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A GENERALIZATION OF MICHAEL FINITE DIMENSIONAL SELECTION THEOREM

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Abstract

In this paper we generalize the classical finite dimensional selection theorem due to Michael [12, theorem 1.2] to the case where the target space is only a Hausdorff uniform space. This also generalizes the zero-dimensional selection theorem of Fakhoury-Gieler [7, 8]. The proof of this generalization utilizes an elegant construction due to Ageev.

The purpose of this paper is to generalize Michael finite dimensional selection theorem [12, theorem 1.2] to the case of a Hausdorff uniform target space. The following notations and definitions will be fixed throughout this paper.

If E is a uniform space, we let $\mathscr{U}(E)$ be the basis of the filter of entourages defining the uniformity of E that consists of the open symmetric entourages [4, II.5] and where each such entourage is of the form $V = \{(x, y) \in E \times E : f(x, y) < a\}$ for some **pseudo-metric** f on E and for some $a > 0$ [5, IX.5, Theorem 1]. $A \subseteq E$ is said to be V-small where $V \in \mathscr{U}(E)$ if $A \times A \subseteq V$.

Let $\mathscr{B}_0(E)$ be the set of non-empty subsets of E. A family $\mathscr{S} \subseteq \mathscr{B}_0(E)$ is said to be equiuniformly-LC^n if $\forall V \in \mathscr{U}(E)$, $\exists W \in \mathscr{U}(E)$ such that for any compact polyhedron K of dimension $\leq n$ [14, p. 142] and for any $A \in \mathscr{S}$ and any continuous map $\varphi : K \to A$ such that $\varphi(K)$ is W-small, then φ extends to a continuous map $\varphi' : \mathrm{Con}(K) \to A$ such that $\varphi'(\mathrm{Con}(K))$ is V-small (where $\mathrm{Con}(K) = K \times [0, 1]/K \times \{1\}$ is the cone over K with the quotient topology and where K is identified to $K \times \{0\} \subseteq \mathrm{Con}(K)$ by the obvious map). Note that this is the same concept as that of a uniformly equi-LC^n family defined in [12], in case of a metric E, but our terminology is more consistent with the theme of this paper.

$A \subseteq E$ is said to be C^n if any continuous map of a compact polyhedron of dimension $\leq n$ into A extends to a continuous map of $\mathrm{Con}(K)$ into A. If A is C^n for all $A \in \mathscr{S} \subseteq \mathscr{B}_0(E)$, we say that $\mathscr{S}$ is C^n. In the above $n \geq -1$, where

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any $\mathscr{S} \subseteq \mathscr{B}_0(E)$ is equiuniformly-LC^{-1} and C^{-1}. $\mathscr{S} \subseteq \mathscr{B}_0(E)$ is said to be equimetrizable [7, 8] if there exists a filter basis $\mathscr{U}_0 = \{V_n : n \geq 1\} \subseteq \mathscr{U}(E)$ for a coarser uniformity on E such that $\forall U \in \mathscr{U}(E)$, $\exists m \geq 1$ such that $A \times A \cap V_m \subseteq U$ for all $A \in \mathscr{S}$ (the family $\mathscr{S}$ is then said to be $\mathscr{U}_0$ equimetrizable).

If X, E are topological spaces, a map $\varphi : X \to \mathscr{B}_0(E)$ is said to be lower semi-continuous (= l.s.c.) if for any U open $\subseteq E$, $\{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X. If $A \subseteq X$, a selection of φ on A is a map $g : A \to E$ such that $g(a) \in \varphi(a)$ for all $a \in A$.

Our generalization of the Michael finite dimensional selection theorem [12, theorem 1.2] is given by:

MAIN THEOREM. *Let E be a Hausdorff uniform space and let X be a paracompact space and A closed $\subseteq X$ such that* $\dim(X \bmod A) \leq n+1$ [13, p. 50] *and let $\varphi : X \to \mathscr{B}_0(E)$ be a l.s.c. map such that $\{\varphi(x) : x \in X\}$ is an equimetrizable, equiuniformly-LC^n family of complete subsets of E, and let $g : A \to E$ be a continuous selection of φ on A. Then g extends to a continuous selection of φ on some open $U \supseteq A$. If $\varphi(x)$ is C^n for all $x \in X$, we may take $U = X$.*

The proof of this theorem depends on the following three lemmas.

LEMMA 1 [see 12, Lemma 11.1]. *Let E be a uniform space and let $\mathscr{S}$ be an equiuniformly-$LC^n \subseteq \mathscr{B}_0(E)$. Then $\forall V \in \mathscr{U}(E)$, $\exists W \in \mathscr{U}(E)$ such that for all $A \in \mathscr{S}$ and any compact polyhedron X of dimension $\leq n+1$ and for any continuous map $k : X \to W(A)$, there exists a continuous map $f : X \to A$ such that $f(x) \in V(k(x))$ for all $x \in X$.*

Proof. Let $Z \in \mathscr{U}(E)$ such that $Z^2 \subseteq V$. By induction using the equiuniformly-LC^n property of $\mathscr{S}$. $\exists S \in \mathscr{U}(E)$, $S \subseteq Z$ such that for all $A \in \mathscr{S}$ and any finite simplicial complex K of dimension $\leq n+1$ and for any map $u : K^0 \to A$ such that $u(\sigma \cap K^0)$ is S-small for all $\sigma \in K$, then u extends to a continuous map $v : K \to A$ such that $v(\sigma)$ is Z-small for all $\sigma \in K$.

Let $W \in \mathscr{U}(E)$ such that $W^3 \subseteq S$ and let X and k be as in the lemma. Passing to a fine barycentric subdivision of X, we may assume that $k(\sigma)$ is W-small $\forall \sigma \in X$. For all $v \in X^0$ let $f(v) \in A$ be such that $k(v) \in W(f(v))$, then f extends to a continuous map over X such that $f(\sigma)$ is Z-small for all $\sigma \in X$. For $x \in X$, we have $x \in \langle v_0, \ldots, v_m \rangle$ and $f(x) \in Z(f(v_0))$, $f(v_0) \in W(k(v_0))$, $k(v_0) \in W(k(x))$ give $f(x) \in V(k(x))$ as desired.

LEMMA 2 [see 12, Lemma 11.2]. *Let E be a locally convex topological vector space (= LCTVS) and let $\mathscr{S}$ be an equiuniformly-$LC^n \subseteq \mathscr{B}_0(E)$. Then $\forall R, T \in \mathscr{U}(E)$, $\exists M \in \mathscr{U}(E)$ depending on R, T and $\exists S \in \mathscr{U}(E)$ depending only on R such that if K is a compact polyhedron of dimension $\leq n$ and if $A \in \mathscr{S}$ then any continuous map $k : K \to M(A)$ such that $k(K)$ is S-small extends to a continuous map $k' : \mathrm{Con}(K) \to T(A)$ such that $k'(\mathrm{Con}(K))$ is R-small. If $\mathscr{S}$ is C^n, then for $R = E \times E$, we may take $S = E \times E$.*

Proof. Let $L \in \mathscr{U}(E)$ such that $L^3 \subseteq R$. By the equiuniformly-LC^n property of $\mathscr{S}$, there exists $S \in \mathscr{U}(E)$ such that if K is a compact polyhedron of dimension $\leq n$ and if $A \in \mathscr{S}$ then any continuous map $k : K \to A$ such that $k(K)$ is S^3-small extends to a continuous map $k' : \mathrm{Con}(K) \to A$ such that $k'(\mathrm{Con}(K))$ is L-small.

Let $Z \in \mathscr{U}(E)$ such that $Z \subseteq T \cap S \cap L$ and $Z = \{(x, y) \in E : p(x - y) < 1\}$ where p is a continuous pseudonorm on E [6]. Now Z determines $M \in \mathscr{U}(E)$ by lemma 1 so that if K is a compact polyhedron of dimension $\leq n$ and if $A \in \mathscr{S}$ and if $k : K \to M(A)$ is any continuous map such that $k(K)$ is S-small then there exists a continuous map $f : K \to A$ such that $f(x) \in Z(k(x))$ for all $x \in K$. Note that $f(K)$ is ZSZ-small or $f(K)$ is S^3-small, hence f extends to a continuous map $f' : \mathrm{Con}(K) \to A$ such that $f'(\mathrm{Con}(K))$ is L-small.

Define $k' : \mathrm{Con}(K) \to E$ by $k'(x, t) = 2t.f(x) + (1 - 2t).k(x)$ for $0 \leq t \leq \frac{1}{2}$ and $k'(x, t) = f'(x, 2t - 1)$ for $\frac{1}{2} \leq t \leq 1$ so that k' is a continuous extension of k and $k'(\mathrm{Con}(K)) \subseteq Z(A) \subseteq T(A)$ and $k'(\mathrm{Con}(K))$ is ZLZ-small, hence it is R-small as desired.

Lemma 3 [see 12, Lemma 11.3 and 1, Lemma 2.8]. *Let E be a uniform space, $V \in \mathscr{U}(E)$, X be a topological space, $\varphi : X \to \mathscr{B}_0(E)$ be a l.s.c. map and let C compact $\subseteq E$. Then $\{x \in X : C \subseteq V(\varphi(x))\}$ is open $\subseteq X$.*

Proof. Let $x_0 \in X$ such that $C \subseteq V(\varphi(x_0))$.

Claim. $\exists S, W \in \mathscr{U}(E)$ such that $SW \subseteq V$ and $C \subseteq W(\varphi(x_0))$.

Proof of Claim. Let $V = \{(x, y) \in E \times E : f(x, y) < a\}$ where f is a pseudometric on E. The map $C \to [0, a)$ defined by $z \to \inf\{f(z, y) : y \in \varphi(x_0)\}$ is upper semi-continuous [4, IV.30, Theorem 4]. Hence $\exists z_0 \in C$ such that $\inf\{f(z_0, y) : y \in \varphi(x_0)\} = \sup\{\inf\{f(z, y) : y \in \varphi(x_0)\} : z \in C\} = a_0 < a$ [4, IV.30, Theorem 3].

Set $W = \{(x, y) \in E \times E : f(x, y) < \frac{1}{2}(a + a_0)\}$, $S = \{(x, y) \in E \times E : f(x, y) < \frac{1}{2}(a - a_0)\}$. Clearly, these satisfy the requirements.

Let $S, W \in \mathscr{U}(E)$ be as given by the above claim. There exists F finite $\subseteq C$ such that $C \subseteq S(F)$. Also $x_0 \in \bigcap_{z \in F}\{x \in X : \varphi(x) \cap W(z) \neq \emptyset\} = O$ open $\subseteq X$. Hence $C \subseteq S(F) \subseteq SW(\varphi(x)) \subseteq V(\varphi(x))$ $\forall x \in O$.

This paper is divided into two sections. Section 1 is devoted to generalizing Ageev construction [1] culminating in theorem 1.4. In section 2 we establish our generalization of Michael finite dimensional selection theorem in theorem 2.2.

1. Ageev construction

The following notations and definitions will be adopted in this section.

Let X and E be topological spaces and let $\varphi : X \to \mathscr{B}_0(E)$ be any map. The graph of $\varphi(= \mathrm{Gr}(\varphi))$ is defined by $\mathrm{Gr}(\varphi) = \{(x, y) \in X \times E : y \in \varphi(x)\}$. A map

$F : X \to \mathscr{B}_0(E)$ is called an (n-)step function from X to E if there exists $\{A_\alpha : \alpha \in I\}$ locally finite open cover of X and K_α compact polyhedron (of dimension n) for all $\alpha \in I$ such that $\mathrm{Gr}(F) = \bigcup\{A_\alpha \times K_\alpha : \alpha \in I\}$ and we denote F by $\{A_\alpha\} \otimes \{K_\alpha\}$. In particular, F is called a contractible (n-)step function if K_α is contractible for all $\alpha \in I$.

Let $F_1 = \{A_\alpha\} \otimes \{K_\alpha\}$, $F_2 = \{B_i\} \otimes \{L_i\}$ be two step functions from X to E: F_1 refines F_2, denoted by $F_1 \le F_2$, if for any α, there exists $i(\alpha)$ such that $A_\alpha \subseteq B_{i(\alpha)}$ and $K_\alpha = L_{i(\alpha)}$, and F_1 star refines F_2, denoted by $F_1 \le^* F_2$, if $A_\alpha \cap B_i \neq \emptyset \Rightarrow K_\alpha \subseteq L_i$ $\forall \alpha, i$. Note that if F_1, F_2, F_3 are step functions from X to E, then

(i) $F_1 \le F_2$ or $F_1 \le^* F_2 \Rightarrow F_1(x) \subseteq F_2(x)$ $\forall x \in X$,

(ii) $F_1 \le^* F_2$ and $F_3 \le F_2 \Rightarrow F_1 \le^* F_3$,

(iii) $F_1 \le^* F_2$ and $F_2 \le^* F_3 \Rightarrow F_1 \le^* F_3$.

Finally, we remark that if E is any infinite dimensional Hausdorff LCTVS and if K is a compact polyhedron and if $f : K \to E$ is a PL–embedding, then $\mathrm{PLEmb}_f(\mathrm{Con}(K), E) = \{g\colon g$ PL-embedding of $\mathrm{Con}(K)$ into $E, g|_K = f\}$ is uniformly dense in $C_f(\mathrm{Con}(K), E) = \{g\colon g$ continuous map of $\mathrm{Con}(K)$ into $E, g|_K = f\}$. Indeed $\mathrm{PL}_f(\mathrm{Con}(K), E) = \{g\colon g$ PL-map of $\mathrm{Con}(K)$ into $E, g|_K = f\}$ is uniformly dense in $C_f(\mathrm{Con}(K), E)$ by using barycentric subdivisions [10, p. 91], and by a general position argument we get that $\mathrm{PLEmb}_f(\mathrm{Con}(K), E)$ is uniformly dense in $\mathrm{PL}_f(\mathrm{Con}(K), E)$ [same argument as 10, p. 94].

The Ageev construction in [1] for the proof of the classical Michael finite dimensional selection theorem is generalized in the following three lemmas.

LEMMA 1.1 [see 1, Lemma 2.7 + Proposition 5.3]. *Let E be a uniform space and let $W, S \in \mathscr{U}(E)$. Then for any paracompact space X and any l.s.c. map $\varphi : X \to \mathscr{B}_0(E)$ and for any continuous map $k : X \to E$ such that $k(x) \in W(\varphi(x))$ for all $x \in X$, there exists a contractible 0-step function $F : X \to \mathscr{B}_0(E)$ such that $F(x) \subseteq S(\varphi(x))$, $F(x)$ is W^4-small and $k(x) \in W^2(F(x))$ for all $x \in X$.*

Proof. Let $b : X \to E$ be a selection of φ such that $k(x) \in W(b(x))$ for each $x \in X$. For all $x \in X$, let $O(x)$ be an open neighborhood of x such that $k(O(x))$ is W-small and $O(x) \times \{b(x)\} \subseteq \mathrm{Gr}(S(\varphi))$ by lemma 3 (where $S(\varphi)$ is the map $X \ni z \to S(\varphi(z)) \in \mathscr{B}_0(E)$). Let $\{A_\alpha : \alpha \in I\}$ be a locally finite open refinement of $\{O(x) : x \in X\}$ and let $I \ni \alpha \to x(\alpha) \in X$ be a refining map and let $K_\alpha = \{b(x(\alpha))\}$. Then $F = \{A_\alpha\} \otimes \{K_\alpha\}$ is a contractible 0-step function from X to E and $F(x) = \bigcup\{K_\alpha : x \in A_\alpha\} \subseteq S(\varphi(x))$. Note that

$$\begin{aligned} x \in A_\alpha \subseteq O(x(\alpha)) &\Rightarrow k(x) \in W(k(x(\alpha)) \quad \text{and} \quad k(x(\alpha)) \in W(b(x(\alpha)) \\ &\Rightarrow k(x) \in W^2(F(x)) \end{aligned}$$

So that if $x \in A_\alpha \cap A_\beta$, then $b(x(\alpha)), b(x(\beta)) \in W^2(k(x))$, hence $b(x(\alpha)) \in W^4(b(x(\beta)))$ and $F(x)$ is W^4-small as desired.

LEMMA 1.2 [see 1, Proposition 3.1 + Proposition 5.4]. *Let E be an infinite dimensional Hausdorff LCTVS and let $\mathscr{S}$ be an equiuniformly-$LC^n \subseteq \mathscr{B}_0(E)$. Then $\forall R, T \in \mathscr{U}(E)$, $\exists M \in \mathscr{U}(E)$ depending on R, T and $\exists S \in \mathscr{U}(E)$ depending only on R such that if X is a paracompact space and if $\varphi : X \to \mathscr{S}$ is a l.s.c. map and if $F_k : X \to \mathscr{B}_0(E)$ is a k-step function such that $F_k(x) \subseteq M(\varphi(x))$ and $F_k(x)$ is S-small for all $x \in X$, where $0 \leq k \leq n$, then there exists $F_{k+1} : X \to \mathscr{B}_0(E)$, a contractible $k+1$-step function such that $F_{k+1}(x) \subseteq T(\varphi(x))$ and $F_{k+1}(x)$ is R^2-small for all $x \in X$, and $G_k \leq^* F_{k+1}$ for some k-step function G_k where $G_k \leq F_k$. If $\mathscr{S}$ is C^n then for $R = E \times E$ we may take $S = E \times E$.*

Proof. For $R, T \in \mathscr{U}(E)$, let $S, M \in \mathscr{U}(E)$ be as given by lemma 2. If $F_k = \{A_\alpha\} \otimes \{K_\alpha\}$, then $F_k(x) = \bigcup\{K_\alpha : x \in A_\alpha\}$ is a compact polyhedron [10, p. 2] of dimension $k \leq n$. By lemma 2 and [4, II.31] $F_k(x)$ extends to an R-small PL-embedding of $\mathrm{Con}(F_k(x))$ in $T(\varphi(x))$. We identify $\mathrm{Con}(F_k(x))$ by its image under this PL embedding.

By lemma 3 and the paracompactness of X [14, p. 70], there exists $\{O(x) : x \in X\}$ an open star refinement of $\{A_\alpha\}$ such that $x \in O(x)$ and $O(x) \times \mathrm{Con}(F_k(x)) \subseteq \mathrm{Gr}(T(\varphi))$ (where $T(\varphi)$ is the map $X \ni z \to T(\varphi(z)) \in \mathscr{B}_0(E)$). Let $\{B_\beta\}$ be a locally finite open refinement of $\{O(x)\}$ and let $B_\beta \subseteq O(x_\beta)$ and $N(B_\beta, \{O(x)\}) = \bigcup\{O(x) : O(x) \cap B_\beta \neq \emptyset\} \subseteq A_{\alpha(\beta)}$.

Define $G_k = \{B_\beta\} \otimes \{K_{\alpha(\beta)}\} \leq F_k$ and $F_{k+1} = \{B_\beta\} \otimes \{\mathrm{Con}(F_k(x_\beta))\}$. Note that

$$\begin{aligned} B_\beta \cap B_s \neq \emptyset &\Rightarrow O(x_\beta) \cap B_s \supseteq B_\beta \cap B_s \neq \emptyset \\ &\Rightarrow x_\beta \in O(x_\beta) \subseteq N(B_s, \{O(x)\}) \subseteq A_{\alpha(s)} \\ &\Rightarrow \mathrm{Con}(F_k(x_\beta)) \supseteq F_k(x_\beta) = \bigcup\{K_\alpha : x_\beta \in A_\alpha\} \supseteq K_{\alpha(s)} \end{aligned}$$

hence $G_k \leq^* F_{k+1}$. Also, $F_{k+1}(x) = \bigcup\{\mathrm{Con}(F_k(x_\beta)) : x \in B_\beta\} \subseteq T(\varphi(x))$ and

$$\begin{aligned} x \in B_s &\Rightarrow \forall B_\beta \ni x \quad \mathrm{Con}(F_k(x_\beta)) \supseteq K_{\alpha(s)} \\ &\Rightarrow F_{k+1}(x) \text{ is } R^2\text{-small.} \end{aligned}$$

LEMMA 1.3 [see 1, Proposition 3.4]. *Let X be a normal topological space of covering dimension $\leq n+1$ and let E be a topological space and let $G_i : X \to \mathscr{B}_0(E)$ be a contractible i-step function for $0 \leq i \leq n+1$ such that $G_0 \leq^* G_1 \leq^* \cdots \leq^* G_n \leq^* G_{n+1}$. Then G_{n+1} admits a continuous selection.*

Proof. Let $G_i = \{A_\alpha(i) : \alpha \in \Lambda(i)\} \otimes \{K_\alpha(i) : \alpha \in \Lambda(i)\}$ for $0 \leq i \leq n+1$. Then there exists $\omega_i = \{W_\gamma(i) : \gamma \in J\}$ a discrete family of closed sets such that ω_i refines $\{A_\alpha(i) : \alpha \in \Lambda(i)\}$ for $0 \leq i \leq n+1$ and $\omega = \bigcup\{\omega_i : 0 \leq i \leq n+1\}$ is a locally finite closed covering of X [5, IX.107, Ex.27]. For each $0 \leq i \leq n+1$, take $\alpha(\gamma) \in \Lambda(i)$ such that $W_\gamma(i) \subseteq A_{\alpha(\gamma)}(i)$ and put $X_i = \bigcup\{W_\gamma(i) : \gamma \in J\}$.

It suffices to construct by induction on $0 \leq k \leq n+1$, continuous maps $s_k : \bigcup\{X_i : 0 \leq i \leq k\} \to E$ such that $s_k(W_\gamma(k)) \subseteq K_{\alpha(\gamma)}(k)$ for $\gamma \in J$, $0 \leq k \leq n$ and $s_k = s_{k-1}$ on $\bigcup\{X_i : 0 \leq i \leq k-1\}$ for $0 < k \leq n$.

Define $s_0 : X_0 \to E$ such that $s_0(W_\gamma(0))$ is an arbitrary point in $K_{\alpha(\gamma)}(0)$ for $\gamma \in J$. Assume that s_j have been defined inductively satisfying hypotheses for $0 \leq j < k \leq n+1$. Note that for $0 \leq j < k$ and $\gamma, \beta \in J$

$$W_\gamma(k) \cap W_\beta(j) \neq \emptyset \Rightarrow K_{\alpha(\beta)}(j) \subseteq K_{\alpha(\gamma)}(k)$$
$$\Rightarrow s_{k-1}(W_\gamma(k) \cap W_\beta(j)) \subseteq K_{\alpha(\gamma)}(k)$$

so that $s_{k-1}(W_\gamma(k) \cap X_j) \subseteq K_{\alpha(\gamma)}(k)$. Define $s_k : \bigcup\{X_i : 0 \leq i \leq k\} \to E$ such that $s_k = s_{k-1}$ on $\bigcup\{X_i : 0 \leq i \leq k-1\}$ and $s_k|_{W\gamma(k)}$ to be any continuous extension of $s_{k-1}|_{W\gamma(k) \cap (\bigcup\{X_i : 0 \leq i \leq k-1\})} : W_\gamma(k) \cap (\bigcup\{X_i : 0 \leq i \leq k-1\}) \to K_{\alpha(\gamma)}(k)$ [9, p. 43, p. 68].

As indicated in [1, p. 4374], a direct extension of these lemmas yields another proof of the following Uspenskij's theorem [15] for paracompact spaces with property C.

[15, Theorem 1.3]: Let X be a paracompact space with property C. Then any map $\varphi : X \to \mathscr{B}_0(E)$ where E is a Hausdorff LCTVS space, $\mathrm{Gr}(\varphi)$ open $\subseteq X \times E$ and $\varphi(x)$ contractible for all $x \in X$ admits a continuous selection.

Indeed, we may assume that E is infinite dimensional by replacing φ, if necessary, by the map $X \to \mathscr{B}_0(E \times l_2)$ defined by $X \ni x \to \varphi(x) \times l_2$, where l_2 is the Hilbert space. Using the facts that $\mathrm{Gr}(\varphi)$ open $\subseteq X \times E$ and $\varphi(x)$ contractible for all $x \in X$, we can construct inductively by the same methods of lemma 1.1 and lemma 1.2, with no approximations required, a sequence $G_0 \leq^* G_1 \leq^* \cdots \leq^* G_n \leq^* \cdots$ where G_i is a contactible i-step function with $G_i(x) \subseteq \varphi(x)$ for all $x \in X$ [1, lemma 2.7, Proposition 3.1, Proposition 4.1]. Using the property C, a continuous selection of φ is established by the same method of lemma 1.3 [1, Proposition 3.4].

Now we get the following theorem.

Theorem 1.4 [see 1, Theorem 5.1]. *Let E be an infinite dimensional Hausdorff LCTVS and let $\mathscr{S}$ be an equiuniformly-$LC^n \subseteq \mathscr{B}_0(E)$. Then $\forall R \in \mathscr{U}(E)$, $\exists W \in \mathscr{U}(E)$ such that if X is a paracompact space of covering dimension $\leq n+1$ and if $\varphi : X \to \mathscr{S}$ is a l.s.c. map and if $k : X \to E$ is a continuous map with $k(x) \in W(\varphi(x))$ for all $x \in X$, it follows that $\forall V \in \mathscr{U}(E)$ there exists a continuous map $f : X \to E$ such that $f(x) \in V(\varphi(x)) \cap R(k(x))$ for all $x \in X$. If $\mathscr{S}$ is C^n then for $R = E \times E$ we may take $W = E \times E$.*

Proof. Let $K \in \mathscr{U}(E)$ such that $K^4 \subseteq R$. By induction using lemma 1.2, $\exists M \in \mathscr{U}(E)$ depending on K, V and $\exists S \in \mathscr{U}(E)$ depending only on K such that if $F_0 : X \to \mathscr{B}_0(E)$ is a contractible 0-step function such that $F_0(x) \subseteq M(\varphi(x))$ and $F_0(x)$ is S-small for all $x \in X$, then there exists G_k contractible k-step function and F_{k+1} contractible $k+1$-step function such that $G_k \leq^* F_{k+1}$, $G_k \leq F_k$ for $0 \leq k \leq n$, $F_{n+1}(x) \subseteq V(\varphi(x))$ and $F_{n+1}(x)$ is K^2-small for all $x \in X$.

Let $W \in \mathscr{U}(E)$ such that $W^3 \subseteq K$ and $W^4 \subseteq S$ and let $k : X \to E$ be a continuous map with $k(x) \in W(\varphi(x))$ for all $x \in X$. Lemma 1.1 provides $F_0 : X \to \mathscr{B}_0(E)$ a contractible 0-step function such that $F_0(x) \subseteq M(\varphi(x))$ and $F_0(x)$ is W^4-small (hence S-small) and with $k(x) \in W^2(F_0(x))$ for all $x \in X$. Applying the above mentioned induction it follows that there exists $G_i : X \to \mathscr{B}_0(E)$ a contractible i-step function for $0 \le i \le n+1$ such that $G_0 \le^* G_1 \le^* \cdots \le^* G_n \le^* G_{n+1}$, $G_{n+1}(x) \subseteq V(\varphi(x))$ and $G_{n+1}(x)$ is K^2-small and with $k(x) \in W^6(G_0(x))$ for all $x \in X$. Now lemma 1.3 shows that there exists a continuous map $f : X \to E$ such that $f(x) \in G_{n+1}(x) \subseteq V(\varphi(x))$ for all $x \in X$. Note that $k(x) \in W^6(G_0(x))$, $G_0 \le^* G_{n+1}$ and $G_{n+1}(x)$ is K^2-small give $k(x) \in K^4(f(x)) \subseteq R(f(x))$ as desired.

2. Main theorem

The following convergence theorem is all what we need to establish our main theorem.

THEOREM 2.1. *Let E be a Hausdorff uniform space and let $\mathscr{S}$ be an equimetrizable, equiuniformly-LC^n family of complete subsets of E. Then $\forall R \in \mathscr{U}(E)$, $\exists W \in \mathscr{U}(E)$ such that if X is a paracompact space of covering dimension $\le n+1$ and if $\varphi : X \to \mathscr{S}$ is a l.s.c. map and if $k : X \to E$ is a continuous map with $k(x) \in W(\varphi(x))$ for all $x \in X$, it follows that there exists a continuous selection $f : X \to E$ of φ such that $f(x) \in R(k(x))$ for all $x \in X$. If $\mathscr{S}$ is C^n then for $R = E \times E$ we may take $W = E \times E$.*

Proof. By [3] we may assume that E is an infinite dimensional Hausdorff LCTVS. Let $\mathscr{U}_0 = \{V_n : n \ge 1\} \subseteq \mathscr{U}(E)$ be a filter basis such that $(V_{n+1})^2 \subseteq V_n$ for all $n \ge 1$ and $\mathscr{S}$ is $\mathscr{U}_0$ equimetrizable. Let $K \in \mathscr{U}(E)$ such that $K^3 \subseteq R$ and let $k \ge 0$ such that $A \times A \cap (V_k)^4 \subseteq K$ for all $A \in \mathscr{S}$ where $V_0 = E$.

By theorem 1.4, for all $n \ge 0$ $V_{n+k} \cap K \in \mathscr{U}(E)$ defines $W_n \in \mathscr{U}(E)$, $W_n \subseteq V_{n+k} \cap K$. Set $W = W_0$. Again theorem 1.4 defines inductively for all $n \ge 0$ continuous maps $f_n : X \to E$ such that $f_0 = k$ and $f_{n+1}(x) \in W_{n+1}(\varphi(x)) \cap (V_{n+k} \cap K(f_n(x)))$ for all $n \ge 0$.

Note that $\emptyset \ne \varphi(x) \cap (V_{n+k}(f_{n+1}(x))) \subseteq \varphi(x) \cap (V_{n+k-1}(f_n(x)))$ for all $n \ge 1$ and since $\varphi(x)$ is complete, we get $\emptyset \ne \bigcap\{(\varphi(x) \cap (V_{n+k-1}(f_n(x))))^- : n \ge 1\}$ and since $\mathscr{S}$ is $\mathscr{U}_0$ equimetrizable, we get $\bigcap\{(\varphi(x) \cap (V_{n+k-1}(f_n(x))))^- : n \ge 1\} = \{f(x)\}$ for all $x \in X$. We have $f(x) \in (V_k(f_1(x))))^- \subseteq (V_k)^-(f_1(x)) \subseteq (V_k)^3(f_1(x))$ [4, II.4, Proposition 2]. Also, $f_1(x) \in W_1(\varphi(x)) \subseteq V_{k+1} \cap K(\varphi(x))$ shows that there exists $z(x) \in \varphi(x)$, $(z(x), f_1(x)) \in K$ such that $(z(x), f(x)) \in (\varphi(x) \times \varphi(x)) \cap (V_k)^4 \subseteq K$ so that $f(x) \in K^3(k(x)) \subseteq R(k(x))$.

To establish the continuity of f, let $x_0 \in X$ and let $U, M \in \mathscr{U}(E)$ such that $M^2 \subseteq U$ and let $m \ge 1$ such that $(\varphi(x) \times \varphi(x)) \cap (V_{m+k})^6 \subseteq M$ for all $x \in X$. Note that $x_0 \in O = \{x \in X : (\varphi(x) \cap (V_{m+k}(f_{m+1}(x))))^- \cap M(f(x_0)) \ne \emptyset\}$ open [11, proposition 2.3 + proposition 2.5] so that

$$x \in O \Rightarrow \exists a \in (\varphi(x) \cap (V_{m+k}(f_{m+1}(x))))^- \cap M(f(x_0)), \text{ therefore}$$
$$(a, f(x)) \in (\varphi(x) \times \varphi(x)) \cap (V_{m+k})^6 \subseteq M$$
$$\Rightarrow f(x) \in M^2(f(x_0)) \subseteq U(f(x_0)).$$

Now we can establish our main theorem.

THEOREM 2.2. *Let E be a Hausdorff uniform space, X a paracompact space, A closed $\subseteq X$ with* $\dim(X \bmod A) \leq n+1$, $\varphi : X \to \mathscr{B}_0(E)$ *l.s.c. map such that* $\{\varphi(x) : x \in X \backslash A\}$ *is an equimetrizable, equiuniformly-LC^n family of complete subsets of E and let $g : A \to E$ be a continuous selection of φ. Then there exists U open $\supseteq A$ and a continuous selection of φ on U extending g. If $\{\varphi(x) : x \in X\}$ is C^n then we may take $U = X$.*

Proof. By [3] we may assume that E is an infinite dimensional Hausdorff LCTVS which is the product of Banach spaces [6]. Let $G : X \to E$ be any continuous extension of g [2, theorem 4.1]. Note that $\mathscr{S} = \{\varphi(x) : x \in X \backslash A\} \cup \{\{z\} : z \in E\}$ is an equimetrizable, equiuniformly-LC^n family of complete subsets of E. Let $\mathscr{U}_0 = \{V_n : n \geq 1\} \subseteq \mathscr{U}(E)$ be a filter basis such that $(V_{n+1})^2 \subseteq V_n$ for all $n \geq 1$ and $\mathscr{S}$ is $\mathscr{U}_0$ equimetrizable. Define $\psi : X \to \mathscr{B}_0(E)$ by $\psi(x) = g(x)$ for $x \in A$ and $\psi(x) = \varphi(x)$ for $x \notin A$, then ψ is l.s.c. [11, example 1.3*].

By theorem 2.1, for all $i \geq 0$, V_i defines $W_i \in \mathscr{U}(E)$, $W_i \subseteq V_i$, $(W_{i+1})^- \subseteq W_i$ where $V_0 = E \times E$. Again by theorem 2.1, for all $i \geq 0$, W_i defines $Z_i \in \mathscr{U}(E)$, $Z_i \subseteq W_i$, $(Z_{i+1})^- \subseteq Z_i$. We have $A \subseteq U_i = \{x \in X : G(x) \in Z_i(\psi(x))\} = \{x \in X : \psi(x) \cap Z_i(G(x)) \neq \emptyset\}$ open [11, proposition 2.5]. Observe that if $x \in \bigcap\{U_i : i \geq 1\}$ then $\bigcap\{\psi(x) \cap Z_i(G(x)) : i \geq 1\} = \{a(x)\}$ for some $a(x) \in E$ and $a(x) = g(x)$ for all $x \in A$. Define, for all $i \geq 0$, $A \subseteq O_i$ open $\subseteq (O_i)^- \subseteq U_i$, $(O_{i+1})^- \subseteq O_i$. If $\{\varphi(x) : x \in X\}$ is C^n then we may take $U_0 = O_0 = E$.

Theorem 2.1 defines, for all $i \geq 0$, $h_i : (O_i)^- \backslash O_i \to E$ a continuous selection of φ such that $h_i(x) \in W_i(G(x))$. Again, theorem 2.1 defines, for all $i \geq 0$ using [11, example 1.3*], $g_i : (O_i)^- \backslash O_{i+1} \to E$ a continuous selection of φ such that $g_i = h_i$ on $(O_i)^- \backslash O_i$, $g_i = h_{i+1}$ on $(O_{i+1})^- \backslash O_{i+1}$ and $g_i(x) \in V_i(G(x))$.

Set $U = O_0$, then $(O_0)^- = \bigcup\{(O_i)^- \backslash O_{i+1} : i \geq 0\} \cup (\bigcap\{(O_i)^- : i \geq 0\})$. Define $f : U \to E$ by $f(x) = g_i(x)$ if $x \in U \cap ((O_i)^- \backslash O_{i+1})$ for some $i \geq 0$ and $f(x) = a(x)$ where $\{a(x)\} = \bigcap\{\psi(x) \cap Z_i(G(x)) : i \geq 1\}$ if $x \in \bigcap\{(O_i)^- : i \geq 0\}$. Clearly f is a selection of ψ and f is continuous on $U \cap (\bigcup\{(O_i)^- \backslash O_{i+1} : i \geq 0\})$. If $x_0 \in \bigcap\{(O_i)^- : i \geq 0\}) = \bigcap\{O_i : i \geq 0\}$ and if $R \in \mathscr{U}(E)$, then there exists $m \geq 1$ such that $(\varphi(x) \times \varphi(x)) \cap (V_m)^4 \subseteq K$ for all $x \in X \backslash A$ where $K \in \mathscr{U}(E)$, $K^2 \subseteq R$. Note that $x_0 \in O = \{x \in X : \varphi(x) \cap (K \cap V_m(a(x_0))) \neq \emptyset\} \cap \{x \in X : G(x) \in V_m(G(x_0))\} \cap O_m$ open. If $x \in O$, let $c \in \varphi(x) \cap (K \cap V_m(a(x_0)))$. Then $(c, a(x_0)) \in K \cap V_m$, $(f(x), G(x)) \in V_m$, $(G(x), G(x_0)) \in V_m$, $(G(x_0), a(x_0)) \in Z_m$. Therefore, $(c, f(x)) \in (\varphi(x) \times \varphi(x)) \cap (V_m)^4 \subseteq K$. Hence, $f(x) \in K^2(a(x_0)) \subseteq R(f(x_0))$.

We have the following generalization of Fahkoury-Gieler theorem [7, 8].

COROLLARY 2.3. *Let E be a Hausdorff uniform space, X a paracompact space, A closed $\subseteq X$ with* $\dim(X \bmod A) = 0$, $\varphi : X \to \mathscr{B}_0(E)$ *l.s.c. map such that* $\{\varphi(x) : x \in X \backslash A\}$ *is an equimetrizable family of complete subsets of E and let $g : A \to E$ be a continuous selection of φ. Then g extends to a continuous selection of φ over X.*

Proof. Put $n = -1$ in theorem 2.2.

The following corollary generalizes [12, Corollary 1.3].

COROLLARY 2.4. *Let G be a Hausdorff topological group and let H be a complete metrizable LC^n subgroup of G such that G/H is paracompact and* $\dim G/H \leq n+1$. *Then the canonical map $p : G \to G/H$ is a locally trivial fibration.*

Proof. We consider the left uniform structure on G, then $\mathscr{S} = \{gH : g \in G\}$ is an equimetrizable, equiuniformly-LC^n [12, Example 2.6] family of complete subsets of G and $\varphi : G/H \to \mathscr{B}_0(G)$ defined by $\varphi(x) = p^{-1}(x)$ for all $x \in G/H$ is l.s.c. [11, Example 1.1*], hence p admits a local cross section by theorem 2.2 and therefore it is a locally trivial fibration.

Similarly we can establish the following corollary.

COROLLARY 2.5. *Let G be a Hausdorff topological group and let H be a complete metrizable LC^∞ and C^∞ subgroup of G. Then the canonical map $p : G \to G/H$ is a Serre fibration.*

REFERENCES

[1] S. M. AGEEV, A Non-polyhedral proof of the Michael finite-dimensional selection theorem, J. Math. Sci. (N. Y.) **144** (2007), 4367–4379.

[2] R. ARENS, Extension of functions in fully normal spaces, Pacific J. Math. **2** (1952), 11–23.

[3] R. F. ARENS AND J. EELLS, On embedding uniform and topological spaces, Pacific J. Math. **6** (1956), 397–403.

[4] N. BOURBAKI, Éléments de mathématique, Topologie générale, Chapitres 14, Hermann, Paris, 1971.

[5] N. BOURBAKI, Éléments de mathématique, Topologie générale, Chapitres 5–10, Hermann, Paris, 1974.

[6] N. BOURBAKI, Éléments de mathématique, Livre V, Espaces vectoriels topologiques, Hermann, Paris, 1966.

[7] M. H. FAKHOURY, Sélections continues dans les espaces uniformes, C.R. Acad. Sci., Paris **280** (1975), Serie A, 213–216.

[8] B. A. GIELER, On continuous selections in uniform spaces, Soviet Math. Dokl. **11** (1970), 1400–1402.

[9] S. T. Hu, Theory of retracts, Wayne State Univ. Press, 1965.
[10] J. F. P. Hudson, Piecewise linear topology, W.A. Benjamin, INC., 1969.
[11] E. Michael, Continuous selections I, Ann. of Math. **63** (1956), 361–382.
[12] E. Michael, Continuous selections II, Ann. of Math. **64** (1956), 562–580.
[13] K. Nagami, Dimension theory, Academic Press, New York and London, 1970.
[14] A. R. Pears, Dimension theory of general spaces, Cambridge University Press, 1975.
[15] V. V. Uspenskij, A selection theorem for C-spaces, Topology and its Applications **85** (1998), 351–374.

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