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LIFTINGS OF HOLOMORPHIC MAPS INTO TEICHMÜLLER SPACES

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Abstract

We study liftings of holomorphic maps into some Teichmüller spaces. We also study the relationship between universal holomorphic motions and holomorphic lifts into Teichmüller spaces of closed sets in the Riemann sphere.

1. Introduction

Throughout this paper, we will use the following notations: $\mathbf{C}$ for the complex plane, $\hat{\mathbf{C}}$ for the Riemann sphere $\mathbf{C} \cup \{\infty\}$, and Δ for the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$.

1.1. Teichmüller space of a plane region. We begin with the usual definition of the Teichmüller space of a plane region. For standard facts about classical Teichmüller spaces, the reader may see any of the following texts: [10], [11], [14], [15], [21].

DEFINITION 1.1. Let Ω be a plane region whose complement $\mathbf{C} \backslash \Omega$ contains at least two points. By definition, two quasiconformal mappings f and g with domain Ω belong to the same Teichmüller class if and only if there is a conformal map h from $f(\Omega)$ onto $g(\Omega)$ such that the self-mapping $g^{-1} \circ h \circ f$ of Ω is isotopic to the identity rel the boundary of Ω. This means that $g^{-1} \circ h \circ f$ extends to a homeomorphism of the closure of Ω onto itself which is isotopic to the identity by an isotopy that fixes the boundary pointwise. The *Teichmüller space* $Teich(\Omega)$ is the set of Teichmüller classes of quasiconformal mappings with domain Ω.

Let $M(\Omega)$ be the open unit ball of the complex Banach space $L^{\infty}(\Omega)$. The *standard projection* Φ of $M(\Omega)$ onto $Teich(\Omega)$ maps $\mu \in M(\Omega)$ to the Teichmüller class of any quasiconformal map whose domain is Ω and whose Beltrami

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coefficient is μ. The *basepoints* of $M(\Omega)$ and $Teich(\Omega)$ are 0 and $\Phi(0)$, respectively. It is a standard fact in Teichmüller theory that $Teich(\Omega)$ is a complex Banach manifold such that Φ is a holomorphic split submersion; see, for example, [21].

1.2. Teichmüller space of a closed set in the sphere. Let E be a closed subset in $\hat{\mathbf{C}}$; we will always assume that E contains the points 0, 1, and ∞. A homeomorphism of $\hat{\mathbf{C}}$ onto itself is called *normalized* if it fixes the points 0, 1, and ∞.

Definition 1.2. Two normalized quasiconformal self-mappings f and g of $\hat{\mathbf{C}}$ are said to be E-equivalent iff $f^{-1} \circ g$ is isotopic to the identity rel E. The Teichmüller space $T(E)$ is the set of E-equivalence classes of normalized quasiconformal self-mappings of $\hat{\mathbf{C}}$. The *basepoint* of $T(E)$ is the E-equivalence class of the identity map.

The following analytic description of $T(E)$ will be more useful for our purposes.

Let $M(\mathbf{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbf{C})$. Each μ in $M(\mathbf{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbf{C}}$ onto itself. The basepoint of $M(\mathbf{C})$ is the zero function.

We define the quotient map $P_E : M(\mathbf{C}) \to T(E)$ by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbf{C})$ to the basepoint of $T(E)$. In [18] Lieb proved that $T(E)$ is a complex Banach manifold such that the map P_E from $M(\mathbf{C})$ to $T(E)$ is a holomorphic split submersion; see also [9] for a complete proof. The space $T(E)$ is intimately related with holomorphic motions of the closed set E; see §2 for more details.

1.3. Two special cases. Let E be a finite set (0, 1, and ∞ belong to E). Its complement $\Omega = \hat{\mathbf{C}} \setminus E$ is a sphere with punctures at the points of E, and there is a natural identification of $T(E)$ with the classical Teichmüller space $Teich(\Omega)$. It is defined by setting $\theta(P_E(\mu))$ equal to the Teichmüller class of the restriction of w^μ to Ω. It is clear that $\theta : T(E) \to Teich(\Omega)$ is a well-defined map. It is easy to see that the map θ is biholomorphic; see Example 3.1 in [19] for the details.

When $E = \hat{\mathbf{C}}$, the space $T(\hat{\mathbf{C}})$ consists of all the normalized quasiconformal self-mappings of $\hat{\mathbf{C}}$, and the map $P_{\hat{\mathbf{C}}}$ from $M(\mathbf{C})$ to $T(\hat{\mathbf{C}})$ is bijective. We use it to identify $T(\hat{\mathbf{C}})$ biholomorphically with $M(\mathbf{C})$.

1.4. Contractibility of $T(E)$. The following fact was proved in §7.13 of [9].

Proposition 1.3. *There is a continuous basepoint preserving map s from $T(E)$ to $M(\mathbf{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

Since $M(\mathbf{C})$ is contractible, it follows that the space $T(E)$ is also contractible.

1.5. Forgetful maps. If E is a subset of the closed set $\hat{E}$ and μ is in $M(\mathbf{C})$, then the $\hat{E}$-equivalence class of w^{μ} is contained in the E-equivalence class of w^{μ}. Therefore, there is a well-defined 'forgetful map' $p_{\hat{E},E}$ from $T(\hat{E})$ to $T(E)$ such that $P_E = p_{\hat{E},E} \circ P_{\hat{E}}$. It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion.

1.6. Changing the basepoint. Let w be a normalized quasiconformal self-mapping of $\hat{\mathbf{C}}$, and let $\tilde{E} = w(E)$. By definition, the *allowable map* g from $T(\tilde{E})$ to $T(E)$ maps the $\tilde{E}$-equivalence class of f to the E-equivalence class of $f \circ w$ for every normalized quasiconformal self-mapping f of $\hat{\mathbf{C}}$.

Lemma 1.4. *The allowable map $g : T(\tilde{E}) \to T(E)$ is biholomorphic. If μ is the Beltrami coefficient of w, then g maps the basepoint of $T(\tilde{E})$ to the point $P_E(\mu)$ in $T(E)$.*

See §7.12 in [9] or §6.4 in [19] for a complete proof.

1.7. Statement of the main theorem. The main purpose of this paper is to give a self-contained proof of the following theorem.

The Main Theorem. *Let $E = \{0, 1, \infty, \zeta_1, \ldots, \zeta_n\}$ where $\zeta_i \neq \zeta_j$ for $i \neq j$, and $\zeta_i \neq 0, 1, \infty$ for all $i = 1, \ldots, n$. Let $\hat{E} = E \cup \{\zeta_{n+1}\}$ where ζ_{n+1} is any point in $\hat{\mathbf{C}} \setminus \{0, 1, \infty\}$ distinct from ζ_i for all $i = 1, \ldots, n$. Then, given any holomorphic map f from Δ into $T(E)$, there exists a holomorphic map $\hat{f}$ from Δ into $T(\hat{E})$, such that $p_{\hat{E},E} \circ \hat{f} = f$.*

Remark. This "lifting problem" was mentioned in §7 of the classic paper [5], and the authors called it "a difficult open problem." With the publication of [22], it became possible to give a quick solution of this problem, using Slodkowski's theorem. We shall discuss this in more details in §4. More recently, Chirka (in [4]) published a new proof of Slodkowski's theorem. See also [3], [6], [12], and [14]. The novelty of our present paper is that we use some ideas of Chirka and a theorem of Nag ([20]) to give a direct proof of the above theorem. Our approach, therefore, also gives a new interpretation of Chirka's methods.

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2. Holomorphic lifts and universal holomorphic motions

In this section we study the interesting relationship between "holomorphic lifts" and "universal holomorphic motions." The main purpose in this section is to prove a result of Bers-Royden (Proposition 4 in [5]) in its fullest generality.

2.1. Holomorphic motions.

Definition 2.1. Let V be a connected complex manifold with a basepoint x_0 and let E be any subset of $\hat{\mathbf{C}}$. A *holomorphic motion* of E over V is a map $\phi : V \times E \to \hat{\mathbf{C}}$ that has the following three properties:
(i) $\phi(x_0, z) = z$ for all z in E,
(ii) the map $\phi(x, \cdot) : E \to \hat{\mathbf{C}}$ is injective for each x in V, and
(iii) the map $\phi(\cdot, z) : V \to \hat{\mathbf{C}}$ is holomorphic for each z in E.

We say that V is a *parameter space* of the holomorphic motion ϕ. We will assume that ϕ is a *normalized* holomorphic motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi(x, \cdot)$ for every x in V.

Definition 2.2. Let V and W be connected complex manifolds with basepoints, and f be a basepoint preserving holomorphic map of W into V. If ϕ is a holomorphic motion of E over V, its *pullback* by f is the holomorphic motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \text{for all } (x, z) \in W \times E$$

of E over W.

If E is a proper subset of $\hat{E}$ and $\phi : V \times E \to \hat{\mathbf{C}}$, $\hat{\phi} : V \times \hat{E} \to \hat{\mathbf{C}}$ are two holomorphic motions, we say that $\hat{\phi}$ *extends* ϕ if $\hat{\phi}(x, z) = \phi(x, z)$ for all (x, z) in $V \times E$.

2.2. Universal holomorphic motion of E. Henceforth, we shall always assume that E is a closed subset of $\hat{\mathbf{C}}$ and that 0, 1, and ∞ belong to E.

Definition 2.3. The *universal holomorphic motion* Ψ_E of E over $T(E)$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^{\mu}(z) \quad \text{for } \mu \in M(\mathbf{C}) \text{ and } z \in E.$$

The definition of P_E in §1 guarantees that Ψ_E is well-defined. It is a holomorphic motion since P_E is a holomorphic split submersion and $\mu \mapsto w^{\mu}(z)$

is a holomorphic map from $M(\mathbf{C})$ to $\hat{\mathbf{C}}$ for every fixed z in $\hat{\mathbf{C}}$ (by Theorem 11 in [2]).

This holomorphic motion is "universal" in the following sense:

THEOREM 2.4. *Let $\phi : V \times E \to \hat{\mathbf{C}}$ be a holomorphic motion. If V is simply connected, then there exists a unique basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see §14 in [19].

In what follows, B is a path-connected topological space. Let $\mathscr{H}(\hat{\mathbf{C}})$ denote the group of homeomorphisms of $\hat{\mathbf{C}}$ onto itself, with the topology of uniform convergence in the spherical metric. As usual, E is a closed set in $\hat{\mathbf{C}}$, and 0, 1, and ∞ are in E. The following two lemmas were proved in [19]. For the reader's convenience, and to make our paper self-contained, we are including the proofs.

LEMMA 2.5. *Let $h : B \to \mathscr{H}(\hat{\mathbf{C}})$ be a continuous map such that $h(x)(e) = e$ for all x in B and for all e in E. If $h(x_0)$ is isotopic to the identity rel E for some fixed x_0 in B, then $h(x)$ is isotopic to the identity rel E for all x in B.*

Proof. Let x be any point in B. Choose a path $\gamma : [0,1] \to B$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. It is clear that the map $(t, z) \mapsto h(\gamma(t))(z)$ from $[0,1] \times \hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$ is an isotopy rel E between $h(x_0)$ and $h(x)$. □

LEMMA 2.6. *Let f and g be two continuous maps from B to $T(E)$, satisfying:*
(i) $\Psi_E(f(x), z) = \Psi_E(g(x), z)$ *for all $z \in E$ and $x \in B$, and*
(ii) $f(x_0) = g(x_0)$ *for some x_0 in B.*
Then, $f(x) = g(x)$ for all x in B.

Proof. By Proposition 1.3, there exists a basepoint preserving continuous map $s : T(E) \to M(\mathbf{C})$ such that $P_E \circ s$ is the identity map on $T(E)$. For each x in B, define $\mu(x) = s(f(x))$ and $\nu(x) = s(g(x))$. We will show that the quasiconformal map $h(x) = (w^{\mu(x)})^{-1} \circ w^{\nu(x)}$ is isotopic to the identity rel E. That will prove our lemma.

Since μ and ν are continuous maps of B into $M(\mathbf{C})$ and $\mathscr{H}(\hat{\mathbf{C}})$ is a topological group, Lemma 17 of [2] implies that h is a continuous map of B into $\mathscr{H}(\hat{\mathbf{C}})$.

By condition (i) and Definition 2.3, we have

$$w^{\mu(x)}(z) = \Psi_E(f(x), z) = \Psi_E(g(x), z) = w^{\nu(x)}(z)$$

for all x in B and z in E. Therefore, $h(x)$ fixes the set E pointwise for each x in B. By condition (ii), $h(x_0)$ is isotopic to the identity rel E. It follows by Lemma 2.5, that $h(x)$ is isotopic to the identity rel E for all x in B. □

Let E and $\hat{E}$ be any two closed subsets of $\hat{\mathbf{C}}$ such that $E \subset \hat{E}$ (as in §1, we assume that 0, 1, and ∞ belong to both E and $\hat{E}$). Recall from §1.5, the forgetful map $p_{\hat{E},E}$ from $T(\hat{E})$ to $T(E)$ such that $P_E = p_{\hat{E},E} \circ P_{\hat{E}}$. The following is a consequence of Lemma 2.6. Here, Ψ_E is the universal holomorphic motion of E and $\Psi_{\hat{E}}$ is the universal holomorphic motion of $\hat{E}$.

Lemma 2.7. *Let V be a connected complex Banach manifold with basepoint, and let f and g be basepoint preserving holomorphic maps from V into $T(E)$ and $T(\hat{E})$ respectively. Then $p_{\hat{E},E} \circ g = f$ if and only if $g^*(\Psi_{\hat{E}})$ extends $f^*(\Psi_E)$.*

See §13 in [19] for the proof.

2.3. A proposition. We prove the following generalization of Proposition 4 in [5]. This is an easy consequence of Theorem 2.4 and Lemma 2.7, and shows the importance of universal holomorphic motions.

Proposition 2.8. *Let V be a simply connected complex Banach manifold with a basepoint. The following statements are equivalent*:

(1) *Every holomorphic motion $\phi : V \times E \to \hat{\mathbf{C}}$ extends to a holomorphic motion $\hat{\phi} : V \times \hat{E} \to \hat{\mathbf{C}}$.*
(2) *For every basepoint preserving holomorphic map $f : V \to T(E)$, there exists a basepoint preserving holomorphic map $g : V \to T(\hat{E})$ such that $f = p_{\hat{E},E} \circ g$.*

Proof. (1) $\Rightarrow$ (2): Let $f : V \to T(E)$ be a basepoint preserving holomorphic map. Then, $f^*(\Psi_E) := \phi$ is a holomorphic motion of E over V. By (1) there exists a holomorphic motion $\hat{\phi} : V \times \hat{E} \to \hat{\mathbf{C}}$ such that $\hat{\phi}$ extends ϕ. By Theorem 2.4, there exists a basepoint preserving holomorphic map $g : V \to T(\hat{E})$ such that $g^*(\Psi_{\hat{E}}) = \hat{\phi}$. Since $\hat{\phi}$ extends ϕ, it follows by Lemma 2.7 that $p_{\hat{E},E} \circ g = f$.

(2) $\Rightarrow$ (1): Let $\phi : V \times E \to \hat{\mathbf{C}}$ be a holomorphic motion. By Theorem 2.4, there exists a basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$. By (2) there exists a basepoint preserving holomorphic map $g : V \to T(\hat{E})$ such that $f = p_{\hat{E},E} \circ g$. Let $g^*(\Psi_{\hat{E}}) := \hat{\phi}$; then, $\hat{\phi}$ is a holomorphic motion of $\hat{E}$ over V. It follows by Lemma 2.7 that $\hat{\phi}$ extends ϕ. □

Recall from §1.3, that when $E = \hat{\mathbf{C}}$, we can identify $T(\hat{\mathbf{C}})$ biholomorphically with $M(\mathbf{C})$. The pullback $\tilde{\Psi}_{\hat{\mathbf{C}}}$ of $\Psi_{\hat{\mathbf{C}}}$ to $M(\mathbf{C})$ by $P_{\hat{\mathbf{C}}}$ satisfies

$$\tilde{\Psi}_{\hat{\mathbf{C}}}(\mu, z) = \Psi_{\hat{\mathbf{C}}}(P_{\hat{\mathbf{C}}}(\mu), z) = w^{\mu}(z)$$

for all $(\mu, z) \in M(\mathbf{C}) \times \hat{\mathbf{C}}$. So, when we use $P_{\hat{\mathbf{C}}}$ to identify $T(\hat{\mathbf{C}})$ with $M(\mathbf{C})$, the universal holomorphic motion of $\hat{\mathbf{C}}$ becomes the map

$$\Psi_{\hat{\mathbf{C}}}(\mu, z) = w^{\mu}(z)$$

for $(\mu, z) \in M(\mathbf{C}) \times \hat{\mathbf{C}}$.

Corollary 2.9. *Let V be a simply connected complex Banach manifold with a basepoint. The following statements are equivalent*:

(1) *Every holomorphic motion $\phi : V \times E \to \hat{\mathbf{C}}$ extends to a holomorphic motion $\hat{\phi} : V \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$.*

(2) *For every basepoint preserving holomorphic map $f : V \to T(E)$, there exists a basepoint preserving holomorphic map $g : V \to M(\mathbf{C})$ such that $f = P_E \circ g$.*

3. Proof of the main theorem

Recall that $E = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$ where $\zeta_i \neq \zeta_j$ for $i \neq j$, and $\zeta_i \neq 0, 1, \infty$ for all $i = 1, \dots, n$. By Lemma 1.4, we may assume that $f : \Delta \to T(E)$ is a basepoint preserving holomorphic map.

For a fixed $0 < r < 1$, let $f_r(z) = f(rz) = [w^{\mu}]_E$. Then we define n holomorphic functions $f_{i,r}(z) = w^{\mu}(\zeta_i)$ for $i = 1, \dots, n$. Let $D = \hat{\mathbf{C}} \backslash \Delta$ be the exterior of Δ. We define n maps on D, which are holomorphic in a neighborhood of D, as

$$g_i(z) = f_{i,r}\left(\frac{1}{z}\right)$$

for $|z| \geq 1$ and for all $1 \leq i \leq n$. Furthermore, we extend g_i to $\hat{\mathbf{C}}$ as follows:

$$g_i(z) = g_i\left(\frac{1}{\bar{z}}\right)$$

for $|z| \leq 1$ and for all $1 \leq i \leq n$. We have the following:

(a) $g_i(\infty) = g_i(0) = \zeta_i$ for $i = 1, \dots, n$;

(b) for any fixed $z \in \hat{\mathbf{C}}$, $g_i(z) \neq g_j(z)$ for $1 \leq i \neq j \leq n$ and $g_i(z) \neq 0, 1, \infty$ for all $i = 1, \dots, n$;

(c) $g_i(z)$ is a bounded function on $\hat{\mathbf{C}}$.

So, there is a constant $C_0 > 0$ such that

$$|g_i(z)| \leq C_0$$

for all $z \in \hat{\mathbf{C}}$ and for all $1 \leq i \leq n$. Moreover, there is a number $\delta > 0$ such that

$$|g_i(z) - g_j(z)| > \delta \tag{3.1}$$

for all $1 \leq i \neq j \leq n$ and for all $z \in \hat{\mathbf{C}}$. Furthermore, $(\partial g_i / \partial \bar{z})(z) = 0$ for $z \in D$ and there is a constant $C_1 > 0$ such that

$$\left| \frac{\partial g_i}{\partial \bar{z}}(z) \right| \leq C_1 \tag{3.2}$$

for all $z \in \hat{\mathbf{C}}$ and for all $1 \leq i \leq n$.

Choose a C^{∞} function $0 \leq \lambda(x) \leq 1$ on $\mathbf{R}^+ = \{x \geq 0\}$ such that $\lambda(0) = 1$ and $\lambda(x) = 0$ for $x \geq \delta/2$. Define

$$\Theta(z, w) = \sum_{i=1}^{n} \lambda(|w - g_i(z)|) \frac{\partial g_i}{\partial \bar{z}}(z), \quad (z, w) \in \hat{\mathbf{C}} \times \mathbf{C}. \tag{3.3}$$

LEMMA 3.1. *The function* $\Theta(z,w)$ *has the following properties*:

(i) *only one term in the sum* (3.3) *defining* $\Theta(z,w)$ *can be nonzero*,

(ii) $\Theta(z,w)$ *is uniformly bounded by* C_1 *on* $\hat{\mathbf{C}} \times \mathbf{C}$,

(iii) $\Theta(z,w) = 0$ *for* $(z,w) \in (D \times \mathbf{C}) \cup (\hat{\mathbf{C}} \times (\overline{\Delta}_R)^c)$ *where* $R = C_0 + \delta/2$, *and* $\overline{\Delta}_R$ *denotes the closure of* $\Delta_R = \{z \in \mathbf{C} : |z| < R\}$,

(iv) $\Theta(z,w)$ *is a Lipschitz function in the* w*-variable with a Lipschitz constant* L *independent of* $z \in \hat{\mathbf{C}}$.

Proof. Item (i) follows, since, if a point w is within distance $\delta/2$ of one of the values $g_i(z)$, it must be at a distance greater that $\delta/2$ from any of the other values $g_j(z)$ (see (3.1)).

Item (ii) follows from (i) because there can be only one term in (3.3) which is nonzero and that term is bounded by the bound on $(\partial g_j/\partial \bar{z})(z)$ by (3.2).

Item (iii) follows because if $z \in D$, then $(\partial g_i/\partial \bar{z})(z) = 0$, and if $w \in \hat{\mathbf{C}} \backslash \Delta_R$, then $|w - g_i(z)| \geq \delta/2$ for all $z \in \hat{\mathbf{C}}$ and for all $i = 1, \dots, n$. Therefore, $\lambda(|w - g_i(z)|) = 0$ for all $i = 1, \dots, n$. Thus, $\Theta(z,w) = 0$ for $(z,w) \in (D \times \mathbf{C}) \cup (\hat{\mathbf{C}} \times (\overline{\Delta}_R)^c)$.

To prove (iv), we note that there is a constant $C_2 > 0$, such that $|\lambda(x) - \lambda(x')| \leq C_2|x - x'|$. Since $|(\partial g_i/\partial \bar{z})(z)| \leq C_1$, we have

$$|\Theta(z,w) - \Theta(z,w')| \leq C_1 C_2 \sum_{i=1}^{n} \big|\, |w - g_i(z)| - |w' - g_i(z)| \,\big|. \tag{3.4}$$

Since only one of the terms in the sum (3.3) for $\Theta(z,w)$ is nonzero and possibly some different term is nonzero in the sum for $\Theta(z,w')$, we obtain

$$|\Theta(z,w) - \Theta(z,w')| \leq 2C_1 C_2 |w - w'|.$$

Thus $L = 2C_1C_2$ is a Lipschitz constant independent of $z \in \hat{\mathbf{C}}$. □

Let $\mathscr{C}(\mathbf{C})$ denote the complex Banach space of bounded, continuous functions ϕ on $\mathbf{C}$ with the norm

$$\|\phi\| = \sup_{z \in \mathbf{C}} |\phi(z)|.$$

As usual, $L^\infty(\mathbf{C})$ denotes the complex Banach space of L^∞ functions on $\mathbf{C}$ with the L^∞-norm denoted by $\|\phi\|_\infty$.

Since $\Theta(z, f(z))$ is an L^∞ function with a compact support in $\overline{\Delta}$ for any $f \in \mathscr{C}(\mathbf{C})$, we can define an operator $\mathscr{Q}$ mapping functions in $\mathscr{C}(\mathbf{C})$ to functions in $L^\infty(\mathbf{C})$ with compact support by

$$\mathscr{Q}f(z) = \Theta(z, f(z)), \quad f(z) \in \mathscr{C}(\mathbf{C}).$$

Since $\Theta(z,w)$ is Lipschitz in the w variable with a Lipschitz constant L independent of $z \in \hat{\mathbf{C}}$, we have

$$|\mathscr{Q}f(z) - \mathscr{Q}g(z)| = |\Theta(z, f(z)) - \Theta(z, g(z))| \leq L|f(z) - g(z)|.$$

Thus,

$$\|\mathscr{Q}f - \mathscr{Q}g\|_\infty \le L\|f - g\|$$

and $\mathscr{Q} : \mathscr{C}(\mathbf{C}) \to L^\infty(\mathbf{C})$ is a continuous operator.

In the theory of quasiconformal mappings, the $\mathscr{P}$-operator is defined by

$$\mathscr{P}f(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{f(t)}{t - z}\, d\xi d\eta, \quad t = \xi + i\eta$$

where $f \in L^\infty(\mathbf{C})$ and has a compact support in $\mathbf{C}$. Then, $\mathscr{P}f(z) \to 0$ as $z \to \infty$. Furthermore, if f is continuous and has compact support, one can show that

$$\frac{\partial(\mathscr{P}f)}{\partial\bar{z}}(z) = f(z), \quad z \in \mathbf{C} \tag{3.5}$$

and using the notion of generalized derivatives, equation (3.5) is still true almost everywhere, if we only know that f has compact support and is in L^p, for $p \ge 1$.

A classical result in the theory of quasiconformal mappings (see [1]) is that $\mathscr{P}$ transforms L^∞ functions with compact support in $\mathbf{C}$ to Hölder continuous functions with Hölder exponent $1 - 2/p$ for every $p > 2$. A stronger result says that $\mathscr{P}$ carries L^∞ functions with compact supports to functions with an $|\varepsilon\log(\varepsilon)|$ modulus of continuity (see [12]). More precisely, for any $R > 0$, there exists a constant $B(R) > 0$, depending on R such that

$$|\mathscr{P}f(z) - \mathscr{P}f(z')| \le B(R)\|f\|_\infty |z - z'| \log\frac{1}{|z - z'|}$$

for all $z, z' \in \Delta_R$, $|z - z'| < \frac{1}{2}$.

This implies that for every $0 < \alpha < 1$, there exists a constant $A(R) > 0$ such that

$$|\mathscr{P}f(z) - \mathscr{P}f(z')| \le A(R)\|f\|_\infty |z - z'|^\alpha \tag{3.6}$$

for all $z, z' \in \Delta_R$, $|z - z'| < \frac{1}{2}$, where $A(R)$ depends only on R and α.

Now consider the operator

$$\mathscr{K} = \mathscr{P} \circ \mathscr{Q}.$$

Clearly, it is a continuous operator from $\mathscr{C}(\mathbf{C})$ into itself.

Lemma 3.2. *There is a constant $C_3 > 0$ such that*

$$\|\mathscr{K}f\| \le C_3 \quad \textit{for all } f \in \mathscr{C}(\mathbf{C}).$$

Proof. Since $\Theta(z,w) = 0$ for $z \in D$ and since $\Theta(z,w)$ is bounded by C_1, we have

$$|\mathcal{K}f(z)| = \left|\frac{1}{\pi}\iint_{\hat{\mathbf{C}}} \frac{\Theta(t,f(t))}{t-z}\,d\xi d\eta\right| = \left|\frac{1}{\pi}\iint_{\Delta} \frac{\Theta(t,f(t))}{t-z}\,d\xi d\eta\right|$$
$$\le \frac{1}{\pi}\iint_{\Delta} \frac{|\Theta(t,f(t))|}{|t-z|}\,d\xi d\eta \le \frac{C_1}{\pi}\iint_{\Delta} \frac{1}{|t-z|}\,d\xi d\eta \le 2C_1 = C_3$$

where $t = \xi + i\eta$. □

Lemma 3.3. *Let $p > 2$ and*

$$0 < \alpha = 1 - \frac{2}{p} < 1.$$

Then, for any $f \in \mathcal{C}(\mathbf{C})$, $\mathcal{K}f$ is α-Hölder continuous with a Hölder constant

$$H = \max\{A(1)C_1, 2^{1+\alpha}C_3\}$$

where H is independent of f.

Proof. From (3.6), we have

$$|\mathcal{K}f(z) - \mathcal{K}f(z')| = |\mathcal{P}(\mathcal{Q}f)(z) - \mathcal{P}(\mathcal{Q}f)(z')|$$
$$\le A(1)\|\mathcal{Q}f\|_\infty |z-z'|^\alpha \le A(1)C_1|z-z'|^\alpha \le H|z-z'|^\alpha$$

where $|z-z'| < \frac{1}{2}$.

When $|z-z'| \ge \frac{1}{2}$, by Lemma 3.2, we have

$$|\mathcal{K}f(z) - \mathcal{K}f(z')| \le 2C_3 \le 2^{1+\alpha}C_3|z-z'|^\alpha \le H|z-z'|^\alpha.$$

This completes the proof. □

Lemma 3.4. *For any $\varepsilon > 0$, there exists an $R > 0$, such that $|\mathcal{K}f(z)| < \varepsilon$ for all $f \in \mathcal{C}(\mathbf{C})$ and $z \in \mathbf{C}$ with $|z| \ge R$.*

Proof. Since $\|\mathcal{Q}f\|_\infty \le C_1$ from (ii) of Lemma 3.1, and the support of $\mathcal{Q}f$ is in $\overline{\Delta}$, by (iii) of the same lemma, when $|z| \ge 2$, we have

$$|\mathcal{K}f(z)| \le \frac{C_1}{\pi}\iint_{\Delta} \frac{d\xi d\eta}{|t-z|} \le \left(\frac{2C_1}{\pi}\iint_{\Delta}\frac{d\xi d\eta}{|t|}\right)\frac{1}{|z|}$$

where $t = \xi + i\eta$. Hence we are done. □

Remark. We know that $|\mathcal{K}f(z)| \to 0$ if $|z| \to \infty$. However, to check compactness of the operator $\mathcal{K}$, we need a kind of uniformity around $z = \infty$, like the existence of $R > 0$ independent of $f \in \mathcal{C}(\mathbf{C})$ in the above lemma.

In fact, from Lemma 3.2, we know that the family $\{\mathcal{K}f\}_{f\in\mathcal{C}(\mathbf{C})}$ is uniformly bounded, and from Lemma 3.3, it follows that the family is equicontinuous.

Therefore, from these lemmas, we merely conclude from the Ascoli-Arzela theorem that the family is relatively compact with respect to *the topology of the uniform convergence on any compact sets* of $\mathbf{C}$, which is weaker than the topology of $\mathscr{C}(\mathbf{C})$. For example, let

$$f_n(z) = \min\left\{\max\left\{1, \frac{|z|}{n}\right\}, 2e^{-|z|+2n}\right\}$$

for $z \in \mathbf{C}$. Then, each f_n is 2-Lipschitz and satisfies $\|f_n\| \leq 2$ and $f_n(z) \to 0$ as $|z| \to \infty$ (and hence it is an α-Hölder function for all $\alpha \in (0,1]$ whose Hölder norm depends only on α). However, the family $\{f_n\}_{n \in \mathbf{N}}$ is not compact in $\mathscr{C}(\mathbf{C})$.

Acknowledgement. We thank the referee for pointing to us the above remark and also the example.

The above lemmas imply that $\mathscr{K} : \mathscr{C}(\mathbf{C}) \to \mathscr{C}(\mathbf{C})$ is a continuous compact operator. Recall from the statement of the main theorem that ζ_{n+1} is any point in $\hat{\mathbf{C}} \backslash \{0, 1, \infty\}$ distinct from $\zeta_1, \dots, \zeta_n$. For ζ_{n+1}, let

$$\mathscr{B} = \{f \in \mathscr{C}(\mathbf{C}) : \|f\| \leq |\zeta_{n+1}| + C_3\}.$$

It is a bounded convex subset in $\mathscr{C}(\mathbf{C})$. The continuous compact operator $\zeta_{n+1} + \mathscr{K}$ maps $\mathscr{B}$ into itself. By Schauder fixed point theorem (see Theorem 2A on page 56 of [24]) $\zeta_{n+1} + \mathscr{K}$ has a fixed point in $\mathscr{B}$. That is, there is a $g_{n+1} \in \mathscr{B}$ such that

$$g_{n+1}(z) = \zeta_{n+1} + \mathscr{K} g_{n+1}(z) \quad \text{for all } z \in \mathbf{C}.$$

Since $\mathscr{Q}f(z)$ has a compact support in $\overline{\Delta}$ for any $g \in \mathscr{C}(\mathbf{C})$, $\mathscr{K} g_{n+1}(z) \to 0$ as $z \to \infty$. So, g_{n+1} can be extended continuously to ∞ such that $g_{n+1}(\infty) = \zeta_{n+1}$.

Lemma 3.5. *The solution $g_{n+1}(z)$ is the unique fixed point of the operator $\zeta_{n+1} + \mathscr{K}$.*

Proof. Suppose $g(z)$ and $\tilde{g}(z)$ are two solutions. Let

$$\phi(z) = g(z) - \tilde{g}(z) = \mathscr{K} g(z) - \mathscr{K} \tilde{g}(z).$$

Then $\phi(z) \to 0$ as $z \to \infty$. Now,

$$\frac{\partial \phi}{\partial \bar{z}}(z) = \frac{\partial g}{\partial \bar{z}}(z) - \frac{\partial \tilde{g}}{\partial \bar{z}}(z) = \Theta(z, g(z)) - \Theta(z, \tilde{g}(z)).$$

By Lemma 3.1, we get

$$\frac{\partial \phi}{\partial \bar{z}}(z) = 0 \quad \text{for all } z \in D.$$

Since $\Theta(z, w)$ is Lipschitz in w-variable with a Lipschitz constant L, we have, for all $z \in \hat{\mathbf{C}}$,

$$\left|\frac{\partial \phi}{\partial \bar{z}}(z)\right| = |\Theta(z, g(z)) - \Theta(z, \tilde{g}(z))| \leq L|g(z) - \tilde{g}(z)| = L|\phi(z)|.$$

Assuming that $\phi(z)$ is not equal to zero, define

$$\psi(z) = -\frac{\frac{\partial \phi}{\partial \bar{z}}(z)}{\phi(z)},$$

and otherwise, define $\psi(z)$ to be equal to zero. Note that, if $\phi(z) = 0$, then, since $\left|\frac{\partial \phi}{\partial \bar{z}}\right| \leq L(\phi(z))$, we have $\frac{\partial \phi}{\partial \bar{z}} = 0$. Then $\psi(z)$ is a L^∞ function with compact support in $\overline{\Delta}$. So we have $\mathscr{P}\psi$ in $\mathscr{C}(\mathbf{C})$ such that

$$\frac{\partial \mathscr{P}\psi}{\partial \bar{z}}(z) = \psi(z).$$

Consider $e^{\mathscr{P}\psi} \cdot \phi$. Then

$$\frac{\partial(e^{\mathscr{P}\psi} \cdot \phi)}{\partial \bar{z}}(z) \equiv 0.$$

This means that $e^{\mathscr{P}\psi} \cdot \phi$ is holomorphic on $\mathbf{C}$.

When $z \to \infty$, $\mathscr{P}\psi \to 0$ and $\phi(z) \to 0$. This implies that $e^{\mathscr{P}\psi} \cdot \phi$ is bounded on $\mathbf{C}$. Thus, $e^{\mathscr{P}\psi} \cdot \phi$ is a constant function. But $\phi(\infty) = 0$, and so $e^{\mathscr{P}\psi} \cdot \phi \equiv 0$. Hence, $\phi(z) \equiv 0$ and $g(z) = \tilde{g}(z)$ for all $z \in \mathbf{C}$. □

Since

$$\frac{\partial g_{n+1}}{\partial \bar{z}}(z) = \Theta(z, g_{n+1}(z))$$

and since $\Theta(z, w) = 0$ for all $z \in D$, we have

$$\frac{\partial g_{n+1}}{\partial \bar{z}}(z) = 0 \quad \text{for all } z \in D.$$

Therefore, $g_{n+1}(z)$ is holomorphic on D.

For ζ_i, $1 \leq i \leq n$, consider

$$\mathscr{K}g_i(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\Theta(t, g_i(t))}{t - z}\, d\xi d\eta$$

where $t = \xi + i\eta$. From the definition of $\Theta(z, w)$, we have

$$\Theta(t, g_i(t)) = \frac{\partial g_i}{\partial \bar{t}}(t).$$

Therefore,

$$\mathscr{K}g_i(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\frac{\partial g_i}{\partial \bar{t}}(t)}{t - z}\, d\xi d\eta.$$

This implies that

$$\frac{\partial \mathscr{K} g_i}{\partial \bar{z}}(z) = \frac{\partial g_i}{\partial \bar{z}}(z)$$

and that

$$\frac{\partial (g_i - \mathscr{K} g_i)}{\partial \bar{z}}(z) \equiv 0.$$

So, $g_i(z) - \mathscr{K} g_i(z)$ is holomorphic on $\mathbf{C}$. When $z \to \infty$, $g_i(z) \to \zeta_i$ and $\mathscr{K} g_i(z) \to 0$. So, $g_i(z) - \mathscr{K} g_i(z)$ is bounded. Therefore it is a constant function. We get

$$g_i(z) = \zeta_i + \mathscr{K} g_i(z).$$

Thus, $g_i(z)$ is the unique solution of the operator $\zeta_i + \mathscr{K}$.

We claim that $g_{n+1}(z) \neq g_i(z)$ for all $z \in \mathbf{C}$ and $1 \leq i \leq n$. To prove this, we assume that there is a point $a \in \hat{\mathbf{C}}$ such that $g_{n+1}(a) = g_i(a)$. It is clear that $a \neq \infty$. Then, we have

$$g_{n+1}(a) - g_i(a) = (\zeta_{n+1} - \zeta_i) + \mathscr{K} g_{n+1}(a) - \mathscr{K} g_i(a).$$

Note that $g_{n+1}(z) = \zeta_{n+1} + \mathscr{K} g_{n+1}(z)$ and $g_i(z) = \zeta_i + \mathscr{K} g_i(z)$. We have

$$\phi(z) = g_{n+1}(z) - g_i(z) = \zeta_{n+1} + \mathscr{K} g_{n+1}(z) - \zeta_i - \mathscr{K} g_i(z).$$

Therefore,

$$\frac{\partial \phi}{\partial \bar{z}}(z) = \frac{\partial g_{n+1}(z)}{\partial \bar{z}} - \frac{\partial g_i(z)}{\partial \bar{z}} = \Theta(z, g_{n+1}(z)) - \Theta(z, g_i(z)).$$

For $z \in D$, we have, by (iii) of Lemma 3.1,

$$\frac{\partial \phi}{\partial \bar{z}}(z) = 0.$$

For all $z \in \hat{\mathbf{C}}$, we have

$$\left| \frac{\partial \phi}{\partial \bar{z}}(z) \right| \leq |\Theta(z, g_{n+1}(z)) - \Theta(z, g_i(z))| \leq L|g_{n+1}(z) - g_i(z)| \leq L|\phi(z)|.$$

Assuming that $\phi(z)$ is not equal to zero, define

$$\psi(z) = -\frac{\dfrac{\partial \phi}{\partial \bar{z}}(z)}{\phi(z)},$$

and if $\phi(z) = 0$, let $\psi(z)$ to be equal to zero. Then, $\psi(z)$ is a L^∞ function with compact support in $\overline{\Delta}$. So, we have $\mathscr{P}\psi$ in $\mathscr{C}(\mathbf{C})$ such that

$$\frac{\partial \mathscr{P}\psi}{\partial \bar{z}}(z) = \psi(z).$$

Consider $e^{\mathscr{P}\psi} \cdot \phi$. Then

$$\frac{\partial(e^{\mathscr{P}\psi} \cdot \phi)}{\partial \bar{z}}(z) \equiv 0.$$

When $z \to \infty$, $\mathscr{P}\psi \to 0$ and $\phi(z) \to \zeta_{n+1} - \zeta_i$. This implies that $e^{\mathscr{P}\psi} \cdot \phi$ is bounded on $\mathbf{C}$. Thus, $e^{\mathscr{P}\psi} \cdot \phi$ is a constant function. But $\phi(\infty) = \zeta_{n+1} - \zeta_i$ and so

$$e^{\mathscr{P}\psi} \cdot \phi \equiv \zeta_{n+1} - \zeta_i \neq 0.$$

By our assumption, $\phi(a) = 0$, which is impossible.

Now, let

$$f_{n+1,r}(z) = g_{n+1}\left(\frac{1}{z}\right) \quad \text{for } |z| < 1.$$

Let

$$M_{n+1} = \{w \in \mathbf{C}^{n+1} : w_i \neq w_j \text{ for } i \neq j \text{ and } w_i \neq 0, 1 \text{ for all } i = 1, \dots, n+1\}.$$

We can define a holomorphic function

$$F_r(z) = (f_{1,r}(z), \dots, f_{n,r}(z), f_{n+1,r}(z)) : \Delta \to M_{n+1}.$$

Recall that $\hat{E} = E \cup \{\zeta_{n+1}\}$.

By a theorem of Nag (see [20]), there exists a holomorphic universal covering map $\pi : T(\hat{E}) \to M_{n+1}$ such that π maps the basepoint in $T(\hat{E})$ to the point $(\zeta_1, \dots, \zeta_{n+1})$. Since Δ is simply connected, there exists a holomorphic map

$$\hat{f}_r : \Delta \to T(\hat{E})$$

such that $\pi \circ \hat{f}_r = F_r$, and we can choose $\hat{f}_r$ to be basepoint preserving.

Recall from the beginning of §3, that $f_r(z) = [w^\mu]_E$. Suppose $\hat{f}_r(z) = [w^\nu]_{\hat{E}}$. Then, by §1.5, we have

$$p_{\hat{E},E}([w^\nu]_{\hat{E}}) = [w^\nu]_E.$$

Consider the two maps $f_r : \Delta \to T(E)$ and $p_{\hat{E},E} \circ \hat{f}_r : \Delta \to T(E)$. They are both basepoint preserving. Furthermore, at each ζ_i, for $i = 1, \dots, n$, we have $w^\mu(\zeta_i) = w^\nu(\zeta_i)$. Therefore, by Lemma 2.6, we conclude that $p_{\hat{E},E} \circ \hat{f}_r = f_r$ on Δ. This proves the lifting of the holomorphic map f_r on Δ_r.

Since $f_{n+1,r}$ misses the points 0, 1, and ∞, the family $\{f_{n+1,r}\}_{0<r<1}$ forms a normal family. Therefore, there exists a convergent subsequence $f_{n+1,r_k} \to f_{n+1}$ when $r_k \to 1$. It is clear that $f_{i,r_k} \to f_i$ when $r_k \to 1$. We claim that

LEMMA 3.6. *For all* $z \in \Delta$, $f_{n+1}(z) \neq f_i(z)$.

See the proof at the end of this section.

For z in Δ, define

$$F(z) = (f_1(z), \dots, f_{n+1}(z)).$$

By §1.3, $T(\hat{E})$ is identified with the classical Teichmüller space $Teich(\hat{\mathbf{C}} \backslash \hat{E})$, which is finite dimensional. Since each $\hat{f}_r(0) = [id] \in T(\hat{E})$ for all $0 < r < 1$, the family $\{\hat{f}_r\}_{0<r<1}$ is relatively compact, because of the completeness of the Kobayashi distance (which is the same as Teichmüller distance) on $T(\hat{E})$ (see Proposition 3 in [16], and also [23]). The holomorphy of the limit function $\hat{f}$ then follows from Weierstrass' theorem, since $T(\hat{E})$ is a bounded domain in $\mathbf{C}^{n+1}$ via Bers embedding. Since $\pi \circ \hat{f}_r = F_r$, we have $\pi \circ \hat{f} = F$, by continuity.

Finally, suppose $f(z) = [w^{\tilde{\mu}}]_E$ and $\hat{f}(z) = [w^{\tilde{\nu}}]_{\hat{E}}$. By §1.5, we have

$$p_{\hat{E},E}([w^{\tilde{\nu}}]_{\hat{E}}) = [w^{\tilde{\nu}}]_E.$$

Consider two maps $f : \Delta \to T(E)$ and $p_{\hat{E},E} \circ \hat{f} : \Delta \to T(E)$. They are both basepoint preserving. Furthermore, at each ζ_i, we have $w^{\tilde{\mu}}(\zeta_i) = w^{\tilde{\nu}}(\zeta_i)$ (because $\pi \circ \hat{f} = F$). It follows by Lemma 2.6 that $p_{\hat{E},E} \circ \hat{f} = f$.

Proof of Lemma 3.6. Consider a set of four points $S = \{z_1, z_2, z_3, z_4\}$ in $\mathbf{C}$. These points are distinct if and only if the cross ratio

$$Cr(S) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is not equal to 0, 1, or ∞.

Consider $S(z) = \{f_i(z), f_j(z), f_{n+1}(z), \infty\}$. The cross ratio

$$Cr(S(z)) = \frac{f_i(z) - f_{n+1}(z)}{f_j(z) - f_{n+1}(z)}.$$

We only need to show that for any fixed $0 < r_0 < 1$, $Cr(S(z))$ is not equal to 0, 1, or ∞ for any $z \in \Delta_{r_0}$ where Δ_{r_0} is the disk centered at zero with radius r_0.

For any $0 < r < 1$, let $S_r(z) = \{f_{i,r}(z), f_{j,r}(z), f_{n+1,r}(z), \infty\}$. Then

$$Cr(S_r(z)) = \frac{f_{i,r}(z) - f_{n+1,r}(z)}{f_{j,r}(z) - f_{n+1,r}(z)}.$$

Since $\mathbf{C} \backslash \{0, 1\}$ is complete hyperbolic and

$$C_r(S_r(0)) = \frac{\zeta_i - \zeta_{n+1}}{\zeta_j - \zeta_{n+1}} \in \mathbf{C} \backslash \{0, 1\}$$

for all $0 < r < 1$, again by Proposition 3 in [16], the family $\{C_r(S_r(z))\}_{0<r<1}$ is relatively compact in the space of holomorphic mappings from Δ to $\mathbf{C} \backslash \{0, 1\}$. Thus, for any $|z| < r_0$ and for any $0 < r < 1$, we obtain

$$|Cr(S_r(z))| \le K$$

for some $K > 0$.

This implies that the cross ratio $Cr(S(z))$ is bounded away from ∞ by K, by letting $r \to 1^-$. Following a similar argument, we can show that the cross ratio $Cr(S(z))$ is also bounded away from 0 and 1 for any $|z| < r_0$. So $f_{n+1}(z) \neq f_i(z)$ for any $1 \le i \le n$ on Δ_{r_0}. Since $0 < r_0 < 1$ is an arbitrary number, we conclude that $f_{n+1}(z) \neq f_i(z)$ on Δ, for any $1 \le i \le n$. This completes the proof. $\square$

4. Some concluding remarks

In their paper [5], Bers and Royden showed the intimate relationship between Teichmüller spaces and holomorphic motions. They noted that the lifting problem in §1.7 is nicely connected with the question of extending holomorphic motions. In fact, in Proposition 2.8 of our paper, let $V = \Delta$ and E and $\hat{E}$ be the two finite sets given in the statement of our main theorem. Then, by our main theorem and Proposition 2.8, it follows that every holomorphic motion of E over Δ extends to a holomorphic motion of $\hat{E}$ (over Δ). By Proposition 1 in [5], it then follows that given any holomorphic motion $\phi : \Delta \times K \to \hat{\mathbf{C}}$, where K is **any** set in $\hat{\mathbf{C}}$ (not necessarily closed), there exists a holomorphic motion $\hat{\phi} : \Delta \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that $\hat{\phi}$ extends ϕ.

It is important to note that the lifting problem that we discuss in our main theorem does not work if Δ is replaced by a domain in $\mathbf{C}^n$ $(n \geq 2)$. In fact, let E and $\hat{E}$ be the two given finite sets in our main theorem, and $n \geq 2$. Then, by our discussion in §1.3, $T(E)$ and $T(\hat{E})$ are the classical Teichmüller spaces of the sphere with punctures at E and $\hat{E}$ respectively. Consider the identity map $i : T(E) \to T(E)$; if it has a holomorphic lift into $T(\hat{E})$, i.e. if there exists a holomorphic map $g : T(E) \to T(\hat{E})$ such that $p_{\hat{E},E} \circ g = i$, then the map g will be a holomorphic section of the map $p_{\hat{E},E}$. This is impossible by a theorem of Earle and Kra; see [7] (also proved by Hubbard in [13]). By Proposition 2.8, that also means that the universal holomorphic motion $\Psi_E : T(E) \times E \to \hat{\mathbf{C}}$ cannot be extended to a holomorphic motion of the set $\hat{E}$.

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