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KODAI MATH. J.
29 (2006), 462–474

ON THE ISHIDA AND DU BOIS COMPLEXES

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Abstract

In [12] Ishida introduces a complex, denoted by $\tilde{\Omega}^{\cdot}_{Y}$, associated to a filtered semi-toroidal variety Y over Spec $\mathbf{C}$ and proves that it is quasi-isomorphic to the Du Bois complex $\underline{\underline{\Omega}}^{\cdot}_{Y}$ ([5]). In this article we regard a filtered semi-toroidal variety Y as an ideally log smooth log scheme over Spec $\mathbf{C}$, and we give an interpretation of the Ishida complex $\tilde{\Omega}^{\cdot}_{Y}$ in terms of logarithmic geometry. Therefore, given a log smooth log scheme X over Spec $\mathbf{C}$, we use this logarithmic interpretation of the Ishida complex to construct the following distinguished triangle in the Du Bois derived category $\mathrm{D_{diff}(X)}$: $I_M\omega^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D \to \cdot$, where $D = X - X_{triv}$ (X_{triv} being the trivial locus for the log structure M on X). Since the complex $I_M\omega^{\cdot}_X$ calculates the De Rham cohomology with compact supports of the smooth analytic space X^{an}_{triv} ([20, Corollary 1.6]), this triangle is useful to give an interpretation of $H^{\cdot}_{DR,c}(X_{triv}/\mathbf{C})$ as the hyper-cohomology of the simple complex $\underline{s}[\underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D]$.

Introduction

Ishida ([12], [13]) generalized the notion of toric polyhedra by introducing varieties with singularities locally isomorphic to those of toric polyhedra, in the étale topology. He called these "semi-toroidal varieties". He proved that, when a semi-toroidal variety X over Spec $\mathbf{C}$ is endowed with a good filtration, it is possible to construct a global dualizing complex for X ([13, Theorem 5.4]). He used this to define a complex, $\tilde{\Omega}^{\cdot}_{X}$, associated to a filtered semi-toroidal variety X, whose cohomology is connected with the cohomological groups of the constant sheaf $\mathbf{C}$ on the singular analytic space X^{an}. This complex is a generalization of one introduced by Danilov ([3]) for a normal variety with toroidal singularities ([3]). It is one of the most important tools in toroidal geometry. In fact, in the semi-toroidal case, it is closely connected to the toric type of singularities of the scheme.

On the other hand, Du Bois ([5], [6]) has defined a category $\mathrm{C_{diff}}(X)$, which can be seen as a "filtered version" of the Herrera-Lieberman category ([10]): the

* *Key words*: Logarithmic Schemes, Toric singularities, Ishida complex, Du Bois complex. *Supported by*: EC, RTN (Research Training Network) Arithmetic Algebraic Geometry (AAG); MIUR, GVA project; Università di Padova PGR "CPDG021784". 2000 *AMS Classification*: 14FXX.

Received February 28, 2005; revised June 5, 2006.

objects of $\mathrm{C_{diff}}(X)$ are filtered complexes, and the morphisms are $\mathcal{O}_X$-linear maps of complexes, which are compatible with the filtration. Working in this filtered category, Du Bois proved that a part of the mixed Hodge structure (namely, the Hodge filtration in cohomology) of a singular variety X over Spec $\mathbf{C}$ can be described by using a complex, $\underline{\underline{\Omega}}^{\cdot}_X$, which belongs to the derived category $\mathrm{D_{diff}}(X)$. This complex is constructed through the use of a proper smooth hyper-covering $\pi. : X. \to X$ of X, by taking the classical De Rham complex $\Omega^{\cdot}_{X_i}$ of each X_i, and defining $\underline{\underline{\Omega}}^{\cdot}_X$ as the direct image $\mathbf{R}\pi._*\Omega^{\cdot}_{X.}$. The filtration on $\underline{\underline{\Omega}}^{\cdot}_X$ comes from the natural Hodge filtration F defined on each complex $\Omega^{\cdot}_{X_i}$.

In the above context, one of the main results of Ishida ([12, Theorem 4.1]) was that, when X is a filtered semi-toroidal variety over Spec $\mathbf{C}$, there exists a natural map from the Ishida complex $\tilde{\Omega}^{\cdot}_X$ to the Du Bois complex $\underline{\underline{\Omega}}^{\cdot}_X$, which is an isomorphism in the Du Bois derived category $\mathrm{D_{diff}}(X)$. Therefore, he can describe a part of the mixed Hodge structure of a semi-toroidal variety X by simply using its own toric structure, without the construction of hyper-resolutions.

In this article, we then begin to compare two different points of view: toric geometry on one hand, and logarithmic geometry on the other. A first approach to the analysis of this comparison was done by K. Kato ([16]): having fixed a logarithmic structure on a scheme X, he defined a toric singularity at a point x of X in terms of the regularity in the logarithmic sense of x. In this direction, we can regard the theory of toroidal embeddings (resp. semi-toroidal varieties) as a theory of varieties with smooth logarithmic structures (resp. with ideally smooth log structures) over Spec $\mathbf{C}$. Then, the Ishida complex $\tilde{\Omega}^{\cdot}_X$ of a toroidal (or semi-toroidal) variety X, can be interpreted merely using the log structure of the variety. Under this interpretation, comparing the Ishida complex with the Du Bois complex $\underline{\underline{\Omega}}^{\cdot}_X$, we are able to find the most important result of this article. Namely, given an fs log smooth log scheme X over Spec $\mathbf{C}$, and $D = X - X_{triv}$ (X_{triv} being the trivial locus for the log structure M on X), there exists the following distinguished triangle in the derived category $\mathrm{D_{diff}}(\mathrm{X})$

$$I_M\omega^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D \to,$$

where $I_M\mathcal{O}_X$ is the ideal of $\mathcal{O}_X$ introduced by Tsuji in [21], and $I_M\omega^{\cdot}_X = I_M\mathcal{O}_X \otimes_{\mathcal{O}_X} \omega^{\cdot}_X$. Since Tsuji had proved that the complex $I_M\omega^{\cdot}_X$ calculates the Analytic De Rham cohomology with compact supports of the smooth analytic space X^{an}_{triv}, this triangle is useful to give a more intrinsic definition of the Algebraic De Rham cohomology with compact supports $H^{\cdot}_{DR,c}(X_{triv}/\mathbf{C})$. In fact, we can define it as the hyper-cohomology $\mathbf{H}^{\cdot}(X, s[\tilde{\Omega}^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D])$, where $s[\tilde{\Omega}^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D]$ is the simple complex associated to $\tilde{\Omega}^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D$, which is isomorphic to $\underline{s}[\underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D]$, in the Du Bois derived category $\mathrm{D_{diff}}(\mathrm{X})$.

I would like to thank Prof. L. Illusie for his helpful suggestions and comments and the University of Paris-Sud Orsay for its kind hospitality.

I have also to thank the anonymous referee for his remarks and suggestions which have allowed me to improve the earlier version.

NOTATION: By S we denote the logarithmic scheme Spec $\mathbf{C}$ endowed with the trivial log structure, and, by a *log scheme*, we mean a logarithmic scheme over S, whose underlying scheme is a separated $\mathbf{C}$-scheme of finite type. Moreover, given a scheme X over S, by $\mathrm{D}^+_{\mathrm{coh}}(\mathrm{X})$ we denote the derived category of complexes of coherent $\mathcal{O}_X$-modules limited below.

1. Ishida complex for ideally log smooth log schemes

We start to give the definition of fs ideally log smooth log scheme over S ([14, Theorem (3.5), Example (3.7)], [20, Theorem 2.23, Corollary 2.24], [8, Propositions II.1.0.11, II.1.0.12]).

DEFINITION 1.1. Let X be a log scheme over S, with fs log structure M. We say X is ideally log smooth over S if, étale locally on X, there exist a toric monoid P, and ideal I of P, a scheme U over S and étale morphisms $\varphi : U \to X$, $\psi : U \to \operatorname{Spec} \mathbf{C}[P]/(I)$, where U is endowed with the log structure $\varphi^* M$, Spec $\mathbf{C}[P]/(I)$ is endowed with the log structure $P \to \mathbf{C}[P]/(I)$, which sends I to $\{0\}$, and $\varphi^* M$ coincides with the log structure associated to $P \to \mathcal{O}_U$.

Let now X be a log smooth log scheme over S. We note that the pair $(X, U = X_{triv})$ is a toroidal embedding in the sense of [12, §5] (X_{triv} being the trivial locus for the log structure M on X). Let $D = X - U$. Ishida defines the $\mathcal{O}_X$-module $\Theta_X(-\log D)$, by $\Gamma(V, \Theta_X(-\log D)) = \{\Delta : \mathcal{O}_V \to \mathcal{O}_V; \Delta \in \mathrm{Der}(\mathcal{O}_V)$ and $\Delta(\mathscr{I}(D)|_V) \subset \mathscr{I}(D)|_V\}$, for each open set $V \subset X$, and $\mathscr{I}(D)$ the ideal of definition of D inside X.

He defines the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D) := \mathscr{H}om_{\mathcal{O}_X}(\Theta_X(-\log D), \mathcal{O}_X)$, and takes $\Omega^p_X(\log D) := \bigwedge^p \Omega^1_X(\log D)$.

LEMMA 1.2. *Denoted by ω^1_X the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log M)$, then there exists an isomorphism*

$$\Omega^1_X(\log D) \cong \omega^1_X$$

Proof. We first construct a global map

$$\Psi : \mathrm{Der}^{\log}_{\mathbf{C}}(\mathcal{O}_X) \to \Theta_X(-\log D) \tag{1}$$

where $\mathrm{Der}^{\log}_{\mathbf{C}}(\mathcal{O}_X)$ is the sheaf of the logarithmic $\mathbf{C}$-derivations on X. Now, since X is log smooth over S, then ω^1_X is locally free $\mathcal{O}_X$-module, and $\mathscr{H}om_{\mathcal{O}_X}(\mathrm{Der}^{\log}_{\mathbf{C}}(\mathcal{O}_X), \mathcal{O}_X) = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\omega^1_X, \mathcal{O}_X), \mathcal{O}_X) \cong \omega^1_X$. Therefore, from (1), we get a global map

$$\Psi^\vee : \Omega^1_X(\log D) := \mathscr{H}om_{\mathcal{O}_X}(\Theta_X(-\log D), \mathcal{O}_X) \to \omega^1_X$$

The map Ψ in (1) is constructed using the definition of logarithmic derivations: indeed, let $(\Delta, \delta) \in \mathrm{Der}^{\log}_{\mathbf{C}}(\mathcal{O}_X)$, where $\Delta : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \mathcal{O}_X$, and $\delta : M \xrightarrow{\mathrm{dlog}} \omega^1_X \to \mathcal{O}_X$. We define $\Psi((\Delta, \delta)) := \Delta : \mathcal{O}_X \to \mathcal{O}_X$. It is easy to show that Δ is in fact an element of $\Theta_X(-\log D)$. Then, working étale locally on X, it is easy to prove that $\Psi^\vee$ is an isomorphism. Indeed, if X is étale locally equal to Spec $\mathbf{C}[P]$ (P toric), endowed with canonical log structure $e : P \hookrightarrow \mathbf{C}[P]$, then ω^1_X and $\Omega^1_X(\log D)$ are both étale locally isomorphic to the free $\mathbf{C}[P]$-module $\mathbf{C}[P] \otimes_{\mathbf{Z}} P^{gp}$ ([12, §2], [8, Proposition 1.2]). ■

Let $\{X_i\}_i$ be the filtration on X (coming from the log structure M), defined by the following closed subschemes X_i of X, for $0 \le i \le n = \dim X$,

$$X_i = \{x \in X : \mathrm{rk}_{\mathbf{Z}}\, \overline{M}^{gp}_x \ge n - i\}$$

where $\overline{M}^{gp}_x := M^{gp}_x / \mathcal{O}^*_{X,x}$. It is an increasing filtration

$$X_0 \subset X_1 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots \subset X_n = X$$

From Definition 1.1, for each point $x \in X$, an étale local model of X at the point x is given by a toric variety $V_x = \mathrm{Spec}\, \mathbf{C}[P_x]$, and we have étale maps $\varphi_x : U_x \to X$ and $\psi_x : U_x \to V_x$ (ψ_x strict). Let $r(x) = \dim V_x$ (which is the dimension of X at the point x), $L_x = P^{gp}_x$, and let σ_x be the rational convex polyhedral cone generated by the toric monoid P_x inside $L_x \otimes_{\mathbf{Z}} \mathbf{R}$. Taking a face τ of the dual cone $\sigma^\vee_x$, let $V(\tau)$ the closure of the orbit $\mathrm{orb}(\tau) := \mathrm{Spec}\, \mathbf{C}[L_x \cap \tau^\perp]$. We know $\mathrm{codim}\, V(\tau) = \dim \tau = \mathrm{rk}_{\mathbf{Z}}\, L_x/(L_x \cap \tau^\perp)$. There is an increasing filtration $\{V_{xi}\}_i$ on V_x, defined by the closed subvarieties

$$V_{xi} = \bigcup_{\dim \tau \ge r(x) - i} V(\tau)$$

It is the natural filtration of the toric polyhedron V_x ([12, §5]).

Denoted by F_x the affine fan Spec P_x associated to the log regular variety V_x, there is an identification $V_{xi} = \bigcup_{y \in F_x;\, \mathrm{codim}\, y \ge r(x)-i} \overline{\{y\}}$, where $\overline{\{y\}}$ is the closure of y in V_x, and $\mathrm{codim}\, y = \mathrm{codim}_{V_x} \overline{\{y\}}$. Moreover, since $\mathrm{rk}_{\mathbf{Z}}\, \overline{P}^{gp}_z \ge r(x) - i$ if and only if $z \in \bigcup_{y \in F_x;\, \mathrm{codim}\, y \ge r(x)-i} \overline{\{y\}}$, there is an identification $V_{xi} = \{z \in V_x : \mathrm{rk}_{\mathbf{Z}}\, \overline{P}^{gp}_z \ge r(x) - i\}$, and so $\varphi^{-1}_x(X_i) = \psi^{-1}_x(V_{xi})$; therefore X, endowed with the filtration $\{X_i\}_i$, is a filtered semi-toroidal variety in the sense of [12, Definition 5.2].

For each point $x \in X$ and integer i, we take the locally closed subscheme $S^{(i)}_x := V_{xi} - V_{xi-1}$ of V_x: it is equal to $\bigcup_{\tau \prec \sigma^\vee_x;\, \dim \tau = r(x)-i} \mathrm{orb}(\tau)$, and it is non singular of pure dimension i. We denote by $V^{(i)}_x$ the normalization of the closure of $S^{(i)}_x$ in V_x: $V^{(i)}_x$ is equal to the disjoint union $\coprod_{\tau \prec \sigma^\vee_x;\, \dim \tau = r(x)-i} V(\tau) = \coprod_{y \in F_x;\, \mathrm{codim}\, y = r(x)-i} \overline{\{y\}}$. The pair $(V^{(i)}_x, S^{(i)}_x)$ is a toroidal embedding of pure dimension i. We denote by $D(\tau)$ the reduced divisor $V(\tau) - \mathrm{orb}(\tau)$, and by $D^{(i)}_x$ the reduced divisor $V^{(i)}_x - S^{(i)}_x = \coprod_{\tau \prec \sigma^\vee_x;\, \dim \tau = r(x)-i} D(\tau)$.

Similarly, for each i, let $U^{(i)} = X_i - X_{i-1}$, and let $X^{(i)}$ be the normalization of the closure of $U^{(i)}$. The pair $(X^{(i)}, U^{(i)})$ is a toroidal embedding of pure dimension i. Let $D^{(i)}$ be the reduced divisor $X^{(i)} - U^{(i)}$.

For each point $x \in X$, Ishida defines the locally free $\mathcal{O}_{V^{(i)}_x}$-modules, of rank equal to the dimension $r(x)$ of X at x, $\Theta_{V^{(i)}_x}(-\log D^{(i)}_x)$, and $\Omega^1_{V^{(i)}_x}(\log D^{(i)}_x)$. He takes $\Omega^p_{V_x^{(i)}}(\log D_x^{(i)}) := \bigwedge^p \Omega^1_{V_x^{(i)}}(\log D_x^{(i)})$. By Lemma 1.2, we have an identification $\Omega^p_{V_x^{(i)}}(\log D_x^{(i)}) \cong \omega^p_{V_x^{(i)}}$. We denote by $\lambda^{(i)}$ the natural maps $\lambda^{(i)} : V_x^{(i)} \to V_x$. Following [12, §5], for each integer p, we define the complex $C^{\cdot}(V_x, \Omega_p^{\vee}(\mathcal{O}))$ in degree $-i$, $0 \le i \le r(x)$, as

$$C^{-i}(V_x, \Omega_p^{\vee}(\mathcal{O})) := \lambda_*^{(i)} \omega^{i-p}_{V_x^{(i)}} \tag{2}$$

We simply denote $C^{-i}(V_x, \Omega_p^{\vee}(\mathcal{O}))$ by $C_p^{-i}(V_x)$. Since

$$V_x^{(i)} = \coprod_{y \in F_x;\, \mathrm{codim}\, y = r(x)-i} \overline{\{y\}}, \quad \text{then } \omega^1_{V_x^{(i)}} = \bigoplus_{y \in F_x;\, \mathrm{codim}\, y = r(x)-i} \omega^1_{\overline{\{y\}}}.$$

For each point $y \in V_x$, $\mathrm{codim}_{V_x}\, y = r(x) - i$, we consider the subgroup of P_x^{gp} generated by $P_x^{gp} \cap \mathcal{O}^*_{V_x, y}$. This is free of rank $\dim \overline{\{y\}} = i$. We denote it by K_y^{gp}. Since, for each point $y \in F_x$, there is an isomorphism $\mathcal{O}_{\overline{\{y\}}} \otimes_{\mathbf{Z}} K_y^{gp} \cong \omega^1_{\overline{\{y\}}}$, then

$$C_p^{-i}(V_x) \cong \bigoplus_{y \in F_x;\, \mathrm{codim}\, y = r(x)-i} \mathcal{O}_{\overline{\{y\}}} \otimes_{\mathbf{Z}} \bigwedge^{i-p} K_y^{gp}$$

The differential $\delta_p^i(V_x) : C_p^{-i}(V_x) \to C_p^{-i+1}(V_x)$ of this complex is defined as the direct sum, over $y, z \in F_x$, codim $y = r(x) - i$, codim $z = r(x) - i + 1$, of the maps $\delta^i_{p,(y,z)}$, where:

- $\delta^i_{p,(y,z)} = 0$ if $z \notin \overline{\{y\}}$;
- if $z \in \overline{\{y\}}$, then $\delta^i_{p,(y,z)} : \mathcal{O}_{\overline{\{y\}}} \otimes_{\mathbf{Z}} \bigwedge^{i-p} K_y^{gp} \to \mathcal{O}_{\overline{\{z\}}} \otimes_{\mathbf{Z}} \bigwedge^{i-p-1} K_z^{gp}$ is the

"Poincaré residue map". This map sends a section $a \otimes_{\mathbf{Z}} k_1 \wedge \cdots \wedge k_{i-p}$ into $a_{|\overline{\{z\}}} \otimes_{\mathbf{Z}} \langle k_1, N \rangle k_2 \wedge \cdots \wedge k_{i-p}$, where N is an element of $L_x^{\vee}$ such that $\langle k, N \rangle = 0$, for $k \in K_z^{gp}$, and $\langle k, N \rangle \ge 0$, for $k \in K_y^{gp} - K_z^{gp}$ (we recall this map does not depend on the choice of this element N, as it is shown in [19]).

We consider now the natural maps $\Lambda^{(i)} : X^{(i)} \to X$. Following [12, §5], for each $p \in \mathbf{Z}$, we define the complex $C^{\cdot}(X, \Omega_p^{\vee}(\mathcal{O}))$ in degree $-i$, $0 \le i \le n = \dim X$, as

$$C^{-i}(X, \Omega_p^{\vee}(\mathcal{O})) := \Lambda_*^{(i)} \omega^{i-p}_{X^{(i)}} \tag{3}$$

For simplicity, we denote $C^{-i}(X, \Omega_p^{\vee}(\mathcal{O}))$ by $C_p^{-i}(X)$.

The differential $\delta_p^i : C_p^{-i}(X) \to C_p^{-i+1}(X)$ of this complex is induced by the Poincaré residue maps, for every $x \in X$. Indeed, we recall that, for every $x \in X$, the étale maps φ_x and ψ_x are toroidal étale morphisms in the sense of [12, §5], so the pairs $(U_x, \psi_x^{-1}(T_x))$, with T_x the torus of V_x, are toroidal embeddings, which correspond to (X, U). Then, since the homomorphisms $\delta_p^i(V_x)$ are naturally defined by the Poincaré residue maps, and since $\psi_x^*(\Theta_{V_x}(-\log D_x)) = \varphi_x^*(\Theta_X(-\log D))$, by étale localization, we can glue together these local maps to get a homomorphism δ_p^i such that $\varphi_x^* C_p^{\cdot}(X) = \psi_x^* C_p^{\cdot}(V_x)$, at each point $x \in X$ ([8, II.3.1]).

Let now Y be an ideally log smooth log scheme over S, with log structure

M. Let $\{Y_i\}_i$ be the filtration on Y defined by the closed subschemes Y_i of Y, for $0 \le i \le s = \dim Y$,

$$Y_i = \{y \in Y : \mathrm{rk}_{\mathbf{Z}}\, \overline{M}_y^{gp} \ge s - i\}$$

It is an icreasing filtration on Y. From Definition 1.1, for each point $y \in Y$, an étale local model of Y at y is given by a toric polyhedron $Y_y = \mathrm{Spec}\, \mathbf{C}[P_y]/(I_y)$, for an ideal I_y of the toric monoid P_y, and there are étale maps $\varphi_y : U_y \to Y$, $\psi_y : U_y \to Y_y$ (where $M_{U_y} = (P_y \to \mathcal{O}_{U_y})^a$).

For $y \in Y$, let $r(y) = \dim Y_y$. Let $\Phi_y = F_y \cap Y_y$ be the star closed subset of the affine fan F_y associated to $\mathrm{Spec}\, \mathbf{C}[P_y]$. There is an increasing filtration $\{Y_{yi}\}_i$ on Y_y, defined by the closed subschemes $Y_{yi} = \bigcup_{z \in \Phi_y;\, \mathrm{codim}\, z \ge r(y)-i} \overline{\{z\}}$. This filtration coincides with the natural filtration of the toric polyhedron Y_y, and $\varphi_y^{-1}(Y_i) = \psi_y^{-1}(Y_{yi})$. Therefore, Y, endowed with the filtration $\{Y_i\}$, is a filtered semi-toroidal variety.

For similar arguments used in the log smooth case, we define, as in (2), (3), the complexes $C^{\cdot}(Y_y, \Omega_p^{\vee}(\mathcal{O}))$ and $C^{\cdot}(Y, \Omega_p^{\vee}(\mathcal{O}))$, using the natural maps $\lambda^{(i)} : Y_y^{(i)} = \coprod_{z \in \Phi_y;\, \mathrm{codim}\, z = r(y)-i} \overline{\{z\}} \to Y_y$, and $\Lambda^{(i)} : Y^{(i)} \to Y$. We simply denote $C^{\cdot}(Y_y, \Omega_p^{\vee}(\mathcal{O}))$ and $C^{\cdot}(Y, \Omega_p^{\vee}(\mathcal{O}))$ by $C_p^{\cdot}(Y_y)$ and $C_p^{\cdot}(Y)$, respectively.

From [12, Theorem 5.4], the complex $C_0^{\cdot}(Y)$ associated to an ideally log smooth log scheme Y is a global dualizing complex, because it is isomorphic in the derived category to $f^!\mathbf{C}$ (where $f : Y \to S$). We simply denote it by $\mathscr{D}_Y^{\cdot}$. Moreover, for each integer p, $\mathscr{E}xt^i_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot}) = 0$, for every $i \ne 0$ ([12, Proposition 6.1]). Then $\mathscr{E}xt^0_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot})$ is equal, in the derived category, to $\mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot})$. Globalizing Lemma 2.8 in [12], we get an isomorphism in the derived category

$$\mathscr{H}om^{\cdot}_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot}) \xrightarrow{\cong} \mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot})$$

Definition 1.3. Let Y be an ideally log smooth log scheme over S. We define the Ishida complex $\tilde{\Omega}_Y^{\cdot}$ associated to Y, in degree p, $0 \le p \le \dim Y$, as

$$\tilde{\Omega}_Y^p := \mathscr{H}^0(\mathscr{H}om^{\cdot}_{\mathcal{O}_Y}(C_p^{\cdot}(Y), \mathscr{D}_Y^{\cdot}))$$

The differential $\tilde{d}_Y^p : \tilde{\Omega}_Y^p \to \tilde{\Omega}_Y^{p+1}$ is induced by that of the log De Rham complex $\omega_Y^{\cdot}$, via the inclusion $\tilde{\Omega}_Y^{\cdot} \hookrightarrow \omega_Y^{\cdot}$.

Remark 1.4. For an ideally log smooth log scheme Y over S, we have defined the sheaf $\tilde{\Omega}_Y^p$ as the Grothendieck dual (with respect to the global dualizing complex $\mathscr{D}_Y^{\cdot}$) of the complex which consists of modules of logarithmic differential p-forms and Poincaré residue maps. In fact, given an étale local model Y_y of Y at the point y, the sheaf $\tilde{\Omega}_Y^p$ locally coincides with a subsheaf $\tilde{\Omega}_{Y_y}^p$ of $\omega_{Y_y}^p$ ([12, §2, Proposition 2.2 and §3]). The differential operator $d : \omega_{Y_y}^p \to \omega_{Y_y}^{p+1}$ is such that $d(\tilde{\Omega}_{Y_y}^p) \subset \tilde{\Omega}_{Y_y}^{p+1}$, for any $p \in \mathbf{N}$. So, the Ishida complex $\tilde{\Omega}_Y^{\cdot}$ is the natural globalization of a subcomplex $\tilde{\Omega}_{Y_y}^{\cdot}$ of $\omega_{Y_y}^{\cdot}$ and the global map $\tilde{\Omega}_Y^{\cdot} \to \omega_Y^{\cdot}$ is in fact an inclusion.

In the case when X is log smooth log scheme over S, we denote by $\tilde{L}^0\omega^{\cdot}_X$ the subcomplex of $\omega^{\cdot}_X$ which is the 0-degree term of the filtration $\tilde{L}^{\cdot}$ on $\omega^{\cdot}_X$ defined by Ogus ([20, Definition 1.2 and Lemma 2.15]). We note that, when X is a smooth scheme over S with fs log smooth log structure given by a normal crossing divisor D, this filtration $\tilde{L}^{\cdot}$ coincides with the Deligne's weight filtration $W^{\cdot}$ ([8, Lemma III.1.0.2]). So, $\tilde{L}^{\cdot}$ is a sort of "generalization" of the weight filtration to a generic log smooth log scheme over S. We recall that the hypercohomology of the complex $\tilde{L}^0\omega^{\cdot}_X$ calculates the analytical cohomology of X^{an} ([20, Theorem 1.4]). We can compare the Ishida complex $\tilde{\Omega}^{\cdot}_X$ with $\tilde{L}^0\omega^{\cdot}_X$.

Proposition 1.5. *Under the previous notations, there exists an isomorphism of complexes, in the category* $\mathrm{C}_{\mathrm{diff}}(\mathrm{X})$,

$$\psi^{\cdot} : \tilde{L}^0\omega^{\cdot}_X \xrightarrow{\cong} \tilde{\Omega}^{\cdot}_X.$$

Proof. We first note that we can view $\tilde{L}^0\omega^{\cdot}_X$ as an object in $\mathrm{C}_{\mathrm{diff}}(\mathrm{X})$, where the filtration $F^{\cdot}$ is defined as $F^q(\tilde{L}^0\omega^p_X) = \tilde{L}^0\omega^p_X$, if $p \geq q$, and $F^q(\tilde{L}^0\omega^p_X) = 0$ if $p < q$.

We want now to prove that $\tilde{L}^0\omega^{\cdot}_X$ and $\tilde{\Omega}^{\cdot}_X$ are isomorphic as complexes. To this aim, we construct a global map $\psi^{\cdot} : \tilde{L}^0\omega^{\cdot}_X \to \tilde{\Omega}^{\cdot}_X$. Now, by [12, Proposition 3.11], we know that $\tilde{\Omega}^{\cdot}_X = v_*\Omega^{\cdot}_V$, where V is the smooth locus of X and $v : V \hookrightarrow X$ is the open immersion. So, to construct a map $\psi^{\cdot} : \tilde{L}^0\omega^{\cdot}_X \to v_*\Omega^{\cdot}_V$ is equivalent to construct a map $\Psi^{\cdot} : v^*\tilde{L}^0\omega^{\cdot}_X \to \Omega^{\cdot}_V$, by adjunction. We note that, on the smooth open V of X, the log structure is given by the normal crossing divisor $D \cap V$, and so $v^*\tilde{L}^0\omega^{\cdot}_X = \tilde{L}^0(v^*\omega^{\cdot}_X) = \tilde{L}^0\omega^{\cdot}_V = \Omega^{\cdot}_V$ (since $\tilde{L}^0\omega^{\cdot}_V = W^0\omega^{\cdot}_V = \Omega^{\cdot}_V \hookrightarrow \omega^{\cdot}_V$). So the map $\Psi^{\cdot}$ is exactly the identity map.

We show that the adjoint map $\psi^{\cdot}$ of $\Psi^{\cdot}$ is an isomorphism. We work étale locally on X, and suppose that $X = \operatorname{Spec} \mathbf{C}[P]$, where P is a toric monoid and the log structure M is given by $e : P \hookrightarrow \mathbf{C}[P]$.

We recall that the Ogus filtration $\tilde{L}^0$ on $\omega^{\cdot}_X$ admits a local graded description. Indeed, we consider the P-graded $\mathbf{Z}$-algebra $\mathbf{C}[P] = \bigoplus_{p \in P} \mathbf{C}e(p)$ and the $\mathbf{Z}$-algebra $\bigwedge^j P^{gp}$, then $\mathbf{C}[P] \otimes_{\mathbf{Z}} \bigwedge^j P^{gp} \cong \omega^j_{\mathbf{C}[P]}$ (which locally represents the sheaf ω^j_X) has a natural structure of a P-graded $\mathbf{C}[P]$-module; its component in degree p is just $\mathbf{C}e(p) \otimes_{\mathbf{Z}} \bigwedge^j P^{gp}$.

Now, given a face F of the monoid P, and an element $p \in P$, let $L^i_p(F) \bigwedge^j P^{gp}$ be the subgroup of $\bigwedge^j P^{gp}$ generated by all the elements of the form $\operatorname{dlog} p_1 \wedge \cdots \wedge \operatorname{dlog} p_j$, such that $p_1, \ldots, p_j \in P$ and there exists $k \in \mathbf{N}$ and $f \in F$ such that $kp + f - (p_1 + \cdots + p_i) \in P$ ([20, Definition 3.2]). In particular, if $F = \{0\}$ and $i = j$, the condition becomes $kp - (p_1 + \cdots + p_j) \in P$. We simply denote by $L_p \bigwedge^j P^{gp}$ the subgroup $L^j_p(\{0\}) \bigwedge^j P^{gp}$. This is such that $L_p \bigwedge^j P^{gp} \subseteq L_q \bigwedge^j P^{gp}$, for any $p, q \in P$, $p \geq q$. Ogus calls such a collection of $\mathbf{Z}$-submodules of $\bigwedge^j P^{gp}$ a P-filtration. The image of $\bigoplus_{p \in P} \mathbf{C}e(p) \otimes_{\mathbf{Z}} L_p \bigwedge^j P^{gp}$ inside $\mathbf{C}[P] \otimes_{\mathbf{Z}} \bigwedge^j P^{gp} \cong \omega^j_{\mathbf{C}[P]}$ is a P-graded $\mathbf{C}[P]$-submodule. Then, the quasi-coherent sheaf on X associated to this $\mathbf{C}[P]$-submodule is nothing but $\tilde{L}^0\omega^{\cdot}_X$ (see [20, Lemma 3.3]).

So, we compare the $\mathbf{C}[P]$-submodule $\mathrm{Im}\{\bigoplus_{p\in P}\mathbf{C}e(p)\otimes_{\mathbf{Z}} L_p^j \bigwedge^j P^{gp} \to \mathbf{C}[P]\otimes_{\mathbf{Z}} \bigwedge^j P^{gp}\}$ of $\omega^j_{\mathbf{C}[P]}$ with the $\mathbf{C}[P]$-module $\tilde{\Omega}^j_{\mathbf{C}[P]} = \bigoplus_{p\in P}\mathbf{C}e(p)\otimes_{\mathbf{Z}} \bigwedge^j P^{gp}[\rho(p)]$, where $\rho(p) = \pi\cap p^{\perp}$ is a face of the cone π, with $\pi^{\vee}$ the cone generated by P. We note that, for each $p\in P$, if $\langle p, F\rangle$ is the face of P generated by p and F, then $L_p^i(F)$ is just the image of the natural map $\bigwedge^i\langle p, F\rangle^{gp}\otimes\bigwedge^{j-i}P^{gp}\to\bigwedge^j P^{gp}$. In our case, $i = j$ and $F = \{0\}$ and so $L^j_{p_j}(\{0\})$ is the image of the map $\bigwedge^j\langle p\rangle^{gp}\to\bigwedge^j P^{gp}$ which is equal to $\bigwedge^j\langle p\rangle^{gp}$. Therefore, for each $p\in P$, the image of $\mathbf{C}e(p)\otimes L^j_p(\{0\})\bigwedge^j P^{gp}$ inside $\mathbf{C}e(p)\otimes\bigwedge^j P^{gp}$ is just $\mathbf{C}e(p)\otimes\bigwedge^j\langle p\rangle^{gp}$. On the other hand, we consider $\mathbf{C}e(p)\otimes\bigwedge^j P^{gp}[\rho(p)]$ and we note that, for each $p\in P$, $P^{gp}\cap(p^{\perp}\cap\pi)^{\perp} = P^{gp}\cap\langle p\rangle^{gp} = \langle p\rangle^{gp}$ if $p\in\pi^{\perp}$, and $P^{gp}\cap\{0\}^{\perp} = P^{gp}$ if $p\in\mathrm{int}(\pi^{\vee})$. So, if we consider the global image $\mathrm{Im}\{\bigoplus_{p\in P}\mathbf{C}e(p)\otimes_{\mathbf{Z}} L_p^j\bigwedge^j P^{gp}\to\mathbf{C}[P]\otimes_{\mathbf{Z}}\bigwedge^j P^{gp}\}$, we see that it coincides with $\tilde{\Omega}^j_{\mathbf{C}[P]}$.

Therefore, $\psi^{\cdot} : \tilde{L}^0\omega^{\cdot}_X \to \tilde{\Omega}^{\cdot}_X$ is an isomorphism of complexes and, since it respects the filtrations $F^{\cdot}$, it is an isomorphism in the category $\mathrm{C_{diff}(X)}$. ■

2. Distinguished Triangle in $D_{\mathrm{diff}}(X)$

In this section we mainly refer to the articles of Ishida [12], Du Bois [5] and Ogus [20].

Let X be a log smooth log scheme over S. Let $U = X_{triv}$, $j : U \hookrightarrow X$ be the open immersion and $D = X - U$. We have just seen in the previous section that the pair (X, U) is a toroidal embedding, and the ideally log smooth log subscheme D of X is a filtered semi-toroidal variety.

We briefly recall the construction of the category $\mathrm{C_{diff}(X)}$ and the complex $\underline{\underline{\Omega}}^{\cdot}_X$ given by Du Bois in [5] for a generic scheme X of finite type over S.

The objects of $\mathrm{C_{diff}(X)}$ are all the triples $(C^{\cdot}, F, d)$, where $(C^{\cdot}, d)$ is a complex of $\mathcal{O}_X$-modules and F is a decreasing filtration on $C^{\cdot}$ such that:

(1) $C^{\cdot}$ is bounded below,

(2) the filtration F is biregular, i.e. for each component C^i of $C^{\cdot}$, there exist integers $k, l\in\mathbf{Z}$ such that $F^kC^i = C^i$ and $F^lC^i = 0$,

(3) d is a differential operator of order at most one which preserves the filtration,

(4) $\mathrm{Gr}^p_F(d) : \mathrm{Gr}^p_F(C^i)\to\mathrm{Gr}^p_F(C^{i+1})$ is $\mathcal{O}_X$-linear for each p and i.

So Du Bois considers the derived category $\mathrm{D_{diff}(X)}$ of $\mathrm{C_{diff}(X)}$ and defines a complex $\underline{\underline{\Omega}}^{\cdot}_X$ like an object of $\mathrm{D_{diff}(X)}$, constructed as follows: let $\alpha. : X. \to X$ be a proper smooth hyper-covering of X (see Deligne [4]). Since each component X_n has the canonical De Rham complex $\Omega^{\cdot}_{X_n}$, we have the complex $\Omega^{\cdot}_{X.}$ on the simplicial scheme $X.$ which consists of $\Omega^{\cdot}_{X_n}$'s. Thus, the complex $\underline{\underline{\Omega}}^{\cdot}_X$ is defined as $\mathbf{R}\alpha._*\Omega^{\cdot}_{X.}$. Du Bois shows that this definition is indipendent on the choice of the proper smooth hyper-covering $X.$ of X. For each integer p, $\mathbf{R}\alpha._*(\Omega^p_{X.})$ is canonically identified with $\mathrm{Gr}^p_F(\underline{\underline{\Omega}}^{\cdot}_X)$ and it is an element of $\mathrm{D^+_{coh}(X)}$, denoted by $\underline{\underline{\Omega}}^p_X$ ([5, Theorem 2.2]).

We recall that, given the toroidal embedding (X, U) and the filtered semi-toroidal variety $D = X - U$, Ishida defines the complexes $\tilde{\Omega}^{\cdot}_X$ and $\tilde{\Omega}^{\cdot}_D$: now, since X is an fs log smooth log scheme over S and its closed subscheme D is an fs ideally log smooth log scheme, we have given, in the previous section, an interpretation of these complexes in logarithmic terms.

Since the Ishida and Du Bois complexes are both compatible with pull-backs by étale morphisms (by definition), we can work étale locally on X.

We recall that, for an fs log smooth log scheme X over S, with log structure M, it is defined a sheaf of ideals I_M of the sheaf of monoids M as follows ([21, Definition 2.4, Lemma 2.5 and Corollaries 2.6, 2.7]): $\Gamma(U, I_M) = \{a \in \Gamma(U, M)$: the image of a in M_x is contained in p, for all points $x \in U$, and all prime $p \in \operatorname{Spec} M_x$ of height $1\}$. We can see that, for $x \in X$, $(I_M)_x = \{a \in M_x$: a is contained in p, for all primes $p \in \operatorname{Spec} M_x$ of height $1\}$. Let $I_M\omega^{\cdot}_X$ denote the complex $I_M\mathcal{O}_X \otimes_{\mathcal{O}_X} \omega^{\cdot}_X$. We regard this complex as an object of $\mathrm{C_{diff}(X)}$, with the filtration defined as $F^q(I_M\omega^p_X) = I_M\omega^p_X$, if $p \geq q$, and $F^q(I_M\omega^p_X) = 0$, if $p < q$.

Let $F = F(X)$ be the fan associated to the log regular log scheme X ([16, Definition (9.3) and §10]). Then $D \cap F$ is a star closed subset of F.

The first claim of this section is to prove the following

Proposition 2.1. *For each integer p, $0 \leq p \leq n = \dim X$, there exists a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \to I_M\omega^p_X \to \tilde{\Omega}^p_X \to \tilde{\Omega}^p_D \to 0, \tag{4}$$

where $\tilde{\Omega}^p_X$ and $\tilde{\Omega}^p_D$ are the sheaves of Definition 1.3, *for the log smooth log scheme X and the ideally log smooth log scheme D, respectively*.

Proof. From a globalization of [12, Lemma 3.6], we get

$$\tilde{\Omega}^p_D \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(D), \mathcal{D}^{\cdot}_X),$$

where $\mathcal{D}^{\cdot}_X$ is the global dualizing complex $C^{\cdot}_0(X)$ of X. Now, from the proof of [12, Lemma 3.9], we have an exact sequence of complexes

$$0 \to C^{\cdot}_p(D) \to C^{\cdot}_p(X) \to \omega^{n-p}_X[n] \to 0. \tag{5}$$

Now, applying $\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(-, \mathcal{D}^{\cdot}_X)$, we get a triangle in $\mathrm{D^+_{coh}(X)}$

$$\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\omega^{n-p}_X[n], \mathcal{D}^{\cdot}_X) \to \mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(X), \mathcal{D}^{\cdot}_X) \to \mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(D), \mathcal{D}^{\cdot}_X) \to.$$

Now, we note that

(1) $\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(X), \mathcal{D}^{\cdot}_X) \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(X), \mathcal{D}^{\cdot}_X) \cong \tilde{\Omega}^p_X$;

(2) Since $C^{\cdot}_p(D)$ is a finite complex of coherent $\mathcal{O}_D$-modules and $C^i_p(D)$ is decomposed as a direct sum of locally free $\mathcal{O}_{\overline{\{y\}}}$-modules, for $y \in F \cap D$, we also have $\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(D), \mathcal{D}^{\cdot}_X) \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(C^{\cdot}_p(D), \mathcal{D}^{\cdot}_X) \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_D}(C^{\cdot}_p(D), \mathcal{D}^{\cdot}_D) \cong \tilde{\Omega}^p_D$.

So, we can write the previous triangle in $\mathrm{D^+_{coh}(X)}$ as follows

$$\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\omega^{n-p}_X[n], \mathcal{D}^{\cdot}_X) \to \tilde{\Omega}^p_X \to \tilde{\Omega}^p_D \to.$$

Since X is Cohen-Macaulay, if $f : X \to S$ is the structure morphism, $f^!\mathbf{C}$ is a dualizing complex on X, and we have a quasi-isomorphism $f^!\mathbf{C} \cong \mathcal{D}^{\cdot}_X$.

Moreover, from [21], $f^!\mathbf{C} \cong I_M\omega_X^n[n]$. We get $\mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_X}(\omega_X^{n-p}[n], \mathscr{D}^{\cdot}_X) \cong \mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_X}(\omega_X^{n-p}[n], I_M\omega_X^n[n]) \cong \mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_X}(\omega_X^{n-p}, I_M\omega_X^n)$.

Now, since (X, M) is an fs log smooth log scheme, the $\mathcal{O}_X$-module ω_X^1 is locally free of finite type and $\mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_X}(\omega_X^{n-p}, I_M\omega_X^n) = \mathbf{R}\mathscr{H}om^{\cdot}_{\mathcal{O}_X}(\omega_X^{n-p}, I_M\mathcal{O}_X \otimes \omega_X^n) \cong (\omega_X^{n-p})^\vee \otimes I_M\mathcal{O}_X \otimes \omega_X^n \cong I_M\mathcal{O}_X \otimes \omega_X^p = I_M\omega_X^p$. So we get a short exact sequence in $\mathrm{D}^+_{\mathrm{coh}}(\mathrm{X})$, $0 \to I_M\omega_X^p \to \tilde{\Omega}_X^p \to \tilde{\Omega}_D^p \to 0$. ■

Ishida in [12, Theorem 4.1, Proposition 4.2, Theorem 6.2] compared the complex $\tilde{\Omega}^{\cdot}_Y$, associated to a filtered semi-toroidal variety Y, with the Du Bois complex $\underline{\underline{\Omega}}^{\cdot}_Y$. Therefore,

Theorem 2.2. *Given an fs ideally log smooth log scheme Y over S, the natural homomorphism of filtered complexes* ([12, §4])

$$\psi_{\cdot} : \tilde{\Omega}^{\cdot}_Y \to \underline{\underline{\Omega}}^{\cdot}_Y \tag{6}$$

is an isomorphism in the derived category $\mathrm{D}_{\mathrm{diff}}(\mathrm{Y})$.

The main result of this article is the following

Main Theorem 2.3. *The short exact sequences* (4) *of Proposition* 2.1, *for variable p, form an exact sequence of complexes*

$$0 \to I_M\omega^{\cdot}_X \to \tilde{\Omega}^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D \to 0. \tag{7}$$

Moreover, by Theorem 2.2, *there exist the isomorphisms $\tilde{\Omega}^{\cdot}_X \cong \underline{\underline{\Omega}}^{\cdot}_X$ and $\tilde{\Omega}^{\cdot}_D \cong \underline{\underline{\Omega}}^{\cdot}_D$ in* $\mathrm{D}_{\mathrm{diff}}(\mathrm{X})$, *so the sequence* (7) *gives the following distinguished triangle in the derived category* $\mathrm{D}_{\mathrm{diff}}(\mathrm{X})$:

$$I_M\omega^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D \to . \tag{8}$$

where, denoted by $i : D \hookrightarrow X$ the closed immersion, from [5, (3.2.1)], *when we write $\underline{\underline{\Omega}}^{\cdot}_X \to \underline{\underline{\Omega}}^{\cdot}_D$ we mean the map $i^* : \underline{\underline{\Omega}}^{\cdot}_X \to \mathbf{R}i_*\underline{\underline{\Omega}}^{\cdot}_D$ in* $\mathrm{D}_{\mathrm{diff}}(\mathrm{X})$ (*which exists by the funtoriality of the Du Bois complex*).

Proof. We have to check that the short exact sequences (4), for variable p, are compatible with the differentials. Since the differential of $\tilde{\Omega}^{\cdot}_X$ (resp. $\tilde{\Omega}^{\cdot}_D$) is induced by that of $\omega^{\cdot}_X$ (resp. $\omega^{\cdot}_D$), and the maps $\tilde{\Omega}^p_X \to \tilde{\Omega}^p_D$ are the natural maps induced by the inclusion $i : D \hookrightarrow X$, we just know that these commute with the differentials.

Moreover, since also the differential of $I_M\omega^{\cdot}_X$ is induced by that of $\omega^{\cdot}_X$, we can conclude that the maps $I_M\omega^p_X \hookrightarrow \tilde{\Omega}^p_X$, for variable $p \in \mathbf{N}$, commute with the differentials. ■

3. De Rham Cohomology with compact supports

Let X be a log smooth log scheme whose underlying scheme is proper over S. Let $U = X_{triv}$, $D = X - U$, and $i_D : D \hookrightarrow X$, like in the previous section.

We have just seen there exists the exact sequence of complexes

$$0 \to I_M\omega_X^{\cdot} \to \tilde{\Omega}_X^{\cdot} \to i_{D*}\tilde{\Omega}_D^{\cdot} \to 0. \tag{9}$$

So, in the category $\mathrm{D_{diff}(X)}$, we have an isomorphism between $I_M\omega_X^{\cdot}$ and the simple complex associated to $[\tilde{\Omega}_X^{\cdot} \to i_{D*}\tilde{\Omega}_D^{\cdot}]$, which we denote by $s[\tilde{\Omega}_X^{\cdot} \to \tilde{\Omega}_D^{\cdot}]$. We note that the functor $s[-] =$ "associate simple complex" from $\mathrm{C_{diff}(X)}$ into $\mathrm{C_{diff}(X)}$, sends filtered quasi-isomorphisms into filtered quasi-isomorphisms, and so it is possible to derive it and get a functor $\underline{\underline{s}}[-] : \mathrm{D_{diff}(X)} \to \mathrm{D_{diff}(X)}$ in the derived category. Therefore, we consider the simple complex $\underline{\underline{s}}[\underline{\underline{\Omega}}_X^{\cdot} \to \underline{\underline{\Omega}}_D^{\cdot}]$ in $\mathrm{D_{diff}(X)}$. We note that there is an isomorphism in $\mathrm{D_{diff}(X)}$ $s[\tilde{\Omega}_X^{\cdot} \to i_{D*}\tilde{\Omega}_D^{\cdot}] \cong \underline{\underline{s}}[\underline{\underline{\Omega}}_X^{\cdot} \to \underline{\underline{\Omega}}_D^{\cdot}]$.

Now, we suppose that there exists a closed immersion of X inside a smooth scheme P over S, and we consider the following closed immersions

$$D \overset{i_D}{\hookrightarrow} X \overset{i}{\hookrightarrow} P$$

In [2, Definition 2.1], and [1, §1], there is a definition of De Rham cohomology with compact supports of U, which is the following

$$H_{DR,c}^{\cdot}(U/\mathbf{C}) = \mathbf{H}^{\cdot}(X, s[\Omega_{P\hat{|}X}^{\cdot} \to i_{D*}\Omega_{P\hat{|}D}^{\cdot}]),$$

where $P\hat{|}X$ and $P\hat{|}D$ are the formal completions of P along the closed subschemes X and D, respectively.

Now, by ([20, Theorem 1.5 and Corollary 1.6] and [21, Definition 3.2]), the hyper-cohomology of the complex $I_M\omega_X^{\cdot}$ is isomorphic to the analytic cohomology with compact supports of U^{an}, $H_c^{\cdot}(U^{an}, \mathbf{C})$, and, by (9), $I_M\omega_X^{\cdot}$ is isomorphic to $s[\tilde{\Omega}_X^{\cdot} \to \tilde{\Omega}_D^{\cdot}]$, in the derived category. So, since $H_c^{\cdot}(U^{an}, \mathbf{C})$ is isomorphic to the algebraic De Rham cohomology with compact supports of U, $H_{DR,c}^{\cdot}(U/\mathbf{C})$, then we can give the following

Definition 3.1. We define the Algebraic De Rham cohomology with compact supports of U as the hyper-cohomology of the simple complex $s[\tilde{\Omega}_X^{\cdot} \to \tilde{\Omega}_D^{\cdot}]$, i.e.

$$H_{DR,c}^{\cdot}(U/\mathbf{C}) := \mathbf{H}^{\cdot}(X, s[\tilde{\Omega}_X^{\cdot} \to \tilde{\Omega}_D^{\cdot}]),$$

where $\mathbf{H}^{\cdot}(X, s[\tilde{\Omega}_X^{\cdot} \to \tilde{\Omega}_D^{\cdot}]) \cong \mathbf{H}^{\cdot}(X, \underline{\underline{s}}[\underline{\underline{\Omega}}_X^{\cdot} \to \underline{\underline{\Omega}}_D^{\cdot}])$.

Remark 3.2. Definition 3.1 is more intrinsic, because it does not depend on the choice of a closed immersion of X into a smooth scheme P over S.

3.1. Applications

We give some applications of the previous definition of algebraic De Rham cohomology with compact supports.

A first simple example consists in a smooth and proper variety X over S, with log structure given by a normal crossing divisor $D \hookrightarrow X$. We consider $U = X_{triv} = X - D$. The De Rham cohomology with compact supports of U can be calculate by using the complex of formal differential forms $s[\Omega_X^{\cdot} \to \Omega_{X\hat{|}D}^{\cdot}]$, as in [1, Definition 1.2]. On the other hand, $\omega_X^{\cdot} = \Omega_X^{\cdot}(\log D)$ and $I_M\omega_X^{\cdot} =$

$\mathscr{I}(D)\Omega^{\cdot}_X(\log D)$, where $\mathscr{I}(D)$ is the ideal of definition of D in $\mathcal{O}_X$. Then, we have an exact sequence of complexes,

$$0 \to \mathscr{I}(D)\Omega^{\cdot}_X(\log D) \to \Omega^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D \to 0.$$

We can use the complex $s[\Omega^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D]$ to calculate $H^{\cdot}_{DR,c}(U/\mathbf{C})$. Indeed, in this case, the Ishida complex $\tilde{\Omega}^{\cdot}_D$ is easy to compute and it is isomorphic to the complex of differential forms "à la Sullivan", which is defined, for any $p \in \mathbf{N}$, as

$$\tilde{\Omega}^p_D = \mathrm{Ker}\{\epsilon_{0*}\Omega^p_{D_0} \rightrightarrows \varepsilon_{1*}\Omega^p_{D_1}\},$$

where D_0 is the normalization of D, $D_1 = D_0 \times_D D_0$, $\varepsilon_i : D_i \to D$, $i = 0, 1$ (see [6, Proposition 7.4]).

Other more interesting applications of Definition 3.1 can be given by log smooth log schemes X over S, whose underlying schemes are proper and quasi-smooth ([3, Definition 14.1]), namely whose all the local models for their toroidal underlying schemes are associated with simplicial cones. For example, we consider X to be the projectivization of the affine cone $\mathscr{C} = V(xy - z^2) \subset \mathbf{A}^3_{\mathbf{C}}$, which has a toric singularity at the origin. Let $U = X_{triv}$. The divisor $D = X - U = D_1 \cup D_2$ consists of two quasi-smooth components D_1, D_2, which intersect quasi-transversally. Locally at the origin, X is the toric variety associated to the simplicial cone $\sigma = \langle a_1, a_2 \rangle \subset \mathbf{R}^2$, generated by $a_1 = (1, 2)$ and $a_2 = (1, 0)$, i.e. $X = X_\sigma = \mathrm{Spec}\, \mathbf{C}[\sigma^\vee \cap P^{gp}] = \mathrm{Spec}\, \mathbf{C}[P]$, where P is the toric monoid associated to σ, and $\sigma^\vee$ is the dual cone of σ, which is generated by $m_1 = (0, 1)$ and $m_2 = (2, -1)$ in $\mathbf{R}^2$. If $m_3 = a_2 = (1, 0)$, we note that $2m_3 = m_1 + m_2$ in P and P^{gp} is free of rank 2.

If we follow the construction of algebraic De Rham cohomology with compact supports of U given in [1], we need to consider a closed immersion of X inside a smooth variety P over S and define $H^{\cdot}_{DR,c}(U/\mathbf{C})$ as the hypercohomology of the complex $s[\Omega^{\cdot}_{\hat{P|X}} \to \Omega^{\cdot}_{\hat{P|D}}]$, but we need to prove that this is independent on the choice of P. On the other hand, by using Definition 3.1, we can calculate $H^{\cdot}_{DR,c}(U/\mathbf{C})$ as the hypercohomology of the complex $s[\tilde{\Omega}^{\cdot}_X \to \tilde{\Omega}^{\cdot}_D]$. In this case, $\tilde{\Omega}^{\cdot}_X \cong v_*\Omega^{\cdot}_V$, where $V = X_{smooth}$ is the (classical) smooth locus of X and $v : V \hookrightarrow X$ is the open immersion. Moreover, by using the Poincaré Residue isomorphism given in [3, §15.7], we get an explicit description of the Ishida complex $\tilde{\Omega}^{\cdot}_D$ in terms of the Ishida complexes of the quasi-smooth normal closed subvarieties D_1, D_2 and $D_1 \cap D_2$.

Therefore, for a log smooth log scheme over S, whose underlying scheme is proper and quasi-smooth, and the divisor $D = X - X_{triv}$ consists of quasi-smooth components $D_1, \ldots, D_N$, that intersect quasi-transversally, we have good descriptions for $\tilde{\Omega}^{\cdot}_X$ and $\tilde{\Omega}^{\cdot}_D$, so it is more convenient to use Definition 3.1 to calculate $H^{\cdot}_{DR,c}(X_{triv}/\mathbf{C})$ instead of [1, Definition 1.2].

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