D. Q. SI
KODAI MATH. J.
35 (2012), 463–484

A FINITENESS THEOREM FOR MEROMORPHIC MAPPINGS SHARING FEW HYPERPLANES

Duc Quang Si

Abstract

In this article, we prove a finiteness theorem for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ sharing $2n+2$ hyperplanes without counting multiplicity.

1. Introduction

In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbf{C}$ cannot have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values [9].

In 1975, H. Fujimoto [5] generalized Nevanlinna's results to the case of meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. He considered two distinct meromorphic maps f and g of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ satisfying the condition that $\nu_{(f,H_j)} = \nu_{(g,H_j)}$ for q hyperplanes $H_1, H_2, \dots, H_q$ of $\mathbf{P}^n(\mathbf{C})$ in general position, where $\nu_{(f,H_j)}$ means the map of $\mathbf{C}^m$ into $\mathbf{Z}$ whose value $\nu_{(f,H_j)}(a)$ $(a \in \mathbf{C}^m)$ is the intersection multiplicity of the images of f and H_j at $f(a)$. He proved the following.

Theorem A [5]. *Let H_i, $1 \le i \le 3n+2$ be $3n+2$ hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in general position, and let f and g be two nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with $f(\mathbf{C}^m) \not\subseteq H_i$ and $g(\mathbf{C}^m) \not\subseteq H_i$ such that $\nu_{(f,H_i)} = \nu_{(g,H_i)}$ for $1 \le i \le 3n+2$, where $\nu_{(f,H_i)}$ and $\nu_{(g,H_i)}$ denote the pull-back of the divisors (H_i) on $\mathbf{P}^n(\mathbf{C})$ by f and g, respectively. Assume that either f or g is linearly non-degenerate over $\mathbf{C}$, i.e., the image does not included in any hyperplane in $\mathbf{P}^n(\mathbf{C})$. Then $f \equiv g$.*

Later on, the finiteness problem of meromorphic mappings sharing hyperplanes without counting multiplicities has been studied very intensively by many authors. Here we formulate some recent results on this problem.

2010 *Mathematics Subject Classification*. Primary 32H30, 32A22; Secondary 30D35.
Key words and phrases. truncated multiplicity, meromorphic mapping, unicity, finiteness.
Received September 30, 2011.

Take a meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ which is linearly nondegenerate over $\mathbf{C}$, a positive integer d and q hyperplanes $H_1, \dots, H_q$ of $\mathbf{P}^n(\mathbf{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \le m-2 \quad (1 \le i < j \le q)$$

and consider the set $\mathscr{F}(f, \{H_i\}_{i=1}^q, d)$ of all linearly nondegenerate over $\mathbf{C}$ meromorphic maps $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the following two conditions:

(a) $\min(\nu_{(f,H_j)}, d) = \min(\nu_{(g,H_j)}, d)$ $(1 \le j \le q)$,

(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

If $d = 1$, we will say that f and g share q hyperplanes $\{H_j\}_{j=1}^q$ without counting multiplicity.

Denote by $\#S$ the cardinality of the set S. In 1999, H. Fujimoto proved a finiteness theorem for meromorphic mappings as follows.

Theorem B. *If $q = 3n+1$ then $\#\mathscr{F}(f, \{H_i\}_{i=1}^q, 2) \le 2$.*

Some recent years, Z. Chen - Q. Yang [2] and S. D. Quang [11] considered the case where $q = 2n+3$ and they gave some uniqueness theorems for such meromorphic mappings. For the case where $q = 2n+2$, in [11] S. D. Quang proved that

Theorem C. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $H_1, \dots, H_{2n+2}$ be $2n+2$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position with*

$$\dim f^{-1}(H_i \cap H_j) \le m-2 \quad (1 \le i < j \le 2n+2).$$

Let g be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ satisfying:

(i) $$\min\{\nu_{(f,H_j),\le n}, 1\} = \min\{\nu_{(g,H_j),\le n}, 1\} \quad \textit{and}$$
$$\min\{\nu_{(f,H_j),\ge n}, 1\} = \min\{\nu_{(g,H_j),\ge n}, 1\} \quad (1 \le j \le 2n+2),$$

(ii) *$f(z) = g(z)$ on $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$.*

If $n \ge 2$ then $f \equiv g$.

However, in Theorem C, the condition (i) means that the multiplicities of the zeros of the functions (f, H_j) and (g, H_j) are considered to level n. Therefore the following question arises naturally: *"Are there any finiteness theorems for meromorphic mappings sharing $2n+2$ hyperplanes without counting multiplicity?"* It seems to us that the techniques used in the proofs of the above mentioned results are not enough to apply to this case.

Our main purpose in this paper is to give a positive answer for the above question. To do so, we will introduce some new techniques.

We would also like to note that in the definition of the family $\mathscr{F}(f, \{H_i\}_{i=1}^q, d)$, the meromorphic mapping g is assumed to be linearly nondegenerate. In this paper we will show that if $q \geq 2n+2$ then each map g satisfying the conditions (a) and (b) is linearly nondegenerate.

Now let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $H_1, \dots, H_q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq q).$$

Let d be an integer. We consider the set $\mathscr{G}(f, \{H_i\}_{i=1}^q, d)$ of all meromorphic maps $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the conditions:

(a) $\min(\nu_{(f,H_j)}, d) = \min(\nu_{(g,H_j)}, d)$ $(1 \leq j \leq q)$,

(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

Our main result is stated as follows.

Theorem 1.1. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $H_1, \dots, H_{2n+2}$ be $2n+2$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position with*

$$\dim f^{-1}(H_i \cap H_j) \leq m-2 \quad (1 \leq i < j \leq 2n+2).$$

If $n \geq 2$ then $\#\mathscr{G}(f, \{H_i\}_{i=1}^{2n+2}, 1) \leq 2$.

In the section §4, we will consider meromorphic mappings with three families of hyperplanes in $\mathbf{P}^n(\mathbf{C})$ and we will give a finiteness theorem for such maps.

Throughout this paper, we always assume that $n \geq 2$.

Acknowledgements. This work was done during a stay of the author at Mathematisches Forschungsinstitut Oberwolfach. He wishes to express his gratitude to this institute. The author would also like to thank Professors Junjiro Noguchi and Do Duc Thai for their valuable suggestions concerning this material. This work was supported in part by a NAFOSTED grant of Vietnam.

2. Basic notions in Nevanlinna theory

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbf{C}^m : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$\sigma(z) := (dd^c\|z\|^2)^{m-1} \quad \text{and} \quad \eta(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \quad \text{on } \mathbf{C}^m \backslash \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\mathscr{D}^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}$. We define the map $\nu_F : \Omega \to \mathbf{Z}$ by

$$\nu_F(z) := \max\{l : \mathscr{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < l\} \quad (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbf{C}^m$ a map $\nu : \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\le m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\le m-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or empty set.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \le m-2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbf{C}^m$ and for a positive integer M or $M = \infty$, we define the counting function of ν by

$$\nu^{(M)}(z) = \min\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \int_{B(t)} \nu(z)\sigma & \text{if } m \ge 2, \\ \sum_{|z| \le t} \nu(z) & \text{if } m = 1. \end{cases}$$

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}}\, dt \quad (1 < r < \infty).$$

For a meromorphic function φ on $\mathbf{C}^m$, we set $N_\varphi(r) = N(r, \nu_\varphi)$ and $N_\varphi^{(M)}(r) = N(r, \nu_\varphi^{(M)})$. We will omit the character ${}^{(M)}$ if $M = \infty$.

2.4. Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $I(f) = \{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r) = \int_{S(r)} \log\|f\|\eta - \int_{S(1)} \log\|f\|\eta.$$

Let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n = 0\}$, where $a := (a_0, \dots, a_n) \neq (0, \dots, 0)$. We set $(f, H) = \sum_{i=0}^n a_i f_i$. It is easy to see that the divisor $\nu_{(f,H)}$ does not depend on the choices of reduced representation of f and coefficients $a_0, \dots, a_n$. Moreover, we define the proximity function of f with respect to H by

$$m_{f,H}(r) = \int_{S(r)} \log \frac{\|f\| \cdot \|H\|}{|(f,H)|}\eta - \int_{S(1)} \log \frac{\|f\| \cdot \|H\|}{|(f,H)|}\eta,$$

where $\|H\| = (\sum_{i=0}^n |a_i|^2)^{1/2}$.

2.5. Let φ be a nonzero meromorphic function on $\mathbf{C}^m$, which is occationally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log^+|\varphi|\eta,$$

where $\log^+ t = \max\{0, \log t\}$ for $t > 0$. The Nevanlinna characteristic function of φ is defined by

$$T(r,\varphi) = N_{1/\varphi}(r) + m(r,\varphi).$$

There is a fact that

$$T_\varphi(r) = T(r,\varphi) + O(1).$$

The meromorphic function φ is said to be small with respect to f iff $\parallel\ T(r,\varphi) = o(T_f(r))$.

Here as usual, by the notation "$\parallel\ P$" we mean the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E dr < \infty$.

The following plays essential roles in Nevanlinna theory (see [10]).

Theorem 2.6 (First main theorem). *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping and let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ such that $f(\mathbf{C}^m) \not\subset H$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T_f(r) \quad (r > 1).$$

Theorem 2.7 (Second main theorem). *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \dots, H_q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then*

$$\parallel\ (q-n-1)T_f(r) \le \sum_{i=1}^{q} N^{(n)}_{(f,H_i)}(r) + o(T_f(r)).$$

Lemma 2.8 (Lemma on logarithmic derivative). *Let f be a nonzero meromorphic function on $\mathbf{C}^m$. Then*

$$\left\| \quad m\left(r, \frac{\mathscr{D}^\alpha(f)}{f}\right) = O(\log^+ T_f(r)) \quad (\alpha \in \mathbf{Z}^m_+). \right.$$

3. Proof of Theorem 1.1

Lemma 3.1. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Suppose that $g \in \mathscr{G}(f, \{H_i\}_{i=1}^q, 1)$ with $q \ge 2n+2$. Then g is linearly nondegenerate.*

Proof. Suppose that there exists a hyperplane H satisfying $g(\mathbf{C}^m) \subset H$. We assume that f has a reduce representation $f = (f_0 : \cdots : f_n)$ and $H =$

$\{(\omega_0 : \cdots : \omega_n) \mid \sum_{i=0}^n a_i\omega_i = 0\}$. We consider function $(f, H) = \sum_{i=0}^n a_i f_i$. Since f is linearly nondegenerate, $(f.H) \not\equiv 0$. On the other hand $(f, H)(z) = (g, H)(z) = 0$ for all $z \in \bigcup_{i=1}^q f^{-1}(H_i)$, hence

$$N_{(f,H)}(r) \geq \sum_{i=1}^q N^{(1)}_{(f,H_i)}(r).$$

It yields that

$$\begin{aligned} \| \ T_f(r) \geq N_{(f,H)}(r) &\geq \sum_{i=1}^q N^{(1)}_{(f,H_i)}(r) \\ &\geq \frac{(q-n-1)}{n} T_f(r) + o(T_f(r)) \geq \frac{n+1}{n} T_f(r) + o(T_f(r)). \end{aligned}$$

Letting $r \to +\infty$, we get $T_f(r) = 0$. This is a contradiction. Hence $g(\mathbf{C}^m)$ can not be contained in any hyperplanes of $\mathbf{P}^n(\mathbf{C})$. Therefore g is linearly nondegenerate. ■

Lemma 3.2. *Suppose $q \geq n+2$. Then*

$$\| \ T_g(r) = O(T_f(r)) \quad \text{and} \quad \| \ T_f(r) = O(T_g(r)) \quad \text{for each } g \in \mathcal{G}(f, \{H_i\}_{i=1}^q, 1).$$

Proof. By the Second Main Theorem, we have

$$\begin{aligned} \| \ (q-n-1)T_g(r) &\leq \sum_{i=1}^q N^{(n)}_{(g,H_i)}(r) + o(T_g(r)) \\ &\leq \sum_{i=1}^q nN^{(1)}_{(g,H_i)}(r) + o(T_g(r)) \leq qnT_f(r) + o(T_g(r)). \end{aligned}$$

Hence $\| \ T_g(r) = O(T_f(r))$. Similarly, we get $\| \ T_f(r) = O(T_g(r))$. ■

Let f^1 and f^2 be two distinct meromorphic mappings in $\mathcal{G}(f, \{H_i\}_{i=1}^q, 1)$. For $t = 1, \ldots, q$, we set

$$\begin{aligned} S^t_{f^1,f^2,>n} &:= \{z \in \mathbf{C}^m; \min\{\nu_{(f^1,H_t)}(z), \nu_{(f^2,H_t)}(z)\} > n\}, \\ S^t_{f^1,f^2,<n} &:= \{z \in \mathbf{C}^m; 0 < \max\{\nu_{(f^1,H_t)}(z), \nu_{(f^2,H_t)}(z)\} < n\}, \\ S^t_{f^1,f^2,n} &:= \overline{(S^t_{f^1,f^2,>n} \cup S^t_{f^1,f^2,<n})}. \end{aligned}$$

Then each $S^t_{f^1,f^2,n}$ is either an empty set or an analytic subset in $\mathbf{C}^m$ of codimension 1. We denote again by $S^t_{f^1,f^2,n}$ the reduced divisor on $\mathbf{C}^m$ with the support $S^t_{f^1,f^2,n}$.

Lemma 3.3. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and let $\{H_i\}_{i=1}^{2n+2}$ be $2n+2$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position.*

If there are two distinct maps f^1 *and* f^2 *in* $\mathcal{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$ *then the following assertions hold*

(i) $\| \ N(r, S^t_{f^1,f^2,n}) = o(T_f(r)), \ \forall t = 1, \dots, 2n+2,$

(ii) $\| \ (q - n - 1)T_{f^l}(r) = \sum_{t=1}^{q} N^{(n)}_{(f^l,H_t)}(r) + o(T_f(r)), \ l = 1, 2,$

(iii) *Moreover, if we assume further that* $\dfrac{(f^1, H_i)}{(f^2, H_i)} \not\equiv \dfrac{(f^1, H_j)}{(f^2, H_j)}$ *for all* $1 \le i, j \le q$, *then*

$$\| \ T_{f^1}(r) + T_{f^2}(r) = \sum_{v=i,j} (N^{(n)}_{(f^1,H_t)} + N^{(n)}_{(f^2,H_t)} - (n+1)N^{(1)}_{(f^1,H_t)}) + \sum_{v=1}^{q} N^{(1)}_{(f^1,H_v)} + o(T_f(r)).$$

Proof. By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f^1, H_1)}{(f^2, H_1)} \equiv \frac{(f^1, H_2)}{(f^2, H_2)} \equiv \cdots \equiv \frac{(f^1, H_{k_1})}{(f^2, H_{k_1})}}_{\text{group } 1} \not\equiv \underbrace{\frac{(f^1, H_{k_1+1})}{(f^2, H_{k_1+1})} \equiv \cdots \equiv \frac{(f^1, H_{k_2})}{(f^2, H_{k_2})}}_{\text{group } 2}$$

$$\not\equiv \underbrace{\frac{(f^1, H_{k_2+1})}{(f^2, H_{k_2+1})} \equiv \cdots \equiv \frac{(f^1, H_{k_3})}{(f^2, H_{k_3})}}_{\text{group } 3} \not\equiv \cdots \not\equiv \underbrace{\frac{(f^1, H_{k_{s-1}+1})}{(f^2, H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f^1, H_{k_s})}{(f^2, H_{k_s})}}_{\text{group } s},$$

where $k_s = 2n + 2$.

For each $1 \le t \le 2n+2$, we set

$$\sigma(t) = \begin{cases} t + n & \text{if } t \le n+2, \\ t + -n - 2 & \text{if } n+2 < t \le 2n+2, \end{cases}$$

and

$$P_t = (f^1, H_t)(f^2, H_{\sigma(t)}) - (f^2, H_t)(f^1, H_{\sigma(t)}).$$

Since $f^1 \not\equiv f^2$, the number of elements of each group is at most n. Hence $\dfrac{(f^1, H_t)}{(f^2, H_t)}$ and $\dfrac{(f^1, H_{\sigma(t)})}{(f^2, H_{\sigma(t)})}$ belong to distinct groups for all $t \le q - 2$. This means that $P_t \not\equiv 0$ $(1 \le t \le 2n+2)$. By changing indices again if necessary, we may assume that $i = 1$ and $j = \sigma(1)$.

Fix an index t with $1 \le t \le q$. For $z \notin I(f^1) \cup I(f^2) \cup \bigcup_{k \ne l} f^{-1}(H_k \cap H_l)$, it is easy to see that:

• If z is a zero point of (f^1, H_t) then z is a zero point of P_t with multiplicity at least $\min\{\nu_{(f^1,H_t)}, \nu_{(f^2,H_t)}\}$. Similarly, if z is a zero point of $(f^1, H_{\sigma(t)})$ then z is a zero point of P_t with multiplicity at least $\min\{\nu_{(f^1,H_{\sigma(t)})}, \nu_{(f^2,H_{\sigma(t)})}\}$.

• If z is a zero point of (f^1, H_v) with $v \notin \{t, \sigma(t)\}$ then z is a zero point of P_t (because $f^1(z) = f^2(z)$).

Thus, we have

$$(3.4)\qquad \nu_{P_t}(z) \geq \min\{\nu_{(f^1,H_t)}, \nu_{(f^2,H_t)}\} + \min\{\nu_{(f^1,H_{\sigma(t)})}, \nu_{(f^2,H_{\sigma(t)})}\} + \sum_{\substack{v=1\\ v\neq t,\sigma(t)}}^{q} \nu^{(1)}_{(f^1,H_v)}(z),$$

for all z outside the analytic set $I(f)\cup I(g)\cup\bigcup_{k\neq l} f^{-1}(H_k\cap H_l)$ of dimension $\leq m-2$.

For two integers a and b, set

$$S_{a,b,n} = \begin{cases} 1 & \text{if } \max\{a,b\} < n, \\ 0 & \text{if } \min\{a,b\} \leq n \leq \max\{a,b\} \\ 1 & \text{if } \min\{a,b\} > n. \end{cases}$$

It is easy to see that

$$\min\{a,b\} \geq \min\{a,n\} + \min\{b,n\} - n + S_{a,b,n}.$$

Therefore, inequality (3.4) implies that

$$(3.5)\qquad \nu_{P_t}(z) \geq \sum_{v=t,\sigma(t)} (\min\{\nu_{(f^1,H_v)}(z), n\} + \min\{\nu_{(f^2,H_v)}(z), n\} - n\min\{\nu_{(f^1,H_v)}(z), 1\} + S^v_{f^1,f^2,n}(z)) + \sum_{\substack{v=1\\ v\neq t,\sigma(t)}}^{q} \nu^{(1)}_{(f^1,H_v)}(z),$$

for all z outside the analytic set $I(f)\cup I(g)\cup\bigcup_{k\neq l} f^{-1}(H_k\cap H_l)$.

By integrating both sides of the above inequality, we get

$$N_{P_t}(r) \geq \sum_{v=t,\sigma(t)} (N^{(n)}_{(f^1,H_v)}(r) + N^{(n)}_{(f^2,H_v)}(r) - nN^{(1)}_{(f^1,H_v)}(r) + N(r, S^v_{f^1,f^2,n})) + \sum_{\substack{v=1\\ v\neq t,\sigma(t)}}^{q} N^{(1)}_{(f^1,H_v)}(r).$$

On the other hand, by Jensen's formula and by the definition of the characteristic function we have

$$\begin{aligned} N_{P_t}(r) &= \int_{S(r)} \log|P_t|\eta + O(1) \\ &\leq \int_{S(r)} \log(|(f^1,H_t)|^2 + |(f^1,H_{\sigma(t)}|^2)^{1/2}\eta \\ &\quad + \int_{S(r)} \log(|(f^2,H_t)|^2 + |(f^2,H_{\sigma(t)}|^2)^{1/2}\eta + O(1) \end{aligned}$$

$$
\begin{aligned}
&\le \int_{S(r)} \log(\|f^1\|(\|H_t\|^2 + \|H_{\sigma(i)}\|^2)^{1/2})\eta \\
&\quad + \int_{S(r)} \log(\|f^2\|(\|H_t\|^2 + \|H_{\sigma(t)}\|^2)^{1/2})\eta + O(1) \\
&= \int_{S(r)} \log\|f^1\|\eta + \int_{S(r)} \log\|f^2\|\eta + O(1) \\
&= T_{f^1}(r) + T_{f^2}(r) + O(1).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(3.6) \quad T_{f^1}(r) + T_{f^2}(r) &\ge \sum_{v=t,\sigma(t)} (N^{(n)}_{(f^1,H_v)}(r) + N^{(n)}_{(f^2,H_v)}(r) - nN^{(1)}_{(f^1,H_v)}(r) \\
&\qquad + N(r, S^v_{f^1,f^2,n})) + \sum_{\substack{v=1 \\ v\ne t,\sigma(t)}}^{2n+2} N^{(1)}_{(f^1,H_v)}(r) + o(T_f(r)) \\
&\ge \sum_{v=t,\sigma(t)} (N^{(n)}_{(f^1,H_v)}(r) + N^{(n)}_{(f^2,H_v)}(r) \\
&\qquad - nN^{(1)}_{(f^1,H_v)}(r) + N(r, S^v_{f^1,f^2,n})) \\
&\quad + \frac{1}{2} \sum_{\substack{v=1 \\ v\ne t,\sigma(t)}}^{2n+2} (N^{(1)}_{(f^1,H_v)}(r) + N^{(1)}_{(f^2,H_v)}(r)) + o(T_f(r)).
\end{aligned}
$$

By summing-up both sides of the above inequality over $t = 1, \dots, 2n+2$ and by the second main theorem we have

$$
\begin{aligned}
(3.7) \quad \| \ &(2n+2)(T_{f^1}(r) + T_{f^2}(r)) \\
&\ge 2 \sum_{v=1}^{2n+2} (N^{(n)}_{(f^1,H_v)}(r) + N^{(n)}_{(f^2,H_v)}(r) + N(r, S^v_{f^1,f^2,n})) + o(T_f(r)) \\
&\ge (2n+2)(T_{f^1}(r) + T_{f^2}(r)) + 2 \sum_{v=1}^{2n+2} N(r, S^v_{f^1,f^2,n}) + o(T_f(r)).
\end{aligned}
$$

The last equality yields that

$$
\| \ N(r, S^v_{f^1,f^2,n}) = o(T_f(r)), \quad \forall v = 1, \dots, 2n+2 \ \text{ and } \ l = 1, 2.
$$

It also yields that inequalities (3.6) and (3.7) become equalities. Hence, we have the followings

(i) $\| \ N(r, S^v_{f^1,f^2,n}) = o(T_f(r))$,

(ii) $\| \ (q - n - 1)T_{f^l}(r) = \sum_{v=1}^{q} N^{(n)}_{(f^l,H_v)}(r) + o(T_f(r)), \ l = 1, 2,$

(iii) $\| \ T_{f^1}(r) + T_{f^2}(r) = \sum_{v=t,\sigma(t)} (N^{(n)}_{(f^1,H_t)} + N^{(n)}_{(f^2,H_t)} - (n+1)N^{(1)}_{(f^1,H_t)})$

$$+ \sum_{v=1}^{2n+2} N^{(1)}_{(f^1,H_v)} + o(T_f(r)).$$

The lemma is proved. ■

Lemma 3.8. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and let $\{H_i\}_{i=1}^{2n+2}$ be $2n+2$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. If there are two distinct maps f^1 and f^2 in $\mathscr{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$ then the following assertions hold.*

(i)
$$\begin{aligned} \| \ & N^{(n)}_{(f^1,H_t)} + N^{(n)}_{(f^2,H_t)} - (n+1)N^{(1)}_{(f^1,H_t)} \\ &= N^{(n)}_{(f^1,H_s)} + N^{(n)}_{(f^2,H_s)} - (n+1)N^{(1)}_{(f^1,H_s)} \\ &\quad + o(T_f(r)), \quad \forall 1 \le t \le s \le 2n+2. \end{aligned}$$

(ii)
$$\begin{aligned} \| \ T_{f^1}(r) + T_{f^2}(r) &= 2(N^{(n)}_{(f^1,H_t)}(r) + N^{(n)}_{(f^2,H_t)}(r) - (n+1)N^{(1)}_{(f^1,H_t)}(r)) \\ &\quad + \sum_{v=1}^{q} N^{(1)}_{(f^1,H_v)}(r) + o(T_f(r)), \quad \forall 1 \le t \le 2n+2. \end{aligned}$$

Proof. Since $f^1 \not\equiv f^2$, for two arbitrary indices t and s in $\{1, \dots, 2n+2\}$ there exists an index $i \in \{1, \dots, 2n+2\}$ such that

$$\frac{(f^1,H_i)}{(f^2,H_i)} \not\equiv \frac{(f^1,H_t)}{(f^2,H_t)}, \quad \text{and} \quad \frac{(f^1,H_i)}{(f^2,H_i)} \not\equiv \frac{(f^1,H_s)}{(f^2,H_s)}.$$

By Lemma 3.3 (iii), we have

$$\begin{aligned} \Big\| \ & \sum_{v=t,i} (N^{(n)}_{(f^1,H_v)}(r) + N^{(n)}_{(f^2,H_v)}(r) - (n+1)N^{(1)}_{(f^1,H_v)}(r)) + \sum_{v=1}^{q} N^{(1)}_{(f^1,H_v)}(r) \\ &= T_{f^1}(r) + T_{f^2}(r) + o(T_f(r)) \\ &= \sum_{v=s,i} (N^{(n)}_{(f^2,H_t)}(r) - (n+1)N^{(1)}_{(f^1,H_t)}(r)) + \sum_{v=1}^{q} N^{(1)}_{(f^1,H_v)}(r) + o(T_f(r)). \end{aligned}$$

This yields that

$$\begin{aligned} \| \ & N^{(n)}_{(f^1,H_t)}(r) + N^{(n)}_{(f^2,H_t)}(r) - (n+1)N^{(1)}_{(f^1,H_t)}(r) \\ &= N^{(n)}_{(f^1,H_s)}(r) + N^{(n)}_{(f^2,H_s)}(r) - (n+1)N^{(1)}_{(f^1,H_s)}(r) + o(T_f(r)). \end{aligned}$$

Therefore the first assertion of the lemma is proved.

It is clear that the second assertion directly follows from Lemma 1.1 (iii) and the first assertion. We complete the proof of the lemma. ■

Proposition 3.9. *Let f^0, f^1, f^2 be three maps in $\mathcal{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$. Assume that there exists an index i such that $\| \ N^{(1)}_{(f,H_i)}(r) = o(T_f(r))$ then*

$$f^0 = f^1 \quad or \quad f^1 = f^2 \quad or \quad f^2 = f^0.$$

Proof. Suppose that f^0, f^1, f^2 are three distinct maps in $\mathcal{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$. By the assumption and by Lemma 3.8(i), we have

$$\| \ N^{(n)}_{(f^k,H_t)}(r) + N^{(n)}_{(f^l,H_t)}(r) - (n+1)N^{(1)}_{(f^k,H_t)}(r) = o(T_f(r)), \tag{3.10}$$

$$N(r, S^t_{f^k,f^l,n}) = o(T_f(r)), \quad \forall 1 \le t \le 2n+2, \tag{3.11}$$

where $\{k,l\} \subset \{0,1,2\}$.

Assume that z is a zero of (f, H_t) satisfying $z \notin S^t_{f^k,f^l,n}$, $\forall k,l$. Then

$$\max\{v_{(f^k,H_t)}(z), v_{(f^l,H_t)}(z)\} \ge n, \quad \forall k,l \in \{0,1,2\}.$$

Therefore, there are at least two indices k,l $(0 \le k < l \le 2)$ such that $v_{(f^k,H_t)}(z) \ge n$ and $v_{(f^l,H_t)}(z) \ge n$. Hence

$$\min\{n, v_{(f^0,H_t)}(z)\} + \min\{n, v_{(f^l,H_t)}(z)\} + \min\{n, v_{(f^2,H_t)}(z)\} \ge 2n+1.$$

This implies that

$$\sum_{k=0}^{2} N^{(n)}_{(f^k,H_t)}(r) \ge (2n+1)N^{(1)}_{(f,H_t)}(r) + o(T_f(r)). \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\begin{aligned} \| \ 3(n+1)N^{(1)}_{(f,H_t)}(r) &= 2\sum_{k=0}^{2} N^{(n)}_{(f^k,H_t)}(r) + o(T_f(r)) \\ &\ge 2(2n+1)N^{(1)}_{(f,H_t)}(r) + o(T_f(r)). \end{aligned}$$

Thus

$$\| \ N^{(1)}_{(f,H_t)}(r) = o(T_f(r)), \quad \forall t = 1, \dots, 2n+2.$$

By the second main theorem, we have

$$(n+1)T_f(r) \le \sum_{t=1}^{2n+2} N^{(n)}_{(f,H_t)}(r) + o(T_f(r)) \le n\sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)) = o(T_f(r)).$$

This is a contradiction. Therefore f^0, f^1, f^2 are not three distinct maps. The lemma is proved. ∎

Proposition 3.13. *Let f and $\{H_i\}_{i=1}^{2n+2}$ be as in Lemma 3.9. Let f^1 and f^2 be two distinct maps in $\mathcal{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$. Assume that there exist two indices $1 \le i < j \le 2n+2$ such that $\dfrac{(f^1, H_i)}{(f^1, H_j)} = \dfrac{(f^1, H_i)}{(f^1, H_j)}$. Then $\| \ N^{(1)}_{(f^1,H_i)}(r) = N^{(1)}_{(f^1,H_j)}(r) = o(T_f(r))$.*

Proof. Since $\frac{(f^1,H_i)}{(f^2,H_j)} = \frac{(f^1,H_i)}{(f^2,H_j)}$, we have $v_{(f^1,H_i)} = v_{(f^2,H_i)}$ and $v_{(f^1,H_j)} = v_{(f^2,H_j)}$. By changing the representations of f^1 and f^2, we may assume that $(f^1,H_i) = (f^2,H_i)$. This yields that $(f^1,H_j) = (f^2,H_j)$. Since $f^1 \not\equiv f^2$, there exits an index k $(k \neq i, k \neq j)$ such that

$$P \overset{Def}{=} (f^1,H_i)(f^2,H_k) - (f^1,H_k)(f^2,H_i) = (f^1,H_i)((f^1,H_k) - (f^2,H_k)) \not\equiv 0.$$

By the assumption $(f^1,H_t) = (f^2,H_t)$ on $\bigcup_{l=1}^q f^{-1}(H_l) \backslash (f^{-1}(H_i) \cap f^{-1}(H_j))$, $1 \le t \le q$, we have

$$\begin{aligned} v_P(z) &\ge v_{(f^1,H_i)}(z) + \min\{1, v_{(f^1,H_i)}\}(z) + \min\{v_{(f^1,H_k)}(z), v_{(f^2,H_k)}(z)\} \\ &\quad + \sum_{\substack{t=1 \\ t \neq i,k}}^q \min\{1, v_{(f^1,H_t)}\}(z) \\ &\ge v_{(f^1,H_i)}(z) + \min\{1, v_{(f^1,H_i)}\}(z) + \min\{n, v_{(f^1,H_k)}(z)\} + \min\{n, v_{(f^2,H_k)}(z)\} \\ &\quad - n\min\{1, v_{(f^1,H_k)}(z)\} + \sum_{\substack{t=1 \\ t \neq i,k}}^q \min\{1, v_{(f^1,H_t)}\}(z), \end{aligned}$$

for all $z \in \bigcup_{s=1}^q f^{-1}(H_s) \backslash (\bigcup_{s \neq t} f^{-1}(H_s) \cap f^{-1}(H_t))$. This implies that

$$(3.14) \qquad \begin{aligned} \| \ N_P(r) &\ge N_{(f^1,H_i)}(r) + N^{(1)}_{(f^1,H_i)}(r) + N^n_{(f^1,H_k)}(r) + N^n_{(f^2,H_k)}(r) \\ &\quad - nN^{(1)}_{(f^1,H_k)}(r) + \sum_{\substack{t=1 \\ t \neq i,j}}^q N^{(1)}_{(f,H_t)}(r) + o(T_f(r)) \\ &\ge \sum_{l=1,2} (N^n_{(f^l,H_k)}(r) + N^n_{(f^l,H_k)}(r) - (n+1)N^{(1)}_{(f^l,H_k)}(r)) \\ &\quad + N^{(1)}_{(f,H_i)}(r) + \sum_{t=1}^q N^{(1)}_{(f,H_t)}(r) + o(T_f(r)). \end{aligned}$$

By Lemma 1.1 (iii), we have

$$(3.15) \qquad \begin{aligned} \| \ N_P(r) &\le T_{f^1}(r) + T_{f^2}(r) \\ &= \sum_{l=1,2} (N^n_{(f^l,H_k)}(r) + N^n_{(f^l,H_k)}(r) - (n+1)N^{(1)}_{(f^l,H_k)}(r)) \\ &\quad + \sum_{t=1}^q N^{(1)}_{(f,H_t)}(r) + o(T_f(r)). \end{aligned}$$

Combining (3.14) and (3.15), we have $\| \ N^{(1)}_{(f,H_i)}(r) = o(T_f(r))$. Similarly, we also have $\| \ N^{(1)}_{(f,H_j)}(r) = o(T_f(r))$. The lemma is proved. ∎

Now for three mappings $f^0, f^1, f^2 \in \mathscr{F}(f, \{H_j\}_{j=1}^{2n+2}, 1)$, we set

$$F_k^{ij} = \frac{(f^k, H_i)}{(f^k, H_j)} \quad (0 \le k \le 2, 1 \le i < j \le 2n+2).$$

For meromorphic functions F, G, H on $\mathbf{C}^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{Z}_+^m$, we put

$$\Phi^\alpha(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{1}{F} & \dfrac{1}{G} & \dfrac{1}{H} \\ \mathscr{D}^\alpha\left(\dfrac{1}{F}\right) & \mathscr{D}^\alpha\left(\dfrac{1}{G}\right) & \mathscr{D}^\alpha\left(\dfrac{1}{H}\right) \end{vmatrix}$$

Lemma 3.16. *Let f^0, f^1, f^2 be three distinct maps in $\mathscr{G}(f, \{H_i\}_{i=1}^{2n+2}, 1)$. Assume that there exist $i, j \in \{1, 2, \dots, q\}$ $(i \ne j)$ such that $\Phi^\alpha := \Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) \equiv 0$ for all $|\alpha| = 1$. Then the following assertions hold*:

(i) $\| \; N^{(1)}_{(f^k, H_i)}(r) = \dfrac{1}{n} N^{(n)}_{(f^k, H_i)}(r) + o(T_f(r)),\ \forall k = 0, 1, 2,$

(ii) $\| \; 2 \sum_{v=i,j} N^{(1)}_{(f, H_v)}(r) \ge \sum_{v=1}^{2n+2} N^{(1)}_{(f, H_v)}(r) + o(T_f(r)),$

(iii) $\| \; N^{(n)}_{(f^k, H_i)}(r) + N^{(n)}_{(f^l, H_i)}(r) \le \dfrac{1}{2}(T_{f^k}(r) + T_{f^l}(r)) + o(T_f(r)).$

Proof. (i) Since f^0, f^1, f^2 are three distinct maps, it follows from Lemma 3.9 and Lemma 3.13 that $F_0^{ij} \not\equiv F_1^{ij} \not\equiv F_2^{ij} \not\equiv F_0^{ij}$. Then we have

$$\Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) = 0 \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ F_0^{ji} & F_1^{ji} & F_2^{ji} \\ \mathscr{D}^\alpha(F_0^{ji}) & \mathscr{D}^\alpha(F_1^{ji}) & \mathscr{D}^\alpha(F_2^{ji}) \end{vmatrix} = 0$$

$$\Leftrightarrow (F_1^{ji} - F_0^{ji}) \cdot \mathscr{D}^\alpha(F_2^{ji} - F_0^{ji}) - (F_2^{ji} - F_0^{ji}) \cdot \mathscr{D}^\alpha(F_1^{ji} - F_0^{ji}) = 0$$

$$\Leftrightarrow \mathscr{D}^\alpha\left(\frac{F_2^{ji} - F_0^{ji}}{F_1^{ji} - F_0^{ji}}\right) = 0.$$

Since the above assertions hold for each $|\alpha| = 1$, then there exists a constant $c \in \mathbf{C} \setminus \{0, 1\}$ such that

$$F_2^{ji} - F_0^{ji} = c(F_1^{ji} - F_0^{ji}) \Leftrightarrow (1 - c)F_0^{ji} + cF_1^{ji} = F_2^{ji}. \tag{3.17}$$

By (3.17), it is easy to see that outside an analytic subset of codim ≥ 2 of $\mathbf{C}^n$ we have

$$\nu_{(f^k, H_j)}(z) \ge \min\{\nu_{(f^l, H_j)}(z), \nu_{(f^t, H_j)}(z)\},$$

where $\{k, l, t\} = \{0, 1, 2\}$.

Therefore, if z is a point of $f^{-1}(H_i) \setminus (\bigcup_{k \ne l} S^i_{f^k, f^l, n})$ then

$$n = \nu_{(f^k, H_i)}(z) = \nu_{(f^l, H_i)}(z) \le \nu_{(f^t, H_i)}(z),$$

where (k,l,t) is a permutation of $(0,1,2)$. This implies that

$$\| \ N^{(n)}_{(f^k,H_i)}(z) = nN^{(1)}_{(f^k,H_i)}(z) + o(T_f(r)), \quad \forall k = 0,1,2.$$

Hence the first assertion is proved.

(ii) Consider the meromorphic mapping $F : \mathbf{C}^m \to \mathbf{P}^1(\mathbf{C})$ with a reduced representation $F = (F_0^{ji}h : F_1^{ji}h)$, where h is a meromorphic function on $\mathbf{C}^m$.

We see that

$$T_F(r) = T\left(r, \frac{F_0^{ji}}{F_1^{ji}}\right) \le T(r, F_0^{ji}) + T\left(r, \frac{1}{F_1^{ji}}\right) + O(1)$$
$$\le T(r, F_0^{ji}) + T(r, F_1^{ji}) + O(1) \le T_{f_0}(r) + T_{f_1}(r) + O(1) = O(T_f(r)).$$

It is also clear that if z is a zero $F_t^{ji}h$ $(0 \le t \le 2)$ then z must be either zero of (f, H_i) or zero of (f, H_j). Therefore

$$N^{(1)}_{F_0^{ji}h}(r) + N^{(1)}_{F_1^{ji}h}(r) + N^{(1)}_{F_2^{ji}h}(r) \le N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r). \tag{3.18}$$

Applying the Second Main Theorem to the map F and the points $\{w_0 = 0\}$, $\{w_1 = 0\}$, $\{(1-c)w_0 + cw_1 = 0\}$ in $\mathbf{P}^1(\mathbf{C})$, we have

$$\| \ T_F(r) \le N^{(1)}_{F_0^{ji}h}(r) + N^{(1)}_{F_1^{ji}h}(r) + N^{(1)}_{F_2^{ji}h}(r) + o(T_F(r))$$
$$\le N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r) + o(T_f(r)).$$

Applying the First Main Theorem to the map F and the hyperplane $\{w_0 - w_1 = 0\}$ in $\mathbf{P}^1(\mathbf{C})$, we have

$$T_F(r) \ge N_{(F_0^{ji} - F_1^{ji})h}(r) \ge \sum_{\substack{v=1 \\ v \ne i,j}}^{2n+2} N^{(1)}_{(f,H_v)}(r).$$

Thus

$$\| \ 2(N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r)) \ge \sum_{\substack{v=1 \\ v \ne i,j}}^{2n+2} N^{(1)}_{(f,H_v)}(r) + o(T_f(r)). \tag{3.19}$$

Hence, the second assertion is proved.

(iii) By Lemma 3.8(ii) and by the Second Main Theorem, we have

$$\| \ T_{f^k}(r) + T_{f^l}(r) = 2(N^{(n)}_{(f^k,H_i)}(r) + N^{(n)}_{(f^l,H_i)}(r) - (n+1)N^{(1)}_{(f,H_i)}(r))$$
$$= \sum_{v=k,l}\left(2N^{(n)}_{(f^v,H_i)}(r) - (n+1)N^{(1)}_{(f^v,H_i)}(r) + \frac{1}{2}\sum_{t=1}^{2n+2} N^{(1)}_{(f^v,H_t)}(r)\right)$$
$$+ o(T_f(r))$$
$$\ge \sum_{v=k,l}\left(\frac{n-1}{n}N^{(n)}_{(f^v,H_i)}(r) + \frac{n+1}{2n}T_{f^v}(r)\right) + o(T_f(r)).$$

Thus

$$\| \ N^{(n)}_{(f^k,H_i)}(r) + N^{(n)}_{(f^l,H_i)}(r) \le T_{f^k}(r) + T_{f^l}(r) + o(T_f(r)).$$

Hence the third assertion is proved. ∎

LEMMA 3.20. *Let f^0, f^1, f^2 be three maps in $\mathscr{G}(f,\{H_i\}_{i=1}^q,1)$. Assume that there exist $i,j \in \{1,2,\dots,q\}$ $(i \neq j)$ and $|\alpha| = 1$ such that $\Phi^\alpha := \Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) \not\equiv 0$. Then, for each $0 \le k \le 2$, the following assertions hold*

(i) $\sum_{k=0}^2 N^{(n)}_{(f^k,H_i)}(r) + 2\sum_{\substack{t=1\\ t\neq i,j}}^q N^{(1)}_{(f,H_t)}(r) - (2n+1)N^{(1)}_{(f,H_i)}(r) \le N_{\Phi^\alpha}(r)$,

(ii) $N_{\Phi^\alpha}(r) \le \sum_{k=0}^2 T_{f^k}(r) - \sum_{k=0}^2 N^{(n)}_{(f^k,H_j)}(r) + nN^{(1)}_{(f,H_j)}(r) + o(T_f(r))$,

(iii) *Moreover, if we assume further that $\Phi^\alpha(F_0^{ji}, F_1^{ji}, F_2^{ji}) \not\equiv 0$ for all $|\alpha| = 1$ then*

$$2(N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r)) \ge \sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

Proof. (i) Denote by S the set of all singularities of $f^{-1}(H_t)$ $(1 \le t \le q)$. Then S is an analytic subset of codimension at least two in $\mathbf{C}^m$. We set

$$I = S \cup \bigcup_{s\neq t} (f^{-1}(H_t) \cap f^{-1}(H_t)).$$

Then I is also an analytic subset of codimension at least two in $\mathbf{C}^m$.

Assume that z_0 is a zero point of (f,H_t) with $t \notin \{i,j\}$ and $z_0 \notin I$. There exists a holomorphic function h on an open neighborhood U of z_0 such that $\nu_h = \min\{1, \nu_{(f,H_t)}\}$ on U. Since $|\alpha| = 1$, we may write

$$\begin{aligned}
(3.21)\qquad &\Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij})\\
&= F_0^{ij}\cdot F_1^{ij}\cdot F_2^{ij} \times \begin{vmatrix} (F_0^{ji} - F_1^{ji}) & (F_0^{ji} - F_2^{ji}) \\ \mathscr{D}^\alpha(F_0^{ji} - F_1^{ji}) & \mathscr{D}^\alpha(F_0^{ji} - F_2^{ji}) \end{vmatrix}\\
&= h^2 F_0^{ij}\cdot F_1^{ij}\cdot F_2^{ij} \times \begin{vmatrix} (h^{-1}(F_0^{ji} - F_1^{ji})) & (h^{-1}(F_0^{ji} - F_2^{ji})) \\ \mathscr{D}^\alpha(h^{-1}(F_0^{ji} - F_1^{ji})) & \mathscr{D}^\alpha(h^{-1}(F_0^{ji} - F_2^{ji})) \end{vmatrix}.
\end{aligned}$$

This yields that

$$(3.22)\qquad \nu_{\Phi^\alpha}(z_0) \ge 2 = 2\min\{1, \nu_{(f,H_t)}(z_0)\}.$$

Now assume that z_0 is a zero point of (f,H_i) and $z \notin I$. Without loss of generality, we may assume that $\nu_{(f^0,H_i)}(z_0) \le \nu_{(f^1,H_i)}(z_0) \le \nu_{(f^2,H_i)}(z_0)$. We may write

$$\Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) = F_0^{ij} \cdot F_1^{ij} \cdot F_2^{ij} \times \begin{vmatrix} (F_0^{ji} - F_1^{ji}) & (F_0^{ji} - F_2^{ji}) \\ \mathscr{D}^\alpha(F_0^{ji} - F_1^{ji}) & \mathscr{D}^\alpha(F_0^{ji} - F_2^{ji}) \end{vmatrix}$$
$$= F_0^{ij}[F_1^{ij}(F_0^{ji} - F_1^{ji})F_2^{ij}\mathscr{D}^\alpha((F_0^{ji} - F_2^{ji}))$$
$$- F_2^{ij}(F_0^{ji} - F_2^{ji})F_1^{ij}\mathscr{D}^\alpha((F_0^{ji} - F_1^{ji}))]. \tag{3.23}$$

It is easy to see that $F_0^{ij}(F_0^{ji} - F_1^{ji})$ and $F_2^{ij}(F_0^{ji} - F_2^{ji})$ are holomorphic on a neighborhood of z_0. We also have

$$v^\infty_{F_2^{ij}\mathscr{D}^\alpha(F_0^{ji} - F_2^{ji})}(z_0) \le 1$$

and

$$v^\infty_{F_1^{ij}\mathscr{D}^\alpha(F_0^{ji} - F_1^{ji})}(z_0) \le 1.$$

Therefore, equality (3.23) implies that

$$v_{\Phi^\alpha}(z_0) \ge v_{(f^0, H_i)}(z_0) - 1$$
$$\ge \sum_{k=0}^{2} \min\{n, v_{(f^k, H_i)}(z_0)\} - (2n+1)\min\{1, v_{(f, H_i)}(z_0)\}. \tag{3.24}$$

Integrating both sides of the above inequality, we obtain that

$$\sum_{k=0}^{2} N^{(n)}_{(f^k, H_i)}(r) + \sum_{\substack{t=1 \\ t \ne i,j}}^{q} N^{(1)}_{(f, H_t)}(r) - (2n+1)N^{(1)}_{(f, H_i)}(r) \le N_{\Phi^\alpha}(r).$$

The first assertion of the lemma is proved.

(ii) We now prove the second assertion of the lemma. We have

$$\Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) = F_0^{ij} \cdot F_1^{ij} \cdot F_2^{ij} \cdot \begin{vmatrix} 1 & 1 & 1 \\ F_0^{ji} & F_1^{ji} & F_2^{ji} \\ \mathscr{D}^\alpha(F_0^{ji}) & \mathscr{D}^\alpha(F_1^{ji}) & \mathscr{D}^\alpha(F_2^{ji}) \end{vmatrix}$$
$$= \begin{vmatrix} F_0^{ij} & F_1^{ij} & F_2^{ij} \\ 1 & 1 & 1 \\ F_0^{ij}\mathscr{D}^\alpha(F_0^{ji}) & F_1^{ij}\mathscr{D}^\alpha(F_1^{ji}) & F_2^{ij}\mathscr{D}^\alpha(F_2^{ji}) \end{vmatrix}$$
$$= F_0^{ij}\left(\frac{\mathscr{D}^\alpha(F_2^{ji})}{F_2^{ji}} - \frac{\mathscr{D}^\alpha(F_1^{ji})}{F_1^{ji}}\right) + F_1^{ij}\left(\frac{\mathscr{D}^\alpha(F_0^{ji})}{F_0^{ji}} - \frac{\mathscr{D}^\alpha(F_2^{ji})}{F_2^{ji}}\right)$$
$$+ F_2^{ij}\left(\frac{\mathscr{D}^\alpha(F_1^{ji})}{F_1^{ji}} - \frac{\mathscr{D}^\alpha(F_0^{ji})}{F_0^{ji}}\right).$$

By the Logarithmic Derivative Lemma, it follows that

$$(3.25)\qquad m(r,\Phi^{\alpha}) \leq \sum_{v=0}^{2} m(r,F_v^{ij}) + 2\sum_{v=0}^{2} m\left(\frac{\mathscr{D}^{\alpha}(F_v^{ji})}{F_v^{ji}}\right) + O(1)$$
$$\leq \sum_{v=0}^{2} m(r,F_v^{ij}) + o(T_f(r)).$$

On the other hand, by (3.22) and (3.24), Φ^{α} is holomorphic at all zeros of (f,H_i). Hence a zero of (f,H_i) is not pole of Φ^{α}. Thus, a pole of Φ^{α} is a zero of (f,H_j). Assume that z_0 is a zero of (f,H_j), and $z \notin I$. Put $d = \min_{0\leq k\leq 2}\{\nu_{(f^k,H_j)}(z)\}$. Choose a holomorphic function h on $\mathbf{C}^m$ with multiplicity d at z such that $F_0^{ji} = h\cdot\varphi_0$, $F_1^{ji} = h\cdot\varphi_1$, $F_2^{ji} = h\cdot\varphi_2$, where φ_v are meromorphic on $\mathbf{C}^m$ and holomorphic on a neighborhood of z. Then

$$\Phi^{\alpha}(F_0^{ij},F_1^{ij},F_2^{ij}) = F_0^{ij}\cdot F_1^{ij}\cdot F_2^{ij}\cdot \begin{vmatrix} 1 & 1 & 1 \\ F_0^{ji} & F_1^{ji} & F_2^{ji} \\ \mathscr{D}^{\alpha}(F_0^{ji}) & \mathscr{D}^{\alpha}(F_1^{ji}) & \mathscr{D}^{\alpha}(F_2^{ji}) \end{vmatrix}$$
$$= F_0^{ij}\cdot F_1^{ij}\cdot F_2^{ij}\cdot \begin{vmatrix} F_1^{ji}-F_0^{ji} & F_2^{ji}-F_0^{ji} \\ \mathscr{D}^{\alpha}(F_1^{ji}-F_0^{ji}) & \mathscr{D}^{\alpha}(F_2^{ji}-F_0^{ji}) \end{vmatrix}$$
$$= F_0^{ij}\cdot F_1^{ij}\cdot F_2^{ij}\cdot h^2\cdot \begin{vmatrix} \varphi_1-\varphi_0 & \varphi_2-\varphi_0 \\ \mathscr{D}^{\alpha}(\varphi_1-\varphi_0) & \mathscr{D}^{\alpha}(\varphi_2-\varphi_0) \end{vmatrix}$$

Hence z is a pole of Φ^{α} with multiplicity at most $\sum_{v=0}^{2}\nu_{(f^v,H_j)}(z) - 2a$. We have

$$(3.26)\qquad \nu_{1/\Phi^{\alpha}}(z_0) \leq \sum_{k=0}^{2}\nu_{(f^k,H_j)}(z_0) - 2d = \sum_{k=0}^{2}\nu_{(f^k,H_j)}(z_0) - 2\min_{0\leq k\leq 2}\{\nu_{(f^k,H_j)}(z_0)\}$$
$$= \sum_{k=0}^{2}\nu_{F_k^{ij}}(z) - \sum_{k=0}^{2}\min\{n,\nu_{(f^k,H_j)}(z_0)\} + n\min\{1,\nu_{(f,H_j)}(z_0)\}.$$

This yields that

$$(3.27)\qquad N_{1/\Phi^{\alpha}}(r) \leq \sum_{k=0}^{2} N_{F_k^{ji}}(r) - \sum_{k=0}^{2} N^{(n)}_{(f^k,H_j)}(r) + nN^{(1)}_{(f,H_j)}(r).$$

From (3.25) and (3.27) we get

$$N_{\Phi^{\alpha}}(r) \leq T(r,\Phi^{\alpha}) + O(1) = m(r,\Phi^{\alpha}) + N_{1/\Phi^{\alpha}}(r) + O(1)$$
$$\leq \sum_{k=0}^{2}(m(r,F_k^{ij}) + N_{F_k^{ji}}(r)) - \sum_{k=0}^{2} N^{(n)}_{(f^k,H_j)}(r) + nN^{(1)}_{(f,H_j)}(r) + o(T_f(r))$$

$$= \sum_{k=0}^{2} T(r, F_k^{ij}) - \sum_{k=0}^{2} N^{(n)}_{(f^k, H_j)}(r) + nN^{(1)}_{(f, H_j)}(r) + o(T_f(r))$$

$$\leq \sum_{k=0}^{2} T_{f^k}(r) - \sum_{k=0}^{2} N^{(n)}_{(f^k, H_j)}(r) + nN^{(1)}_{(f, H_j)}(r) + o(T_f(r)).$$

This implies the second assertion of the lemma.

(iii) Now we assume that $\Phi^{\alpha}(F_0^{ji}, F_1^{ji}, F_2^{ji}) \not\equiv 0$. By the second assertion of the lemma, we have

$$\begin{aligned}\sum_{k=0}^{2} T_{f^k}(r) \geq \sum_{k=0}^{2} &(N^{(n)}_{(f^k, H_i)}(r) + N^{(n)}_{(f^k, H_j)}(r)) + 2\sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) \\ &- (2n+3)N^{(1)}_{(f, H_i)}(r) - (n+2)N^{(1)}_{(f, H_j)}(r) + o(T_f(r))\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2} T_{f^k}(r) \geq \sum_{k=0}^{2} &(N^{(n)}_{(f^k, H_i)}(r) + N^{(n)}_{(f^k, H_j)}(r)) + 2\sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) \\ &- (2n+3)N^{(1)}_{(f, H_j)}(r) - (n+2)N^{(1)}_{(f, H_i)}(r) + o(T_f(r)).\end{aligned}$$

Summing-up both sides of these above inequalities, we get

$$\begin{aligned}(3.28)\qquad 2\sum_{k=0}^{2} T_{f^k}(r) &\geq 2\sum_{k=0}^{2}(N^{(n)}_{(f^k, H_i)}(r) + N^{(n)}_{(f^k, H_j)}(r)) + 4\sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) \\ &\quad - (3n+5)N^{(1)}_{(f, H_i)}(r) - (3n+5)N^{(1)}_{(f, H_j)}(r) + o(T_f(r)) \\ &= \sum_{0 \leq k < l \leq 2} \Bigg(\sum_{v=i,j} (N^{(n)}_{(f^k, H_v)}(r) + N^{(n)}_{(f^l, H_v)}(r) \\ &\qquad - (n+1)N^{(1)}_{(f, H_v)}) + \sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) \Bigg) \\ &\quad - \sum_{v=i,j} 2N^{(1)}_{(f, H_v)} + \sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) + o(T_f(r)).\end{aligned}$$

From Lemma 3.7(iv) and the inequality (3.28), it follows that

$$2\sum_{k=0}^{2} T_{f^k}(r) \geq 2\sum_{k=0}^{2} T_{f^k}(r) - 2\sum_{v=i,j} N^{(1)}_{(f, H_v)}(r) + \sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) + o(T_f(r)).$$

Thus

$$2\sum_{v=i,j} N^{(1)}_{(f,H_v)}(r) \geq \sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

The third assertion is proved. ■

Proof of Theorem 1.1. Suppose that there exist three distinct maps f^0, f^1, f^2 in $\mathcal{G}(f,\{H_i\}_{i=1}^{2n+2},1)$. By Lemma 3.16(ii) and Lemma 3.20(iii), we always have

$$2(N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r)) \geq \sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)), \quad \forall 1 \leq i < j \leq 2n+2.$$

Summing-up both sides of the above inequality over all $1 \leq i < j \leq 2n+2$, we get

$$2(2n+1)\sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) \geq (n+1)(2n+1)\sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

Thus

$$\sum_{t=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) = o(T_f(r)).$$

By the Second Main Theorem, we have

$$\begin{aligned} \| \ (n+1)T_f(r) &\leq \sum_{i=1}^{2n+2} N^{(n)}_{(f,H_t)}(r) + o(T_f(r)) \\ &\leq n\sum_{i=1}^{2n+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)) = o(T_f(r)). \end{aligned}$$

This is a contradiction.

Hence $\#\mathcal{G}(f,\{H_i\}_{i=1}^{2n+2},1) \leq 2$. We complete the proof of the theorem. ■

4. Meromorphic mappings and three families of hyperplanes

Let f^0, f^1, f^2 be three distinct meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{H_i^k\}_{i=1}^{2n+2}$ $(k=0,1,2)$ be three families of hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Each hyperplane H_i^k is given by

$$H_i^k = \left\{(\omega_0 : \cdots : \omega_n) \,\middle|\, \sum_{v=0}^{n} a_{iv}^k \omega_v = 0\right\}$$

Let $f^k = (f_0^k : \cdots : f_n^k)$ be a reduced representation of f^k $(k = 0, 1, 2)$. We set

$$(f^k, H_i^k) = \sum_{v=0}^{n} a_{iv}^k f_v^k.$$

In this section, we will prove a finiteness theorem for meromorphic mappings with three families of hyperplanes as follows.

THEOREM 4.1. *Let f^0, f^1, f^2 and $\{H_i^k\}_{i=1}^{2n+2}$ $(k = 0, 1, 2)$ be as above. Assume that f^0 is linearly nondegenerate and*

(a) $\dim(f^0)^{-1}(H_i^0) \cap (f^0)^{-1}(H_j^0) \leq m - 2$ $\forall 1 \leq i < j \leq 2n+2$,

(b) $(f^k)^{-1}(H_i^k) = (f^0)^{-1}(H_i^i)$, *for* $k = 1, 2$, *and* $i = 1, \ldots, 2n+2$,

(c) $\dfrac{(f^k, H_v^k)}{(f^k, H_j^k)} = \dfrac{(f^0, H_v^0)}{(f^0, H_j^0)}$ *on* $\bigcup_{i=1}^{2n+2} (f^0)^{-1}(H_i^0) \backslash (f^0)^{-1}(H_j^0)$, *for* $1 \leq v, j \leq 2n+2$.

If $n \geq 2$ then there exist two distinct indices $t, l \in \{0, 1, 2\}$ and a linearly projective transformation $\mathscr{L}$ such that $\mathscr{L}(f^t) = f^l$ and $\mathscr{L}(H_i^t) = H_i^l$ for all $i = 1, \ldots, 2n+2$.

Proof. Fix an index $k \in \{1, 2\}$. Since $H_1^k, H_2^k, \ldots, H_{n+1}^k$ are $n+1$ hyperplanes in general position, we consider the linearly projective transformation $\mathscr{L}^k$ of $\mathbf{P}^n(\mathbf{C})$ is given by $\mathscr{L}^k((z_0 : \cdots : z_n)) = (\omega_0 : \cdots : \omega_n)$ with

$$\begin{pmatrix} \omega_0 \\ \vdots \\ \omega_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_{10} & \cdots & b_{1n} \\ \vdots & \cdots & \vdots \\ b_{(n+1)0} & \cdots & b_{(n+1)n} \end{pmatrix}}_{B} \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix},$$

where

$$B = \underbrace{\begin{pmatrix} a_{10}^0 & \cdots & a_{1n}^0 \\ \vdots & \cdots & \vdots \\ a_{(n+1)0}^0 & \cdots & a_{(n+1)n}^0 \end{pmatrix}}_{A_0}{}^{-1} \cdot \underbrace{\begin{pmatrix} a_{10}^k & \cdots & a_{1n}^k \\ \vdots & \cdots & \vdots \\ a_{(n+1)0}^k & \cdots & a_{(n+1)n}^k \end{pmatrix}}_{A_k}$$

We set

$$(c_{i0}^k, \ldots, c_{in}^k) = (a_{i0}^k, \ldots, a_{in}^k) \cdot B^{-1}, \quad \text{for } i = 1, \ldots, 2n+2.$$

Since $A_0 \circ B = A_k$, then

$$(c_{i0}^k, \ldots, c_{in}^k) = (a_{i0}^0, \ldots, a_{in}^0), \quad \forall i = 1, \ldots, n+1.$$

Suppose that there exists an index $i_0 \in \{n+2, \ldots, 2n+2\}$ such that

$$(c_{i_0 0}^k, \ldots, c_{i_0 n}^k) \neq (a_{i_0 0}^0, \ldots, a_{i_0 n}^0).$$

We consider the following function

$$F = \sum_{j=0}^{n} (c_{i_0 j}^k - a_{i_0 j}^0) f_j^0.$$

Since f^0 is linearly nondegenerate, F is a nonzero meromorphic function on $\mathbf{C}^m$. For $z \in \bigcup_{i=1}^{2n+2} (f^0)^{-1}(H_i^0) \backslash I(f^0)$, without loss of generality we may assume that $(f^0, H_1^0)(z) \neq 0$, then

$$\begin{aligned}
F(z) &= \sum_{j=0}^{n} (c_{i_0 j}^k - a_{i_0 j}^0) f_j^0(z) = (a_{i_0 0}^k, \dots, a_{i_0 n}^k) \cdot B^{-1}(f^0)(z) - (f^0, H_{i_0}^0)(z) \\
&= (a_{i_0 0}^k, \dots, a_{i_0 n}^k) \cdot A_k^{-1} \circ A_0(f^0)(z) - (f^0, H_{i_0}^0)(z) \\
&= \frac{(a_{i_0 0}^k, \dots, a_{i_0 n}^k) \cdot A_k^{-1} \circ A_0(f^0)(z) - (f^0, H_{i_0}^0)(z)}{(f^0, H_1^0)(z)} \cdot (f^0, H_1^0)(z) \\
&= \frac{(a_{i_0 0}^k, \dots, a_{i_0 n}^k) \cdot A_k^{-1} \circ A_k(f^k)(z) - (f^k, H_{i_0}^k)(z)}{(f^k, H_1^k)(z)} \cdot (f^0, H_1^0)(z) \\
&= \frac{(a_{i_0 0}^k, \dots, a_{i_0 n}^k)(f^k)(z) - (f^k, H_{i_0}^k)(z)}{(f^k, H_1^k)(z)} \cdot (f^0, H_1^0)(z) \\
&= \frac{(f^k, H_{i_0}^k)(z) - (f^k, H_{i_0}^k)(z)}{(f^k, H_1^k)(z)} \cdot (f^0, H_1^0)(z) = 0.
\end{aligned}$$

Therefore, it follows that

$$N_F(r) \geq \sum_{i=1}^{2n+2} N_{(f^0, H_i^0)}^{(1)}(r).$$

On the other hand, by Jensen formula we have that

$$N_F(r) = \int_{S(r)} \log|F(z)|\eta + O(1) \leq \int_{S(r)} \log\|f^0(z)\|\eta + O(1) = T_{f^0}(r) + o(T_{f^0}(r)).$$

By using the Second Main Theorem, we obtain

$$\begin{aligned}
\| \ (n+1)T_{f^0}(r) &\leq \sum_{i=1}^{2n+2} N_{(f^0, H_i^0)}^{(n)}(r) + o(T_{f^0}(r)) \\
&\leq n \sum_{i=1}^{2n+2} N_{(f^0, H_i^0)}^{(1)}(r) + o(T_{f^0}(r)) \leq nT_{f^0}(r).
\end{aligned}$$

It implies that $\| \ T_{f^0}(r) = 0$. This is a contradiction to the fact that f^0 is linearly nondegenerate. Therefore we have

$$(c_{i0}^k, \dots, c_{in}^k) = (a_{i0}^0, \dots, a_{in}^0), \quad \forall i = 1, \dots, 2n+2.$$

Hence $\mathscr{L}^k(H_i^k) = H_i^0$ for all $i = 1, \dots, 2n+2$.

We set $\tilde{f}^k = \mathscr{L}^k(f^k)$, $k = 1,2$. Then f^0, $\tilde{f}^1$, $\tilde{f}^2$ belong to $\mathscr{G}(f^0, \{H_i^0\}_{i=1}^{2n+2}, 1)$. By Theorem 1.1, one of the following assertions holds

(i) $f^0 = \tilde{f}^1$, i.e. $f^0 = \mathscr{L}^1(f^1)$ and $\mathscr{L}^1(H_i^1) = H_i^0$ for all $i = 1, \dots, 2n+2$,

(ii) $f^0 = \tilde{f}^2$, i.e. $f^0 = \mathscr{L}^2(f^2)$ and $\mathscr{L}^2(H_i^2) = H_i^0$ for all $i = 1, \dots, 2n+2$,

(iii) $\tilde{f}^1 = \tilde{f}^2$, i.e. $f^1 = (\mathscr{L}^1)^{-1} \circ \mathscr{L}^2(f^2)$ and $(\mathscr{L}^1)^{-1} \circ \mathscr{L}^2(H_i^1) = H_i^2$ for all $i = 1, \dots, 2n+2$.

We complete the proof of the theorem. ■

References

[1] Y. Aihara, Finiteness theorem for meromorphic mappings, Osaka J. Math. **35** (1998), 593–61.

[2] Z. Chen and Q. Yan, Uniqueness theorem of meromorphic mappings into $\mathbf{P}^N(\mathbf{C})$ sharing $2N+3$ hyperplanes regardless of multiplicities, Internat. J. Math. **20** (2009), 717–726.

[3] G. Dethloff and T. V. Tan, Uniqueness theorems for meromorphic mappings with few hyperplanes, Bull. Sci. Math. **133** (2009), 501–514.

[4] P. H. Ha and S. D. Quang and D. D. Thai, Unicity theorems with truncated multiplicities of meromorphic mappings in several complex variables sharing small identical sets for moving targets, Intern. J. Math. **21** (2010), 1095–1120.

[5] H. Fujimoto, The uniqueness problem of meromorphic maps into the complex projective space, Nagoya Math. J. **58** (1975), 1–23.

[6] H. Fujimoto, Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into $\mathbf{P}^{N_1}(\mathbf{C}) \times \cdots \times \mathbf{P}^{N_k}(\mathbf{C})$, Japanese J. Math. **11** (1985), 233–264.

[7] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, Nagoya Math. J. **152** (1998), 131–152.

[8] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, II, Nagoya Math. J. **155** (1999), 161–188.

[9] R. Nevanlinna, Einige Eideutigkeitssätze in der Theorie der meromorphen Funktionen, Acta. Math. **48** (1926), 367–391.

[10] J. Noguchi and T. Ochiai, Introduction to geometric function theory in several complex variables, Trans. Math. Monogr. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[11] S. D. Quang, Unicity of meromorphic mappings sharing few hyperplanes, Ann. Polon. Math. **102** (2011), 255–270.

[12] S. D. Quang and T. V. Tan, Uniqueness theorem of meromorphic mappings with few hyperplanes, Ann. Univ. Mariae Curie-Skłodowska **97** (2008), 97–110.

[13] M. Ru, A uniqueness theorem with moving targets without counting multiplicity, Proc. Amer. Math. Soc. **129** (2001), 2701–2707.

[14] L. Smiley, Geometric conditions for unicity of holomorphic curves, Contemp. Math. **25** (1983), 149–154.

[15] D. D. Thai and S. D. Quang, Uniqueness problem with truncated multiplicities of meromorphic mappings in several compex variables, Internat. J. Math. **17** (2006), 1223–1257.

Duc Quang Si
Department of Mathematics
Hanoi University of Education
Cau Giay, Hanoi
Vietnam
E-mail: ducquang.s@gmail.com