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CONSERVATION OF THE MASS FOR SOLUTIONS TO A CLASS OF SINGULAR PARABOLIC EQUATIONS

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Abstract

In this paper we deal with the Cauchy problem associated to a class of quasilinear singular parabolic equations with L^∞ coefficients, whose prototypes are the p-Laplacian $\left(\frac{2N}{N+1} < p < 2\right)$ and the Porous medium equation $\left(\left(\frac{N-2}{N}\right)_+ < m < 1\right)$. In this range of the parameters p and m, we are in the so called fast diffusion case. We prove that the initial mass is preserved for all the times.

1. Introduction

Let us consider the following homogeneous quasilinear parabolic equation

$$
\begin{cases} u_t = \operatorname{div} A(x,t,u,Du), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u(0,x) = \mu \end{cases} \tag{1.1}
$$

where μ is a nonnegative Radon measure with finite mass and compact support and the functions $A := (A_1,\dots,A_N)$ are assumed to be only measurable in $(x,t) \in \mathbf{R}^N \times [0,+\infty)$, continuous with respect to u and Du for almost all (x,t). For p-Laplacian type equation we let A satisfy the following structure conditions:

$$
\begin{cases} A(x,t,u,\eta)\cdot\eta \geq c_0|\eta|^p, \\ |A(x,t,u,\eta)| \leq c_1|\eta|^{p-1}, \end{cases} \tag{1.2}
$$

for almost all $(x,t) \in \mathbf{R}^N \times [0,+\infty)$ and $(u,\eta) \in \mathbf{R} \times \mathbf{R}^N$ with

$$
\frac{2N}{N+1} < p < 2 \tag{1.3}
$$

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(supercritical range of the fast diffusion case) and c_0, c_1 are given positive constants. Moreover, we assume that there exists $L > 0$ such that

$$\begin{cases} (A(x,t,u,\eta_1) - A(x,t,u,\eta_2)) \cdot (\eta_1 - \eta_2) \geq 0, \\ |A(x,t,u_1,\eta) - A(x,t,u_2,\eta)| \leq L|u_1 - u_2|(1 + |\eta|^{p-1}), \end{cases} \tag{1.4}$$

for almost all $(x,t) \in \mathbf{R}^N \times [0,+\infty)$ and all $u, u_i \in \mathbf{R}$ and $\eta, \eta_i \in \mathbf{R}^N$, $i = 1,2$. For Porous medium type equation we follow the notation of [6] chapter 7, section 5. Let

$$u_t = \operatorname{div} A(x,t,u,D(|u|^{m-1}u)), \quad (x,t) \in \mathbf{R}^N \times [0,+\infty), \tag{1.5}$$

where A is required to satisfy the following conditions

$$\begin{cases} A(x,t,u,\eta) \cdot \eta \geq c_0'|\eta|^2, \\ |A(x,t,u,\eta)| \leq c_1'|\eta|, \end{cases} \tag{1.6}$$

for almost all $(x,t) \in \mathbf{R}^N \times [0,+\infty)$ and $(u,\eta) \in \mathbf{R} \times \mathbf{R}^N$ with

$$\left(\frac{N-2}{N}\right)_+ < m < 1. \tag{1.7}$$

We assume the following monotonicity and Lipschitz conditions

$$\begin{cases} (A(x,t,u,\eta_1) - A(x,t,u,\eta_2)) \cdot (\eta_1 - \eta_2) \geq 0, \\ |A(x,t,u_1,\eta) - A(x,t,u_2,\eta)| \leq L' \big| |u_1|^{m-1}u_1 - |u_2|^{m-1}u_2 \big| (1 + |\eta|), \end{cases} \tag{1.8}$$

that are sufficient to have a comparison principle and to preserve the positivity of solutions.

Actually, we remark that hypotheses (1.4) and (1.8) not only imply a comparison principle for weak solution of (1.1), but also guarantee the existence of the solution. See, for instance, [10] and [14]). We recall that in [14] it is proved the existence of the weak solutions of (1.1) with a Dirac mass instead of a generic Radon measure as here and, in that specific case, sharp pointwise estimates from above and from below are proved.

The aim of this paper is to prove that under these assumptions the mass is preserved. This result is known for the prototype equations with initial data a Dirac mass (where the explicit solutions are known) but, as far we know, not for general equations.

Let us recall the fundamental solutions of the prototype equations.

First, let us consider the p-Laplacian equation. It is known that for some positive constant C_p (see, for instance, chapter 11, section 4 of [15] and [16]) the function

$$\mathscr{B}_p = t^{-N/\lambda}\left[C_p + \gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right]^{-(p-1)/(2-p)} \tag{1.9}$$

with

$$\gamma_p = \left(\frac{1}{\lambda}\right)^{1/(p-1)} \frac{2-p}{p} \quad \text{and} \quad \lambda = N(p-2) + p \tag{1.10}$$

is the solution of the following Cauchy problem in $\mathbf{R}^N \times (t > 0)$

$$\begin{cases} u_t = \operatorname{div}(|Du|^{p-2}Du), \\ u(x,0) = \delta(0). \end{cases}$$

Analogously, for the Porous Medium equation it is known that for some positive constant C_m (see, for instance, chapter 2, section 1 of [15] and [16]), the function

$$\mathscr{B}_m = t^{-N/\kappa}\left[C_m + \gamma_m\left(\frac{|x|}{t^{1/\kappa}}\right)^2\right]^{-1/(1-m)} \tag{1.11}$$

with

$$\gamma_m = \left(\frac{1}{\kappa}\right)\frac{1-m}{2} \quad \text{and} \quad \kappa = N(m-1)+2$$

is the solution of the following Cauchy problem in $\mathbf{R}^N \times (t > 0)$

$$\begin{cases} u_t = \Delta(u^m), \\ u(x,0) = \delta(0). \end{cases}$$

Let us now introduce the usual definition of a weak solution. A locally bounded, non-negative function $u(x,t)$ is a solution of (1.1) in $\mathbf{R}^N \times \mathbf{R}^+$, if

$$u \in C(\mathbf{R}^+; L^2(\mathbf{R}^N)) \cap L^p(\mathbf{R}^+; W^{1,p}(\mathbf{R}^N)),$$

and for every subinterval $[t_1, t_2] \subset \mathbf{R}^+$

$$\int_{\mathbf{R}^N} u\phi\, dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbf{R}^N}(-u\phi_t + A(x,t,u,Du)\cdot D\phi)\, dxdt = 0,$$

for all test functions $\phi \in W^{1,2}(\mathbf{R}^+; L^2(\mathbf{R}^N)) \cap L^p(\mathbf{R}^+; W^{1,p}(\mathbf{R}^N))$.

We use this definition of solution, because u_t may have a modest degree of regularity and in general has meaning only in the sense of distributions (see, for instance, [5] and [6]).

Notice that the explicit fundamental solutions have less regularity than what we required in the previous definition. In general, when the initial datum is a measure, the gradient of the solution belongs only to the Marcinkiewicz space of order $\dfrac{N(p-1)}{N-1}$. However, the gradient of the solution raised to the power $(p-1)$ belongs to the Marcinkiewicz space of order $\dfrac{N}{N-1}$ and therefore, to L^1. Hence, a distributional solution is well defined. For a more refined theory, see [1] and [2] for the definition of entropy solutions, and see [4] and [12] for the definition of renormalized solutions.

Keeping this in mind and following the approach of [9] and [13], we define the notion of weak solution of (1.1) in the case of an initial datum measure.

A non-negative function $u(x,t)$ is a weak solution of (1.1) if the following assumptions are satisfied

- $u \in C(\mathbf{R}^+; L^1(\mathbf{R}^N))$,
- for any $s > 0$, $u(x,t)$ is a weak solution of (1.1) in $\mathbf{R}^N \times [s, +\infty)$
- for any $\phi \in C^\infty(\mathbf{R}^N)$ and with compact support, we have

$$\lim_{t \to 0} \int_{\mathbf{R}^N} u(x,t)\phi(x)\,dx = \int_{\mathbf{R}^N} \phi(x)\,d\mu.$$

We are now able to state our main Theorem concerning p-Laplacian type equations:

THEOREM 1.1 (p-Laplacian type). *Let u be a nonnegative solution of*

$$\begin{cases} u_t = \operatorname{div} A(x,t,u,Du), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u(x,0) = \mu, & x \in \mathbf{R}^N, \end{cases} \tag{1.12}$$

with A satisfying (1.2), (1.4), *p in the supercritical range* (1.3) *and μ a nonnegative Radon measure with finite mass and with compact support. Then for any $t > 0$*

$$\int_{\mathbf{R}^N} u(x,t)\,dx = \int_{\mathbf{R}^N} d\mu. \tag{1.13}$$

Let us consider Porous medium type equation. Analogously to p-Laplacian type equation, we can introduce the definitions of weak solution. For the sake of readability of the paper, we omit these definitions and we refer the reader to [6], [13], [14] and [16] for all the necessary details.

THEOREM 1.2 (Porous medium type). *Let u be a nonnegative solution of*

$$\begin{cases} u_t = \operatorname{div} A(x,t,u,Du^m), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u(x,0) = \mu, & x \in \mathbf{R}^N, \end{cases} \tag{1.14}$$

with A satisfying (1.6), (1.8), *m in the supercritical range* (1.7) *and μ a nonnegative Radon measure with finite mass and with compact support. Then for any $t > 0$*

$$\int_{\mathbf{R}^N} u(x,t)\,dx = \int_{\mathbf{R}^N} d\mu. \tag{1.15}$$

The existence of weak solutions to such kind of equations is guaranteed by Theorem 1.3 proved in [14].

The proofs in this paper are based on L^1-L^1 estimates and L^∞-L^1 estimates. We recall that the L^1-L^1 estimates are not only a kind of integral Harnack estimates, but they also give a sharp quantitative estimate on the speed of the propagation of the solution.

We will prove only Theorem 1.1, i.e. we will consider only the p-Laplacian type case. We will refer the reader to the recent monograph [6], in order to see how to extend these results to the case of Porous medium type equation.

Note that below the critical values, i.e. for $1 < p \leq \dfrac{2N}{N+1}$ and $0 < m \leq \left(\dfrac{N-2}{N}\right)_+$, the extinction of the solution occurs, so we don't have conservation of the mass.

In the supercritical range, i.e. $\dfrac{2N}{N+1} < p < 2$ and $\left(\dfrac{N-2}{N}\right)_+ < m < 1$ the initial datum needs to have a finite mass, otherwise the solution becomes $+\infty$ at any positive time (for these results, see for instance [15]).

In the degenerate case, i.e. $p > 2$ and $m > 1$ there is the finite propagation of the support, so the mass is clearly conserved.

As for the uniqueness of the solutions to (1.12) and (1.14), we recall that this issue was considered for the p-Laplacian equation in [9] for $p > 2$ and for the Porous medium equation in [8] in the case $N = 1$ and in [13] for any N and in a more general setting. The extension of the uniqueness result to our case seems to be not trivial at all, also because the uniqueness would depend on the choice of the definition of solution (renormalized, entropy, distributional).

The paper is organized in this way: in §2, we collect some known results to be used in the proofs of our results.

In §3, estimates from above are derived for the solutions of problems (1.12) and (1.14) Then we apply these estimates to deduce Theorem 1.1 and Theorem 1.2.

We stress that throughout the paper with γ we will denote constants depending only upon the data, i.e. for equation (1.12) depending only upon N, p, c_0, c_1 and for equation (1.14) upon N, m, c_0', c_1'.

2. Preliminaries

Let $B_\rho(x)$ denote as usual the euclidean ball in $\mathbf{R}^N$ with center at x and radius ρ, and set $B_\rho(0) = B_\rho$.

We use the following results along the paper:

THEOREM 2.1 (Local L^1 form of the Harnack inequality, [7]). *Let*

$$u \in C_{loc}(\mathbf{R}^+; L^2_{loc}(\mathbf{R}^N)) \cap L^p_{loc}(\mathbf{R}^+; W^{1,p}_{loc}(\mathbf{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbf{R}^N \times [0, +\infty)$ *and* $1 < p < 2$. *There exists a constant γ depending only upon the data, such that for all cylinders* $B_{2\rho}(y) \times [s, t] \subset \mathbf{R}^N \times [0, +\infty)$,

$$\sup_{s \leq \tau \leq t} \int_{B_\rho(y)} u(x, \tau)\, dx \leq \gamma \inf_{s \leq \tau \leq t} \int_{B_{2\rho}(y)} u(x, \tau)\, dx + \gamma \left(\frac{t-s}{\rho^\lambda}\right)^{1/(2-p)},$$

where $\lambda = N(p-2) + p$.

THEOREM 2.2 (L^1-L^∞ estimates, [7]). *Let*

$$u \in C_{loc}(\mathbf{R}^+; L^2_{loc}(\mathbf{R}^N)) \cap L^p_{loc}(\mathbf{R}^+; W^{1,p}_{loc}(\mathbf{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbf{R}^N \times [0, +\infty)$ *and assume* (1.3) *holds. There exists a constant* γ_1 *depending only upon the data such that for all cylinders* $B_{2\rho}(y) \times [s - (t-s), s + (t-s)] \subset \mathbf{R}^N \times [0, +\infty)$,

$$\sup_{B_\rho(y)\times[s,t]} u(x,\tau) \le \frac{\gamma_1}{(t-s)^{N/\lambda}} \left(\inf_{2s-t\le\tau\le t} \int_{B_{2\rho}(y)} u(x,\tau)\, dx \right)^{p/\lambda} + \gamma_1 \left(\frac{t-s}{\rho^p} \right)^{1/(2-p)}. \tag{2.1}$$

Note that this Theorem claims that if a solution is in L^1 at a certain time t_0, it is in L^∞ for any time $s > t_0$. Note that if a function v is the solution of (1.12) and it can be approximated by regular problems, by (2.1) one easily derives that $v \in C([s,\infty); L^2(\mathbf{R}^N)) \cap L^p((s,\infty); W^{1,p}(\mathbf{R}^N))$ for any $s > 0$.

The results of this section hold for Porous medium type equations (see Appendix B of the monograph [6]).

3. The case of compact support

The aim of this section is to get an estimate from above for the solution of the problem (1.12) assuming (1.2), (1.3) and (1.4) that there exists $R > 0$ such that the support of μ is contained in B_R.

Fix a positive time $T > 0$ and $x \in \mathbf{R}^N$ such that $|x| > 2R$ and $|x| \ge \left(\frac{T}{\gamma_2}\right)^{1/\lambda}$ where $\gamma_2 := \left(\frac{1}{2\gamma}\right)^{2-p}$ and γ is the constant defined in Theorem 2.1.

Lemma 3.1. *Let u be a weak solution of* (1.12) *under* (1.2)–(1.4). *Let $T > 0$. For all x with*

$$|x| > \left(\frac{T}{\gamma_2}\right)^{1/\lambda} \vee 2R, \tag{3.1}$$

we have

$$u(x,T) \le \gamma_1 4^{(2p-1)/2(2-p)} \left(\frac{T}{|x|^p}\right)^{1/(2-p)}. \tag{3.2}$$

Proof. Apply Theorem 2.2 with $s = \frac{T+\varepsilon}{2}$, $t = T$, $\rho = \frac{|x|}{4}$ to get

$$u(x,T) \le \sup_{y\in B_\rho(x)} u(y,T) \le \gamma_1 2^{N/\lambda} (T-\varepsilon)^{-N/\lambda} \left(\inf_{\varepsilon\le t\le T} \int_{B_{2\rho}(x)} u(y,t)\, dy \right)^{p/\lambda} + \gamma_1 2^{-1/(2-p)} \left(\frac{T-\varepsilon}{\rho^p} \right)^{1/(2-p)}. \tag{3.3}$$

Taking into account that the function μ has the mass concentrated in the ball B_R and that by hypothesis (3.1) the balls $B_{2\rho}(x)$ and B_R have empty intersection, we get that $\lim_{\varepsilon\to 0}\left(\int_{B_{2\rho}(x)} u(y,\varepsilon)\,dy\right)^{p/\lambda} = 0$. Therefore letting $\varepsilon \to 0$ we get (3.2).

Proof of Theorem 1.1. Fix $R_0 > 0$. By equation (1.12)

$$\frac{d}{dt}\int_{B_{R_0}} u(x,t)\,dx = \int_{B_{R_0}} \operatorname{div} A(x,t,u,Du)\,dx = \int_{\partial B_{R_0}} \langle A(\sigma,t,u,Du),\nu\rangle\,d\sigma \tag{3.4}$$

where ν is the external normal vector and the last equality comes from Gauss's theorem.

Applying (1.2) to (3.4) we get

$$\sup_{0\le t\le T}\left|\int_{B_{R_0}} u(x,t)\,dx - \int_{B_{R_0}} d\mu\right| \le c_1 \int_{\partial B_{R_0}}\int_0^T |Du(\sigma,t)|^{p-1}\,dtd\sigma$$

and hence

$$\sup_{0\le t\le T}\left|\int_{B_{R_0}} u(x,t)\,dx - \int_{B_{R_0}} d\mu\right| \le c_1\omega_N^{1/p}R_0^{(N-1)/p}T^{1/p}\left(\int_{\partial B_{R_0}}\int_0^T |Du(\sigma,t)|^p\,dtd\sigma\right)^{(p-1)/p}, \tag{3.5}$$

where ω_N is the surface area of unit sphere $\partial B(0,1)$, i.e., $\omega_N = \dfrac{N\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)}$.

Let $\zeta \in C^\infty(\mathbf{R}^N)$ be a cutoff function such that $0 \le \zeta \le 1$, $\zeta = 1$ in $B_{2R_0}\backslash B_{R_0/2}$, $\zeta = 0$ in $B_{R_0/4}$ and $\mathbf{R}^N\backslash B_{3R_0}$ and $|\nabla\zeta| \le \gamma_3\frac{1}{R_0}$.

Assume that $\frac{R_0}{4} > R$. If we multiply equation (1.12) by $\zeta^p u$ and integrate it in $\mathbf{R}^N \times [0,T]$, we get

$$\int_{\mathbf{R}^N}\int_0^T u_t\zeta^p u\,dsdx = \int_{\mathbf{R}^N}\int_0^T \operatorname{div} A(x,s,u,Du)\zeta^p u\,dsdx.$$

Using integration by parts, we have

$$\int_{\mathbf{R}^N}\zeta^p\int_0^T \frac{1}{2}\frac{d}{ds}(u^2)\,dsdx = -\int_{\mathbf{R}^N}\int_0^T A(x,s,u,Du)D(\zeta^p u)\,dsdx.$$

Then we can write,

$$\begin{aligned}\frac{1}{2}\int_{\mathbf{R}^N}\zeta^p u^2(x,T)\,dx-\frac{1}{2}\int_{\mathbf{R}^N}\zeta^p\lim_{t\to 0}u^2(x,t)\,dx\\=-\int_{\mathbf{R}^N}\int_0^T A(x,s,u,Du)\zeta^p Du\,dsdx\\-p\int_{\mathbf{R}^N}\int_0^T A(x,s,u,Du)u\zeta^{p-1}D\zeta\,dsdx.\end{aligned}$$

This is equivalent to the following if we consider the properties of ζ:

$$\begin{aligned}\int_{\mathbf{R}^N}\int_0^T A(x,s,u,Du)\zeta^p Du\,dsdx=\frac{1}{2}\lim_{t\to 0}\int_{\mathbf{R}^N}\zeta^p u^2(x,t)\,dx-\frac{1}{2}\int_{\mathbf{R}^N}\zeta^p u^2(x,T)\,dx\\-p\int_{B_{3R_0}\setminus B_{2R_0}}\int_0^T A(x,s,u,Du)u\zeta^{p-1}D\zeta\,dsdx\\-p\int_{B_{R_0/2}\setminus B_{R_0/4}}\int_0^T A(x,s,u,Du)u\zeta^{p-1}D\zeta\,dsdx.\end{aligned}$$

Now if we use (1.2), we obtain

$$\begin{aligned}c_0\int_{\mathbf{R}^N}\int_0^T|Du|^p\zeta^p\,dsdx\le\frac{1}{2}\lim_{t\to 0}\int_{\mathbf{R}^N}\zeta^p u^2(x,t)\,dx\\+pc_1\int_{B_{3R_0}\setminus B_{2R_0}}\int_0^T|Du|^{p-1}\zeta^{p-1}|u|D\zeta\,dsdx\\+pc_1\int_{B_{R_0/2}\setminus B_{R_0/4}}\int_0^T|Du|^{p-1}\zeta^{p-1}|u|D\zeta\,dsdx.\end{aligned}$$

Applying Young inequality on the right hand side, we get

$$\begin{aligned}c_0\int_{\mathbf{R}^N}\int_0^T|Du|^p\zeta^p\,dsdx\le\frac{1}{2}\lim_{t\to 0}\int_{\mathbf{R}^N}\zeta^p u^2(x,t)\,dx\\+\frac{c_0}{2}\int_{B_{3R_0}\setminus B_{2R_0}}\int_0^T|Du|^p\zeta^p\,dsdx\\+\left(\frac{2c_1(p-1)}{c_0}\right)^{p-1}\int_{B_{3R_0}\setminus B_{2R_0}}\int_0^T|u|^p|D\zeta|^p\,dsdx\\+\frac{c_0}{2}\int_{B_{R_0/2}\setminus B_{R_0/4}}\int_0^T|Du|^p\zeta^p\,dsdx\\+\left(\frac{2c_1(p-1)}{c_0}\right)^{p-1}\int_{B_{R_0/2}\setminus B_{R_0/4}}\int_0^T|u|^p|D\zeta|^p\,dsdx.\end{aligned}$$

Since $|\nabla\zeta| \le \gamma_3 \dfrac{1}{R_0}$, we have

$$c_0 \int_{\mathbf{R}^N} \int_0^T |Du|^p \zeta^p \, dsdx \le \frac{1}{2} \lim_{t\to 0} \int_{\mathbf{R}^N} \zeta^p u^2(x,t)\, dx + \frac{c_0}{2} \int_{\mathbf{R}^N} \int_0^T |Du|^p \zeta^p \, dsdx$$
$$+ \left(\frac{2c_1(p-1)}{c_0}\right)^{p-1} \frac{\gamma_3^p}{R_0^p}$$
$$\times \left[\int_{B_{3R_0} \setminus B_{2R_0}} \int_0^T |u|^p \, dsdx + \int_{B_{R_0/2} \setminus B_{R_0/4}} \int_0^T |u|^p \, dsdx \right].$$

Then,

$$\frac{c_0}{2} \int_{\mathbf{R}^N} \int_0^T |Du|^p \zeta^p \, dsdx \le \frac{1}{2} \lim_{t\to 0} \int_{\mathbf{R}^N} \zeta^p u^2(x,t)\, dx$$
$$+ \left(\frac{2c_1(p-1)}{c_0}\right)^{p-1} \frac{\gamma_3^p}{R_0^p}$$
$$\times \left[\int_{B_{3R_0} \setminus B_{2R_0}} \int_0^T |u|^p \, dsdx + \int_{B_{R_0/2} \setminus B_{R_0/4}} \int_0^T |u|^p \, dsdx \right].$$

Multiply both sides with $\dfrac{2}{c_0}$,

$$\int_{\mathbf{R}^N} \int_0^T |Du|^p \zeta^p \, dsdx \le \frac{1}{c_0} \lim_{t\to 0} \int_{\mathbf{R}^N} \zeta^p u^2(x,t)\, dx$$
$$+ \frac{2}{c_0} \left(\frac{2c_1(p-1)}{c_0}\right)^{p-1} \frac{\gamma_3^p}{R_0^p}$$
$$\times \left[\int_{B_{3R_0} \setminus B_{2R_0}} \int_0^T |u|^p \, dsdx + \int_{B_{R_0/2} \setminus B_{R_0/4}} \int_0^T |u|^p \, dsdx \right].$$

Since $\int_{B_{2R_0} \setminus B_{R_0/2}} \int_0^T |Du|^p \zeta^p \, dsdx \le \int_{\mathbf{R}^N} \int_0^T |Du|^p \zeta^p \, dsdx$ and $\zeta = 1$ in $B_{2R_0} \setminus B_{R_0/2}$,

$$\int_{B_{2R_0} \setminus B_{R_0/2}} \int_0^T |Du|^p \, dsdx \le \frac{1}{c_0} \lim_{t\to 0} \int_{\mathbf{R}^N} u^2(x,t)\, dx$$
$$+ \frac{2}{c_0} \left(\frac{2c_1(p-1)}{c_0}\right)^{p-1} \frac{\gamma_3^p}{R_0^p}$$
$$\times \left[\int_{B_{3R_0} \setminus B_{2R_0}} \int_0^T |u|^p \, dsdx + \int_{B_{R_0/2} \setminus B_{R_0/4}} \int_0^T |u|^p \, dsdx \right].$$

Let $\gamma_4 := \max\left(\frac{2}{c_0}, \frac{2}{c_0}\left(\frac{2c_1(p-1)}{c_0}\right)^{p-1}\gamma_3^p\right)$. Then we obtain,

$$\int_{B_{2R_0}\setminus B_{R_0/2}}\int_0^T |Du(x,s)|^p\,dsdx$$
$$\leq \frac{\gamma_4}{2}\lim_{t\to 0}\int_{\mathbf{R}^N} u^2(x,t)\,dx$$
$$+\frac{\gamma_4}{R_0^p}\left[\int_{B_{3R_0}\setminus B_{2R_0}}\int_0^T |u(x,s)|^p\,dsdx + \int_{B_{R_0/2}\setminus B_{R_0/4}}\int_0^T |u(x,s)|^p\,dsdx\right].$$

If we assume that $\frac{R_0}{4} > \left(\frac{T}{\gamma_2}\right)^{1/\lambda} \vee 2R$ we can apply Lemma 3.1 to get

$$\int_{B_{2R_0}\setminus B_{R_0/2}}\int_0^T |Du(x,s)|^p\,dsdx$$
$$\leq \frac{\gamma_5}{R_0^p}\left[\int_{B_{3R_0}\setminus B_{R_0/4}}\int_0^T \gamma_1^p 4^{(2p-1)p/2(2-p)}\left(\frac{s}{|R_0|^p}\right)^{p/(2-p)} dsdx\right]$$

and hence,

$$\int_{B_{2R_0}\setminus B_{R_0/2}}\int_0^T |Du(x,s)|^p\,dsdx \leq \gamma_6 T^{2/(2-p)} R_0^{N-2p/(2-p)}. \tag{3.6}$$

Therefore there exists $\tilde{R} \in \left[\frac{R_0}{2}, 2R_0\right]$ such that

$$\int_{\partial B_{\tilde{R}}}\int_0^T |Du(\sigma,s)|^p\,dsd\sigma \leq \gamma_6 T^{2/(2-p)} R_0^{N-1-2p/(2-p)}. \tag{3.7}$$

Combining together (3.5) and (3.7), we have

$$\sup_{0\leq t\leq T}\left|\int_{B_{\tilde{R}}} u(x,t)\,dx - \int_{B_{\tilde{R}}} d\mu\right| \leq \gamma_7 T^{1/(2-p)} R_0^{N-1-2(p-1)/(2-p)} \tag{3.8}$$

Therefore

$$\sup_{0\leq t\leq T}\left|\int_{\mathbf{R}^N} u(x,t)\,dx - \int_{\mathbf{R}^N} d\mu\right|$$
$$\leq \gamma_8\left[T^{1/(2-p)} R_0^{N-1-2(p-1)/(2-p)} + \int_{\mathbf{R}^N\setminus B_{R_0/2}}\int_0^T \left(\frac{t}{|x|^p}\right)^{1/(2-p)} dtdx\right].$$

As $N - 1 - \frac{2(p-1)}{2-p} < 0$, when $R_0 \to \infty$, we have that for any $0 < t < T$

$$\int_{\mathbf{R}^N} u(x,t)\, dx = \int_{\mathbf{R}^N} d\mu$$

and this implies Theorem 1.1 in the case of a measure μ with compact support.

Remark. Note that to prove the conservation of the mass we did not used condition (1.4). This condition is necessary only to have the existence of the solutions.

Remark. **The case of an initial datum in L^1 not necessarily with compact support**.

Let us consider the case of u nonnegative solution of

$$\begin{cases} u_t = \operatorname{div} A(x,t,u,Du), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u(x,0) = f, & x \in \mathbf{R}^N, \end{cases}$$

with A satisfying (1.2), (1.4), and p in the supercritical range (1.3) and $f \in L^1$ (not necessarily with compact support). In order to apply the previous result, for each $n \in \mathbf{N}$ let us introduce the function $I_n(x)$ equal to 1 if $x \in B_n$ and equal to 0 otherwise. Let $f_n(x) = f(x)I_n(x)$ and define u_n as the solutions of

$$\begin{cases} (u_n)_t = \operatorname{div} A(x,t,u_n,Du_n), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u_n(x,0) = f_n, & x \in \mathbf{R}^N, \end{cases}$$

By assumptions (1.2) and (1.4), for each $n \in \mathbf{N}$ there exists a unique solution. Moreover by the comparison principle the sequence $\{u_n\}_{n=1}^{\infty}$ is an increasing sequence that converges to a function u_∞. By classical regularity results (see, for instance, [5]) u_n are equi-Hölder continuous. As the operator is monotone, we can apply the Minty's lemma (see [11]) to have that u_∞ is the solution of

$$\begin{cases} (u_\infty)_t = \operatorname{div} A(x,t,u_\infty,Du_\infty), & (x,t) \in \mathbf{R}^N \times [0,+\infty), \\ u_\infty(x,0) = f, & x \in \mathbf{R}^N, \end{cases}$$

As $u_n \nearrow u_\infty$ we have that for each $t > 0$ and for each $n \in \mathbf{N}$

$$\int_{\mathbf{R}^N} u_\infty(x,t)\, dx \geq \int_{\mathbf{R}^N} u_n(x,t)\, dx = \int_{\mathbf{R}^N} f_n(x)\, dx$$

and letting $n \to \infty$ we deduce

$$\int_{\mathbf{R}^N} u_\infty(x,t)\, dx \geq \int_{\mathbf{R}^N} f(x)\, dx.$$

To prove the reverse inequality, consider $m \in \mathbf{N}$. In B_m the function u_m converges uniformly to u_∞. Therefore

$$\int_{\mathbf{R}^N} f(x)\,dx \geq \lim_{n\to\infty} \int_{B_m} u_n(x,t)\,dx = \int_{B_m} u_\infty(x,t)\,dx$$

and letting $m \to \infty$ we deduce

$$\int_{\mathbf{R}^N} u_\infty(x,t)\,dx \leq \int_{\mathbf{R}^N} f(x)\,dx.$$

Therefore also for u_∞ holds the conservation of mass property.

Remark. **Application of the results to Fokker-Planck equation**.

Let us consider the following problem

$$\begin{cases} w_t = \operatorname{div}(A(x,t,w,Dw) + \operatorname{div}(xw), & (x,t) \in \mathbf{R}^N \times (t>0), \\ w(x,0) = \mu, \quad x \in \mathbf{R}^N, \end{cases} \tag{3.9}$$

where the operator A satisfies conditions (1.2), (1.3) and (1.4) or (1.8). As proved by Carrillo-Toscani [3] (see also [15] and references therein), equation (3.9) can be transformed in equation (1.1) by the change of variables

$$w(x,t) = \alpha(t)^N u(\alpha(t)x, \beta(t)),$$

where $\alpha(t) = e^t$ and $\beta(t) = \dfrac{1}{k}(e^{kt} - 1)$.

From the estimates on the p-Laplacian type equation (respectively the Porous medium type equation) we can deduce that if μ is a nonnegative Radon measure with finite mass and with compact support, also equation (3.9) enjoys the property of the conservation of the mass.

Remark. As already noticed in §2, these results can be proved for the Porous medium case following the same arguments we applied for the p-Laplacian type case.

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