Y. SANO
KODAI MATH. J.
31 (2008), 431–440

ON THE HOLOMORPHIC INVARIANTS FOR GENERALIZED KÄHLER-EINSTEIN METRICS

Yuji Sano

Abstract

In [9], Mabuchi extended the notion of Kähler-Einstein metrics to the case of Fano manifolds with novanishing Futaki invariant. We call them generalized Kähler-Einstein metrics. He defined the holomorphic invariant α_M in terms of the extremal Kähler vector field, which is the obstruction for the existence of generalized Kähler-Einstein metrics. The purpose of this short paper is to show that the above obstruction is actually equivalent to the vanishing of the holomorphic invariant of Futaki's type defined by Futaki [4] (see also [8]). As its corollary, we can show that $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ admits generalized Kähler-Einstein metrics by the method using multiplier ideal sheaves in [6].

1. Introduction

Let (M, Ω) be an n-dimensional Kähler manifold with Kähler class Ω. For a Kähler form $\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,\bar{j}} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$, we denote its Ricci form by $\mathrm{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \det(g_{i\bar{j}})$. The Kähler form ω is called a Kähler-Einstein form when its Ricci form is proportional to ω. Since the Ricci form represents the first Chern class of M, when we consider the problems concerning Kähler-Einstein metrics we should assume that the Kähler class Ω is proportional to $c_1(M)$, that is to say, $c_1(M)$ has a definite sign. Although the existence problem of Kähler-Einstein metrics for the case when $c_1(M)$ is negative definite is solved by Aubin and Yau independently and the case when $c_1(M)$ is zero is solved by Yau, the existence problem for the case of $c_1(M) > 0$ (the Fano case) is still open. In this paper, we assume $(M, c_1(M))$ be a Fano manifold. For a Kähler form ω, we denote the Einstein discrepancy function by h_ω such that

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} h_\omega, \quad \int_M e^{h_\omega} \omega^n = \int_M \omega^n.$$

1991 *Mathematics Subject Classification*. Primary 53C55, Secondary 53C21, 55N91.

Key words and phrases. Kähler-Einstein metric, Futaki invariant, multiplier Hermitian structure, multiplier ideal sheaf.

Received December 11, 2007; revised May 1, 2008.

One of the well-known obstructions for the existence of Kähler-Einstein metrics on Fano manifolds is the vanishing of the holomorphic invariant $F : \mathfrak{h}(M) \to \mathbf{C}$ introduced by Futaki [2] (which is often called Futaki invariant)

$$F(X) := \int_M X h_g \omega_g^n,$$

where $\mathfrak{h}(M)$ is the Lie algebra of holomorphic vector fields. In [9], Mabuchi extended the notion of Kähler-Einstein metrics to the case of Fano manifolds with non vanishing Futaki invariant F as follows; the Kähler form ω_g is "Kähler-Einstein" in the sense of [9] if and only if the complex gradient vector field

$$\mathrm{grad}_\omega^{\mathbf{C}}(1 - e^{h_g}) := \frac{1}{\sqrt{-1}} \sum_{ij} g^{i\bar{j}} \frac{\partial(1 - e^{h_g})}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$$

of $1 - e^{h_g}$ is holomorphic. To avoid confusion, we call it a *generalized Kähler-Einstein form* throughout this paper. Note that generalized Kähler-Einstein forms are just the ordinary Kähler-Einstein forms when Futaki invariant vanishes. Mabuchi introduced the holomorphic invariant α_M as an obstruction of Futaki's type for generalized Kähler-Einstein metrics, which is described in terms of the extremal Kähler vector field introduced by Futaki and Mabuchi [5]. To be self-contained, let us recall the definition of the extremal Kähler vector field. Let

$$\tilde{\mathfrak{k}}_\omega := \left\{ \varphi \in C_{\mathbf{R}}^\infty(M) \mid \mathrm{grad}_\omega^{\mathbf{C}} \varphi \in \mathfrak{h}(M), \int_M \varphi \omega^n = 0 \right\}.$$

Let $\mathrm{pr} : L^2(M, \omega)_{\mathbf{R}} \to \tilde{\mathfrak{k}}_\omega$ be the orthogonal projection, where $L^2(M, \omega)_{\mathbf{R}}$ is the Hilbert space of all real-valued L^2-functions on (M, ω). Then, we define the extremal Kähler vector field by

$$v_\omega := \mathrm{grad}_\omega^{\mathbf{C}} \mathrm{pr}(s(\omega) - \hat{s}) = \mathrm{grad}_\omega^{\mathbf{C}} \mathrm{pr}(1 - e^{h_\omega}), \tag{1}$$

where $s(\omega)$ is the scalar curvature and $\hat{s}$ is its average. (cf. [5] and Theorem 2.1 in [9] for the second equality in (1).) Now we let

$$\alpha_M := \max_M \mathrm{pr}(1 - e^{h_\omega}),$$

which is independent of the choice of ω. Mabuchi (Theorem 3.1 in [9]) proved the following obstruction for generalized Kähler-Einstein metrics as the analogue of Futaki's obstruction.

THEOREM 1.1 (Mabuchi). *Let $(M, c_1(M))$ be a Fano manifold. If M admits generalized Kähler-Einstein metrics, then $\alpha_M < 1$.*

On the other hand, Mabuchi extended the notion of generalized Kähler-Einstein metrics as Einstein multiplier Hermitian metrics of some type σ in [10] with respect to the Hamiltonian holomorphic vector field $X = -v_\omega$. We will recall it in Section 2. Futaki ([4], also Li [8]) defined the holomorphic invariant F_X^σ as the generalization of Futaki invariant for Einstein multiplier Hermitian metrics. Its definition also will be illustrated in Section 2. Then, our main result is

Theorem 1.2. *Let (M, g) be a Fano manifold with a Kähler metric g which represents $c_1(M)$. Let v_{ω_g} be the extremal Kähler vector field. For the multiplier Hermitian structure of type of $\sigma(s) = -\log(s + c)$ for some $c > \alpha_M$ with respect to $X = -v_{\omega_g}$, the holomorphic invariant F_X^σ vanishes if and only if $\alpha_M < 1$.*

Kähler-Ricci solitons can be regarded as one of Einstein multiplier Hermitian metrics. (As for Einstein multiplier Hermitian metrics, see Section 2.) Futaki and the author extended Nadel's multiplier ideal sheaves to the case of Kähler-Ricci solitons in order to show their existence ([6]). We can extend this method also to the case of Einstein multiplier Hermitian metrics of any σ (Section 3). Combining Theorem 1.2 we have

Corollary 1.3. $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ *admit generalized Kähler-Einstein metrics in $c_1(M)$.*

This example is found by Mabuchi by solving an ODE (see Example 5.8 [9]). We remark that the above example is a non Kähler-Einstein manifold due to Matsushima's obstruction [11], so the above generalized Kähler-Einstein metrics are not the ordinary Kähler-Einstein metrics. As for other examples of generalized Kähler-Einstein manifolds, see Section 5 in [9]. The organization of this paper is as follows. In Section 2, we recall Einstein multiplier Hermitian metrics and the definition of the holomorphic invariant F_X^σ. Then we complete the proof of Theorem 1.2. In Section 3, we explain the extension of multiplier ideal sheaves to the case of Einstein multiplier Hemitian metrics. In Section 4, we complete the proof of Corollary 1.3.

Acknowledgements. The author thanks Yanir Rubinstein for his detailed and useful remarks and comments. This work is supported by JSPS-EPDI fellowship and done during the author's stay in Institut des Hautes Études Scientifiques. He thanks IHES for its kindly hospitality.

2. Einstein multiplier Hermitian metrics

In this section, we shall recall Eintein multiplier Hermitian metrics and complete the proof of the main theorem. Firstly let us recall multiplier Hermitian metrics. Let (M, g) be a Fano manifold. For a non trivial holomorphic vector field X, let

$$\mathscr{K}_X := \{\omega \in c_1(M) \mid L_{X_\mathbf{R}}\omega = 0\}$$

where $X_{\mathbf{R}} := X + \overline{X}$ and $L_{X_{\mathbf{R}}}$ is the Lie derivative along $X_{\mathbf{R}}$. Suppose that X is Hamiltonian, that is to say, to each $\omega \in \mathscr{K}_X$ we can associate a real-valued function $\theta_{X,\omega} \in C^{\infty}_{\mathbf{R}}(M)$ such that

$$X = \operatorname{grad}^{\mathbf{C}}_{\omega} \theta_{X,\omega}, \qquad \int_M \theta_{X,\omega}\omega^n = 0.$$

Let $l_{\min,X} := \min \theta_{X,\omega}$, $l_{\max,X} := \max \theta_{X,\omega}$ which are independent of the choice of $\omega \in \mathscr{K}_X$ ([5]). Let $\sigma(s)$ be a real-valued smooth function defined on the interval $[l_{\min,X}, l_{\max,X}]$ satisfying one of the following conditions:

(a) $\dot{\sigma} \leq 0 \leq \ddot{\sigma}$

(b) $\ddot{\sigma} \geq 0$

where $\dot{\sigma}$ and $\ddot{\sigma}$ are the first derivative and the second derivative of σ respectively. To each $\omega \in \mathscr{K}_X$, Mabuchi denotes a multiplier Hermitian metric of type σ by a conformally Kähler metric $\tilde{\omega} := \omega \exp(-\sigma(\theta_{X,\omega})/n)$. Since the multiplier Hermitian metric $\tilde{\omega}$ can be regarded as an Hermitian metric on the holomorphic tangent bundle TM of M, it induces an Hermitian connection

$$\tilde{\nabla} = \nabla - \frac{\partial(\sigma(\theta_{X,\omega}))}{n}\,\mathrm{id}_{TM},$$

where ∇ is the natural connection with respect to ω. Then the Ricci form $\mathrm{Ric}^{\sigma}(\omega)$ of $(\tilde{\omega}, \tilde{\nabla})$ equals to

$$\mathrm{Ric}(\omega) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\sigma(\theta_{X,\omega}),$$

and we call a form $\tilde{\omega}$ satisfying $\mathrm{Ric}^{\sigma}(\omega) = \omega$ an Einstein multiplier Hermitian form. Futaki defined the holomorphic invariant for Einstein metrics in the sense of multiplier Hermitian metrics in [4] as follows;

$$F^{\sigma}_X(Y) := \int_M Y(h_\omega + \sigma(\theta_{X,\omega}))e^{-\sigma(\theta_{X,\omega})}\omega^n, \qquad Y \in \mathfrak{h}(M). \tag{2}$$

As the ordinary Futaki invariant, F^{σ}_X is also independent of the choice of g and a Lie algebra character, and the obstruction for the existence of Einstein multiplier Hermitian metrics.

Now let us see that generalized Kähler-Einstein metrics are Einstein multiplier Hermitian metrics. Firstly, take the extremal Kähler vector field v_ω as $-X$. Next let $\sigma(s) = -\log(s + c)$ where c is a constant strictly greater than α_M. Then, since

$$\begin{aligned}\left(\frac{\sqrt{-1}}{2\pi}\right)^{-1}(\mathrm{Ric}^{\sigma}(\omega) - \omega) &= \partial\bar{\partial}(h_\omega - \log(-\mathrm{pr}(1 - e^{h_\omega}) + c)) \\ &= -\partial\bar{\partial}\log(e^{-h_\omega}(-\mathrm{pr}(1 - e^{h_\omega}) + c)),\end{aligned}$$

so we find that $\tilde{\omega}$ is an Einstein multiplier Hermitian form if and only if

$$\mathrm{pr}(1 - e^{h_\omega}) = c - e^{c' + h_\omega} \tag{3}$$

for some constant c'. Since $\int_M \mathrm{pr}(1 - e^{h_\omega})\omega^n = 0$ and $\int_M e^{h_\omega}\omega^n = \int_M \omega^n$, we have $e^{c'} = c = 1$. So (3) implies

$$\mathrm{grad}_\omega^{\mathbf{C}}(1 - e^{h_\omega}) = \mathrm{grad}_\omega^{\mathbf{C}}(\mathrm{pr}(1 - e^{h_\omega})) \in \mathfrak{h}(M),$$

that is to say, ω is a generalized Kähler-Einstein form. Now let us complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us consider F_X^σ for generalized Kähler-Einstein metrics, i.e.,

$$\begin{aligned} X &= -v_\omega \\ \theta_{X,\omega} &= -\mathrm{pr}(1 - e^{h_\omega}) \\ \sigma(s) &= -\log(s + c), \quad c > \alpha_M. \end{aligned}$$

To each holomorphic vector field Y, we can associate a function $\tilde{\theta}_{Y,\omega} \in C_{\mathbf{C}}^\infty(M)$ satisfying

$$\mathrm{grad}_\omega^{\mathbf{C}}\, \tilde{\theta}_{Y,\omega} = Y, \quad \int_M \tilde{\theta}_{Y,\omega} e^{h_\omega}\omega^n = 0.$$

Then, we have

$$\frac{1}{\sqrt{-1}}\tilde{\theta}_{Y,\omega} = \frac{1}{\sqrt{-1}}\Delta_\omega\tilde{\theta}_{Y,\omega} - Yh_\omega, \tag{4}$$

where $\Delta_\omega := -\sum_{i,j} g^{i\bar{j}}\dfrac{\partial^2}{\partial z^i \partial \bar{z}^j}$ is the complex Laplacian with respect to ω. Remark that this Laplacian has the opposite sign of the ordinary one. The equality (4) comes from the proof of Theorem 2.4.3 in p. 41 [3] (also (2.1) in [15]). To see it, for any point $x \in M$ let us choose a local coordinate near x such that $g_{i\bar{j}} = \delta_{i\bar{j}}$ at x. Then, at x the Ricci identity implies

$$\begin{aligned} \partial_{\bar{j}}(\tilde{\theta}_{Y,\omega} - \Delta_\omega\tilde{\theta}_{Y,\omega} + \sqrt{-1}\,Yh_\omega) &= (\tilde{\theta}_{Y,\omega})_{\bar{j}} + (\tilde{\theta}_{Y,\omega})_{i\bar{i}\bar{j}} + \{(\tilde{\theta}_{Y,\omega})_{\bar{i}}(h_\omega)_i\}_{\bar{j}} \\ &= (\tilde{\theta}_{Y,\omega})_{\bar{j}} + (\tilde{\theta}_{Y,\omega})_{i\bar{i}\bar{j}} + (\tilde{\theta}_{Y,\omega})_{\bar{i}}(h_\omega)_{i\bar{j}} \\ &= (\tilde{\theta}_{Y,\omega})_{\bar{j}} + (\tilde{\theta}_{Y,\omega})_{i\bar{i}\bar{j}} + (\tilde{\theta}_{Y,\omega})_{\bar{i}}(\mathrm{Ric}_{i\bar{j}} - g_{i\bar{j}}) \\ &= (\tilde{\theta}_{Y,\omega})_{i\bar{i}\bar{j}} + (\tilde{\theta}_{Y,\omega})_{\bar{i}}\,\mathrm{Ric}_{i\bar{j}} \\ &= (\tilde{\theta}_{Y,\omega})_{\bar{i}\bar{j}i} - (\tilde{\theta}_{Y,\omega})_{\bar{k}}\,\mathrm{Ric}_{k\bar{j}} + (\tilde{\theta}_{Y,\omega})_{\bar{i}}\,\mathrm{Ric}_{i\bar{j}} = 0, \end{aligned}$$

hence $\tilde{\theta}_{Y,\omega} - \Delta_\omega\tilde{\theta}_{Y,\omega} + \sqrt{-1}\,Yh_\omega$ is constant on M. Furthermore, we find that $\tilde{\theta}_{Y,\omega} - \Delta_\omega\tilde{\theta}_{Y,\omega} + \sqrt{-1}\,Yh_\omega$ equals to zero, because the definition of $\tilde{\theta}_{Y,\omega}$ and an integration by parts imply

$$\begin{aligned} \int_M (\tilde{\theta}_{Y,\omega} - \Delta_\omega\tilde{\theta}_{Y,\omega} + \sqrt{-1}\,Yh_\omega)e^{h_\omega}\omega^n &= \int_M (-\Delta_\omega\tilde{\theta}_{Y,\omega} + \sqrt{-1}\,Yh_\omega)e^{h_\omega}\omega^n \\ &= 0. \end{aligned}$$

From (4), we have

$$
\begin{aligned}
F_X^\sigma(Y) &= \int_M Y(h_\omega - \log(\theta_{X,\omega} + c)) \cdot (\theta_{X,\omega} + c)\omega^n \\
&= c\int_M Yh_\omega\omega^n + \int_M (Yh_\omega)\theta_{X,\omega}\omega^n - \int_M Y\theta_{X,\omega}\omega^n \\
&= cF(Y) + \frac{1}{\sqrt{-1}}\int_M (-\tilde{\theta}_{Y,\omega} + \Delta_\omega\tilde{\theta}_{Y,\omega})\theta_{X,\omega}\omega^n - \int_M Y\theta_{X,\omega}\omega^n \\
&= cF(Y) + \frac{1}{\sqrt{-1}}\int_M (-\tilde{\theta}_{Y,\omega} + \Delta_\omega\tilde{\theta}_{Y,\omega})\theta_{X,\omega}\omega^n \\
&\quad - \frac{1}{\sqrt{-1}}\int_M (\Delta_\omega\tilde{\theta}_{Y,\omega})\theta_{X,\omega}\omega^n \\
&= cF(Y) - \frac{1}{\sqrt{-1}}\int_M \tilde{\theta}_{Y,\omega}\theta_{X,\omega}\omega^n \\
&= (c-1)F(Y), \qquad (5)
\end{aligned}
$$

where F is the ordinary Futaki invariant. The equality (5) comes from Theorem C in [5] (see also [7]). To see it, let $\theta_{Y,\omega} \in C^\infty_{\mathbf{C}}(M)$ be the function satisfying $\operatorname{grad}^{\mathbf{C}}_\omega \theta_{Y,\omega}$ and $\int_M \theta_{Y,\omega}\omega^n = 0$. Since $\theta_{Y,\omega} - \tilde{\theta}_{Y,\omega}$ is constant and $\theta_{X,\omega} \in \tilde{\mathfrak{k}}_\omega$, it is sufficient to show

$$F(Y) = \frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}\theta_{X,\omega}\omega^n.$$

In fact,

$$
\begin{aligned}
F(Y) &= \int_M Yh_\omega\omega^n = \int_M (\operatorname{grad}^{\mathbf{C}}_\omega \theta_{Y,\omega})h_\omega\omega^n \\
&= \frac{1}{\sqrt{-1}}\int_M \sum_{i,j} g^{i\bar{j}}\frac{\partial h_\omega}{\partial z^i}\frac{\partial\theta_{Y,\omega}}{\partial \bar{z}^j}\omega^n \\
&= \frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}(\Delta_\omega h_\omega)\omega^n = -\frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}(s(\omega) - n)\omega^n \\
&= -\frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}(\mathrm{pr}(s(\omega) - n))\omega^n = -\frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}(\mathrm{pr}(1 - e^{h_\omega}))\omega^n \\
&= \frac{1}{\sqrt{-1}}\int_M \theta_{Y,\omega}\theta_{X,\omega}\omega^n.
\end{aligned}
$$

If $\alpha_M < 1$, then letting $c = 1$ we find that the holomorphic invariant F_X^σ vanishes. Conversely, if F_X^σ vanishes, then $c = 1$ (i.e., $\alpha_M < 1$) or the ordinary Futaki invariant vanishes (i.e., $\alpha_M = 0$, see [9]). □

3. Multiplier ideal sheaves

In [6], Futaki and the author extended Nadel's multiplier ideal sheaves ([12]) to the case of Kähler-Ricci solitons, which are Einstein multiplier Hermitian metrics of type $\sigma(s) = -s + (\text{constant})$. In this section, we shall explain the extension of the multiplier ideal sheaves to the case of Einstein multiplier Hermitian metrics of all types. In general the multiplier ideal sheaves are constructed as follows. (In this paper, we adopt the classical definition in [12]. As for the more general definition, see [1].) Let $(M, c_1(M))$ be an n-dimensional Fano manifold with a compact subgroup G of the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of M. Let $S = \{u_i\}$ be a sequence of G-invariant Kähler potentials in $c_1(M)$ such that for any $\alpha \in \left(\frac{n}{n+1}, 1\right)$,

$$\sup u_i = 0, \quad \lim_{i\to\infty} \int_M \exp(-\alpha u_i)\omega_0^n = \infty, \tag{6}$$

and for some non-empty open subset $U \subset M$

$$\int_U \exp(-u_i)\omega_0^n \leq O(1). \tag{7}$$

(Note that there always exists U satisfying (7) due to the property of plurisubharmonic functions. For the detailed proof, see Theorem 3.1 in p. 236 [14].) For S as above, Nadel constructed a $G^{\mathbf{C}}$-invariant coherent ideal sheaf $\mathcal{I}_S \subset \mathcal{O}_M$, which is called the *multiplier ideal sheaf*, satisfying

- $\mathcal{I}_S \neq \mathcal{O}_M$, $\mathcal{I}_S \neq 0$
- For a non-negative Hermitian line bundle L,

$$H^q(M, L \otimes \mathcal{I}_S) = 0 \quad (q > 0), \tag{8}$$

where $G^{\mathbf{C}}$ denotes the complexification of G. Remark that we can construct a coherent ideal sheaf satisfying the same property as in the Kähler-Einstein case replacing the constant $\frac{n}{n+1}$ by any positive constant $c < 1$. Let us consider the case of multiplier Hermitian structures. Let X be a holomorphic Hamiltonian vector field whose potential function is denoted by $\theta_X \in C^\infty_{\mathbf{R}}(M)$ with respect to ω. The Kähler form $\omega_\varphi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ is an Einstein multiplier Hermitian metric if and only if φ satisfies

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = \exp(h_g - \varphi + \sigma(\theta_X)). \tag{9}$$

We consider the continuity method for (9)

$$\frac{\det(g_{i\bar{j}} + \varphi_{t,i\bar{j}})}{\det(g_{i\bar{j}})} = \exp(h_g - t\varphi + \sigma(\theta_{X,t})), \tag{10}$$

where $\theta_{X,t}$ is the potential function of X with respect to $\omega_t = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$. Since the right hand in (10) at $t = 0$ is uniformly bounded, (10) is solvable at $t = 0$ (cf. Appendix 4 in [10]). By the implicit function theorem, the space $\mathscr{T} = \{t' \in [0,1] \mid (10)$ is solvable at $t = t'\}$ is open. By the standard theory, the closedness of $\mathscr{T}$ is equivalent to the uniform C^0 estimate of φ_t. Let $b := l_{\max,X} - l_{\min,X} > 0$. Mabuchi ([10]) studied in detail that the multiplier Hermitian structures have similar properties to the ordinary Kähler structures. So, by the similar arguments in [6] we can prove

Proposition 3.1 (Proposition 4.1 in [6]). *If there is a real constant* $\alpha_0 \in \left(\frac{n+b}{n+1+b}, 1\right)$ *and a uniform constant* C *such that*

$$\int_M \exp(-\alpha_0(\varphi_t - \sup\varphi_t))e^{-\sigma(\theta_{X,0})}\omega_0^n \le C \tag{11}$$

for $t \in [0, t_\infty)$, *then* (10) *is solvable at* $t = t_\infty$. *In particular, if* (10) *is not solvable at* $t_\infty \in (0,1]$, *then there is a sequence* $\{t_i\}$ *such that* $t_i \to t_\infty$ *and*

$$\int_M \exp(-\alpha(\varphi_{t_i} - \sup\varphi_{t_i}))\omega_0^n \to \infty$$

as $i \to \infty$ *for any constant* $\alpha \in \left(\frac{n+b}{n+1+b}, 1\right)$.

From this proposition, we find that if M does not admit any Einstein multiplier Hermitian structure of type σ the divergence of $\{\varphi_{t_i} - \sup\varphi_{t_i}\}$ produces the $G^{\mathbf{C}}$-invariant multiplier ideal sheaf $\mathscr{I}^\sigma$ as defined by Nadel in [12]. This multiplier ideal sheaf $\mathscr{I}^\sigma$ satisfies the Nadel's cohomology vanishing theorem (8), because $\frac{n+b}{n+1+b} < 1$. Nadel showed the relationship between the subvariety $\mathscr{V} \subset M$ cut out by the multiplier ideal sheaf $\mathscr{I}$ induced by the continuity method and Futaki invariant ([13]). Futaki and the author extended his result to the case of Kähler-Ricci solitons (Theorem 1.4 in [6]). Following the similar arguments, it is easy to check that Nadel's result can be extended to the case of Einstein multiplier Hermitian metrics as follows.

Proposition 3.2 ([13], [6]). *Let* (M,g) *be a Fano manifold with a* $X_{\mathbf{R}}$-*invariant Kähler metric* g *whose Kähler form represents* $c_1(M)$ *and a compact subgroup* G *of* $\mathrm{Aut}(M)$. *Suppose that closedness does not hold for the continuity method of* (10), *so that we get a* $G^{\mathbf{C}}$-*invariant multiplier ideal subvariety* $\mathscr{V}^\sigma \subset M$. *Let* $Y \in \mathfrak{h}(M)$ *be a holomorphic vector field on* M *such that* $F_X^\sigma(Y) = 0$. *Then we have* $\mathscr{V}^\sigma \not\subset Z^+(Y)$.

Here $Z(Y) \subset M$ denotes the zero set of Y and

$$Z^{+}(Y) := \{p \in Z(Y) \mid \mathrm{Re}(\mathrm{div}(Y)(p)) > 0\},$$

where $\mathrm{div}(Y) = (L_Y vol_g)/vol_g$ and vol_g is the volume form induced by g. In general $\mathrm{div}(Y)$ will depend on our choice of volume form. However it is easy to check that at points where Y vanishes, $\mathrm{div}(Y)$ does not depend on our choice of the volume form. Therefore, $Z^{+}(Y)$ is a well-defined set.

4. Proof of Corollary 1.3

In this section, we shall show Corollary 1.3 by using Proposition 3.2. Firstly, to apply Proposition 3.2, it is necessary to investigate the possible multiplier ideal subvarieties. The possible multiplier ideal subvarieties on the surface given by the blow-up of $\mathbf{CP}^2$ at one point are determined by using its symmetry. Let us see it. Let M be the surface obtained by blowing up $\mathbf{CP}^2$ at $p_0 = [1:0:0]$. Since M is a toric Fano manifold and its moment polytope has the $\mathbf{Z}_2$-symmetry, M has the action of the compact group G generated by the compact torus action and the $\mathbf{Z}_2$-action. Let $p_1 = [0:1:0]$ and $p_2 = [0:0:1]$. Let E be the exceptional divisor of the above blow up. Since the proper transform $\overline{p_0 p_1}$ on the blow-up of $\mathbf{CP}^2$ at p_0 has the self-intersection zero, we can translate it in M. In fact, since

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

fixes $(1,0,0) \in \mathbf{C}^3$, so we find that $\tau_1 \in \mathrm{Aut}(M)$ and the complexification $G_1^{\mathbf{C}}$ of $G_1 := \tau_1 G \tau_1^{-1}$ gives rise to the continuous translations of the proper transform $\overline{p_0 p_1}$ in X. By the same way, we have a compact group G_2 whose complexification $G_2^{\mathbf{C}}$ gives rise to the continuous translations of $\overline{p_0 p_2}$. Let G' be the compact subgroup of $\mathrm{Aut}(M)$ generated by G, G_1 and G_2. From the invariance of MIS under $(G')^{\mathbf{C}}$-action, we can reduce the possible MIS to the following two cases; the exceptional divisor E or the $(+1)$-curve L which does not intersect with E.

Next, to apply Proposition 3.2, it is necessary to show the vanishing of the holomorphic invariant F_X^{σ}. For that purpose, we use the following result of Zhou and Zhu ([16]).

Proposition 4.1 (Proposition 5.1 in [16]). *Let* $\underline{\mathrm{pr}(1-e^{h_\omega})}$ *be the potential function of the extremal Kähler vector field. On* $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$, *we have*

$$-2 < \mathrm{pr}(1-e^{h_\omega}) < 1.$$

Now, we are in position to complete the proof of Corollary 1.3.

Proof of Corollary 1.3. Firstly, combining Theorem 1.2 and Proposition 4.1, we find that the holomorphic invariant $F^{\sigma}_{-v_\omega}(Y) = 0$ for all $Y \in \mathfrak{h}(M)$ where $\sigma(s) = -\log(s+1)$ on $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$. It is easy to check that there is a holomorphic vector field $Y \in \mathfrak{h}(M)$ whose real part $Y_{\mathbf{R}}$ generates a flow which fixes both of E and L, and flows from E towards L. Since $E \subset Z^{+}(Y)$ and $L \subset Z^{+}(-Y)$, Proposition 3.2 implies that the continuity method (10) does not produce any $(G')^{\mathbf{C}}$-invariant multiplier ideal sheaves on $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$. Hence, $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ admits generalized Kähler-Einstein metrics. $\square$

References

[1] J. P. Demailly and J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. École Norm. Sup. (4) **34** (2001), 525–556.

[2] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, Invent. Math. **73** (1983), 437–443.

[3] A. Futaki, Kähler-Einstein metrics and integral invariants, Lecture notes in mathematics **1314**, 1988.

[4] A. Futaki, Some invariant and equivariant cohomology classes of the space of Kähler metrics, Proc. Japan Acad. Ser. A Math. Sci. **78** (2002), 27–29.

[5] A. Futaki and T. Mabuchi, Bilinear forms and extremal Kähler vector fields associated with Kähler classes, Math. Ann. **301** (1995), 199–210.

[6] A. Futaki and Y. Sano, Multiplier ideal sheaves and integral invariants on toric Fano manifolds, arXiv:0711.0614, 2007.

[7] C. LeBrun and S. R. Simanca, Extremal Kähler metrics and complex deformation theory, Geom. Funct. Anal. **4** (1994), 298–336.

[8] Q. Li, On multiplier Hermitian structures on compact Kähler manifolds, arXiv:0704.2646, 2007.

[9] T. Mabuchi, Kähler-Einstein metrics for manifolds with nonvanishing Futaki character, Tohoku Math. J. (2) **53** (2000), 171–182.

[10] T. Mabuchi, Multiplier Hermitian structures on Kähler manifolds, Nagoya Math. J. **170** (2003), 73–115.

[11] Y. Matsushima, Sur la structure du group d'homémomorphismes analytiques d'une certaine variété kähleriennes, Nagoya Math. J. **11** (1957), 145–150.

[12] A. M. Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. (2) **132** (1990), 549–596.

[13] A. M. Nadel, Multiplier ideal sheaves and Futaki's invariant, Geometric Theory of Singular Phenomena in Partial Differential Equations, Cortona, 1995, Sympos. Math. **XXXVIII**, Cambridge Univ. Press, Cambridge, 1998, 7–16.

[14] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, Invent. Math. **89** (1987), 225–246.

[15] G. Tian and X. Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comment. Math. Helv **77** (2002), 297–325.

[16] B. Zhou and X. Zhu, Relative K-stability and modified K-energy on toric manifolds, arXiv:math.DG/0603237, 2006.

Yuji Sano
Institut des Hautes Études Scientifiques
Le Bois-Marie, 35, route de Chatres
F-91440 Bures-sur-Yvette
France
E-mail: sano@ihes.fr