore, relation (2.18) implies $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following Proposition is proved

PROPOSITION 4.1. *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relations* (1.3) *and* (1.4) *is Hopf.*

Due to the above Proposition we have that M is a Hopf hypersurface and relations of Theorem 2.2 and Remark 2.6 hold. Relation (4.1) for $X = \varphi W$, $Y = W$ and $Z = \varphi W$ due to (2.7) implies

$$\alpha(\lambda - \nu)g(\nabla_{\varphi W} W, \varphi W) = 0.$$

Suppose that $\alpha(\lambda - \nu) \neq 0$ then $g(\nabla_{\varphi W} W, \varphi W) = 0$. Moreover, relation (4.1) for $X = W$, $Y = \varphi W$ and $Z = W$ implies $g(\nabla_W \varphi W, W) = 0$. The inner product of Codazzi equation for $X = \xi$ and $Y = W$ with φW because of (2.5) implies $g(\nabla_\xi W, \varphi W) = \frac{\alpha}{2}$. Combination of the Gauss equation with the formula for the Riemannian curvature for $X = W$, $Y = \varphi W$ and $Z = W$ taking into account

$$\nabla_{\varphi W} W = \nu\xi, \quad \nabla_W \varphi W = -\lambda\xi, \quad \text{and} \quad \nabla_{\varphi W}\varphi W = \nabla_W W = 0,$$

implies $\lambda\nu = -\frac{c}{4}$ and because of (2.5) we have that $\alpha(\lambda + \nu) = -c$. Thus λ, ν are constant and since $\lambda \neq \nu$ we have that M is locally congruent to a real hypersurface of type (B).

Let $\alpha(\lambda - \nu) = 0$. If $\alpha = 0$ in case of $\mathbf{C}P^2$ we have two cases: 1) if $\lambda \neq \nu$ then M is locally congruent to a non-homegeneous real hypersurface considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve, 2) if $\lambda = \nu$ then M is locally congruent to a geodesic hypersphere of radius $r = \frac{\pi}{4}$. In case of $\mathbf{C}H^2$ M is a Hopf hypersurface with $A\xi = 0$ (for the construction of such real hypersurfaces see [5]).

If $\alpha \neq 0$ then $\lambda = \nu$. The latter implies that

$$(A\varphi - \varphi A)X = 0$$

for any X tangent to M. So due to Theorem 2.1 M is locally congruent to a real hypersurface of type (A).

Conversely, it will be proved that real hypersurfaces of type (A), real hypersurfaces of type (B) and Hopf hypersurfaces with $A\xi = 0$ in $M_2(c)$ satisfy relation (1.4).

Let M be a real hypersurface of type (A) in $M_2(c)$, then the shape operator of M is given by

$$A\xi = \alpha\xi \quad \text{and} \quad AZ_1 = \rho Z_1, \quad \text{for any } Z_1 \in \mathbf{D},$$

where α and ρ are constants. Then relation (2.3) for $X = \xi$ and $X = Z_1$, $Z_1 \in \mathbf{D}$ due to the above relation of the shape operator implies

$$l\xi = 0 \quad \text{and} \quad lZ_1 = \left(\frac{c}{4} + \alpha\rho\right)Z_1, \quad \text{for any } Z_1 \in \mathbf{D}.$$

Relation (1.4) yields relation (4.1). The last one taking into account the above relation for the structure Jacobi operator is satisfied for all the possible combinations of X, Y and $Z \in \mathbf{D}$.

Let M be a real hypersurface of type (B), then the shape operator of M with respect to an orthonormal basis $\{\xi, Z_1, Z_2\}$ is given by

$$A\xi = \alpha\xi, \quad AZ_1 = s_1 Z_1 \quad \text{and} \quad AZ_2 = s_2 Z_2,$$

where α, s_1 and s_2 are constants. Relation (2.3) for $X = \xi$, $X = Z_1$ and $X = Z_2$ because of the above relation of the shape operator implies

$$l\xi = 0, \quad lZ_1 = \left(\frac{c}{4} + \alpha s_1\right) Z_1 \quad \text{and} \quad lZ_2 = \left(\frac{c}{4} + \alpha s_2\right) Z_2.$$

Relation (4.1) taking into account the above relation for the structure Jacobi operator is satisfied for all the possible combinations of X, Y and $Z \in \{Z_1, Z_2\}$.

Let M be a Hopf hypersurface with $A\xi = 0$ in $M_2(c)$, then the shape operator of M at some point P is given by

$$A\xi = 0, \quad AZ_1 = t_1 Z_1 \quad \text{and} \quad AZ_2 = t_2 Z_2,$$

where t_1, t_2 are non-constant functions. Then relation (2.3) for $X = \xi$ and $X \in \{Z_1, Z_2\}$ because of the above relation yields

$$l\xi = 0, \quad lZ_1 = \frac{c}{4} Z_1 \quad \text{and} \quad lZ_2 = \frac{c}{4} Z_2.$$

Then following the same combinations as in the case of real hypersurfaces of type (A) it is proved that Hopf hypersurfaces with $A\xi = 0$ satisfy relation (1.4).

4.1. Real hypersurfaces in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator sastisfies relation (1.4)

4.1.1. Hopf hypersurfaces in non-flat complex space forms

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator satisfies relation (1.4). Relation (4.1) also holds. We consider two cases.

Case I: $\alpha^2 + c \neq 0$.

In this case relations of Theorem 2.2 and remark 2.3 hold. Following similar steps to those of the case of three dimensional real hypersurfaces we obtain

$$g(\nabla_{\varphi W} W, \varphi W) = \kappa_2 = 0 \quad \text{and} \quad g(\nabla_W \varphi W, W) = \kappa_1 = 0.$$

The inner product of Codazzi equation for $X = \xi$ and $Y = W$ with φW due to (2.5) yields

$$\kappa_3 = g(\nabla_\xi W, \varphi W) = \frac{\alpha}{2}.$$

The inner product of Codazzi equation for $X = W$ and $Y = \varphi W$ with any $W_1 \in \mathbf{D}_W$, if $n \geq 3$, which is the orthogonal complement of the $span\{W, \varphi W, \xi\}$ yields

$$(4.4)\qquad vg(\nabla_W \varphi W, W_1) - \lambda g(\nabla_{\varphi W} W, W_1) = g(A\nabla_W \varphi W - A\nabla_{\varphi W} W, W_1).$$

Furthermore, the inner product of (4.1) for $X = W$, $Y = \varphi W$ and $Z = W_1$ due to relation (4.4) and relation (2.3) for $X = \nabla_W \varphi W$ and $X = \nabla_{\varphi W} W$, since $\lambda \neq v$ leads to

$$g(\nabla_{\varphi W} W, W_1) = 0, \quad \text{for any } W_1 \in \mathbf{D}_W.$$

Because of the latter and the fact that $\nabla_{\varphi W} W$ has no component on W and φW we conclude that

$$(4.5)\qquad \nabla_{\varphi W} W = v\xi \quad \text{and} \quad \nabla_{\varphi W} \varphi W = 0.$$

The inner product of (4.1) for $X = \varphi W$, $Y = W$ and $Z = W_1$ due to relation (4.4) and relation (2.3) for $X = \nabla_W \varphi W$ and $X = \nabla_{\varphi W} W$, since $\lambda \neq v$ leads to

$$g(\nabla_W \varphi W, W_1) = 0, \quad \text{for any } W_1 \in \mathbf{D}_W.$$

Due to the latter and the fact that $\nabla_W \varphi W$ has no component on W and φW we conclude that

$$(4.6)\qquad \nabla_W \varphi W = -\lambda\xi \quad \text{and} \quad \nabla_W W = 0.$$

The Riemannian curvature tensor is given by the relation

$$(4.7)\qquad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Relation (4.7) for $X = W$, $Y = \varphi W$ and $Z = W$ because of relations (4.5) and (4.6) yields

$$(4.8)\qquad R(W, \varphi W)W = (Wv)\xi + v\lambda\varphi W + (\lambda + v)\nabla_\xi W.$$

On the other hand from the Gauss equation for $X = W$, $Y = \varphi W$ and $Z = W$ we obtain

$$(4.9)\qquad R(W, \varphi W)W = -(\lambda v + c)\varphi W.$$

The combination of the inner product of the above two relations with $W_1 \in \mathbf{D}_W$ implies

$$(\lambda + v)g(\nabla_\xi W, W_1) = 0.$$

Suppose that $g(\nabla_\xi W, W_1) \neq 0$ then the above relation gives $\lambda + v = 0$. Combining the inner product of (4.8) and (4.9) with φW results in $\lambda v = -\frac{c}{2}$. Substituting the previous two relations in (2.4) leads to $c = 0$, which is a contradiction. Therefore, on M we have that $g(\nabla_\xi W, W_1) = 0$, which implies that $\nabla_\xi W$ has no component on $\mathbf{D}_W$. Therefore, since $\kappa_3 = \frac{\alpha}{2}$ the following relations hold

$$\nabla_\xi W = \frac{\alpha}{2}\varphi W \quad \text{and} \quad \nabla_\xi \varphi W = -\frac{\alpha}{2} W. \tag{4.10}$$

The combination of the inner product of relations (4.8) and (4.9) with φW because of (4.10) implies $2\lambda\nu + \frac{\alpha}{2}(\lambda + \nu) = -c$. The last one due to (2.4) results in $\lambda\nu = -\frac{c}{4}$. Substitution of the previous one in (2.4) implies $\alpha(\lambda + \nu) = -c$. Therefore, the real hypersurface has at least three distinct constant principal curvatures. Substitution of the principal curvatures in $\lambda\nu = -\frac{c}{4}$ implies that only type (B) satisfies this relation.

If $\alpha(\mu - \nu) = 0$ holds, which implies that either M is a Hopf hypersurface with $\alpha = 0$ or $\mu = \nu$. The last one results in $(A\varphi - \varphi A)X = 0$, X tangent to M.

Case II: $\alpha^2 + c = 0$.

In this case the ambient space is $\mathbf{C}H^n$ and $\alpha \neq 0$. Suppose that $\lambda \neq \frac{\alpha}{2}$ then $A\varphi W = \nu\varphi W$ and (2.5) results in $\nu = \frac{\alpha}{2}$. Following similar steps as in the previous case gives a contradiction.

Therefore, $\lambda = \frac{\alpha}{2}$ is the only eigenvalue for all vector fields in $\mathbf{D}$ and M is locally congruent to a horosphere. Therefore, due to Theorem 1.2, relation $(A\varphi - \varphi A)X = 0$ and Theorem 2.1 we obtain

Proposition 4.2. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator satisfies relation* (1.4). *Then, M is locally congruent*

i) *to a real hypersurface of type (A)*

ii) *or to a Hopf hypersurface with $A\xi = 0$,*

iii) *or to a real hypersurface of type (B), i.e. in case of $\mathbf{C}P^n$ a tube of radius $r \in \left(0, \frac{\pi}{4}\right)$ around the complex quadric Q^{n-1} and in case of $\mathbf{C}H^n$ a tube of some radius r around the canonically (totally geodesic) embedded n-dimensional real hyperbolic space.*

4.2. Ruled hypersurfaces in non-flat complex space forms

Let M be a ruled real hypersurface whose structure Jacobi operator satisfies relation (1.4) and also relation (4.1) holds. Furthermore, the relations (3.6)–(3.12) are satisfied. Relation (4.1) for $X = U$, $Y = \varphi U$ and $Z = U$ due to (3.11) and (3.12), results in $\kappa_1 = 0$. Combination of the Gauss equation for $X = U$, $Y = \varphi U$ and $Z = U$ due to $\kappa_1 = 0$ with the definition of the Riemannian curvature $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$ implies $c = 0$, which is impossible. Thus we have proved the following Proposition.

Proposition 4.3. *There are no ruled real hypersurfaces in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator satisfies relation* (1.4).

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