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HOMOGENEOUS REINHARDT DOMAINS CONTAINING NO COORDINATE HYPERPLANES

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Abstract

As is well-known, a homogeneous Reinhardt domain in $\mathbf{C}^*$ coinsides with $\mathbf{C}^*$. In this paper, generalizing this fact, we show that a pseudoconvex homogeneous Reinhardt domain in $(\mathbf{C}^*)^n$ coinsides with $(\mathbf{C}^*)^n$ itself.

1. Introduction

In the study of Reinhardt domains D, investigating the structures of their holomorphic automorphism groups $\mathrm{Aut}(D)$ has fundamental importance. When D is bounded, the structure of $\mathrm{Aut}(D)$ was clarified by, for example, Sunada [6], Shimizu [2], [3]. But, in the general case, or when D is not necessarily bounded, a little is known about the structure of $\mathrm{Aut}(D)$. For instance, related to the investigation of the structure of $\mathrm{Aut}(D)$, we have the fundamental problem of determining a homgeneous Reinhardt domain, that is, the problem that when a Reinhardt domain D admits a transitive action by $\mathrm{Aut}(D)$, what form does D have? When D is bounded, the following result is shown [3] (for the definition of the algebraic equivalence relation between Reinhardt domains, see Section 2 below):

Theorem 1.1. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$. If D is homogeneous, then D is algebraically equivalent to the direct product $B_{n_1} \times \cdots \times B_{n_k}$ of balls, where B_{n_i} denotes the unit ball in $\mathbf{C}^{n_i}$.*

On the other hand, as for the general case, there is a conjecture as follows:
Conjecture. For every homogeneous pseudoconvex Reinhardt domain D in $\mathbf{C}^n$, there exist k positive integers $n_1, \dots, n_k$ (k may be 0) and non-negative integers l, m such that $n = n_1 + \cdots + n_k + l + m$ and D is algebraically equivalent

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to the direct product $B_{n_1} \times \cdots \times B_{n_k} \times \mathbf{C}^l \times (\mathbf{C}^*)^m$, where $\mathbf{C}^*$ denotes the puctured complex plane.

Theorem 1.1 implies that this conjecture is true when D is bounded. But it remains open in the unbounded case. The purpose of this paper is to give a partial answer in such case by applying the method given in Shimizu [5]. Namely, we prove the following:

Theorem 1.2. *Let D be a pseudoconvex Reinhardt domain in $(\mathbf{C}^*)^n$. If D is homogeneous, then D coincides with $(\mathbf{C}^*)^n$.*

Note that the above theorem gives a higher-dimensional generalization of the classical fact that a Reinhardt domain D in $\mathbf{C}^*$ is inhomogeneous except when D coincides with $\mathbf{C}^*$ itself.

This paper is organized as follows. In Section 2, we recall basic concepts and results on Reinhardt domains. In particular, we collect some preliminary results used for proving Theorem 1.2 mainly from Shimizu [5]. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

We first collect some notations and terminology. As a general notational convention, we denote elements of $\mathbf{Z}^n$, $\mathbf{R}^n$, or $\mathbf{C}^n$ by column vectors. When dealing with matrices, we denote by I_p and O the unit matrix of degree p and the zero matrix, respectively. The set of non-zero complex numbers is denoted by $\mathbf{C}^*$. The multiplicative group of complex numbers of absolute value 1 is denoted by $U(1)$. An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them.

We now recall basic concepts and results on Reinhardt domains (cf. Shimizu [2], [3]). Write $T = (U(1))^n$. The group T acts as a group of automorphisms on $\mathbf{C}^n$ by the standard rule

$$\alpha \cdot z = {}^t(\alpha_1 z_1, \ldots, \alpha_n z_n) \quad \text{for } \alpha = {}^t(\alpha_1, \ldots, \alpha_n) \in T \text{ and } z = {}^t(z_1, \ldots, z_n) \in \mathbf{C}^n.$$

By definition, a Reinhardt domain D in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ which is stable under the action of T, that is, such that $\alpha \cdot D \subset D$ for all $\alpha \in T$. The group T then acts as a group of automorphisms on D. The subgroup of $\mathrm{Aut}(D)$ induced by the action of T is denoted by $T(D)$.

An automorphism φ of $(\mathbf{C}^*)^n$ is called an algebraic automorphism of $(\mathbf{C}^*)^n$ if the components of φ are given by Laurent monomials, that is, φ is of the form

$$\varphi : (\mathbf{C}^*)^n \ni {}^t(z_1, \ldots, z_n) \mapsto {}^t(w_1, \ldots, w_n) \in (\mathbf{C}^*)^n,$$
$$w_i = \alpha_i z_1^{a_{i1}} \cdots z_n^{a_{in}}, \quad i = 1, \ldots, n,$$

where $(a_{ij}) \in GL(n, \mathbf{Z})$ and $(\alpha_i) \in (\mathbf{C}^*)^n$. The set $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ of all algebraic automorphisms of $(\mathbf{C}^*)^n$ forms a subgroup of $\mathrm{Aut}((\mathbf{C}^*)^n)$.

Let φ be an algebraic automorphisms of $(\mathbf{C}^*)^n$ and write $\varphi(z) = {}^t(\varphi_1(z), \dots, \varphi_n(z))$. In general, the components $\varphi_1, \dots, \varphi_n$ have zeroes or poles along each coordinate hyperplane. If, for two domains D and D' in $\mathbf{C}^n$ not necessarily contained in $(\mathbf{C}^*)^n$, they have no poles on D and $\varphi : D \to \mathbf{C}^n$ maps D biholomorphically onto D', then we say that φ induces a biholomorphic mapping of D onto D'.

Consider a biholomorphic mapping $\varphi : D \to D'$ between two Reinhardt domains D onto D' in $\mathbf{C}^n$. The following proposition gives a necessary and sufficient condition for φ to be equivariant with respect to the T-actions.

Proposition 2.1 (cf. [3, Section 2]). *φ is induced by an algebraic automorphism of $(\mathbf{C}^*)^n$ if and only if it has the property that $\varphi T(D) \varphi^{-1} = T(D')$.*

Biholomorphic mappings between Reinhardt domains equivariant with respect to the T-actions may be considered as natural isomorphisms in the category of Reinhardt domains. In view of this observation, we say that two Reinhardt domains in $\mathbf{C}^n$ are algebraically equivalent if there is a biholomorphic mapping between them induced by an algebraic automorphism of $(\mathbf{C}^*)^n$.

There is a useful correspondence between Reinhardt domains and tube domains. A tube domain T_Ω in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ given by $T_\Omega = \Omega + \sqrt{-1}\mathbf{R}^n$, where Ω is a domain in $\mathbf{R}^n$. We call Ω the base of T_Ω. For each element η of $\mathbf{R}^n$, we define an automorphism σ_η of T_Ω given as a translation of $\mathbf{C}^n$ by $\sigma_\eta(\zeta) = \zeta + \sqrt{-1}\eta$ for $\zeta \in T_\Omega$. Now consider a mapping $\mathrm{ord} : (\mathbf{C}^*)^n \to \mathbf{R}^n$ defined by

$$\mathrm{ord}({}^t(z_1, \dots, z_n)) = {}^t\!\left(-\frac{1}{2\pi}\log|z_1|, \dots, -\frac{1}{2\pi}\log|z_n|\right) \quad \text{for } {}^t(z_1, \dots, z_n) \in (\mathbf{C}^*)^n.$$

If D is a Reinhardt domain in $(\mathbf{C}^*)^n$, then $\mathrm{ord}(D)$ is a domain in $\mathbf{R}^n$, and it is well-known that D is pseudoconvex if and only if $\mathrm{ord}(D)$ is a convex domain in $\mathbf{R}^n$. To each Reinhardt domain D in $(\mathbf{C}^*)^n$, there is associated a tube domain T_Ω in $\mathbf{C}^n$ with $\Omega = \mathrm{ord}(D)$. The tube domain T_Ω naturally becomes a covering manifold of D. Indeed, introduce a covering $\varpi : \mathbf{C}^n \to (\mathbf{C}^*)^n$ defined by

$$\varpi({}^t(\zeta_1, \dots, \zeta_n)) = {}^t(e^{-2\pi\zeta_1}, \dots, e^{-2\pi\zeta_n}) \quad \text{for } {}^t(\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n.$$

Then we have $T_\Omega = \varpi^{-1}(D)$, and the restriction $\varpi : T_\Omega \to D$ is a covering projection. The covering transformation group for $\varpi : T_\Omega \to D$ is given by $\{\sigma_\eta \mid \eta \in \mathbf{Z}^n\}$. We call T_Ω the covering tube domain of D and $\varpi : T_\Omega \to D$ the canonical covering projection. Note that if D is pseudoconvex, then $\varpi : T_\Omega \to D$ gives the universal covering of D. Indeed, if D is pseudoconvex, then T_Ω is simply connected, because Ω is convex, and consequently simply connected.

Let D be a Reinhardt domain in $(\mathbf{C}^*)^n$ and write $\Omega = \mathrm{ord}(D)$. Suppose that D is pseudoconvex. It follows that, for the convex domain Ω, there exists an affine transformation f of $\mathbf{R}^n$ such that

$$f(\Omega) = \Xi^{(1)} \times \mathbf{R}^l, \tag{2.1}$$

where l is an integer between 0 and n and $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^{n-l}$ containing no complete straight lines (cf. [1]). This implies that if, for each point ξ of Ω, we denote by V_ξ the maximal vector subspace of $\mathbf{R}^n$ such that $\xi + V_\xi \subset \Omega$, then the vector subspaces $V_\xi, \xi \in \Omega$ coincide to each other, which we denote by $V(D)$, and its dimension is equal to l. As a consequence, l is independent of the choice of f satisfying (2.1). Therefore the integer l is an invariant associated with D, which we denote by $l(D)$.

Here are some observations about $l(D)$. Let D be a pseudoconvex Reinhardt domain in $(\mathbf{C}^*)^n$ and write $\Omega = \operatorname{ord}(D)$. When $l(D) = 0$, the domain D is algebraically equivalent to a bounded Reinhardt domain in $(\mathbf{C}^*)^n$ (cf. [1]). On the other hand, when $l(D) > 0$, write an affine transformation f of $\mathbf{R}^n$ satisfying (2.1) as $f(\xi) = L\xi + b$ for $\xi \in \mathbf{R}^n$, where $L \in GL(n, \mathbf{R})$ and $b \in \mathbf{R}^n$. If we define an affine transformation F of $\mathbf{C}^n$ by $F(\zeta) = L\zeta + b$ for $\zeta \in \mathbf{C}^n$, then we have

$$F(T_\Omega) = T_{f(\Omega)} = T_{\Xi^{(1)} \times \mathbf{R}^{l(D)}} = T_{\Xi^{(1)}} \times T_{\mathbf{R}^{l(D)}} = T_{\Xi^{(1)}} \times \mathbf{C}^{l(D)},$$

and hence T_Ω is holomorphically equivalent to $T_{\Xi^{(1)}} \times \mathbf{C}^{l(D)}$. Note that $l(D) = n$ if and only if $D = (\mathbf{C}^*)^n$.

The following lemma is easily proved by using the notion of Liouville foliation introduced in Shimizu [4] (cf. [5]).

Lemma 2.1. *Let $E \times \mathbf{C}^l$ and $E' \times \mathbf{C}^{l'}$ be two domains in $\mathbf{C}^n$, where E and E' are domains in $\mathbf{C}^{n-l}$ and $\mathbf{C}^{n-l'}$, respectively, that are holomorphically equivalent to bounded domains. Suppose that there is a biholomorphic mapping Φ of $E \times \mathbf{C}^l$ onto $E' \times \mathbf{C}^{l'}$. Then l and l' coincide. Moreover, if each point $w \in \mathbf{C}^n = \mathbf{C}^{n-l} \times \mathbf{C}^l$ is written as*

$$w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix}, \quad w^{(1)} \in \mathbf{C}^{n-l}, w^{(2)} \in \mathbf{C}^l,$$

then Φ has the form

$$\Phi : E \times \mathbf{C}^l \ni w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} \Phi^{(1)}(w^{(1)}) \\ \Phi^{(2)}(w) \end{pmatrix} \in E' \times \mathbf{C}^{l'},$$

where $\Phi^{(1)} : E \ni w^{(1)} \mapsto \Phi^{(1)}(w^{(1)}) \in E'$ gives a biholomorphic mapping of E onto E'.

As an immediate consequence of this lemma, we see that $l(D)$ is a biholomorphic invariant:

Corollary 2.1. *If two pseudoconvex Reinhardt domains D and D' in $(\mathbf{C}^*)^n$ are holomorphically equivalent, then $l(D)$ and $l(D')$ coincide.*

We denote by $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ the group of all complex affine transformations of $\mathbf{C}^n$ whose linear parts belong to $GL(n,\mathbf{Z})$. We discuss the relation between $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ and $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$. Let Φ be any element of $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ and write $\Phi(\zeta) = A\zeta + \beta$ for $\zeta \in \mathbf{C}^n$, where $A = (a_{ij}) \in GL(n,\mathbf{Z})$ and $\beta = (\beta_i) \in \mathbf{C}^n$. Then we can define an element φ of $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ by

$$\varphi : (\mathbf{C}^*)^n \ni {}^t(z_1, \dots, z_n) \mapsto {}^t(w_1, \dots, w_n) \in (\mathbf{C}^*)^n,$$
$$w_i = e^{-2\pi\beta_i} z_1^{a_{i1}} \cdots z_n^{a_{in}}, \quad i = 1, \dots, n.$$

The mapping $\rho : GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n \to \mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ sending Φ to φ is a group homomorphism of $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ onto $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$. Note that the kernel of ρ is given by $\{\sigma_\eta \mid \eta \in \mathbf{Z}^n\} \subset \mathrm{Aut}(\mathbf{C}^n)$, and that Φ, $\rho(\Phi)$, and the covering projection $\varpi : \mathbf{C}^n \to (\mathbf{C}^*)^n$ commute in the following sense:

$$\varpi \circ \Phi = \rho(\Phi) \circ \varpi \quad \text{for every } \Phi \in GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n. \tag{2.2}$$

If D and D' are Reinhardt domains in $(\mathbf{C}^*)^n$ with the covering tube domains T_Ω and $T_{\Omega'}$, respectively, and if Φ is an element of $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$, then, by (2.2), we have $\rho(\Phi)(D) = D'$ precisely when $\Phi(T_\Omega) = T_{\Omega'}$. As a consequence of this, we see that if there exists an element Φ of $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ such that $\Phi(T_\Omega) = T_{\Omega'}$, then D and D' are algebraically equivalent.

To discuss the correspondence of biholomorphic mappings between Reinhardt domains with biholomorphic mappings between tube domains, let D and D' be two Reinhardt domains in $(\mathbf{C}^*)^n$ and let T_Ω and $T_{\Omega'}$ denote the covering tube domains of D and D', respectively. Suppose $\Phi : T_\Omega \to T_{\Omega'}$ is a biholomorphic mapping between T_Ω and $T_{\Omega'}$ and satisfies the condition that, for some $A \in GL(n,\mathbf{Z})$, we have

$$\Phi(\zeta + \sqrt{-1}m) = \Phi(\zeta) + \sqrt{-1}Am \quad \text{for every } \zeta \in T_\Omega \text{ and every } m \in \mathbf{Z}^n. \tag{2.3}$$

Then, since the covering transformation groups for $\varpi : T_\Omega \to D$ and $\varpi : T_{\Omega'} \to D'$ are given by $\{\sigma_\eta \mid \eta \in \mathbf{Z}^n\} \subset \mathrm{Aut}(T_\Omega)$ and $\{\sigma_\eta \mid \eta \in \mathbf{Z}^n\} \subset \mathrm{Aut}(T_{\Omega'})$, respectively, it follows that there exists a biholomorphic mapping $\varphi : D \to D'$ between D and D' such that $\varpi \circ \Phi = \varphi \circ \varpi$. Conversely, when D and D' are pseudoconvex, every biholomorphic mapping $\varphi : D \to D'$ between D and D' has a lifting $\Phi : T_\Omega \to T_{\Omega'}$, or a biholomorphic mapping Φ of T_Ω onto $T_{\Omega'}$ such that $\varpi \circ \Phi = \varphi \circ \varpi$ and satisfying (2.3) for some $A \in GL(n,\mathbf{Z})$, because $\varpi : T_\Omega \to D$ and $\varpi : T_{\Omega'} \to D'$ are the universal coverings of D and D', respectively. As a consequence of these observations and [2, Section 6, Corollary to Theorem 2], we have the following proposition, which gives a useful tool in our investigation.

Proposition 2.2. *Let $\Phi : T_\Omega \to T_{\Omega'}$ be a biholomorphic mapping between two tube domains T_Ω and $T_{\Omega'}$ in $\mathbf{C}^n$ whose bases Ω and Ω' have the convex hulls containing no complete straight lines. Suppose that there exist elements A and B*

of $GL(n, \mathbf{R})$ *such that*

$$\Phi(\zeta + \sqrt{-1}Am) = \Phi(\zeta) + \sqrt{-1}Bm \quad \textit{for every } \zeta \in T_\Omega \textit{ and every } m \in \mathbf{Z}^n. \tag{2.4}$$

Then Φ *is an affine transformation of* $\mathbf{C}^n$ *whose linear part belongs to* $GL(n, \mathbf{R})$.

Proof. The proof of this proposition is given in Shimizu [5]. Although there are overlaps with that, we carry out the proof in detail for the sake of completeness and self-containedness.

We define automorphisms F_A and F_B of $\mathbf{C}^n$ given as linear transformations by

$$F_A(w) = Aw \quad \text{for } w \in \mathbf{C}^n \qquad \text{and} \qquad F_B(\omega) = B\omega \quad \text{for } \omega \in \mathbf{C}^n.$$

Then the domains $F_A^{-1}(T_\Omega)$ and $F_B^{-1}(T_{\Omega'})$ in $\mathbf{C}^n$ are tube domains. Indeed, writing $\Xi = A^{-1}\Omega$ and $\Xi' = B^{-1}\Omega'$, we have $F_A^{-1}(T_\Omega) = T_\Xi$ and $F_B^{-1}(T_{\Omega'}) = T_{\Xi'}$. Note that, since Ω and Ω' have the convex hulls containing no complete straight lines, Ξ and Ξ' also have the convex hulls containing no complete straight lines. We set $E = \varpi(T_\Xi)$ and $E' = \varpi(T_{\Xi'})$. By the definition of ϖ, the domains E and E' are Reinhardt domains in $(\mathbf{C}^*)^n$, and T_Ξ and $T_{\Xi'}$ are the covering tube domains of E and E', respectively. An application of [2, Section 6, Corollary to Theorem 2] to E and E' yields that every biholomorphic mapping of E onto E' is induced by an algebraic automorphism of $(\mathbf{C}^*)^n$.

Now consider a biholomorphic mapping $\Psi : T_\Xi \to T_{\Xi'}$ between T_Ξ and $T_{\Xi'}$ given by $\Psi = F_B^{-1} \circ \Phi \circ F_A$. Then we see from (2.4) that Ψ satisfies the condition that

$$\Psi(w + \sqrt{-1}m) = \Psi(w) + \sqrt{-1}m \quad \text{for every } w \in T_\Xi \text{ and every } m \in \mathbf{Z}^n.$$

Therefore, as observed above, there exists a biholomorphic mapping $\psi : E \to E'$ between E and E' such that $\varpi \circ \Psi = \psi \circ \varpi$. By the result of the preceding paragraph, ψ is induced by an algebraic automorphism of $(\mathbf{C}^*)^n$. This implies that we can find an element Ψ_0 of $GL(n, \mathbf{Z}) \ltimes \mathbf{C}^n$ such that $\Psi_0(T_\Xi) = T_{\Xi'}$ and $\varpi \circ \Psi_0 = \psi \circ \varpi$ on T_Ξ. Note that both $\Psi : T_\Xi \to T_{\Xi'}$ and $\Psi_0 : T_\Xi \to T_{\Xi'}$ are liftings of ψ. Replacing, if necessary, Ψ_0 by $\sigma_\eta \circ \Psi_0$ for some $\eta \in \mathbf{Z}^n$, we may assume that $\Psi(w_0) = \Psi_0(w_0)$ for a point w_0 of T_Ξ. Then we see by the uniqueness of lifting that $\Psi = \Psi_0$, or $\Phi = F_B \circ \Psi_0 \circ F_A^{-1}$. Since $F_B \circ \Psi_0 \circ F_A^{-1}$ is an affine transformation of $\mathbf{C}^n$, this completes the proof of the proposition. □

To apply the above proposition to our proof of Theorem 1.2, we need a lemma.

LEMMA 2.2. *Let* $\Phi : T_\Omega \to T_{\Omega'}$ *be a biholomorphic mapping between two tube domains* T_Ω *and* $T_{\Omega'}$ *in* $\mathbf{C}^n$. *Suppose that there exists a real* $n \times n$ *matrix* A *such that the condition* (2.3) *holds. Then the matrix* A *is non-singular.*

Proof. We denote by Ker A the kernel of the linear transformation of $\mathbf{R}^n$ determined by A. Suppose that Ker $A \neq \{0\}$ and we shall derive a contradiction.

Consider first the case where $\operatorname{Ker} A \cap \mathbf{Z}^n \neq \{0\}$. Then, taking a non-zero element m of $\operatorname{Ker} A \cap \mathbf{Z}^n$, we see from (2.3) that

$$\Phi(\zeta + \sqrt{-1}m) = \Phi(\zeta) + \sqrt{-1}Am = \Phi(\zeta) \quad \text{for } \zeta \in T_\Omega,$$

which contradicts the assumption that Φ is injective.

Consider next the case where $\operatorname{Ker} A \cap \mathbf{Z}^n = \{0\}$. Let H be the closure $\overline{A\mathbf{Z}^n}$ of $A\mathbf{Z}^n$ in $\mathbf{R}^n$. Then H is a closed subgroup of the vector group $A\mathbf{R}^n$. Consequently, H is a Lie subgroup of the abelian Lie group $A\mathbf{R}^n$. Note that the identity component of H is a linear subspace of $A\mathbf{R}^n$.

When $\dim H = 0$, the group H is a lattice with

$$\text{rank } H \leq \dim A\mathbf{R}^n < n. \tag{2.5}$$

On the other hand, H contains the subgroup $A\mathbf{Z}^n$, which is isomorphic to the lattice $\mathbf{Z}^n$, because $\operatorname{Ker} A \cap \mathbf{Z}^n = \{0\}$ by the assumption. Therefore we have rank $H \geq n$. This contradicts (2.5).

Suppose that $\dim H > 0$. Fix a point ζ_0 of T_Ω and take a neighborhood W of ζ_0 such that $W \cap (\zeta_0 + \sqrt{-1}\mathbf{Z}^n) = \{\zeta_0\}$. Since $\Phi : T_\Omega \to T_{\Omega'}$ is a biholomorphic mapping, $\Phi(W)$ is a neighborhood of $\Phi(\zeta_0)$ and we have

$$\Phi(W) \cap \Phi(\zeta_0 + \sqrt{-1}\mathbf{Z}^n) = \{\Phi(\zeta_0)\}. \tag{2.6}$$

By $H = \overline{A\mathbf{Z}^n}$ and $\dim H > 0$, there exists a non-zero element of $A\mathbf{Z}^n$ arbitrarily close to 0. Therefore, since $\Phi(\zeta_0 + \sqrt{-1}m) = \Phi(\zeta_0) + \sqrt{-1}Am$ by (2.3), there exists an element of $\Phi(\zeta_0 + \sqrt{-1}\mathbf{Z}^n)$ arbitrarily close to $\Phi(\zeta_0)$ which does not coincide with $\Phi(\zeta_0)$ itself. This contradicts (2.6), and the lemma is proved. □

3. Proof of Theorem 1.2

Let D be a pseudoconvex homogeneous Reinhardt domain in $(\mathbf{C}^*)^n$. Then, D coinsides with $(\mathbf{C}^*)^n$ if and only if $l(D) = n$. Therefore, suppose $l(D) < n$ and we shall derive a contradiction.

If $l(D) = 0$, then D is algebraically equivalent to a bounded Reinhardt domain. By Theorem 1.1, every homogeneous bounded Reinhardt domain is algebraically equivalent to the direct product $D' := B_{n_1} \times \cdots \times B_{n_k}$ of balls, which contains the origin. Hence there exists a holomorphic isomorphism of D onto D' induced by an algebraic automorphism $(z_i) \mapsto (w_i)$ of $(\mathbf{C}^*)^n$ of the form $w_i = \alpha_i z_{\tau(i)}$, $i = 1, \ldots, n$, where τ is a permutation of $\{1, 2, \ldots, n\}$ and $(\alpha_1, \ldots, \alpha_n) \in (\mathbf{C}^*)^n$. Consequently, D contains the origin and this contradicts the assumption that $D \subset (\mathbf{C}^*)^n$.

Now we suppose $0 < l(D) < n$ and set $k = n - l(D)$. We divide the proof into four steps.

Step 1. Let T_Ω be the covering tube domain of D and let $\Gamma := \{\sigma_\eta \mid \eta \in \mathbf{Z}^n\}$ be its covering transformarion group. We denote by $\tilde{\varphi} \in \mathrm{Aut}(T_\Omega)$ a lifting of $\varphi \in \mathrm{Aut}(D)$. The set G of all liftings $\tilde{\varphi}$ forms a subgroup of $\mathrm{Aut}(T_\Omega)$ and is given as the normalizer of Γ in $\mathrm{Aut}(T_\Omega)$. Since the covering transformation group Γ is isomorphic to the additive group $\mathbf{Z}^n$, for every $\tilde{\varphi} \in G$ there exists a unique $A \in GL(n, \mathbf{Z})$ such that

$$\tilde{\varphi} \circ \sigma_\eta \circ \tilde{\varphi}^{-1} = \sigma_{A\eta}. \tag{3.1}$$

From this, we have a linear representation ρ of G on $GL(n, \mathbf{Z})$ such that $G \ni \tilde{\varphi} \mapsto A \in GL(n, \mathbf{Z})$. By the assumption, $\mathrm{Aut}(D)$ acts on D transitively. Therefore G acts on T_Ω transitively.

Next, we shall see what influences a permutation of coordinates has on the linear representation ρ of G. We consider a linear transformation of $\mathbf{C}^n$ represented by a matrix $P_\tau := (\delta_{\tau(i)j}) \in GL(n, \mathbf{Z})$, where τ is a permutation of $\{1, 2, \dots, n\}$. Then, the universal covering $\varpi : T_\Omega \to D$ is replaced by the covering $P_\tau \circ \varpi \circ P_\tau^{-1} : T_{P_\tau(\Omega)} \to P_\tau(D)$, and a lifting of $P_\tau \circ \varphi \circ P_\tau^{-1} \in \mathrm{Aut}(P_\tau(D))$ is given by $P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1} \in \mathrm{Aut}(T_{P_\tau(\Omega)})$. The equation (3.1) means that

$$\tilde{\varphi}(\zeta + \sqrt{-1}m) = \tilde{\varphi}(\zeta) + \sqrt{-1}Am \tag{3.2}$$

for every $\zeta \in T_\Omega$ and for every $m \in \mathbf{Z}^n$. Hence we have

$$\begin{aligned} P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1}(\zeta + \sqrt{-1}m) &= P_\tau \circ \tilde{\varphi}(P_\tau^{-1}(\zeta) + \sqrt{-1}P_\tau^{-1}(m)) \\ &= P_\tau(\tilde{\varphi}(P_\tau^{-1}(\zeta)) + \sqrt{-1}AP_\tau^{-1}(m)) \\ &= P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1}(\zeta) + \sqrt{-1}P_\tau A P_\tau^{-1} m \end{aligned}$$

for every $\zeta \in T_{P_\tau(\Omega)}$ and for every $m \in \mathbf{Z}^n$. Consequently the linear representation $\rho : G \to GL(n, \mathbf{Z})$ is just replaced by $\rho' : P_\tau G P_\tau^{-1} \ni P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1} \mapsto P_\tau A P_\tau^{-1} \in GL(n, \mathbf{Z})$.

Step 2. We would like to represent the domain $\Omega = \mathrm{ord}(D)$ in $\mathbf{R}^n$ as simple as possible. By means of a linear transformation L on $\mathbf{R}^n$ induced by a suitable permutation of coordinates, we make $V(D)$ parallel to some coordinate axes.

For simplicity, write $l := l(D)$. Since $\dim V(D) = l$, there exists a basis $\{v_1, \dots, v_l\}$ of $V(D)$ over $\mathbf{R}$. We write $V := (v_1, \dots, v_l)$, which is an $n \times l$ matrix consisting of column vectors $v_1, \dots, v_l$. As rank $V = l$, doing a suitable permutation of coordinates, we have

$$V = \begin{pmatrix} V^{(1)} \\ V^{(2)} \end{pmatrix},$$

where $V^{(1)}$ is a $k \times l$ matrix and $V^{(2)}$ is a non-singular $l \times l$ matrix. Moreover, by means of elementary transformations to column vectors of V if necessary, we have $V^{(2)} = I_l$ and

$$V = \begin{pmatrix} V^{(1)} \\ I_l \end{pmatrix}.$$

We define a matrix L by

$$L := \begin{pmatrix} I_k & -V^{(1)} \\ O & I_l \end{pmatrix} \in GL(n, \mathbf{R}). \tag{3.3}$$

Then we see

$$LV = \begin{pmatrix} I_k & -V^{(1)} \\ O & I_l \end{pmatrix} \begin{pmatrix} V^{(1)} \\ I_l \end{pmatrix} = \begin{pmatrix} O \\ I_l \end{pmatrix}.$$

Writing $\Xi := L\Omega$, we have

$$\Xi = \Xi^{(1)} \times \mathbf{R}^l, \tag{3.4}$$

where $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^k$ containing no complete straight lines. If we consider L as a linear transformation of $\mathbf{C}^n$, then $L(T_\Omega) = T_{L(\Omega)} = T_\Xi = T_{\Xi^{(1)}} \times \mathbf{C}^l$, and $\Phi := L \circ \tilde{\varphi} \circ L^{-1} \in \mathrm{Aut}(T_\Xi)$ satisfies

$$\Phi(w + \sqrt{-1}Lm) = \Phi(w) + \sqrt{-1}LAm \tag{3.5}$$

for every $w \in T_\Xi$ and for every $m \in \mathbf{Z}^n$. Indeed, by (3.2),

$$\begin{aligned} \Phi(w + \sqrt{-1}Lm) &= L \circ \tilde{\varphi} \circ L^{-1}(w + \sqrt{-1}Lm) \\ &= L \circ \tilde{\varphi}(L^{-1}w + \sqrt{-1}m) \\ &= L(\tilde{\varphi}(L^{-1}w) + \sqrt{-1}Am) \\ &= \Phi(w) + \sqrt{-1}LAm. \end{aligned}$$

Note that LGL^{-1} acts on T_Ξ transitively.

STEP 3. We consider a holomorphic autmorphism:

$$\Phi : T_{\Xi^{(1)}} \times \mathbf{C}^l \ni w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} \Phi^{(1)}(w) \\ \Phi^{(2)}(w) \end{pmatrix} \in T_{\Xi^{(1)}} \times \mathbf{C}^l,$$

where $w^{(1)}, \Phi^{(1)}(w) \in T_{\Xi^{(1)}}$ and $w^{(2)}, \Phi^{(2)}(w) \in \mathbf{C}^l$. Since $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^k$ containing no complete straight lines, $T_{\Xi^{(1)}}$ is holomorphically equivalent to a bounded domain in $\mathbf{C}^k$. Hence, Lemma 2.1 implies that the first component $\Phi^{(1)}$ depends only on $w^{(1)}$ and $\Phi^{(1)} : T_{\Xi^{(1)}} \ni w^{(1)} \mapsto \Phi^{(1)}(w^{(1)}) \in T_{\Xi^{(1)}}$ is a holomorphic automorphism.

We shall see a more precise form of $\Phi^{(1)}$. For every $A = \rho(\tilde{\varphi})$ with $\tilde{\varphi} \in G$, set

$$LA =: \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix}, \tag{3.6}$$

where $A^{(1)}$ is a $k \times k$ matrix, and $A^{(2)}$, $A^{(3)}$, $A^{(4)}$ are $k \times l$, $l \times k$, $l \times l$ matrices respectively. Note that the matrix L defined by (3.3) is independent of A. We see that $A^{(1)}$ is an element of $GL(k, \mathbf{R})$ later. Set

$$L^{(1)} := (I_k, -V^{(1)}), \quad (LA)^{(1)} := (A^{(1)}, A^{(2)}). \tag{3.7}$$

Then, by (3.5), $\Phi^{(1)}$ satisfies

$$\Phi^{(1)}(w^{(1)} + \sqrt{-1}L^{(1)}m) = \Phi^{(1)}(w^{(1)}) + \sqrt{-1}(LA)^{(1)}m \tag{3.8}$$

for every $w^{(1)} \in T_{\Xi^{(1)}}$ and for every $m \in \mathbf{Z}^n$. In particular, putting

$$m = {}^t(m_1, \dots, m_k, 0, \dots, 0),$$

and writing $m^{(1)} := {}^t(m_1, \dots, m_k)$, by (3.7) we have

$$L^{(1)}m = m^{(1)}, \quad (LA)^{(1)}m = A^{(1)}m^{(1)}.$$

Hence, (3.8) implies that

$$\Phi^{(1)}(w^{(1)} + \sqrt{-1}m^{(1)}) = \Phi^{(1)}(w^{(1)}) + \sqrt{-1}A^{(1)}m^{(1)} \tag{3.9}$$

for every $w^{(1)} \in T_{\Xi^{(1)}}$ and for every $m^{(1)} \in \mathbf{Z}^k$, and, by Lemma 2.2, the matrix $A^{(1)}$ is non-singular. By Proposition 2.2, we see $\Phi^{(1)} \in GL(k, \mathbf{R}) \ltimes \mathbf{C}^k$, that is, there exist $B^{(1)} \in GL(k, \mathbf{R})$ and $\beta^{(1)} \in \mathbf{C}^k$ such that

$$\Phi^{(1)}(w^{(1)}) = B^{(1)}w^{(1)} + \beta^{(1)}. \tag{3.10}$$

Substituting (3.10) into (3.9) yields $B^{(1)}m^{(1)} = A^{(1)}m^{(1)}$ for every $m^{(1)} \in \mathbf{Z}^k$. Consequently, we see that $B^{(1)} = A^{(1)}$, or

$$\Phi^{(1)}(w^{(1)}) = A^{(1)}w^{(1)} + \beta^{(1)} \quad \text{for every } w^{(1)} \in T_{\Xi^{(1)}}. \tag{3.11}$$

Note that $G^{(1)} := \{\Phi^{(1)} \mid \Phi \in LGL^{-1}\}$ acts on $T_{\Xi^{(1)}}$ transitively.

In (3.11), we can decompose $\Phi^{(1)}$ into real and imaginary components, since $A^{(1)} \in GL(k, \mathbf{R})$. Namely, write

$$w^{(1)} =: \xi^{(1)} + \sqrt{-1}\eta^{(1)}, \quad \beta^{(1)} =: a^{(1)} + \sqrt{-1}b^{(1)},$$

where $\xi^{(1)}$, $\eta^{(1)}$, $a^{(1)}$, $b^{(1)}$ are elements of $\mathbf{R}^k$. Then, we have

$$\Phi^{(1)}(w^{(1)}) = A^{(1)}\xi^{(1)} + a^{(1)} + \sqrt{-1}(A^{(1)}\eta^{(1)} + b^{(1)})$$

and the real component

$$\mathrm{Re}\,\Phi^{(1)}(\xi^{(1)}) := A^{(1)}\xi^{(1)} + a^{(1)} \tag{3.12}$$

gives an affine automorphism of the domain $\Xi^{(1)}$. Then, $H^{(1)} := \{\mathrm{Re}\,\Phi^{(1)} \mid \Phi^{(1)} \in G^{(1)}\}$ acts on $\Xi^{(1)}$ transitively.

Step 4. We see that in (3.12), the translational part $a^{(1)}$ is uniquely determined by the linear part $A^{(1)}$:

Lemma 3.1. *If $A^{(1)}\xi^{(1)} + a^{(1)}$ and $A^{(1)}\xi^{(1)} + b^{(1)}$ are elements of $H^{(1)}$, then $a^{(1)} = b^{(1)}$.*

Proof. By the assumptions, we have

$$A^{(1)}(\Xi^{(1)}) + a^{(1)} = \Xi^{(1)}, \quad A^{(1)}(\Xi^{(1)}) + b^{(1)} = \Xi^{(1)}.$$

Since $\Xi^{(1)} - a^{(1)} = A^{(1)}(\Xi^{(1)})$, it follows that

$$\Xi^{(1)} + (b^{(1)} - a^{(1)}) = (\Xi^{(1)} - a^{(1)}) + b^{(1)} = A^{(1)}(\Xi^{(1)}) + b^{(1)} = \Xi^{(1)}.$$

Since $\Xi^{(1)}$ is a convex domain containning no complete straight lines, this can only happen when $b^{(1)} - a^{(1)} = 0$. □

Note that in (3.6), $A^{(1)}$ is the $k \times k$ principal matrix of LA, where $L \in GL(n, \mathbf{R})$ is the fixed matrix determined by the domain D and $A \in \rho(G) \subset GL(n, \mathbf{Z})$. By Lemma 3.1, we have a surjection of $\rho(G)$ onto $H^{(1)}$. Since $\rho(G)$ is at most countable, so is $H^{(1)}$. This contradicts the fact that $H^{(1)}$ acts on $\Xi^{(1)}$ transitively, and the proof of Theorem 1.2 completely finished.

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