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A NOTE ON NORMAL TRIPLE COVERS OVER $\mathbf{P}^2$ WITH BRANCH DIVISORS OF DEGREE 6

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Abstract

Let S and T be reduced divisors on $\mathbf{P}^2$ which have no common components, and $\Delta = S + 2T$. We assume $\deg \Delta = 6$. Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with branch divisor Δ, i.e. π is ramified along S (resp. T) with the index 2 (resp. 3). In this note, we show that X is either a $\mathbf{P}^1$-bundle over an elliptic curve or a normal cubic surface in $\mathbf{P}^3$. Consequently, we give a necessary and sufficient condition for Δ to be the branch divisor of a normal triple cover over $\mathbf{P}^2$.

Introduction

The first systematic study on triple covers was done by Miranda [8]. Afterwards, triple covers are studied by many mathematicians (e.g. [2, 3, 14, 15]). Yet it is difficult to deal with general triple covers. For example, the following fundamental problem still remains as an open problem.

Problem 0.1. Let $\Delta = S + 2T$ be a divisor on $\mathbf{P}^2 = \mathbf{P}^2_{\mathbf{C}}$, where S and T are reduced divisors which have no common components. Give a necessary and sufficient condition for Δ to be the branch divisor of a normal triple cover over $\mathbf{P}^2$ (see below for the notation).

The above problem is an analogy to [5, Question 1.1]. The difference between Problem 0.1 and [5, Question 1.1] is whether a condition of ramification is given, or not. In some cases, Problem 0.1 was solved by some mathematicians, mainly Tokunaga, as follows:

If $S = 0$, then a normal triple cover $\pi : X \to \mathbf{P}^2$ with branch divisor Δ must be a Galois cover, hence one can see an answer of Problem 0.1 from [8]. In the cases where $(\deg S, \deg T) = (2,1)$, $(2,2)$, $(4,0)$ and $(4,1)$ (i.e. $\deg(S+T) \le 5$), Tokunaga solved Problem 0.1 by using his theory of dihedral covers in [16] and [19]. Moreover, Yasumura showed that, if $\pi : X \to \mathbf{P}^2$ is a normal triple cover with branch divisor Δ and $(\deg S, \deg T) = (4,1)$, then X is a cubic surface in $\mathbf{P}^3$

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and π is a projection centered at a point of $\mathbf{P}^3 \setminus X$ ([21]). In the case where $T = 0$ and S is a sextic curve with at most simple singularities, Ishida and Tokunaga showed that X is either a quotient of an abelian surface by an involution or a normal cubic surface in $\mathbf{P}^3$, and gave an answer of Problem 0.1 ([5]).

The author is inspired by these results to do this study. The aim is to characterize normal triple covers over $\mathbf{P}^2$ with branch divisors of degree 6, and to give an answer of Problem 0.1 in the case $\deg \Delta = 6$ without any assumptions. However, it seems difficlut to do that by the same way of these results because of the following facts:

If $\deg(S + T) \leq 5$, then the double cover over $\mathbf{P}^2$ branched along S is rational. In [16] and [19], this fact plays important role. In [5], Ishida–Tokunaga showed the result by explicit computation of the minimal resolution of each singularity of X.

The new idea of this paper is to use Miranda's theory on triple covers [8] in order to answer Problem 0.1. This idea gives a simple proof and generalization of the known results. We introduce notation based on [8].

Notation. The base field is the field of complex numbers $\mathbf{C}$ throughout this note. We call a finite flat morphism $\pi : X \to Y$ from a scheme X to a variety Y a *cover*. If, in addition, X and Y are normal varieties, we call π a *normal cover*. If the degree of a cover (resp. a normal cover) is three, we call it a *triple cover* (resp. a *normal triple cover*). Let $\pi : X \to Y$ be a normal triple cover. We denote the branch locus in Y of π by Δ_π. Suppose Y is non-singular. Then Δ_π has purely codimension 1 in Y. Hence we can regard Δ_π as a reduced divisor. Moreover we can decompose Δ_π into $S_\pi + T_\pi$, where π is ramified along S_π (resp. T_π) with the index 2 (resp. 3). We denote $S_\pi + 2T_\pi$ by $\overline{\Delta}_\pi$ and call it the *branch divisor* of π. We say that $P \in Y$ is a *total branched point* of π if $\pi^{-1}(P)$ consists of one point.

Remark 0.2. Since normal singularities of surfaces are Cohen-Macaulay, a finite surjective morphism from a normal surface to a smooth surface is a normal cover (cf. [7]). In [1, 5, 15, 16, 18, 19, 21], a normal triple cover over a smooth surface are simply called a "triple cover".

Main theorem. To state the main theorem, we introduce some notation. We denote the dual space of $\mathbf{P}^2$ by $\check{\mathbf{P}}^2$. Let F be the flag variety of pairs of points and lines in $\mathbf{P}^2$, and $p : F \to \mathbf{P}^2$ and $q : F \to \check{\mathbf{P}}^2$ the canonical projections. For an irreducible curve $\Gamma \subset \mathbf{P}^2$, we denote the dual curve of Γ in $\check{\mathbf{P}}^2$ by $\Gamma^\vee$. We will show the following theorem based on Miranda's theory.

Theorem 0.3. *Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with* $\deg \overline{\Delta}_\pi = 6$. *Then $\pi : X \to \mathbf{P}^2$ satisfies one of the following two conditions*;

(i) *S_π is a sextic curve with 9 cusps (hence $\Delta_\pi = S_\pi$ and $S_\pi^\vee$ is smooth), $X \cong q^{-1}(S_\pi^\vee) \subset F$, and π is the restriction of p to X; or*

(ii) *X is a cubic surface in $\mathbf{P}^3$, and π is a projection centered at a point of $\mathbf{P}^3 \backslash X$.*

Furthermore, $\pi : X \to \mathbf{P}^2$ satisfies (i) *if and only if S_π is a sextic curve with* 9 *cusps and the* 9 *cusps are total branched points of π.*

Remark 0.4. Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover satisfying the condition (i) in the above theorem. Let $\tilde{\pi} : \tilde{X} \to \mathbf{P}^2$ be the $\mathbf{L}$-normalization of $\mathbf{P}^2$, where $\mathbf{L}$ is the Galois closure of the extension of the rational function fields $\mathbf{C}(X)/\mathbf{C}(\mathbf{P}^2)$. Then it is easy to see that $\tilde{X}$ is isomorphic to $\Delta_\pi^\vee \times \Delta_\pi^\vee$ (cf. [13]), and $\tilde{\pi}$ is the $\mathscr{S}_3$-cover in [18, Example 6.3], where $\mathscr{S}_3$ is the symmetric group of degree 3. Moreover similar covers to π are used to construct families of Galois closure curves in [12].

Remark 0.5. Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with $\deg \overline{\Delta}_\pi = 6$. Ishida and Tokunaga showed that, if Δ_π is a sextic curve with at most simple singularities, then either X is a quotient of an abelian surface by an involution or π satisfies the condition (ii) in Theorem 0.3 ([5]). Yasumura showed that, if $(\deg S_\pi, \deg T_\pi) = (4, 1)$, then π satisfies the condition (ii) in Theorem 0.3 ([21]). (In the case where $(\deg S_\pi, \deg T_\pi) = (2, 2)$, no characterization of π was known.) Theorem 0.3 is a generalization of these results without any assumptions.

Consequently, we will show the following corollary, which is a generalization of [5, Theorem 1.1].

Corollary 0.6. *Let Δ be a divisor of degree* 6 *on $\mathbf{P}^2$. Then there is a normal triple cover π with $\overline{\Delta}_\pi = \Delta$ if and only if there are homogeneous polynomials $G_i(x_0, x_1, x_2)$ of degree i for $i = 1, 2$ with the following three conditions*:

(1) *$G_2^3 + G_3^2 = 0$ defines Δ;*

(2) *$G_2 \notin \mathfrak{m}_E$ or $G_3 \notin \mathfrak{m}_E^2$ for any prime divisor E, where $\mathfrak{m}_E$ is the maximal ideal of the local ring $\mathcal{O}_E$ at E; and*

(3) *$G_2 \in \mathfrak{m}_E$ or $G_2^3 + G_3^2 \notin \mathfrak{m}_E^2$ for any prime divisor E.*

Remark 0.7. Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with $\deg \overline{\Delta}_\pi = 6$.

(i) If $\deg T_\pi = 3$, then π is a cyclic triple cover since $\mathbf{P}^2$ is simply connected. Conversely, for a reduced cubic curve $\Gamma \subset \mathbf{P}^2$, there is a cyclic triple cover whose branch divisor is 2Γ.

(ii) In the cases where $\deg T_\pi$ is 2 and 1, Tokunaga determined the types of Δ_π in [16] and [18], respectively.

(iii) If a reduce sextic curve Δ is defined as (1) in Corollary 0.6, then the pair (G_2, G_3) satisfies (2) and (3) in Corollary 0.6. In this case, Δ is called a $(2,3)$-*torus sextic* (see [6]). Such curves are studied by Oka ([9, 10, 11]).

(iv) If Δ_π is a reduced sextic curve with at most simple singularities, Ishida and Tokunaga showed that Δ_π is a $(2,3)$-torus sextic ([5]). Corollary 0.6 is a generalization of this result without such assumption.

1. Preliminary

In this section, we recall the theory of triple covers based on Miranda's work [8] and some facts for locally free sheaves of rank 2 on $\mathbf{P}^2$.

1.1. Triple covers. See [3], [8] and [14] for details and proofs. Let Y be a non-singular variety for simplicity.

1.1.1. Let $\pi : X \to Y$ be a triple cover. We denote the kernel of the trace map $\pi_* \mathcal{O}_X \to \mathcal{O}_Y$ by $\mathcal{T}_\pi$, which is the locally free $\mathcal{O}_Y$-module of rank two called the *Tschirnhausen module* for $\pi : X \to Y$. Then we have $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{T}_\pi$ ([8, Theorem 3.6]).

1.1.2. Given a locally free sheaf $\mathcal{E}$ of rank two on Y, the $\mathcal{O}_Y$-algebra structures of $\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{E}$ giving triple covers with $\mathcal{T}_\pi = \mathcal{E}$ are in one-to-one correspondence with $\mathcal{O}_Y$-linear maps $\Phi : S^3\mathcal{E} \to \det \mathcal{E}$ ([8, Theorem 3.6]).

1.1.3. We precisely describe the above correspondence. We do this locally on Y. Hence we assume that Y is affine and $\mathcal{E}$ is free. Let $\{z, w\}$ be a basis of $\mathcal{E}$ over $\mathcal{O}_Y$.

1) Let $\phi : S^2\mathcal{E} \to \mathcal{A}$ be the map induced by the multiplication of $\mathcal{A}$. Then ϕ is of the following form:

$$\phi(z^2) = 2A + az + bw,$$

$$\phi(zw) = -B - dz - aw,$$

$$\phi(w^2) = 2C + cz + dw,$$

where a, b, c and d are in $\mathcal{O}_Y$, and $A = a^2 - bd$, $B = ad - bc$ and $C = d^2 - ac$. In particular, $b \neq 0$ and $c \neq 0$ if $\mathcal{A}$ is an integral domain.

2) Define $\Phi : S^3\mathcal{E} \to \det \mathcal{E}$ by $\Phi(z^3) = -b(z \wedge w)$, $\Phi(z^2w) = a(z \wedge w)$, $\Phi(zw^2) = -d(z \wedge w)$ and $\Phi(w^3) = c(z \wedge w)$. This definition does not depend on the choice of the basis $\{z, w\}$ of $\mathcal{E}$ and gives the correspondence in (1.1.2).

1.1.4. Let $S(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$ and $\mathbf{V}(\mathcal{E}) = Spec_Y\, S(\mathcal{E})$. This is identified with the total space of the dual vector bundle of $\mathcal{E}$. Then $X = Spec_Y(\mathcal{A})$ is embedded in $\mathbf{V}(\mathcal{E})$ as a closed subvariety by the natural surjection $S(\mathcal{E}) \to \mathcal{A}$. The local description of X over Y is as follows:

Let z, w, a, b, c, d, A, B and C be as in (1.1.3). Then z, w are fiber coordinate of $\mathbf{V}(\mathcal{E}) \cong \mathbf{A}^2_Y$, and X is defined by

$$z^2 - \phi(z^2) = zw - \phi(zw) = w^2 - \phi(w^2) = 0,$$

where ϕ's are the polynomials as in (1.1.3). Moreover, X is Cohen-Macaulay.

1.1.5. Assume that $\Phi : S^3\mathcal{E} \to \det \mathcal{E}$ gives a normal triple cover $\pi : X \to Y$ with $\mathcal{T}_\pi = \mathcal{E}$ as above. Then the branch divisor $\overline{\Delta}_\pi$ is locally given by

$$D := B^2 - 4AC = 0,$$

where A, B and C are as in (1.1.3) ([8, Lemma 4.5] and [14, Theorem 1.3]). Moreover, the line bundle associated to $\overline{\Delta}_\pi$ is $(\det \mathscr{T}_\pi)^{-2}$ ([8, Proposition 4.7]).

1.1.6. Let $\pi : X \to Y$ be a normal triple cover. If $\mathscr{T}_\pi \cong \mathscr{L}^{-1} \oplus \mathscr{M}^{-1}$, where $\mathscr{L}$ and $\mathscr{M}$ are line bundles on Y, then $a \in H^0(\mathscr{L})$, $b \in H^0(\mathscr{L}^2 \otimes \mathscr{M}^{-1})$, $c \in H^0(\mathscr{L}^{-1} \otimes \mathscr{M}^2)$ and $d \in H^0(\mathscr{M})$. Hence $\mathscr{L}^2 \geq \mathscr{M}$ and $\mathscr{M}^2 \geq \mathscr{L}$ ([8, Section 6]).

1.1.7. Let $\pi : X \to Y$ be a triple cover. Suppose that π is not étale. Then X is a triple section in the total space of a line bundle $\mathscr{L}$ over Y if and only if $\mathscr{T}_\pi \cong \mathscr{L}^{-1} \oplus \mathscr{L}^{-2}$ ([3]).

1.2. Locally free sheaves of rank 2 on $\mathbf{P}^2$. By a result of Grothendieck, each locally free sheaf of rank 2 on the projective line $\mathbf{P}^1$ is isomorphic to a direct sum $\mathscr{O}_{\mathbf{P}^1}(k_1) \oplus \mathscr{O}_{\mathbf{P}^1}(k_2)$, where integers k_1, k_2 are determined up to a permutation.

Let $\mathscr{E}$ be a locally free sheaf of rank 2 on $\mathbf{P}^2$, and we denote the restriction of $\mathscr{E}$ to a line L on $\mathbf{P}^2$ by $\mathscr{E}_L$. Then $\mathscr{E}_L$ splits $\mathscr{E}_L \cong \mathscr{O}_L(k_{1,L}) \oplus \mathscr{O}_L(k_{2,L})$ as above. We put $d(\mathscr{E}_L) = |k_{1,L} - k_{2,L}|$ for a line L and $d(\mathscr{E}) = \min\{d(\mathscr{E}_L) \mid L \text{ is a line on } \mathbf{P}^2\}$. It is a consequence of the semi-continuity theorems for proper flat morphisms that the set of lines L with $d(\mathscr{E}_L) = d(\mathscr{E})$ forms a Zariski-open set in the dual space $\check{\mathbf{P}}^2$ of $\mathbf{P}^2$. If $d(\mathscr{E}_L) > d(\mathscr{E})$ for a line L, it is called a *jumping line* of $\mathscr{E}$. If $\mathscr{E}$ has no jumping lines, $\mathscr{E}$ is said to be *uniform*.

2. Proofs

In this section, let $(x_0 : x_1 : x_2)$ be a system of homogeneous coordinates of $\mathbf{P}^2$, $U \subset \mathbf{P}^2$ the open set given by $x_0 \neq 0$, and put $u_1 = x_1/x_0$ and $u_2 = x_2/x_0$. We first show the following lemma.

LEMMA 2.1. *Let $\pi : X \to \mathbf{P}^2$ be a normal cover. Then π^*L is connected for any line $L \subset \mathbf{P}^2$.*

Proof. Let X_F be the fiber product of X and F over $\mathbf{P}^2$, and $p_X : X_F \to X$ and $\pi_F : X_F \to F$ the projections. Note that p_X induces the isomorphism $(q \circ \pi_F)^* L \xrightarrow{\sim} \pi^* L$ for any line $L \subset \mathbf{P}^2$ since p induces the isomorphism $q^* L \xrightarrow{\sim} L$. Here we regard lines on $\mathbf{P}^2$ as points of $\check{\mathbf{P}}^2$. By the Stein factorization of $q \circ \pi_F$, we have a finite morphism $\pi' : X' \to \check{\mathbf{P}}^2$ and a projective morphism $q' : X_F \to X'$ with connected fiber such that $q \circ \pi_F = \pi' \circ q'$.

$$
\begin{array}{ccccc}
X & \xleftarrow{p_X} & X_F & \xrightarrow{q'} & X' \\
{\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_F}\downarrow & & \downarrow{\scriptstyle \pi'} \\
\mathbf{P}^2 & \xleftarrow[p]{} & F & \xrightarrow[q]{} & \check{\mathbf{P}}^2
\end{array}
$$

If there is a line L_0 such that π^*L is disconnected, then $\deg \pi' > 1$, thus π^*L is disconnected for a general $L \in \check{\mathbf{P}}^2$, which is a contradiction to X irreducible. $\square$

Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with $\deg \bar{\Delta}_\pi = 6$. Then $\det \mathscr{T}_\pi \cong \mathcal{O}_{\mathbf{P}^2}(-3)$ by (1.1.5). For a general line L on $\mathbf{P}^2$, the restriction $\mathscr{T}_{\pi,L}$ of $\mathscr{T}_\pi$ to L is isomorphic to $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)$ by (1.1.6) since $\pi|_{\pi^*L} : \pi^*L \to L$ is a normal triple cover whose branch divisor is degree 6. We show that $\mathscr{T}_\pi$ is uniform.

PROPOSITION 2.2. *Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with* $\deg \bar{\Delta}_\pi = 6$. *Then $\mathscr{T}_\pi$ is uniform.*

Proof. Suppose that $\mathscr{T}_\pi$ has a jumping line L. Then $\mathscr{T}_{\pi,L} \cong \mathcal{O}_L(-m-2) \oplus \mathcal{O}_L(m-1)$ for some integer $m > 0$. We may assume that L is defined by $x_1 = 0$. Let $\Phi : S^3\mathscr{T}_\pi \to \det \mathscr{T}_\pi$ be the map corresponding to $\pi : X \to \mathbf{P}^2$. The restriction of Φ to L gives sections a_L, b_L, c_L and d_L of $\mathcal{O}_L(m+2)$, $\mathcal{O}_L(3m+3)$, $\mathcal{O}_L(-3m)$ and $\mathcal{O}_L(1-m)$, respectively. In particular, $c_L = 0$ and d_L is constant. We may assume that a_L (resp. b_L) vanishes at $m+2$ (resp. $3m+3$) points of $L \cap U$ if $a_L \neq 0$ (resp. $b_L \neq 0$). By choosing a basis $\{z, w\}$ of $\mathscr{E}$ on U, Φ is described as in (1.1.3) such that the restrictions of a, b, c and d to $L \cap U$ are a_L, b_L, c_L and d_L, respectively.

Suppose $d_L \neq 0$. Since $c_L = 0$, π^*L is locally defined by

$$\begin{aligned} z_L^2 - a_L z_L - b_L w_L - 2(a_L^2 - b_L d_L) &= 0, \\ z_L w_L + d_L z_L + a_L w_L + a_L d_L &= 0, \\ (w_L - 2d_L)(w_L + d_L) &= 0, \end{aligned}$$

where z_L and w_L are the restrictions of z and w to L, respectively. Hence π^*L is disconnected, which is contradiction to Lemma 2.1. Thus $d_L = 0$.

Since $c_L = d_L = 0$, c and d have u_1 as their factor, say $c = u_1 c_1$ and $d = u_1 d_1$. Then we have $D = u_1 D_1$, where

$$D_1 = u_1(ad_1 - bc_1)^2 - 4(a^2 - u_1 bd_1)(u_1 d_1^2 - ac_1).$$

Since $D = 0$ defines a divisor of degree 6 on $\mathbf{P}^2$, $D_1 = 0$ defines one of degree 5. Therefore $a^3 c_1$ vanishes along L since $a_L \in H^0(\mathcal{O}_L(m+2))$.

Suppose $a_L \neq 0$ on L, then c_1 has u_1 as its factor, say $c_1 = u_1 c_2$. We have $D = u_1^2 D_2$, where

$$D_2 = (ad_1 - u_1 bc_2)^2 - 4(a^2 - u_1 bd_1)(d_1^2 - ac_2)$$

Hence $a^2 d_1^2 - 4a^2(d_1^2 - ac_2)$ vanishes along L since $D_2 = 0$ defines a quartic curve on $\mathbf{P}^2$ and $a_L \in H^0(\mathcal{O}_L(m+2))$. Then D has u_1^3 as its factor, which is a contradiction to (1.1.5). Thus $a_L = 0$ on L, and a has u_1 as its factor, say $a = u_1 a_1$. We have $D = u_1^2 D_3$, where

$$D_3 = (u_1 a_1 d_1 - bc_1)^2 - 4u_1(u_1 a_1^2 - bd_1)(d_1^2 - a_1 c_1).$$

As above argument, we can see that $b^2c_1^2$ vanishes along L. Therefore D has u_1^3 as its factor, which is a contradiction, and $\mathcal{T}_\pi$ has no jumping lines. $\square$

From the theorem of [20] and (1.1.5), we have the following corollary.

Corollary 2.3. *For a normal triple cover* $\pi : X \to \mathbf{P}^2$ *with* $\deg \overline{\Delta}_\pi = 6$, $\mathcal{T}_\pi$ *is either* $\mathcal{O}_{\mathbf{P}^2}(-2) \oplus \mathcal{O}_{\mathbf{P}^2}(-1)$ *or* $\Omega_{\mathbf{P}^2}$, *where* $\Omega_{\mathbf{P}^2}$ *is the cotangent sheaf of* $\mathbf{P}^2$.

We first consider the case where $\mathcal{T}_\pi \cong \mathcal{O}_{\mathbf{P}^2}(-2) \oplus \mathcal{O}_{\mathbf{P}^2}(-1)$.

Proposition 2.4. *Let* $\pi : X \to \mathbf{P}^2$ *be a normal triple cover with* $\mathcal{T}_\pi \cong \mathcal{O}_{\mathbf{P}^2}(-2) \oplus \mathcal{O}_{\mathbf{P}^2}(-1)$. *Then* X *is a normal cubic surface in* $\mathbf{P}^3$, *and* π *is identified with a projection centered at a point of* $\mathbf{P}^3 \backslash X$.

Proof. Since $\mathbf{P}^2$ is simply connected, π is not étale. By (1.1.7), X is a triple section of the total space of the line bundle $\mathcal{O}_{\mathbf{P}^2}(1)$. Note that the total space of $\mathcal{O}_{\mathbf{P}^2}(1)$ is isomorphic to $\mathbf{P}^3 \backslash \{P\}$ with the projection centered at P for a point $P \in \mathbf{P}^3$ over $\mathbf{P}^2$. Thus X is a cubic surface in $\mathbf{P}^3$, and π is identified with a projection centered at a point of $\mathbf{P}^3$. $\square$

Next we show that X is a $\mathbf{P}^1$-bundle over an elliptic curve if $\mathcal{T}_\pi \cong \Omega_{\mathbf{P}^2}$. Let V be a vector space of dimension 3, and v_0, v_1, v_2 a basis of V. We regard $\mathbf{P}^2$ as the set of 1-dimensional subspaces of V, $\mathbf{P}(V)$. Then $\check{\mathbf{P}}^2 = \mathbf{P}(V^*)$, where V^* is the dual space of V, and we can regard x_0, x_1 and x_2 as the dual of v_0, v_1 and v_2, respectively. Note that V and V^* are naturally identified with $H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}^2}(1))$ and $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, respectively.

Lemma 2.5. *There is a natural isomorphism of vector spaces*

$$\theta : H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}^2}(3)) \xrightarrow{\sim} H^0(\mathbf{P}^2, (S^3\Omega_{\mathbf{P}^2})^* \otimes \det \Omega_{\mathbf{P}^2}).$$

Furthermore θ *is defined as follows*:

Let f *be a global section of* $\mathcal{O}_{\check{\mathbf{P}}^2}(3)$ *as follows*:

$$f = t_1 v_0^3 + 3t_2 v_0^2 v_1 + 3t_3 v_0^2 v_2 + 3t_4 v_0 v_1^2 + 3t_5 v_0 v_1 v_2 + 3t_6 v_0 v_2^2 + t_7 v_1^3 + 3t_8 v_1^2 v_2 + 3t_9 v_1 v_2^2 + t_{10} v_2^3,$$

where $t_1, \ldots, t_{10} \in \mathbf{C}$. *Then* $\theta(f)$ *is locally*

$$-b_f(z^3)^* \otimes (z \wedge w) + a_f(z^2w)^* \otimes (z \wedge w) - d_f(zw^2)^* \otimes (z \wedge w) + c_f(w^3)^* \otimes (z \wedge w),$$

where

$$a_f = -t_1 u_2 u_1^2 + 2t_2 u_2 u_1 + t_3 u_1^2 - t_4 u_2 - t_5 u_1 + t_8,$$

$$b_f = t_1 u_1^3 - 3t_2 u_1^2 + 3t_4 u_1 - t_7,$$

$$c_f = -t_1u_2^3 + 3t_3u_2^2 - 3t_6u_2 + t_{10},$$

$$d_f = t_1u_2^2u_1 - t_2u_2^2 - 2t_3u_2u_1 + t_5u_2 + t_6u_1 - t_9,$$

and z and w are the differential forms du_1 and du_2, respectively.

Proof. Note that the $\mathbf{P}^1$-bundle $\mathbf{P}(\Omega_{\mathbf{P}^2})$ over $\mathbf{P}^2$ is isomorphic to the flag variety F, and the projection $\mathbf{P}(\Omega_{\mathbf{P}^2}) \to \mathbf{P}^2$ coincides with $p : F \to \mathbf{P}^2$. The canonical embedding $F \hookrightarrow \mathbf{P}^2 \times \check{\mathbf{P}}^2$ is given by the surjection α of the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-1) \to V \otimes \mathcal{O}_{\mathbf{P}^2} \xrightarrow{\alpha} \Omega^*_{\mathbf{P}^2}(-1) \to 0,$$

where α is locally defined by $\alpha(v_0) = -u_1z^*/x_0 - u_2w^*/x_0$, $\alpha(v_1) = z^*/x_0$ and $\alpha(v_2) = w^*/x_0$ (cf. [4, II, Proposition 7.12 and the proof of II, Theorem 8.13]). Let $\mathcal{O}_F(1)$ be an invertible sheaf on F such that $p_*\mathcal{O}_F(1) \cong \Omega^*_{\mathbf{P}^2}(-1)$. Then α induces an isomorphism $q^*\mathcal{O}_{\check{\mathbf{P}}^2}(3) \xrightarrow{\sim} \mathcal{O}_F(3)$. In particular, since $H^0(F, \mathcal{O}_F(3))$ and $H^0(F, q^*\mathcal{O}_{\check{\mathbf{P}}^2}(3))$ are identified with $H^0(\mathbf{P}^2, S^3(\Omega^*_{\mathbf{P}^2}(-1)))$ and S^3V, respectively, the symmetric product of α gives an isomorphism

$$S^3\alpha : H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}}(3)) \xrightarrow{\sim} H^0(\mathbf{P}^2, S^3(\Omega^*_{\mathbf{P}^2}(-1))).$$

Note that there is the natural isomorphism $\kappa : (S^3\Omega_{\mathbf{P}^2})^* \otimes \det \Omega_{\mathbf{P}^2} \xrightarrow{\sim} S^3(\Omega^*_{\mathbf{P}^2}(-1))$, which is locally defined by

$$(z^3)^* \otimes (z \wedge w) \mapsto (z^*)^3/x_0^3, \quad (z^2w)^* \otimes (z \wedge w) \mapsto 3(z^*)^2w^*/x_0^3,$$

$$(w^3)^* \otimes (z \wedge w) \mapsto (w^*)^3/x_0^3, \quad (zw^2)^* \otimes (z \wedge w) \mapsto 3z^*(w^*)^2/x_0^3.$$

Therefore we obtain a natural isomorphism $\theta = \kappa^{-1} \circ S^3\alpha$. We can see the second assertion by direct computation. □

Let $f = f(v_0, v_1, v_2)$ be a global section of $\mathcal{O}_{\check{\mathbf{P}}^2}(3)$ as in Lemma 2.5. Put $\delta_f = \delta_f(u_1, u_2)$ the discriminant of $f(-u_1v_1 - u_2v_2, v_1, v_2)$ with respect to v_1 and v_2. We denote D for $\theta(f)$ in (1.1.5) by D_f.

LEMMA 2.6. *Let f, δ_f and D_f be as above. Then $\delta_f = -27D_f$.*

Proof. The global section f of $\mathcal{O}_{\mathbf{P}^2}(3)$ is as in Lemma 2.5. We have $D_f = B_f^2 - 4A_fC_f$, where $A_f = a_f^2 - b_fd_f$, $B_f = a_fd_f - b_fc_f$ and $C_f = d_f^2 - a_fc_f$. By direct computation, we obtain $\delta_f + 27D_f = 0$. □

We can identify $H^0(\mathbf{P}^2, (S^3\Omega_{\mathbf{P}^2})^* \otimes \det \Omega_{\mathbf{P}^2})$ with $H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}^2}(3))$ by the isomorphism θ.

PROPOSITION 2.7. *Let $\pi : X \to \mathbf{P}^2$ and $\pi' : X' \to \mathbf{P}^2$ be normal triple covers corresponding to $f, f' \in H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}^2}(3))$, respectively. Then $\Delta_\pi = \Delta_{\pi'}$ if and only if $f' = \lambda f$ for some non-zero constant λ. In particular, there is an isomorphism $\sigma : X \to X'$ such that $\pi = \pi' \circ \sigma$ in this case.*

Proof. We first show that f is irreducible. Assume that f is reducible. We may assume that f has v_0 as its factor (i.e. $t_i = 0$ for $i = 7, \dots, 10$), and $b_f \neq 0$. Then z satisfies the following equation (cf. [8, p. 1128]):

$$z^3 - 3A_f z + (b_f B_f - 2a_f A_f) = 0,$$

where $A_f = a_f^2 - b_f d_f$ and $B_f = a_f d_f - b_f c_f$. The above polynomial is divided by $z - t_2 u_1 u_2 + t_3 u_1^2 + 2t_4 u_2 - t_5 u_1$. Thus X is reducible, which is a contradiction.

Hence we may assume that f is irreducible. Let $\Gamma^\vee$ be the curve on $\check{\mathbf{P}}^2$ defined by $f = 0$. Then $\delta_f = 0$ defines a divisor of degree 6 on $\mathbf{P}^2$ whose support is the union of the dual curve of $\Gamma^\vee$ and the lines corresponding to the singular points of $\Gamma^\vee$. Therefore, by Lemma 2.6, $\Delta_\pi = \Delta_{\pi'}$ if and only if $f' = \lambda f$ for some $\lambda \in \mathbf{C}^*$. □

The above proposition enables us to distinguish normal covers for $\Omega_{\mathbf{P}^2}$ by their branch loci.

Proposition 2.8. *Let $\pi : X \to \mathbf{P}^2$ be a triple cover for $f \in H^0(\check{\mathbf{P}}^2, \mathcal{O}_{\check{\mathbf{P}}^2}(3))$. Then X is normal if and only if the curve $\Gamma^\vee \subset \check{\mathbf{P}}^2$ defined by $f = 0$ is smooth. Moreover, if X is normal, then Δ_π is the dual curve of $\Gamma^\vee$, X is isomorphic to $q^{-1}(\Gamma^\vee) \subset F$, and π is identified with the restriction of $p : F \to \mathbf{P}^2$ to $q^{-1}(\Gamma^\vee)$.*

Proof. Suppose X is normal. By the proof of Proposition 2.7, $\Gamma^\vee$ is reduced and irreducible. Put $X' = q^{-1}(\Gamma^\vee)$. Then $p|_{X'} : X' \to \mathbf{P}^2$ is a triple cover with $\mathcal{T}_{p|_{X'}} \cong \Omega_{\mathbf{P}^2}$ ([8, Proposition 8.1]) whose branch locus is given by $\delta_f = 0$. Thus there is an isomorphism $\sigma : X \to X'$ such that $\pi = p|_{X'} \circ \sigma$ by Proposition 2.7. If $\Gamma^\vee$ is singular, then X' is not normal since X' has 1-dimensional singular locus over the singularity of $\Gamma^\vee$. Therefore $\Gamma^\vee$ is smooth. Conversely, we can see that X is normal if $\Gamma^\vee$ is smooth as above argument. □

Proof of Theorem 0.3. Let $\pi : X \to \mathbf{P}^2$ be a normal triple cover with $\deg \overline{\Delta}_\pi = 6$. We can see the first assertion of Theorem 0.3 from Corollary 2.3, Proposition 2.4 and 2.8. We show the second assertion. We can easily check that, if $\pi : X \to \mathbf{P}^2$ satisfies (i) in Theorem 0.3, the 9 cusps of Δ_π are total branched points of π.

Assume that Δ_π is a sextic curve with 9 cusps, and the 9 cusps are total branched points of π. Suppose that X is a normal cubic surface. Then we may assume that there are homogeneous polynomials $G_i(x_0, x_1, x_2)$ of degree i for $i = 2, 3$ such that X is defined by $x_3^3 + 3G_2 x_3 + 2G_3 = 0$, and π is the projection centered at $P = (0 : 0 : 0 : 1)$. Here we regard $(x_0 : x_1 : x_2 : x_3)$ as a system of homogeneous coordinates of $\mathbf{P}^3$. In this case, $\overline{\Delta}_\pi = \Delta_\pi$ is defined by $G_2^3 + G_3^2 = 0$. Since Δ_π has just 9 cusps as its singularities, $G_2 = 0$ and $G_3 = 0$ define reduced curves Γ_2 and Γ_3, respectively, such that they intersect transversally each other. Then it is easy to see that the total branched points of π are just 6 intersection points of Γ_2 and Γ_3, which is a contradiction. Hence X is a subvariety of F. □

Our proof of Corollary 0.6 is almost the same as the proof of [5, Theorem 1.1].

Proof of Corollary 0.6. Suppose that Δ is defined by the equation $G_2^3 + G_3^2 = 0$ with the conditions in the corollary. Let X be the cubic surface in $\mathbf{P}^3$ given by $x_3^3 + 3G_2x_3 + 2G_3 = 0$, and $P = (0:0:0:1) \in \mathbf{P}^3$. Then X is smooth in codimension one by [8, Lemma 5.1], and hence X is normal. Moreover the projection $\pi_P : X \to \mathbf{P}^2$ centered at P is a normal triple cover with $\overline{\Delta}_{\pi_P} = \Delta$.

Conversely, we suppose that $\pi : X \to \mathbf{P}^2$ is a normal triple cover with $\overline{\Delta}_\pi = \Delta$. If Δ is a reduced sextic curve with 9 cusps, then it is known that Δ is a $(2,3)$-torus curve (cf. [17]). Hence suppose that X is a normal cubic surface, and π is a projection centered at $P \in \mathbf{P}^3 \backslash X$. We may assume that $P = (0:0:0:1)$, and X is given by an equation $x_3^3 + 3G_2x_3 + 2G_3 = 0$ for some homogeneous polynomials G_i of degree i $(i = 2,3)$ (cf. [1, Proposition 3.17]). Then Δ is given by $G_2^3 + G_3^2 = 0$. Since X is normal, G_2 and G_3 satisfy the conditions (2) and (3) in the corollary. □

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