M. NISHIO, N. SUZUKI AND M. YAMADA
KODAI MATH. J.
32 (2009), 501–520

WEIGHTED BEREZIN TRANSFORMATIONS WITH APPLICATION TO TOEPLITZ OPERATORS OF SCHATTEN CLASS ON PARABOLIC BERGMAN SPACES

Masaharu Nishio, Noriaki Suzuki and Masahiro Yamada

Abstract

In the setting of α-parabolic Bergman spaces, we consider weighted versions of averaging functions and Berezin transformations. Related norm equivalence relations are shown. They are very useful to study our Bergman spaces. As an application, we characterize the Schatten classes of compact Toeplitz operators.

1. Introduction

We consider the α-parabolic operator

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^{\alpha}$$

on the upper half space $\boldsymbol{R}_+^{n+1}$, where $\Delta_x := \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ denotes the Laplacian on the x-space $\boldsymbol{R}^n$ and $0 < \alpha \le 1$. Here we denote by $X = (x,t)$ and $Y = (y,s)$ points in $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0,\infty)$. The parabolic Bergman space $(\boldsymbol{b}_\alpha^2, \langle \cdot, \cdot \rangle)$ under consideration is a Hilbert space defined by

$$\boldsymbol{b}_\alpha^2 := \{u \in L^2(V); L^{(\alpha)}\text{-harmonic on } \boldsymbol{R}_+^{n+1}\},$$

where V denotes the $(n+1)$-dimensional Lebesgue measure on $\boldsymbol{R}_+^{n+1}$.

Since for $X \in \boldsymbol{R}_+^{n+1}$ the point evaluation $u \mapsto u(X) : \boldsymbol{b}_\alpha^2 \to \boldsymbol{R}$ is bounded (see [5]), the orthogonal projection from $L^2(V)$ onto $\boldsymbol{b}_\alpha^2$ is represented as an integral operator by a kernel R_α, which is called the α-parabolic Bergman kernel. For a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, we define the Toeplitz operator T_μ with symbol μ by

$$(T_\mu u)(X) := \int R_\alpha(X,Y)u(Y)\,d\mu(Y) \quad (u \in \boldsymbol{b}_\alpha^2).$$

2000 *Mathematics Subject Classification*: Primary 35K05; Secondary 26D10, 31B10.

Keywords and phrases: parabolic operator of fractional order, Bergman space, Berezin transformation, parabolic similarity, Toeplitz operator, compact operator of Schatten class.

This work was supported in part by Grant-in-Aid for Scientific Research (C) No. 18540168, No. 18540169, and No. 19540193 Japan Society for the Promotion of Science.

Received November 11, 2008; revised March 5, 2009.

In the study of Toeplitz operators, the following averaging function $\hat{\mu}^{(\alpha)}$ and Berezin transformation $\tilde{\mu}^{(\alpha)}$ are very useful (see [1], [6], [7], [10]):

$$\hat{\mu}^{(\alpha)}(X) := \mu(Q^{(\alpha)}(X))/V(Q^{(\alpha)}(X)),$$

$$\tilde{\mu}^{(\alpha)}(X) := \int R_\alpha(Y,X)^2\, d\mu(Y) \Big/ \int R_\alpha(Y,X)^2\, dV(Y),$$

where $Q^{(\alpha)}(X)$ is an α-parabolic Carleson box, defined by

$$(1.1)\quad Q^{(\alpha)}(X) := \{(y_1,\dots,y_n,s); t \le s \le 2t, |x_j - y_j| \le 2^{-1}t^{1/2\alpha}, j = 1,\dots,n\}.$$

Among our previous studies, we take up here the following compactness result ([7, Theorem 1]): Let $\mu \ge 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$ satisfying

$$(1.2)\qquad \int (1 + t + |x|^{2\alpha})^{-\delta}\, d\mu(x,t) < \infty$$

for some $\delta \in \boldsymbol{R}$. Then the following statements are equivalent:

(i) T_μ is compact on $\boldsymbol{b}_\alpha^2$,

(ii) $\lim_{Y\to\mathscr{A}} \hat{\mu}^{(\alpha)}(Y) = 0$,

(iii) $\lim_{Y\to\mathscr{A}} \tilde{\mu}^{(\alpha)}(Y) = 0$,

where $\mathscr{A}$ is the point of infinity of the one point compactification of $\boldsymbol{R}_+^{n+1}$.

From now on we denote by V^* the weighted measure

$$(1.3)\qquad dV^*(X) = t^{-(n/2\alpha+1)}\, dV(X).$$

Note that V^* is an invariant measure with respect to α-parabolic similarities (see (3.2) below). This invariant measure plays an imprtant role in our argument (see Remark 1 below).

In this paper, we define the weighted versions of averaging functions and Berezin transformations and give some norm estimates for them. As the result, the following new relation between $\hat{\mu}^{(\alpha)}$ and $\tilde{\mu}^{(\alpha)}$ is established.

Theorem 1. *Let $1 \le \sigma < \infty$. Then for a Radon measure $\mu \ge 0$ on $\boldsymbol{R}_+^{n+1}$, $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$ if and only if $\tilde{\mu}^{(\alpha)} \in L^\sigma(V^*)$.*

It will be also shown that if $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$ (so that $\tilde{\mu}^{(\alpha)} \in L^\sigma(V^*)$) then the Toeplitz operator T_μ is compact. This fact brings us a classification of compact Toeplitz operators. For $1 \leqq \sigma < \infty$, we denote by l^σ the set of all σ-summable real sequences.

Definition 1. A compact operator T on a Hilbert space $\mathscr{H}$ is said to be of Schatten σ-class if the sequence of all singular values $(\lambda_j)_{j=0}^\infty$ of T belongs to l^σ where the singular values λ_j of T mean the eigenvalues of $|T| := \sqrt{T^*T}$. Here $(\lambda_j)_{j=0}^\infty$ is arranged in decreasing order and repeated according to multiplicity (if there are only a finite number N of non-zero singular values, we consider $\lambda_j = 0$ for $j > N$). Denote by $\mathscr{S}^\sigma(\mathscr{H})$ the totality of compact operators on $\mathscr{H}$ of Schatten σ-class.

There is the following relation between singular values and eigenvalues. Let $(\mu_j)_{j=0}^{\infty}$ be the sequence of eigenvalues of a compact operator T, repeated according to multiplicity, and in decreasing order of absolute values. Then for every $m \geq 0$,

$$\sum_{j=0}^{m} |\mu_j|^{\sigma} \leq \sum_{j=0}^{m} \lambda_j^{\sigma}$$

holds (see [3, p. 1093]). Note also that if T is a positive definite self-adjoint operator, then $T = |T|$, so that $\lambda_j = \mu_j$ for all j.

We can characterize Toeplitz operators of Schatten class.

THEOREM 2. *Let $1 \leq \sigma < \infty$. For a Radon measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$ satisfying* (1.2) *for some $\delta \in \boldsymbol{R}$, the Toeplitz operator T_μ on $\boldsymbol{b}_\alpha^2$ is in the Schatten σ-class $\mathscr{S}^{\sigma}(\boldsymbol{b}_\alpha^2)$ if and only if $\hat{\mu}^{(\alpha)} \in L^{\sigma}(V^*)$.*

We mention here that in the classical setting (for spaces of holomorphic or harmonic functions), there are some forerunning deep works (e.g. [4], [2], [1]).

This paper will be organized as follows: In section 2, we review the definition of $L^{(\alpha)}$-harmonic functions and some properties of the α-parabolic Bergman kernel. In section 3, we recall the α-parabolic similarity, which enables us to introduce mean functions. Norm estimates of mean operators on Orlicz spaces are proved in section 4. In section 5, we define weighted averaging functions and weighted Berezin transformations. Since both are examples of mean functions, some norm relations are deduced from estimates proved in the previous section. We discuss them and give a proof of Theorem 1 in section 6. In section 7, we apply our norm relations to characterization of Schatten class of compact Toeplitz operator. Theorem 2 is proved in this section. We note that our relations are also useful to study Carleson inequalities on parabolic Bergman spaces (see [9]). The last section is an appendix, where we discuss a property of the space of Schatten class operators of Orlicz type for the sake of completeness.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

Throughout this paper, we denote by $C_c^{\infty}(\boldsymbol{R}_+^{n+1})$, resp. $C_0(\boldsymbol{R}_+^{n+1})$, the set of all infinitely differentiable functions on $\boldsymbol{R}_+^{n+1}$ which have compact support, resp. the set of all continuous functions which tends to zero at the point of infinity $\mathscr{A}$.

A continuous funcion u on $\boldsymbol{R}_+^{n+1}$ is said to be $L^{(\alpha)}$-harmonic, if $L^{(\alpha)}u = 0$ in the sense of distribution, i.e.,

$$\int u(X) \cdot \widetilde{L^{(\alpha)}}\varphi(X)\, dV(X) = 0$$

for every $\varphi \in C_c^\infty(\boldsymbol{R}_+^{n+1})$, where

$$\widetilde{L^{(\alpha)}}\varphi(x,t) := -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}(\varphi(x+y,t)-\varphi(x,t))|y|^{-n-2\alpha}\,dy$$

and

$$c_{n,\alpha} = -4^\alpha\pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0.$$

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is given by

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n}\displaystyle\int_{R^n}\exp(-t|\xi|^{2\alpha}+\sqrt{-1}x\cdot\xi)\,d\xi & t>0 \\ 0 & t\le 0. \end{cases}$$

Then it has the following homogeneity:

$$\partial_x^\beta\partial_t^k W^{(\alpha)}(s^{1/2\alpha}x, st) = s^{-((n+|\beta|)/2\alpha+k)}(\partial_x^\beta\partial_t^k W^{(\alpha)})(x,t), \tag{2.1}$$

where $\beta = (\beta_1,\dots,\beta_n)\in\boldsymbol{N}_0^n$ be a multi-index and $k\ge 0$ be an integer. Here $\boldsymbol{N}_0 = \boldsymbol{N}\cup\{0\}$ denotes the set of all nonnegative integers. We use the following estimate frequently: There exists a constant $C>0$ such that

$$|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)| \le C(t+|x|^{2\alpha})^{-((n+|\beta|)/2\alpha+k)} \tag{2.2}$$

for all $(x,t)\in\boldsymbol{R}_+^{n+1}$ (see [5]). The α-parabolic Bergman kernel $R_\alpha(X,Y) = R_\alpha(x,t;y,s)$ (i.e., the reproducing kernel of the Hilbert space $\boldsymbol{b}_\alpha^2$) is given by

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y,t+s).$$

We also use a kernel $R_\alpha^{\beta,m}$ for $(\beta,m)\in\boldsymbol{N}_0^n\times\boldsymbol{N}_0$:

$$R_\alpha^{\beta,m}(X,Y) := c_{\beta,m}s^{(|\beta|/2\alpha+m)}\partial_y^\beta\partial_s^m R_\alpha(X,Y),$$

where $c_{\beta,m} = (-1)^{|\beta|}(-2)^m/m!$. When $\beta=0$, $R_\alpha^m := R_\alpha^{0,m}$ as well as R_α has the following reproducing property: For $1\le p<\infty$ and for every $u\in\boldsymbol{b}_\alpha^p$, $u = R_\alpha^m u$,

$$u(X) = R_\alpha^m u(X) := \int R_\alpha^m(X,Y)u(Y)\,dV(Y) \tag{2.3}$$

holds. Remark that if $\beta\ne 0$, then $R_\alpha^{\beta,m}$ may not necessarily have a reproducing property.

The following elementary fact is very useful in our later argument: Let $\lambda,\delta\in\boldsymbol{R}$. If $-1<\lambda<\delta-\left(\dfrac{n}{2\alpha}+1\right)$, then there exists a constant $C>0$ such that

$$\int t^\lambda(s+t+|x-y|^{2\alpha})^{-\delta}\,dV(x,t) = Cs^{\lambda-\delta+(n/2\alpha+1)} \tag{2.4}$$

for every $(y,s)\in\boldsymbol{R}_+^{n+1}$. By (2.2) there exists a constant $C>0$ such that

$$|R_\alpha^{\beta,m}(x,t;y,s)| \le Cs^{(|\beta|/2\alpha+m)}(t+s+|x-y|^{2\alpha})^{-(n/2\alpha+1)-(|\beta|/2\alpha+m)}. \tag{2.5}$$

Hence (2.1) and (2.4) show that if $\dfrac{|\beta|}{2\alpha}+m>\left(\dfrac{n}{2\alpha}+1\right)\left(\dfrac{1}{p}-1\right)$, then

$$\|R_{\alpha}^{\beta,m}(\cdot,Y)\|_{L^p(V)}=Cs^{(n/2\alpha+1)(1/p-1)} \tag{2.6}$$

for every $Y=(y,s)\in \boldsymbol{R}_+^{n+1}$, where $C>0$ is independent of Y.

3. α-parabolic similarities and mean functions

In order to define a mean function, we recall α-parabolic similarities ([8]). For $t>0$, the mapping $\tau_t^{(\alpha)}:(y,s)\mapsto(t^{1/2\alpha}y,ts)$ is called an α-parabolic dilation. A transformation on $\boldsymbol{R}^{n+1}$ is said to be an α-parabolic similarity if it is a composition of α-parabolic dilations and translations. Evidently, the equation $L^{(\alpha)}u=0$ is invariant under α-parabolic similarities. Let $X_0=(0,1)$ be taken as a reference point in $\boldsymbol{R}_+^{n+1}$. Then for every $X=(x,t)\in\boldsymbol{R}_+^{n+1}$, there exists a unique α-parabolic similarity Φ_X which maps the reference point X_0 to X and is bijective on $\boldsymbol{R}_+^{n+1}$. In fact, Φ_X is given by

$$\Phi_{(x,t)}(y,s)=T_x\circ\tau_t^{(\alpha)}(y,s)=(t^{1/2\alpha}y+x,ts),$$

where $T_x:(y,s)\mapsto(y+x,s)$ for $x\in\boldsymbol{R}^n$. We remark that $\{\Phi_X;X\in\boldsymbol{R}_+^{n+1}\}=\{\tau_t^{(\alpha)};t>0\}\ltimes\{T_x;x\in\boldsymbol{R}^n\}$ is a semi-direct product as a transformation group, so that $\boldsymbol{R}_+^{n+1}$ has a group structure by $\Phi_X\Phi_Y=\Phi_{X\cdot Y}$, where

$$X\cdot Y:=(t^{1/2\alpha}y+x,ts)=\Phi_X(Y). \tag{3.1}$$

Let f be a Borel measurable function on $\boldsymbol{R}_+^{n+1}$. It is easily seen that for every $Y\in\boldsymbol{R}_+^{n+1}$, we have

$$\int f(X)t^{-(n/2\alpha+1)}\,dV(X)=\int f(Y\cdot X)t^{-(n/2\alpha+1)}\,dV(X). \tag{3.2}$$

This means that $dV^*(X)=t^{-(n/2\alpha+1)}\,dV(X)$ is a left-invariant measure. Note that $t^{-1}\,dV(X)$ is a right-invariant measure (see [8]).

For a Radon measure ρ on $\boldsymbol{R}_+^{n+1}$, we put

$$\mathscr{I}_\rho f(X):=\int f(X\cdot Y)\,d\rho(Y). \tag{3.3}$$

If ρ is absolutely continuous with respect to V, i.e., $d\rho(X)=\rho(X)\,dV(X)$, then, by change of variables,

$$\mathscr{I}_\rho f(X)=t^{-(n/2\alpha+1)}\int f(Y)\rho(X^{-1}\cdot Y)\,dV(Y)$$

holds. Thus for a measure μ, we may define

$$\mathscr{I}_\rho\mu(X):=t^{-(n/2\alpha+1)}\int\rho(X^{-1}\cdot Y)\,d\mu(Y). \tag{3.4}$$

Note that if $d\mu(X) = f(X)\,dV(X)$, then we have

$$\mathscr{I}_\rho(f\,dV)(X) = \mathscr{I}_\rho f(X). \tag{3.5}$$

When ρ is an absolutely continuous probability measure, we call $\mathscr{I}_\rho\mu$ a mean function of μ with respect to ρ.

Using the group structure (3.1), we can consider convolution of measures. Let ρ_1 and ρ_2 be Radon measures on $\boldsymbol{R}_+^{n+1}$. Then the convolution $\rho_1 * \rho_2$ is a Radon measure defined by

$$\int f\,d(\rho_1 * \rho_2) := \iint f(X \cdot Y)\,d\rho_1(X)\,d\rho_2(Y). \tag{3.6}$$

We make some remarks. For a Borel measurable function f, we have

$$\mathscr{I}_{\rho_1}\mathscr{I}_{\rho_2}f = \mathscr{I}_{\rho_1*\rho_2}f \tag{3.7}$$

whenever both sides are defined. If ρ_2 is absolutely continuous with respect to the Lebesgue measure V, then $\rho_1 * \rho_2$ is also absolutely continuous. In this case, we obtain

$$\mathscr{I}_{\rho_1}\mathscr{I}_{\rho_2}\mu = \mathscr{I}_{\rho_1*\rho_2}\mu. \tag{3.8}$$

Moreover if both ρ_1 and ρ_2 are absolutely continuous with respect to V, then the density of $\rho_1 * \rho_2$ is given by

$$\rho_1 * \rho_2(X) = \int \rho_1(Y)\rho_2(Y^{-1} \cdot X)s^{-(n/2\alpha+1)}\,dV(Y). \tag{3.9}$$

By change of variables, we also obtain

$$\rho_1 * \rho_2(X) = \int \rho_1(X \cdot Y^{-1})\rho_2(Y)s^{-1}\,dV(Y). \tag{3.10}$$

4. Boundedness of mean operators on Orlicz spaces

Let Ψ be the set of all convex and strictly increasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$ and $\lim_{s\to\infty}\psi(s) = \infty$. By the convexity of ψ, we have

$$s_0\psi(t) \le \psi(s_0 t) \tag{4.1}$$

for any $s_0 \ge 1$ and $t \ge 0$. From now on, we use the following notation. For $\eta \in \boldsymbol{R}$, we denote by V_η the weighted measure

$$dV_\eta(X) = t^\eta\,dV(X). \tag{4.2}$$

Hence $V^* = V_{-(n/2\alpha+1)}$.

The Orlicz space with respect to $\psi \in \Psi$ and V_η is a Banach space defined by

$$L^\psi(V_\eta) := \{f;\ \text{Borel measurable on } \boldsymbol{R}_+^{n+1}, \|f\|_{L^\psi(V_\eta)} < \infty\},$$

where

$$\|f\|_{L^{\psi}(V_\eta)} := \inf\left\{\tau > 0; \int \psi\left(\frac{|f|}{\tau}\right) dV_\eta \le 1\right\}.$$

When $\psi(t) = t^\sigma$ with $\sigma \ge 1$, the corresponding space is nothing but the usual $L^\sigma(V_\eta)$ and

$$\|f\|_{L^{t^\sigma}(V_\eta)} = \|f\|_{L^\sigma(V_\eta)} := \left(\int |f(X)|^\sigma \, dV_\eta(X)\right)^{1/\sigma}.$$

Proposition 1. *Let ρ be a probability measure on $\boldsymbol{R}_+^{n+1}$ and $\eta \in \boldsymbol{R}$. Then we have the following inequality: For every Borel measurable function $f \ge 0$ on $\boldsymbol{R}_+^{n+1}$,*

$$\int \psi(\mathscr{I}_\rho f(X)) \, dV_\eta(X) \le \left(\int s^{-\eta-1} \, d\rho(Y)\right) \int \psi(f(X)) \, dV_\eta(X).$$

In particular, if ρ satisfies

$$\int s^{-\eta-1} \, d\rho(Y) < \infty, \tag{4.3}$$

then the mean operator $\mathscr{I}_\rho : L^\psi(V_\eta) \mapsto L^\psi(V_\eta)$ is bounded.

Proof. By the Jensen inequality, we have

$$\psi(\mathscr{I}_\rho f(X)) \le \int \psi(f(X \cdot Y)) \, d\rho(Y)$$

so that

$$\iint \psi(f(X \cdot Y)) \, d\rho(Y) \, dV_\eta(X) = \iint s^{-\eta-1} \psi(f(Z)) \, dV_\eta(Z) \, d\rho(Y)$$

gives us the desired inequality. Now suppose (4.3) and $f \in L^\psi(V_\eta)$. Then putting $s_0 := \max\{1, \int s^{-\eta-1} \, d\rho(Y)\}$ and taking $\tau > 0$ such that $\int \psi(|f|/\tau) \, dV_\eta \le 1$, we have

$$\begin{aligned}\int \psi\left(\frac{|\mathscr{I}_\rho f|(X)}{s_0 \tau}\right) dV_\eta(X) &\le \left(\int s^{-\eta-1} \, d\rho(Y)\right) \int \psi\left(\frac{|f|(X)}{s_0 \tau}\right) dV_\eta(X) \\ &\le \int \psi\left(\frac{|f|(X)}{\tau}\right) dV_\eta(X) \le 1,\end{aligned}$$

which implies that $\mathscr{I}_\rho f \in L^\psi(V_\eta)$ and $\|\mathscr{I}_\rho f\|_{L^\psi(V_\eta)} \le s_0 \|f\|_{L^\psi(V_\eta)}$. □

In the case of $\psi(t) = t^\sigma$ with $\sigma \ge 1$, the condition that ρ is probability measure is not necessary.

Proposition 2. *Let* $1 \le \sigma \le \infty$. *For* $\eta \in \boldsymbol{R}$ *and a Radon measure* ρ *on* $\boldsymbol{R}_+^{n+1}$, *we have*

$$\|\mathscr{I}_\rho f\|_{L^\sigma(V_\eta)} \le \left(\int s^{-(\eta+1)/\sigma}\, d|\rho|(Y)\right)\|f\|_{L^\sigma(V_\eta)}$$

for every $f \in L^\sigma(V_\eta)$.

Proof. Use the Minkowski inequality instead of the Jensen inequalty in the above proof. □

5. Weighted averaging functions and weighted Berezin transformations

In this section, we consider weighted versions of averaging functions and Berezin transformations. As before, we use the notation $dV_\lambda(X) = t^\lambda\, dV(X)$. Let μ be a positive Radon measure on $\boldsymbol{R}_+^{n+1}$. For a Borel set S in $\boldsymbol{R}_+^{n+1}$ of finite and positive V_λ-volume, we define a weighted averaging function for μ by

$$A_{S,\lambda}\mu(X) := \frac{1}{V_\lambda(\Phi_X(S))}\int_{\Phi_X(S)} s^\lambda\, d\mu(Y). \tag{5.1}$$

If $S = Q^{(\alpha)}(X_0)$ with $X_0 = (0,1)$ and $\lambda = 0$, then $A_{S,0}\mu$ is nothing but the original averaging function $\hat{\mu}^{(\alpha)}$.

Also let $(\beta, m) \in \boldsymbol{N}_0^n \times \boldsymbol{N}_0$ be a multi-index, $0 < p < \infty$ and $\lambda \in \boldsymbol{R}$. If

$$-1 < \lambda < \left(\frac{n}{2\alpha} + 1\right)(p-1) + \left(\frac{|\beta|}{2\alpha} + m\right)p, \tag{5.2}$$

then $R_\alpha^{\beta,m}(\cdot, X) \in L^p(V_\lambda)$ and by (2.6)

$$\int |R_\alpha^{\beta,m}(Y,X)|^p\, dV_\lambda(Y) = C_1 t^{-\kappa},$$

where C_1 is a constant and

$$\kappa := (p-1)\left(\frac{n}{2\alpha} + 1\right) - \lambda.$$

In this case, we define a weighted Berezin transformation of μ by

$$B_{\beta,m,p,\lambda}\mu(X) := \frac{\int |R_\alpha^{\beta,m}(Y,X)|^p s^\lambda\, d\mu(Y)}{\int |R_\alpha^{\beta,m}(Y,X)|^p\, dV_\lambda(Y)} = \frac{t^\kappa}{C_1}\int |R_\alpha^{\beta,m}(Y,X)|^p s^\lambda\, d\mu(Y), \tag{5.3}$$

If $(\beta, m, p, \lambda) = (0,0,2,0)$, then $B_{0,0,2,0}\mu$ is the original Berezin transformation $\tilde{\mu}^{(\alpha)}$.

These functions in (5.1) and (5.3) are mean functions discussed in section 3. In fact, let

$$\rho_{S,\lambda}(X) := \frac{t^\lambda 1_S(x,t)}{V_\lambda(S)},$$

where 1_S is the characteristic function of S. Then since

$$V_\lambda(\Phi_X(S)) = t^{n/2\alpha+1+\lambda} V_\lambda(S), \tag{5.4}$$

we see easily $A_{S,\lambda}\mu(X) = \mathscr{I}_{\rho_{S,\lambda}}\mu(X)$. Also, by the homogeneity (2.1), we have

$$R_\alpha^{\beta,m}(Y,X) = t^{-(n/2\alpha+1)} R_\alpha^{\beta,m}(X^{-1}\cdot Y, X_0). \tag{5.5}$$

Hence putting $\rho_{\beta,m,p,\lambda}(X) := (t^\lambda/C_1)|R_\alpha^{\beta,m}(X,X_0)|^p$, we obtain

$$\begin{aligned}
B_{\beta,m,p,\lambda}\mu(X) &= \frac{t^\kappa}{C_1}\int |R_\alpha^{\beta,m}(Y,X)|^p s^\lambda \, d\mu(Y) \\
&= \frac{t^\kappa}{C_1}\int |t^{-(n/2\alpha+1)} R_\alpha^{\beta,m}(X^{-1}\cdot Y, X_0)|^p s^\lambda \, d\mu(Y) \\
&= t^{-(n/2\alpha+1)} \int \frac{1}{C_1}\left(\frac{s}{t}\right)^\lambda |R_\alpha^{\beta,m}(X^{-1}\cdot Y, X_0)|^p \, d\mu(Y) \\
&= t^{-(n/2\alpha+1)} \int \rho_{\beta,m,p,\lambda}(X^{-1}\cdot Y)\, d\mu(Y) \\
&= \mathscr{I}_{\rho_{\beta,m,p,\lambda}}\mu(X).
\end{aligned}$$

For a Borel measurable function f on $\boldsymbol{R}_+^{n+1}$, we also set

$$A_{S,\lambda}f(X) := \mathscr{I}_{\rho_{S,\lambda}}(f\,dV)(X) \quad \text{and} \quad B_{\beta,m,p,\lambda}f(X) := \mathscr{I}_{\rho_{\beta,m,p,\lambda}}(f\,dV)(X).$$

Then, as in (3.5), we see

$$A_{S,\lambda}f(X) = \mathscr{I}_{\rho_{S,\lambda}}f(X) = \frac{1}{V_\lambda(\Phi_X(S))}\int_{\Phi_X(S)} f(Y)\,dV_\lambda(Y),$$

and

$$B_{\beta,m,p,\lambda}f(X) = \mathscr{I}_{\rho_{\beta,m,p,\lambda}}f(X) = \frac{\int |R_\alpha^{\beta,m}(Y,X)|^p f(Y)\,dV_\lambda(Y)}{\int |R_\alpha^{\beta,m}(Y,X)|^p\,dV_\lambda(Y)}.$$

6. Norm estimates on Orlicz spaces

In this section we give some norm estimates of the weighted averaging function and the weighted Berezin transformation on Orlicz spaces. Let ψ be a fixed convex function which belongs to Ψ. The following lemmas follow from Proposition 1.

LEMMA 1. *Let $\lambda_1, \lambda_2, \eta \in \boldsymbol{R}$ and let U and V be two relatively compact non-empty open sets in $\boldsymbol{R}_+^{n+1}$. Then for a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, $A_{U,\lambda_1}\mu \in L^\psi(V_\eta)$ if and only if $A_{V,\lambda_2}\mu \in L^\psi(V_\eta)$.*

Proof. Take a relatively compact open set K such that $U \subset K \cdot V := \{Z \cdot Y; Z \in K, Y \in V\}$. Then $\rho_{K,\lambda_2} * \rho_{V,\lambda_2} > 0$ on $K \cdot V$ so that there exists a constant $C > 0$ such that $C\rho_{K,\lambda_2} * \rho_{V,\lambda_2} \geq \rho_{U,\lambda_1}$. Hence it follows from (3.9) that

$$A_{U,\lambda_1}\mu(X) = \mathscr{I}_{\rho U,\lambda_1}\mu(X) \leq C\mathscr{I}_{\rho_{K,\lambda_2} * \rho_{V,\lambda_2}}\mu(X) = C\mathscr{I}_{\rho_{K,\lambda_2}}(A_{V,\lambda_2}\mu)(X). \tag{6.1}$$

By Proposition 1, $\mathscr{I}_{\rho_{K,\lambda_2}}$ is bounded on $L^\psi(V_\eta)$, and hence $A_{V,\lambda_2}\mu \in L^\psi(V_\eta)$ implies $A_{U,\lambda_1}\mu \in L^\psi(V_\eta)$. The converse is similarly proved. □

We assume (5.2). Then

$$\bar{\rho}_{\beta,m,p,\lambda}(x,t) := \frac{t^\lambda}{C_2}(1 + t + |x|^{2\alpha})^{-p(n/2\alpha+1+|\beta|/2\alpha+m)},$$

where C_2 is a constant such that $\int \bar{\rho}_{\beta,m,p,\lambda}(X)\,dV(X) = 1$, defines the mean operator

$$\bar{B}_{\beta,m,p,\lambda} := \mathscr{I}_{\bar{\rho}_{\beta,m,p,\lambda}}.$$

Lemma 2. *Let $(\beta,m) \in \boldsymbol{N}_0^n \times \boldsymbol{N}_0$, $0 < p < \infty$ and $\lambda, \eta \in \boldsymbol{R}$. If (5.2) and*

$$\frac{n}{2\alpha} + 1 + \lambda - p\left(\frac{n}{2\alpha} + 1 + \frac{|\beta|}{2\alpha} + m\right) < \eta + 1 < \lambda + 1 \tag{6.2}$$

hold, then there exists a constant $C > 0$ such that

$$\|\bar{B}_{\beta,m,p,\lambda}f\|_{L^\psi(V_\eta)} \leq C\|f\|_{L^\psi(V_\eta)} \tag{6.3}$$

for every $f \in L^\psi(V_\eta)$.

Proof. By (2.4) and (6.2), we have

$$\begin{aligned}&\int t^{-\eta-1}\bar{\rho}_{\beta,m,p,\lambda}(X)\,dV(X)\\ &\quad= \frac{1}{C_2}\int t^{-\eta-1}t^\lambda(1 + t + |x|^{2\alpha})^{-p(n/2\alpha+1+|\beta|/2\alpha+m)}\,dV(x,t) < \infty,\end{aligned}$$

so that (6.3) follows from Proposition 1. □

Lemma 3. *Let $(\beta,m) \in \boldsymbol{N}_0^n \times \boldsymbol{N}_0$, $0 < p < \infty$ and $\lambda, \tau, \eta \in \boldsymbol{R}$. We assume (5.2). Then for a compact set K of positive Lebesgue measure and a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, we have*

$$\|A_{K,\tau}\mu\|_{L^\psi(V_\eta)} \leq C\|B_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)}$$

with some constant $C > 0$ independent of μ.

Proof. We may assume that $\|B_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)} < \infty$. Take a relatively compact open set $U_0 \neq \emptyset$ and $\delta > 0$ such that

$$|R_\alpha^{\beta,m}(\cdot, X_0)| \geq \delta \quad \text{on } U_0.$$

By (5.5), if $Y^{-1} \cdot X \in U_0$, then $|R_\alpha^{\beta,m}(X,Y)| \geq s^{-(n/2\alpha+1)}\delta$ so that

$$\begin{aligned} B_{\beta,m,p,\lambda}\mu(X) &= \frac{t^\kappa}{C_1}\int |R_\alpha^{\beta,m}(Y,X)|^p t^\lambda \, d\mu(Y) \\ &\geq \frac{1}{C_1} t^\kappa t^{-p(n/2\alpha+1)}\delta^p \int_{\Phi_X(U_0)} s^\lambda \, d\mu(Y) \\ &= \frac{\delta^p}{C_1} t^{-(n/2\alpha+1)-\lambda} V_\lambda(\Phi_X(U_0)) A_{U_0,\lambda}\mu(Y). \end{aligned}$$

Hence by (5.4), we have

$$B_{\beta,m,p,\lambda}\mu \geq \frac{\delta^p}{C_1} V_\lambda(U_0) A_{U_0,\lambda}\mu \geq C A_{U_0,\lambda}\mu.$$

We also take a relatively compact open set U such that $K \cdot U_0^{-1} \subset U$. Then as in (6.1) we have

$$\mathscr{I}_{U,\lambda} B_{\beta,m,p,\lambda}\mu \geq C \mathscr{I}_{U,\lambda} A_{U_0,\lambda}\mu \geq C A_{K,\lambda}\mu \geq C A_{K,\tau}\mu.$$

Hence Proposition 1 again shows

$$\|A_{K,\tau}\mu\|_{L^\psi(V_\eta)} \leq C\|\mathscr{I}_{U,\lambda} B_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)} \leq C\|B_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)}. \qquad \square$$

We also obtain the opposite inequality.

Lemma 4. *Let* $(\beta,m) \in \boldsymbol{N}_0^n \times \boldsymbol{N}_0$, $0 < p < \infty$ *and* $\lambda, \tau, \eta \in \boldsymbol{R}$, *and let* $U \neq \emptyset$ *be a relatively compact open set in* $\boldsymbol{R}_+^{n+1}$. *If* (5.2) *and* (6.2) *hold, then*

$$\|\bar{B}_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)} \leq C\|A_{U,\tau}\mu\|_{L^\psi(V_\eta)}$$

with some constant $C > 0$ *independent of* μ.

Proof. We first remark that by (3.8) $\bar{B}_{\beta,m,p,\lambda} A_{U,\tau}\mu = \mathscr{I}_{\bar{\rho}_{\beta,m,p,\lambda} * \rho_{U,\tau}}\mu$, and also by (3.10)

$$\begin{aligned} \bar{\rho}_{\beta,m,p,\lambda} * \rho_{U,\tau}(x,t) &= \int \bar{\rho}_{\beta,m,p,\lambda}(X \cdot Y^{-1})\rho_{U,\tau}(Y)s^{-1} \, dV(Y) \\ &= \frac{1}{C_2}\int_U \left(\frac{t}{s}\right)^\lambda (1 + s^{-1}t + |x - t^{1/2\alpha}s^{-1/2\alpha}y|^{2\alpha})^{-p(n/2\alpha+1+|\beta|/2\alpha+m)} \\ &\quad \times \frac{\rho_{U,\tau}(y,s)}{s} \, dV(y,s). \end{aligned}$$

This convolution is bounded below by $C\bar{\rho}_{\beta,m,p,\lambda}(x,t)$ with some constant $C > 0$, because

$$1 + s^{-1}t + t|t^{-1/2\alpha}x - s^{-1/2\alpha}y|^{2\alpha} \leq C(1 + t + |x|^{2\alpha}) \tag{6.4}$$

for $(x,t) \in \boldsymbol{R}_+^{n+1}$ and $(y,s) \in \operatorname{supp}(\rho_{U,\lambda})$ with some constant C. In fact, since $s^{-1/2\alpha}y$ is bounded on $\operatorname{supp}(\rho_{U,\lambda})$, if $t^{-1/2\alpha}|x| \le 1$, then $1 + s^{-1}t + t|t^{-1/2\alpha}x - s^{-1/2\alpha}y|^{2\alpha} \le 1 + Ct$. Otherwise,

$$|t^{-1/2\alpha}x - s^{-1/2\alpha}y| \le t^{-1/2\alpha}|x| + s^{-1/2\alpha}|y| \le Ct^{-1/2\alpha}|x| \le C$$

implies (6.4) too. Hence by Lemma 2, we have

$$C^{-1}\|\bar{B}_{\beta,m,p,\lambda}\mu\|_{L^\psi(V_\eta)} \le \|\bar{B}_{\beta,m,p,\lambda}A_{U,\tau}\mu\|_{L^\psi(V_\eta)} \le C\|A_{U,\tau}\mu\|_{L^\psi(V_\eta)},$$

which shows the lemma. □

We make a remark for the case of $\psi(t) = t^\sigma$ $(1 \le \sigma < \infty)$. In this case, the assumption (5.2) is not necessary and the assumption (6.2) can be replaced by

$$\frac{n}{2\alpha} + 1 + \lambda - p\left(\frac{n}{2\alpha} + 1 + \frac{|\beta|}{2\alpha} + m\right) < \frac{\eta+1}{\sigma} < \lambda + 1. \tag{6.5}$$

In fact, for a positive Radon measure $\mu \ge 0$, we put

$$\tilde{B}_{\beta,m,p,\lambda}\mu(X) := t^\kappa \int |R_\alpha^{\beta,m}(Y,X)|^p s^\lambda \, d\mu(Y),$$

where $\kappa = (p-1)\left(\dfrac{n}{2\alpha} + 1\right) - \lambda$, and we consider

$$\tilde{\rho}_{\beta,m,p,\lambda}(x,t) = t^\lambda(1 + t + |x|^{2\alpha})^{-p(n/2\alpha+1+|\beta|/2\alpha+m)}.$$

Then $\tilde{B}_{\beta,m,p,\lambda}\mu(X)$ and $\mathscr{I}_{\tilde{\rho}_{\beta,m,p,\lambda}}\mu(X)$ can be defined without (5.2). Moreover $\tilde{B}_{\beta,m,p,\lambda}\mu(X) \le C\mathscr{I}_{\tilde{\rho}_{\beta,m,p,\lambda}}\mu(X)$ by (2.5) and (5.5). If (6.5) holds, then $\int s^{-(\eta+1)/\sigma}\tilde{\rho}_{\beta,m,p,\lambda}(Y)\,dY < \infty$, and as in Proposition 2, we also see

$$\begin{aligned}\left(\int (\mathscr{I}_{\tilde{\rho}_{\beta,m,p,\lambda}} f(X))^\sigma \, dV_\eta(X)\right)^{1/\sigma} &\le \int \left(\int f(X\cdot Y)^\sigma \, dV_\eta(X)\right)^{1/\sigma} \tilde{\rho}_{\beta,m,p,\lambda}(Y)\,dY \\ &= \left(\int s^{-(\eta+1)/\sigma}\tilde{\rho}_{\beta,m,p,\lambda}(Y)\,dY\right)\|f\|_{L^\sigma(V_\eta)}.\end{aligned}$$

This argument also holds for $\sigma = \infty$. From these observations, we have the following proposition.

Proposition 3. *Let $(\beta,m) \in \boldsymbol{N}_0^n \times \boldsymbol{N}_0$, $0 < p \le \infty$ and $\lambda, \tau \in \boldsymbol{R}$. Then, for a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, we have the following assertions.*

(1) *Let $1 \le \sigma \le \infty$ and $\eta \in \boldsymbol{R}$. Then for a compact set K of positive Lebesgue measure on $\boldsymbol{R}_+^{n+1}$, we have*

$$\|A_{K,\tau}\mu\|_{L^\sigma(V_\eta)} \le C\|\tilde{B}_{\beta,m,p,\lambda}\mu\|_{L^\sigma(V_\eta)}$$

with some constant $C > 0$ independent of μ.

(2) *If* $1 \le \sigma \le \infty$ *and* $\eta \in \boldsymbol{R}$ *satisfy* (6.5), *then for a relatively compact open set* $U \neq \emptyset$ *on* $\boldsymbol{R}_+^{n+1}$,

$$\|\tilde{B}_{\beta,m,p,\lambda}\mu\|_{L^\sigma(V_\eta)} \le C\|A_{U,\tau}\mu\|_{L^\sigma(V_\eta)}$$

with some constant $C > 0$ *independent of* μ.

The following theorem is one of our main results in this paper. Theorem 1 is obtained as a corollary.

Theorem 3. *Let* $-1 < \lambda \le 0 < p < \infty$, *and take an integer* $m \ge \left(\frac{2}{p} - 1\right)\left(\frac{n}{2\alpha} + 1\right)$. *Then there exists a constant* $C > 0$ *such that for a Radon measure* $\mu \ge 0$ *on* $\boldsymbol{R}_+^{n+1}$,

$$C^{-1}\|B_{0,m,p,\lambda}\mu\|_{L^\psi(V^*)} \le \|\hat{\mu}^{(\alpha)}\|_{L^\psi(V^*)} \le C\|B_{0,m,p,\lambda}\mu\|_{L^\psi(V^*)}. \tag{6.6}$$

In particular, $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$ *if and only if* $\tilde{\mu}^{(\alpha)} \in L^\psi(V^*)$.

Proof. By assumption, (5.2) and (6.2) hold for $\beta = 0$ and $\eta = -\left(\frac{n}{2\alpha} + 1\right)$. Hence Lemmas 1, 2, 3, and 4 show our assertion. □

7. Schatten class of compact Toeplitz operators

We extend the definition of Schatten class operators to the Orlicz type. In this section, let $\psi \in \Psi$ be fixed again.

Definition 2. A compact operator T on a Hilbert space $\mathscr{H}$ is said to be of Schatten ψ-class if the sequence of all singular values $(\lambda_j)_{j=0}^\infty$ of T belongs to the sequence space l^ψ of Orlicz type, and then we write $T \in \mathscr{S}^\psi(\mathscr{H})$. Here $(\lambda_j)_{j=0}^\infty \in l^\psi$ means

$$\sum_{j=0}^\infty \psi\left(\frac{\lambda_j}{\tau}\right) < \infty$$

with some constant $\tau > 0$. We put

$$\|T\|_{\mathscr{S}^\psi(\mathscr{H})} := \inf\left\{\tau > 0; \sum_{j=0}^\infty \psi\left(\frac{\lambda_j}{\tau}\right) \le 1\right\}.$$

In Appendix we will verify that $\mathscr{S}^\psi(\mathscr{H})$ is a vector space and is complete with respect to $\|\cdot\|_{\mathscr{S}^\psi(\mathscr{H})}$.

As we mentioned in section 1, we know a necessary and sufficient condition that the Toeplitz operator T_μ is compact. From this fact we obtain another sufficient condition.

Proposition 4. *Let $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$. If $\hat{\mu}^{(\alpha)} \in L^{\psi}(V^*)$, then the Toeplitz operator T_μ is compact.*

For the proof we prepare the following lemma.

Lemma 5. *Let $\varphi \in C_c^{\infty}(\boldsymbol{R}_+^{n+1})$ and $f \in L^{\psi}(V^*)$. Then $\mathscr{I}_{\varphi} f \in C_0(\boldsymbol{R}_+^{n+1})$.*

Proof. We may assume that $\varphi \geq 0$, $\int \varphi(X)\, dV(X) = 1$ and $f \geq 0$. Let $\tau > 0$ be a constant such that $\int \psi(f(X)/\tau)\, dV^*(X) < \infty$. For any $\varepsilon > 0$, choose a compact set K in $\boldsymbol{R}_+^{n+1}$ such that

$$\int_{\boldsymbol{R}_+^{n+1}\setminus K} \psi(f(X)/\tau)\, dV^*(X) < \frac{\psi(\varepsilon)}{\|\varphi^*\|_\infty},$$

where we denote $\varphi^*(X) := t^{n/2\alpha+1}\varphi(X)$ and $\|\varphi^*\|_\infty := \sup_{X\in \boldsymbol{R}_+^{n+1}} |\varphi^*(X)|$. Put

$$K_0 := K \cdot (\mathrm{supp}(\varphi))^{-1} = \{X \in \boldsymbol{R}_+^{n+1}; (\Phi_X(\mathrm{supp}(\varphi))) \cap K \neq \emptyset\}.$$

Then K_0 is compact. Since

$$\mathscr{I}_{\varphi} f(X) = \int \varphi(Y) f(X\cdot Y)\, dV(Y) = \int \varphi^*(Y) f(X\cdot Y)\, dV^*(Y),$$

for each $X \in \boldsymbol{R}_+^{n+1}\setminus K_0$, the Jensen inequality gives us

$$\begin{aligned}\psi\left(\frac{\mathscr{I}_{\varphi} f(X)}{\tau}\right) &\leq \int_{\mathrm{supp}(\varphi)} \psi\left(\frac{f(X\cdot Y)}{\tau}\right)\varphi^*(Y)\, dV^*(Y)\\ &\leq \|\varphi^*\|_\infty \int_{\Phi_X(\mathrm{supp}(\varphi))} \psi\left(\frac{f(Y)}{\tau}\right) dV^*(Y)\\ &\leq \|\varphi^*\|_\infty \int_{\boldsymbol{R}_+^{n+1}\setminus K} \psi\left(\frac{f(Y)}{\tau}\right) dV^*(Y)\\ &< \psi(\varepsilon),\end{aligned}$$

because V^* is invariant under α-parabolic similarities. Hence we have $\mathscr{I}_{\varphi} f(X) < \varepsilon\tau$, which shows the lemma. □

Proof of Proposition 4. Take $\varphi \in C_c^{\infty}(\boldsymbol{R}_+^{n+1})$ such that $\varphi \geq 0$, $\int \varphi(X)\, dV(X) = 1$ and $\varphi(X_0) > 0$. Then by the assumption and Lemma 5, we have $\mathscr{I}_{\varphi}\hat{\mu}^{(\alpha)} \in C_0(\boldsymbol{R}_+^{n+1})$. Since $\hat{\mu}^{(\alpha)} = \mathscr{I}_{\rho}\mu$ and $\varphi * \rho \geq C\rho$ with some constant $C > 0$, where $\rho = \rho_{Q^{(\alpha)}(X_0),0}$, we have

$$\mathscr{I}_{\varphi}\hat{\mu}^{(\alpha)} = \mathscr{I}_{\varphi}\mathscr{I}_{\rho}\mu = \mathscr{I}_{\varphi*\rho}\mu \geq C\mathscr{I}_{\rho}\mu = C\hat{\mu}^{(\alpha)}.$$

This implies $\lim_{X\to\mathscr{A}} \hat{\mu}^{(\alpha)}(X) = 0$, and hence the compactness of T_μ follows. □

We remark a relation between α-parabolic Bergman kernel and the left-invariant measure.

Remark 1. $\|R_\alpha^X\|_{L^2(V)}^2\,dV(X)$ is a left-invariant measure, where we put

$$R_\alpha^X(Y) := R_\alpha(X, Y). \tag{7.1}$$

In fact, we can write $R_\alpha(X,Y) = \sum_{j=0}^\infty e_j(X)e_j(Y)$ in pointwise and $R_\alpha^X = \sum_{j=0}^\infty e_j(X)e_j$ in $\boldsymbol{b}_\alpha^2$ for any complete orthonormal system $(e_j)_{j=0}^\infty$ of $\boldsymbol{b}_\alpha^2$. Then $\|R_\alpha^X\|_{L^2(V)}^2 = R_\alpha(X,X)$. By the homogeneity, $R_\alpha(X,X) = t^{-(n/2\alpha+1)}R_\alpha(X_0,X_0)$, so that

$$\|R_\alpha^X\|_{L^2(V)}^2\,dV(X) = R_\alpha(X,X)\,dV(X) = R_\alpha(X_0,X_0)\,dV^*(X). \tag{7.2}$$

The following theorem is another main result of this paper. Theorem 2 is obtained as a corollary.

THEOREM 4. *Let $\mu \ge 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (1.2) *for some $\delta \in \boldsymbol{R}$. Then $T_\mu \in \mathscr{S}^\psi(\boldsymbol{b}_\alpha^2)$ if and only if $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$. Moreover, we have a norm inequality*

$$C^{-1}\|\hat{\mu}^{(\alpha)}\|_{L^\psi(V^*)} \le \|T_\mu\|_{\mathscr{S}^\psi(\boldsymbol{b}_\alpha^2)} \le C\|\hat{\mu}^{(\alpha)}\|_{L^\psi(V^*)} \tag{7.3}$$

with some constant $C \ge 1$ independent of μ.

Proof. We first show that $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$ implies $T_\mu \in \mathscr{S}^\psi(\boldsymbol{b}_\alpha^2)$. Let $m \ge \dfrac{n}{2\alpha} + 1$ be an integer and $-1 < \lambda < 0$. Since $B_{0,m,1,\lambda}\mu \in L^\psi(V^*)$ by Theorem 3, we can take $\tau > 0$ with

$$\int \psi\left(\frac{B_{0,m,1,\lambda}\mu(X)}{\tau}\right) dV^*(X) \le 1.$$

Clearly T_μ is a positive definite self-adjoint operator. By Proposition 4, it is compact. Hence we can take a comlete orthonormal system $(e_j)_{j=0}^\infty$ of $\boldsymbol{b}_\alpha^2$ which consists of eigenvectors of T_μ such that the corresponding eigenvalues are given by $\lambda_j = \lambda_j(T_\mu) = \langle T_\mu e_j, e_j\rangle = \int |e_j(X)|^2\,d\mu(X)$. Then by the Schwarz inequality,

$$\begin{aligned}
|e_j(X)|^2 &= \left(\int s^{\lambda/2}s^{-\lambda/2}e_j(Y)R_\alpha^m(X,Y)\,dV(Y)\right)^2 \\
&\le \left(\int s^\lambda |R_\alpha^m(X,Y)|\,dV(Y)\right)\left(\int s^{-\lambda}|e_j(Y)|^2|R_\alpha^m(X,Y)|\,dV(Y)\right) \\
&\le C_{m,\lambda}t^\lambda \int s^{-\lambda}|e_j(Y)|^2|R_\alpha^m(X,Y)|\,dV(Y),
\end{aligned}$$

where $C_{m,\lambda} = \int s^{\lambda}|R_{\alpha}^{m}(X_0, Y)|\, dV(Y) < \infty$ by (2.4) and (2.5). Therefor by the Fubini theorem and the Jensen inequality we have

$$\begin{aligned}\lambda_j &\le C_{m,\lambda}\int\left(s^{-\lambda}\int|R_{\alpha}^{m}(X,Y)|t^{\lambda}\,d\mu(X)\right)|e_j(Y)|^2\,dV(Y)\\ &= C\int B_{0,m,1,\lambda}\mu(Y)|e_j(Y)|^2\,dV(Y)\end{aligned}$$

and

$$\psi\left(\frac{\lambda_j}{s_0\tau C}\right) \le \int\psi\left(\frac{B_{0,m,1,\lambda}\mu(Y)}{s_0\tau}\right)|e_j(Y)|^2\,dV(Y),$$

where $s_0 := \max\{1, R_{\alpha}(X_0, X_0)\}$. Therefore by (7.2)

$$\begin{aligned}\sum_{j=0}^{\infty}\psi\left(\frac{\lambda_j}{s_0\tau C}\right) &\le \sum_{j=0}^{\infty}\int\psi\left(\frac{B_{0,m,1,\lambda}\mu(Y)}{s_0\tau}\right)|e_j(Y)|^2\,dV(Y)\\ &= \int\psi\left(\frac{B_{0,m,1,\lambda}\mu(Y)}{s_0\tau}\right)R_{\alpha}(Y,Y)\,dV(Y)\\ &\le \int\psi\left(\frac{B_{0,m,1,\lambda}\mu(Y)}{s_0\tau}\right)s_0\,dV^*(Y)\\ &\le \int\psi\left(\frac{B_{0,m,1,\lambda}\mu(Y)}{\tau}\right)dV^*(Y) \le 1,\end{aligned}$$

which implies that $T_{\mu} \in \mathscr{S}^{\psi}(\boldsymbol{b}_{\alpha}^2)$ and $\|T_{\mu}\|_{\mathscr{S}^{\psi}(\boldsymbol{b}_{\alpha}^2)} \le s_0 C\|B_{0,m,1,\lambda}\mu\|_{L^{\psi}(V^*)}$. Hence the latter inequality in (7.3) follows from Theorem 3.

Next we assume $T_{\mu} \in \mathscr{S}^{\psi}(\boldsymbol{b}_{\alpha}^2)$. As in the proof of the if part, we can take a complete orthonormal system $(e_j)_{j=0}^{\infty}$ of eigenvectors of T_{μ}. Now, take $\tau > 0$ with $\sum_{j=0}^{\infty}\psi(\lambda_j/\tau) \le 1$, where $\lambda_j = \langle T_{\mu}e_j, e_j\rangle$. Then by the spectral mapping theorem,

$$\psi\left(\frac{T_{\mu}}{\tau}\right)u = \sum_{k=0}^{\infty}\psi\left(\frac{\lambda_k}{\tau}\right)\langle u, e_k\rangle e_k$$

for $u \in \boldsymbol{b}_{\alpha}^2$, so that

$$\left\langle e_j(X)e_j, \psi\left(\frac{T_{\mu}}{\tau}\right)R_{\alpha}^X\right\rangle = \left\langle e_j(X)e_j, \sum_{k=0}^{\infty}\psi\left(\frac{\lambda_k}{\tau}\right)e_k(X)e_k\right\rangle = \psi\left(\frac{\lambda_j}{\tau}\right)|e_j(X)|^2 \ge 0$$

and $\psi(\lambda_j/\tau) = \int\langle e_j(X)e_j, \psi(T_{\mu}/\tau)R_{\alpha}^X\rangle\,dV(X)$. Hence we have the following trace formula

$$\sum_{j=0}^{\infty}\psi\left(\frac{\lambda_j}{\tau}\right)=\int\sum_{j=0}^{\infty}\left\langle e_j(X)e_j,\psi\left(\frac{T_\mu}{\tau}\right)R_\alpha^X\right\rangle dV(X)=\int\left\langle R_\alpha^X,\psi\left(\frac{T_\mu}{\tau}\right)R_\alpha^X\right\rangle dV(X)$$
$$=\int\left\langle r_\alpha^X,\psi\left(\frac{T_\mu}{\tau}\right)r_\alpha^X\right\rangle\|R_\alpha^X\|_{L^2(V)}^2\,dV(X)$$

where $r_\alpha^X := R_\alpha^X/\|R_\alpha^X\|_{L^2(V)}$. Since

$$\tilde{\mu}^{(\alpha)}(X)=\frac{\int R_\alpha(X,Y)^2\,d\mu(Y)}{\int R_\alpha(X,Y)^2\,dV(Y)}=\frac{\langle T_\mu R_\alpha^X,R_\alpha^X\rangle}{\|R_\alpha^X\|_{L^2(V)}^2}=\langle r_\alpha^X,T_\mu r_\alpha^X\rangle,$$

the Jensen inequality gives us

$$\psi\left(\frac{\tilde{\mu}^{(\alpha)}(X)}{\tau}\right)=\psi\left(\frac{\langle r_\alpha^X,T_\mu r_\alpha^X\rangle}{\tau}\right)=\psi\left(\sum_{j=0}^{\infty}\frac{\lambda_j}{\tau}\langle r_\alpha^X,e_j\rangle^2\right)$$
$$\le\sum_{j=0}^{\infty}\psi\left(\frac{\lambda_j}{\tau}\right)\langle r_\alpha^X,e_j\rangle^2=\left\langle r_\alpha^X,\psi\left(\frac{T_\mu}{\tau}\right)r_\alpha^X\right\rangle.$$

Hence putting $s_1 := \min\{1,R_\alpha(X_0,X_0)\}$, by (7.2) we have

$$\int\psi\left(s_1\frac{\tilde{\mu}^{(\alpha)}(X)}{\tau}\right)dV^*(X)\le\int s_1\psi\left(\frac{\tilde{\mu}^{(\alpha)}(X)}{\tau}\right)dV^*(X)$$
$$\le\int\psi\left(\frac{\tilde{\mu}^{(\alpha)}(X)}{\tau}\right)\|R_\alpha^X\|_{L^2(V)}^2\,dV(X)$$
$$\le\int\left\langle r_\alpha^X,\psi\left(\frac{T_\mu}{\tau}\right)r_\alpha^X\right\rangle\|R_\alpha^X\|_{L^2(V)}^2\,dV(X)$$
$$=\sum_{j=0}^{\infty}\psi\left(\frac{\lambda_j}{\tau}\right)\le 1,$$

which implies that $\tilde{\mu}^{(\alpha)}\in L^\psi(V^*)$ and $\|\tilde{\mu}^{(\alpha)}\|_{L^\psi(V^*)}\le\frac{1}{s_1}\|T_\mu\|_{\mathscr{S}^\psi(\boldsymbol{b}_\alpha^2)}$. Theorem 3 again shows the former inequality in (7.3). □

8. Appendix

Let $\psi\in\Psi$, that is, $\psi:[0,\infty)\to[0,\infty)$ be a convex and strictly increasing function satisfying that $\psi(0)=0$ and $\lim_{s\to\infty}\psi(s)=\infty$. For a Hilbert space $\mathscr{H}$, we denote by $\mathscr{L}(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$ and $\mathscr{S}^\psi(\mathscr{H})$ the set of all compact operators on $\mathscr{H}$ of Schatten ψ-class, respectively.

It is known in general theory that $\mathscr{S}^{\psi}(\mathscr{H})$ is a linear space, and is complete (cf. [3, pp. 1088–1095], see also [11]), however, we give precise statements in our case for the sake of completeness.

For a compact operator T on $\mathscr{H}$, we denote by $\lambda_j(T)$ the j-th $(j \geq 0)$ singular value of T. The following min-max principle plays an important role.

$$\lambda_j(T) = \min_{\substack{H \subset \mathscr{H}:\text{subspace} \\ \dim H \leq j}} \max_{u \in H^{\perp} \setminus \{0\}} \frac{\|Tu\|}{\|u\|} = \max_{\substack{H \subset \mathscr{H}:\text{subspace} \\ \dim H \geq j+1}} \min_{u \in H \setminus \{0\}} \frac{\|Tu\|}{\|u\|} \tag{8.1}$$

(cf. [3, Lemma 2 in §9 of XI]). In particular,

$$\lambda_0(T) = \|T\|. \tag{8.2}$$

It also follows from (8.1) that for two compact operators T_1 and T_2 on $\mathscr{H}$,

$$\lambda_{j+k}(T_1 + T_2) \leq \lambda_j(T_1) + \lambda_k(T_2) \tag{8.3}$$

for every $j \geq 0$ and $k \geq 0$. Using this inequality and (8.2), we obtain

$$|\lambda_j(T_1) - \lambda_j(T_2)| \leq \|T_1 - T_2\|. \tag{8.4}$$

Moreover, for any $A, B \in \mathscr{L}(H)$, we see

$$\lambda_j(ATB) \leq \|A\| \, \|B\| \lambda_j(T). \tag{8.5}$$

Now we can state the following fundamental results.

Proposition 5. (i) *$\mathscr{S}^{\psi}(\mathscr{H})$ is a vector space.*
(ii) $\|T_1 + T_2\|_{\mathscr{S}^{\psi}(\mathscr{H})} \leq 4(\|T_1\|_{\mathscr{S}^{\psi}(\mathscr{H})} + \|T_2\|_{\mathscr{S}^{\psi}(\mathscr{H})})$.
(iii) *$\mathscr{S}^{\psi}(\mathscr{H})$ is complete in the sense that if $(T_k)_{k=1}^{\infty}$ is a sequence in $\mathscr{S}^{\psi}(\mathscr{H})$ such that $\lim_{k,\ell\to\infty} \|T_k - T_\ell\|_{\mathscr{S}^{\psi}(\mathscr{H})} = 0$, then there exists $T \in \mathscr{S}^{\psi}(\mathscr{H})$ such that $\lim_{k\to\infty} \|T_k - T\|_{\mathscr{S}^{\psi}(\mathscr{H})} = 0$.*

Proof. Let $T_1, T_2 \in \mathscr{S}^{\psi}(\mathscr{H})$ and $c \in \boldsymbol{R}$ be arbitrary. Since $\lambda_j(cT_1) = |c|\lambda_j(T_1)$, $cT_1 \in \mathscr{S}^{\psi}(\mathscr{H})$. To see $T_1 + T_2 \in \mathscr{S}^{\psi}(\mathscr{H})$, we will show (ii). For $k = 1, 2$, take any $\tau_k > 0$ with $\|T_k\|_{\mathscr{S}^{\psi}(\mathscr{H})} < \tau_k/4$. Then

$$\sum_{j=0}^{\infty} \psi\left(\frac{4\lambda_j(T_k)}{\tau_k}\right) \leq 1.$$

By (8.3), we have $\lambda_{2j}(T_1 + T_2) \leq \lambda_j(T_1) + \lambda_j(T_2)$. Since $\psi \in \Psi$ satisfies $4\psi(s + t) \leq \psi(4s) + \psi(4t)$, we have

$$\sum_{j=0}^{\infty} \psi\left(\frac{\lambda_{2j}(T_1 + T_2)}{\tau_1 + \tau_2}\right) \leq \sum_{j=0}^{\infty} \frac{1}{4}\left(\psi\left(\frac{4\lambda_j(T_1)}{\tau_1 + \tau_2}\right) + \psi\left(\frac{4\lambda_j(T_2)}{\tau_1 + \tau_2}\right)\right) \leq \frac{1}{2}.$$

Similarly, $\lambda_{2j+1}(T_1+T_2) \le \lambda_j(T_1) + \lambda_{j+1}(T_2)$ gives us

$$\sum_{j=0}^{\infty} \psi\left(\frac{\lambda_{2j+1}(T_1+T_2)}{\tau_1+\tau_2}\right) \le \frac{1}{2}.$$

Thus we obtain $\|T_1+T_2\|_{\mathscr{S}^{\psi}(\mathscr{H})} \le \tau_1+\tau_2$. Arbitrariness of τ_1 and τ_2 implies (ii). Unfortunately we do not know whether the constant 4 in (ii) can be removed or not.

Finally, we assume $\lim_{k,\ell\to\infty}\|T_k - T_\ell\|_{\mathscr{S}^{\psi}(\mathscr{H})} = 0$. Then $(T_k)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathscr{L}(\mathscr{H})$, so that there exists a compact operator T such that $\lim_{k\to\infty} T_k = T$ in $\mathscr{L}(\mathscr{H})$. Then by (8.4), $\lim_{k\to\infty} \lambda_j(T_k) = \lambda_j(T)$ for $j \ge 0$. For any $\varepsilon > 0$, there exists k_0 such that

$$\sum_{j=0}^{\infty} \psi\left(\frac{\lambda_j(T_k - T_\ell)}{\varepsilon}\right) \le 1$$

if $k, \ell \ge k_0$. Hence, by (8.4) again and the Fatou theorem, we have

$$\sum_{j=0}^{\infty} \psi\left(\frac{\lambda_j(T_k - T)}{\varepsilon}\right) = \sum_{j=0}^{\infty} \lim_{\ell\to\infty} \psi\left(\frac{\lambda_j(T_k - T_\ell)}{\varepsilon}\right) \le \liminf_{\ell\to\infty} \sum_{j=0}^{\infty} \psi\left(\frac{\lambda_j(T_k - T_\ell)}{\varepsilon}\right) \le 1,$$

which implies $T_k - T \in \mathscr{S}^{\psi}(\mathscr{H})$, i.e., $T \in \mathscr{S}^{\psi}(\mathscr{H})$, and $\|T_k - T\|_{\mathscr{S}^{\psi}(\mathscr{H})} \le \varepsilon$ whenever $k \ge k_0$. This shows (iii). □

References

[1] B. R. Choe, H. Koo and H. Yi, Positive Toeplitz operators between the harmonic Bergman spaces, Potential Analysis **17** (2002), 307–335.

[2] B. R. Choe, H. Koo and Y. J. Lee, Positive Schatten(-Herz) class Toeplitz operators on the half-space, Potential Analysis **27** (2007), 73–100.

[3] N, Dunford and J. Schwartz, Linear operators, Part II, Interscience, 1963.

[4] D. H. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. **73** (1987), 345–368.

[5] M. Nishio, K. Shimomura and N. Suzuki, α-parabolic Bergman spaces, Osaka J. Math. **42** (2005), 133–162.

[6] M. Nishio, N. Suzuki and M. Yamada, Toeplitz operators and Carleson measures on parabolic Bergman spaces, Hokkaido Math. J. **36** (2007), 563–583.

[7] M. Nishio, N. Suzuki and M. Yamada, Compact Toeplitz operators on parabolic Bergman spaces, Hiroshima Math. J. **38** (2008), 177–192.

[8] M. Nishio, N. Suzuki and M. Yamada, Parabolic dilations with application to Toeplitz operators on parabolic Bergman spaces, Proceedings of the 15th ICFIDCAA, Osaka Municipal Universities Press, 2008, 307–312.

[9] M. Nishio, N. Suzuki and M. Yamada, Carleson inequalities on parabolic Bergman spaces, preprint.

[10] M. Nishio and M. Yamada, Carleson type measures on parabolic Bergman spaces, J. Math. Soc. Japan **58** (2006), 83–96.

[11] K. Zhu, Operator theory in function spaces, 2nd ed., Math. Surveys **138**, AMS, 1990.

Masaharu Nishio
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi
Osaka 558-8585
Japan
E-mail: nishio@sci.osaka-cu.ac.jp

Noriaki Suzuki
Department of Mathematics
Meijo University
Tenpaku-ku, Nagoya 468-8502
Japan
E-mail: suzukin@ccmfs.meijo-u.ac.jp

Masahiro Yamada
Department of Mathematics
Faculty of Education
Gifu University
Yanagido 1-1, Gifu 501-1193
Japan
E-mail: yamada33@cc.gifu-u.ac.jp