

## HARMONIC TOTAL CHERN FORMS AND STABILITY

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### Abstract

In this paper we will perturb the scalar curvature of compact Kähler manifolds by incorporating it with higher Chern forms, and then show that the perturbed scalar curvature has many common properties with the unperturbed scalar curvature. In particular the perturbed scalar curvature becomes a moment map, with respect to a perturbed symplectic structure, on the space of all complex structures on a fixed symplectic manifold, which extends the results of Donaldson and Fujiki on the unperturbed case.

### 1. Introduction

Many works have been done on the relationship between the existence of constant scalar curvature Kähler metrics and stability in the sense of geometric invariant theory. A way of seeing this relationship is through the moment map picture of an infinite dimensional set up as done by Donaldson [7] and Fujiki [9]. They showed that the set of all Kähler metrics with constant scalar curvature becomes the zero set of the moment map for the action of the group of Hamiltonian symplectomorphisms on the space of all compatible complex structures on a fixed symplectic manifold. Recall that for a Hamiltonian action of a compact Lie group  $K$  on a compact Kähler manifold, having a zero of the moment map along an orbit of the complexified group  $K^c$ -action is equivalent to the stability of the orbit of the reductive group  $K^c$  (c.f. [8], section 6.5). Applying this fact in finite dimensions to the infinite dimensional space of all compatible complex structures we see a relationship between the existence of constant scalar curvature Kähler metrics and infinite dimensional symplectic-GIT stability.

The purpose of this paper is to perturb the scalar curvature by incorporating it with higher Chern classes, and show that the perturbed scalar curvature shares many common properties with the unperturbed scalar curvature. Especially the set of all Kähler metrics with constant perturbed scalar curvature is the zero set of the moment map with respect to a perturbed symplectic form on the space of

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all compatible complex structures on a fixed symplectic manifold. This extends the earlier results of Donaldson and Fujiki in the unperturbed case.

Let  $M$  be a compact symplectic manifold with a fixed symplectic form  $\omega$  and of dimension  $2m$ . Let  $\mathcal{J}$  be the set of all  $\omega$ -compatible integrable complex structures. Then for each  $J \in \mathcal{J}$ ,  $(M, \omega, J)$  becomes a Kähler manifold. For a pair  $(J, t)$  of a complex structure  $J$  and a small real number  $t$ , define a smooth function  $S(J, t)$  on  $M$  by

$$(1) \quad \frac{S(J, t)}{2m\pi} \omega^m = c_1(J) \wedge \omega^{m-1} + tc_2(J) \wedge \omega^{m-2} + \dots + t^{m-1}c_m(J)$$

where  $c_i(J)$  is the  $i$ -th Chern form with respect to the Kähler structure  $(\omega, J)$  on  $M$ , i.e. they are defined by

$$(2) \quad \det\left(I + \frac{i}{2\pi}t\Theta\right) = 1 + tc_1(J) + \dots + t^m c_m(J),$$

$\Theta$  being the curvature matrix of the Levi-Civita connection. Note that  $S(J, 0)$  is equal to the trace of the Ricci curvature  $g^{i\bar{j}}R_{i\bar{j}}$  which is one half of the Riemannian scalar curvature. But since  $S(J, 0)$  more often appears in the computations in Kähler geometry than the Riemannian scalar curvature does, we will call  $S(J, 0)$  the scalar curvature in this paper. We also call  $S(J, t)$  the *perturbed scalar curvature*. As mentioned above the main result of this paper is to show that the perturbed scalar curvature becomes a moment map on  $\mathcal{J}$  with respect to some symplectic structure (Theorem 2.2 in the next section).

This paper is organized as follows. In section 2, we will prove Theorem 2.2. We will give two proofs along the lines of [7] and [21]. In section 3, we study the analogy to extremal Kähler metrics in our perturbed case. We will see that the perturbed extremal Kähler metrics are critical points of the functional on  $\mathcal{J}$  given by the squared  $L^2$ -norm of the perturbed scalar curvature but not critical points of the functional on the space of Kähler forms given by the same integral. In section 4 we will recall Bando’s result [1] on the obstructions to the existence of Kähler metrics with harmonic higher Chern classes and study the relevant Mabuchi functional in the perturbed case. In section 5, we will give a deformation theory of extremal Kähler metrics to the perturbed extremal Kähler metrics extending earlier results of LeBrun and Simanca [18], [19].

## 2. Perturbed symplectic structure on the space of complex structures

Let  $(M, \omega)$  be a compact symplectic manifold of dimension  $2m$  and  $\mathcal{J}$  the space of all  $\omega$ -compatible complex structures on  $M$ . This means that  $J \in \mathcal{J}$  if and only if  $\omega(JX, JY) = \omega(X, Y)$  for all vector fields  $X$  and  $Y$ , and  $\omega(X, JX) > 0$  for all non-zero  $X$ . For later purposes it is convenient to assume that  $J$  acts on the cotangent bundle rather than the tangent bundle. Fixing  $J \in \mathcal{J}$ , we decompose the complexified cotangent bundle into holomorphic and anti-holomorphic parts, i.e.  $\pm\sqrt{-1}$ -eigenspaces of  $J$ :

$$(3) \quad T^*M \otimes \mathbf{C} = T_{J'}^*M \oplus T_{J''}^*M, \quad T_{J''}^*M = \overline{T_{J'}^*M}.$$

Taking arbitrary  $J' \in \mathcal{J}$  we also have the decomposition with respect to  $J'$

$$(4) \quad T^*M \otimes \mathbf{C} = T_{J'}^*M \oplus T_{J''}^*M, \quad T_{J''}^*M = \overline{T_{J'}^*M}.$$

If  $J'$  is sufficiently close to  $J$  then  $T_{J'}^*M$  can be expressed as a graph over  $T_{J''}^*M$  as

$$(5) \quad T_{J'}^*M = \{\alpha + \mu(\alpha) \mid \alpha \in T_{J''}^*M\}$$

for some endomorphism  $\mu$  of  $T_{J''}^*M$  into  $T_{J''}^*M$ :

$$(6) \quad \begin{aligned} \mu &\in \Gamma(\text{End}(T_{J''}^*M, T_{J''}^*M)) \\ &\cong \Gamma(T_{J'}^*M \otimes T_{J''}^*M) \cong \Gamma(T_{J'}^*M \otimes T_{J'}^*M) \end{aligned}$$

where in the last identification we used the Kähler metric defined by the pair  $(\omega, J)$ . This can be expressed in the notation of tensor calculus with indices as

$$\mu_{\bar{k}}^i \mapsto g^{j\bar{k}} \mu_{\bar{k}}^i =: \mu^{ij}$$

where we chose a local holomorphic coordinate system  $(z^1, \dots, z^m)$  and wrote  $\omega$  as  $\omega = \sqrt{-1}g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ .

LEMMA 2.1. *With the above identification understood,  $\mu$  lies in the symmetric part  $\Gamma(\text{Sym}(T_{J'}^*M \otimes T_{J'}^*M))$  of  $\Gamma(T_{J'}^*M \otimes T_{J'}^*M)$ .*

*Proof.* The symplectic form  $\omega$  gives a natural identification between the tangent bundle and the cotangent bundle. This identification then gives a natural symplectic structure on the cotangent bundle, which we denote by  $\omega^{-1}$ . If  $\omega$  is  $J$ -invariant, then  $\omega^{-1}$  is also  $J$ -invariant. For the complex structure  $J$ ,  $\omega^{-1}$  is expressed in terms of the Kähler metric of the Kähler structure  $(\omega, J)$  as

$$\omega^{-1} = -\sqrt{-1}g^{i\bar{j}} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial \bar{z}^{\bar{j}}},$$

where we used the local expression of  $\omega$  as above. Since  $\omega^{-1}$  is  $J$ -invariant and any 1-forms  $\alpha$  and  $\beta$  in  $T_{J'}^*M$  are eigenvectors of  $J$  belonging to  $\sqrt{-1}$ , we have

$$\omega^{-1}(\alpha, \beta) = 0.$$

Similarly we have

$$\omega^{-1}(\mu\alpha, \mu\beta) = 0$$

and, since  $\omega^{-1}$  is also  $J'$ -invariant, we also have

$$\omega^{-1}(\alpha + \mu\alpha, \beta + \mu\beta) = 0.$$

Thus we obtain

$$(7) \quad \omega^{-1}(\alpha, \mu\beta) = \omega^{-1}(\beta, \mu\alpha)$$

which implies that  $\mu \in \Gamma(T'_J M \otimes T'_J M)$  is symmetric because in the local expression,

$$(8) \quad \mu^{ji} \alpha_i \beta_j = \mu^{ij} \alpha_i \beta_j,$$

as desired. □

Considered infinitesimally, the tangent space  $T_J \mathcal{J}$  to  $\mathcal{J}$  at  $J$  is a subspace of  $\text{Sym}(T'_J M \otimes T'_J M)$ .

Then the  $L^2$ -inner product on  $\text{Sym}(T'_J M \otimes T'_J M)$  gives  $\mathcal{J}$  a Kähler structure. But we perturb this Kähler structure in the following way. Let  $t$  be a small real number. For  $\mu$  and  $\nu$  in the tangent space  $T_J \mathcal{J}$ , we define

$$(9) \quad (\nu, \mu)_t = \int_M c_m \left( \bar{v}_{jk} \mu_{\bar{l}}^i \frac{\sqrt{-1}}{2\pi} dz^k \wedge d\bar{z}^{\bar{l}}, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right)$$

where  $c_m$  is the polarization of the determinant viewed as a  $GL(m, \mathbf{C})$ -invariant polynomial, i.e.  $c_m(A_1, \dots, A_m)$  is the coefficient of  $m! t_1 \dots t_m$  in  $\det(t_1 A_1 + \dots + t_m A_m)$ , where  $I$  denotes the identity matrix and  $\Theta = \bar{\partial}(g^{-1} \partial g)$  is the curvature form of the Levi-Civita connection, and where  $u_{jk} \mu_{\bar{l}}^i$  should be understood as the endomorphism of  $T'_J M$  which sends  $\partial/\partial z^j$  to  $u_{jk} \mu_{\bar{l}}^i \partial/\partial z^i$ .

Note that

$$c_m(A, \dots, A) = \det A.$$

This is similar to the wedge product

$$\alpha_1 \wedge \dots \wedge \alpha_m$$

for the type  $(1, 1)$ -forms  $\alpha_1, \dots, \alpha_m$ . For we have

$$\alpha \wedge \dots \wedge \alpha = \det(a_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^m \wedge d\bar{z}^m$$

when  $\alpha = \sum a_{i\bar{j}} dz^i \wedge d\bar{z}^j$ . Therefore there is a symmetry between the endomorphism part and the form part in the integration of (9). This symmetry will be used in this work and was used in the work of Bando [1] quoted in the next section.

When  $t = 0$ ,  $(\cdot, \cdot)_t$  gives the standard  $L^2$ -inner product which is anti-linear in the first factor  $\nu$  and linear in the second factor  $\mu$ . If the real number  $t$  is sufficiently small,  $(\cdot, \cdot)_t$  is still positive definite.

Let  $\mathcal{G}$  be the group of all Hamiltonian symplectomorphisms of  $(M, \omega)$ . The Lie algebra of  $\mathcal{G}$  is isomorphic to the Poisson algebra  $C_0^\infty(M)$  of all smooth functions on  $M$  with average 0:

$$C_0^\infty(M) = \left\{ u \in C^\infty(M) \mid \int_M u \omega^m = 0 \right\}.$$

$\mathcal{G}$  acts on  $\mathcal{J}$  as holomorphic isometries.

**THEOREM 2.2.** *For each fixed small real number  $t$ ,  $S(J, t)/2m\pi$  gives an equivariant moment map on  $\mathcal{J}$  if we consider  $S(J, t)/2m\pi$  as an element of the dual space of  $C_0^\infty(M)$  by the pairing*

$$\left\langle \frac{S(J, t)}{2m\pi}, u \right\rangle = \int_M u \frac{S(J, t)}{2m\pi} \omega^m.$$

The case  $t=0$  is due to Donaldson ([7]) and Fujiki ([9]), and a mildly different proof in this case was also given in Tian’s book [21].

To prove the theorem, let us consider two operators

$$P : C_0^\infty \rightarrow T_J \mathcal{J},$$

$$Q : T_J \mathcal{J} \rightarrow C_0^\infty(M),$$

where  $P$  represents the infinitesimal action of the Lie algebra  $C_0^\infty$  on  $\mathcal{J}$  via Hamiltonian action and  $Q$  represents the derivative of the map which associates to  $J \in \mathcal{J}$  the perturbed scalar curvature  $\frac{1}{2m\pi}S(J, t)$  of the Kähler manifold  $(M, \omega, J)$ . We need to show

$$\Re(P(u), \sqrt{-1}\mu)_t = \langle Q(\mu), u \rangle$$

To compute  $P(u)$ , we have only to compute  $L_X J$  for a smooth vector field  $X$ .

**LEMMA 2.3.** *For a smooth vector field  $X = X' + X''$  we have*

$$L_X J = 2\sqrt{-1}\nabla_J'' X' - 2\sqrt{-1}\nabla_J' X''.$$

*In particular, if  $X_u$  is the Hamiltonian vector field of  $u$ ,*

$$P(u) = 2\sqrt{-1}\nabla_J'' X_u'.$$

*Proof.* Since  $(L_X J)\alpha = L_X(J\alpha) - JL_X\alpha$ , if  $\alpha$  is a type  $(1, 0)$ -form,

$$(10) \quad (L_X J)\alpha = \sqrt{-1}(L_X\alpha - (L_X\alpha)^{1,0} + (L_X\alpha)^{0,1}) = 2\sqrt{-1}(L_X\alpha)^{0,1}.$$

On the other hand

$$(11) \quad L_X\alpha = d(\alpha(X')) + i(X)(\partial_J\alpha + \bar{\partial}_J\alpha).$$

Thus

$$(12) \quad (L_X\alpha)^{0,1} = \bar{\partial}_J(\alpha(X')) + i(X')\bar{\partial}_J\alpha.$$

But

$$\begin{aligned} \bar{\partial}_J(\alpha(X')) &= \nabla_J''(\alpha(X')) \\ &= (\nabla_J''\alpha)(X') + \alpha(\nabla_J''X') = (\bar{\partial}_J\alpha)(X') + \alpha(\nabla_J''X'). \end{aligned}$$

This implies

$$(13) \quad \bar{\partial}_J(\alpha(X')) + i(X')(\bar{\partial}_J\alpha) = \alpha(\nabla_J''X')$$

From (10), (12) and (13) we get

$$(14) \quad (L_X J)\alpha = \alpha(2\sqrt{-1}\nabla_J'' X').$$

Similarly, if  $\alpha$  is a  $(0, 1)$ -form, then

$$(15) \quad (L_X J)\alpha = \alpha(-2\sqrt{-1}\nabla_J' X'').$$

From (14) and (15) we get the lemma. This completes the proof. □

From this lemma we get for the real function  $u$

$$(16) \quad \Re(P(u), \sqrt{-1}\mu)_t = 2\Re(\nabla_J'' X'_u, \mu)_t \\ = 2\Re \int_M mc_m \left( u_{jk} \mu_{\bar{l}}^i \frac{\sqrt{-1}}{2\pi} dz^k \wedge d\bar{z}^{\bar{l}}, \right. \\ \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right).$$

Next we need to compute  $Q$ . We will do this in two ways along the lines of [7] and [21]. First we follow the arguments of [7] just word for word.

If identify  $T_{J'}^* M$  with  $T_J^* M$  through  $\alpha + \mu\alpha \mapsto \alpha$ , this identification induces identifications of differential forms with all degrees, which we denote by  $\iota : \Omega_{J'}^{p,q} \rightarrow \Omega_J^{p,q}$ .

LEMMA 2.4. *With the above identification we have the following.*

- (a) *If a 1-form  $\gamma = \alpha + \bar{\beta} \in T_{J'}^* M \oplus T_J^* M$  is written also as  $\gamma = \alpha' + \mu\alpha' + \bar{\beta}' + \mu\bar{\beta}' \in T_{J'}^* M \oplus T_J^* M$  then*

$$\bar{\beta}' = \bar{\beta} - \mu\alpha$$

*up to first order in  $\mu$ . Namely*

$$\iota(\overline{\beta' + \mu\beta'}) = \bar{\beta} - \mu\alpha$$

*up to first order in  $\mu$ .*

- (b) *If a fixed 2-form  $\chi = \chi^{2,0} + \chi^{1,1} + \chi^{0,2} \in \Omega_J^{2,0} \oplus \Omega_J^{1,1} \oplus \Omega_J^{0,2}$  has  $\chi^{1,1}$  as a  $(1, 1)$ -component with respect to  $J'$ , then*

$$\iota(\chi^{1,1}) = \chi^{1,1} - \mu\chi^{2,0} - \mu\chi^{0,2}$$

*up to first order in  $\mu$ , where we extended the operation of  $\mu$  to higher degree tensors in the obvious way.*

Hereafter we use the notation  $\equiv$  to mean “up to first order in  $\mu$ ”.

*Proof.* (a) From  $\alpha' = \alpha - \overline{\mu\beta'}$  we see

$$\bar{\beta}' = \bar{\beta} - \mu\alpha' = \bar{\beta} - \mu(\alpha - \overline{\mu\beta'}) \equiv \bar{\beta} - \mu\alpha.$$

(b) If a fixed 2-form is written also as  $\chi = (1 + \mu)\alpha_1 \wedge (1 + \mu)\alpha_2 + (1 + \mu)\alpha_3 \wedge (1 + \mu)\beta_1 + (1 + \mu)\beta_2 \wedge (1 + \mu)\beta_3 \in \Omega_{J'}^{2,0} \oplus \Omega_{J'}^{1,1} \oplus \Omega_{J'}^{0,2}$ , then a similar computation as in the proof of (a) shows

$$\begin{aligned} \alpha_3 \wedge \bar{\beta}_1 &\equiv \chi^{1,1} - \alpha_1 \wedge \mu\alpha_2 - \mu\alpha_1 \wedge \alpha_2 - \overline{\mu\beta_1 \wedge \beta_2} - \overline{\beta_1 \wedge \mu\beta_2} \\ &\equiv \chi^{1,1} - \mu\chi^{2,0} - \mu\chi^{0,2}. \end{aligned}$$

This completes the proof. □

**COROLLARY 2.5.** *Let  $E \rightarrow M$  be a vector bundle. If  $\nabla$  is a fixed connection of  $E$  and  $\nabla = \nabla'_J + \nabla''_J$  with respect to the complex structure  $J$ , then by the identification above  $\nabla''_J$  is identified with  $\nabla''_J - \mu\nabla'_J$  up to first order in  $\mu$ .*

*Proof of Theorem 2.2.* The identification  $\iota : T_{J'}^*M \rightarrow T_J^*M$  is a Hermitian isometry up to first order in  $\mu$ , and we can consider the Levi-Civita connections  $\nabla_J$  and  $\nabla_{J'}$  as two unitary connections on the same bundle. If  $J$  is fixed and  $\nabla''$  is varied by  $\sigma \in \Omega^{0,1}(\text{End}(T^*M))$  then the connection changes by  $\sigma - \sigma^*$ . On the other hand, if a connection  $\nabla = \nabla'_J + \nabla''_J$  is fixed and  $J$  varies to  $J'$  by  $\mu$ , then the new  $\nabla''_{J'}$  is identified with  $\nabla''_J - \mu\nabla'_J$  up to first order in  $\mu$  by Corollary 2.5.

Now we compute  $\nabla''_{J'}\alpha$  for a 1-form  $\alpha$  of  $T_{J'}^*M$ , which is strictly speaking equal to  $\iota \circ \nabla''_J \circ \iota^{-1}(\alpha)$ . But  $\nabla''_J \circ \iota^{-1}(\alpha)$  is  $\Omega_{J'}^{0,1}$ -part of  $d(\alpha + \mu\alpha)$  up to first order in  $\mu$ . From this and Lemma 2.4, (b), we get

$$(17) \quad \nabla''_{J'}\alpha \equiv \nabla''_J\alpha + \nabla'_J(\mu\alpha) - \mu(\nabla'_J\alpha).$$

On  $T_{J'}^*M \otimes T_{J'}^*M$ ,  $\mu$  acts as a derivation. To make the notations clear we will denote by  $\mu_1$  (resp.  $\mu_2$ ) the action of  $\mu$  on the first (resp. second) factor. So, on  $T_{J'}^*M \otimes T_{J'}^*M$ , we have  $\mu = \mu_1 \otimes 1 + 1 \otimes \mu_2$ . With these notations the right hand side of (17) is equal to

$$(18) \quad \begin{aligned} \nabla''_J\alpha + \mu_2\nabla'_J\alpha + (\nabla'_J\mu)\alpha - \mu(\nabla'_J\alpha) &\equiv \nabla''_J\alpha - \mu_1\nabla'_J\alpha + (\nabla'_J\mu)\alpha \\ &\equiv (\nabla''_J - \mu\nabla'_J)\alpha + (\nabla'_J\mu)\alpha. \end{aligned}$$

By Corollary 2.5,  $\nabla''_J - \mu\nabla'_J$  is the expression under our identification of  $J'$ -(0,1)-component of a fixed connection  $\nabla_J$ . Thus the variation of the Levi-Civita connection is  $\sigma - \sigma^*$  where  $\sigma = \nabla'_J\mu$ . Notice that  $\sigma$  must be a (0,1)-form with values in  $\text{End}(T_J^*M)$ . So, in local expressions

$$\nabla'_J\mu = (\nabla_j\mu_j^i dz^{\bar{j}})$$

with  $i$  column index,  $j$  row index. Since it is convenient to distinguish the covariant derivative as the endomorphism part from the covariant exterior derivative as the form part, we shall write  $\nabla_J$  to denote the covariant derivative as the endomorphism part and  $d^{\nabla_J}$  to denote the covariant exterior derivative as the form part. Thus, under the variation  $\delta J = \mu$  of the complex structure, the variation  $\delta\Theta$  of the curvature matrix  $\Theta$  is

$$\delta\Theta = d^{\nabla_J}(\sigma - \sigma^*).$$

Its (1, 1)-part is

$$(\delta\Theta)^{1,1} = d^{\nabla'_J}(\nabla'_J\mu) - (d^{\nabla'_J}(\nabla'_J\mu))^*.$$

Since the exterior covariant derivative  $d^{\nabla'_J} \left( \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right)$  of  $\omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta$  vanishes, we have

$$\begin{aligned} & \delta \int_M u \frac{S(J, t)}{2m\pi} \omega^m \\ &= 2\Re \int_M umc_m \left( \frac{\sqrt{-1}}{2\pi} d^{\nabla'_J}(\nabla'_J\mu), \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right) \\ &= -2\Re \int_M mc_m \left( \frac{\sqrt{-1}}{2\pi} d^{\nabla''_J}u \wedge \nabla'_J\mu, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right) \end{aligned}$$

Now the invariant polynomial  $c_m$  takes determinant for the endomorphism part, and therefore we may interchange the roles of the form part and the endomorphism part in the integration above. Thus by the vanishing of  $d^{\nabla'_J} \left( \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right)$  again we can use integration by parts for the covariant derivative of the endomorphism part. Hence we have

$$\begin{aligned} & \delta \int_M u \frac{S(J, t)}{2m\pi} \omega^m \\ &= 2\Re \int_M mc_m \left( \frac{\sqrt{-1}}{2\pi} \nabla''_J d^{\nabla''_J}u \wedge \mu, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right). \end{aligned}$$

where the term  $\frac{\sqrt{-1}}{2\pi} \nabla''_J d^{\nabla''_J}u \wedge \mu$  is expressed in local coordinates

$$\frac{\sqrt{-1}}{2\pi} u_{kj} dz^k \wedge \mu^i_{\bar{l}} dz^{\bar{l}},$$

where  $u_{kj} = \nabla_j \nabla_{\bar{k}} u$ . This coincides with (16), completing the proof of Theorem 2.2.

*Alternate proof of Theorem 2.2.* We only need to show that  $\langle Q(\mu), u \rangle$  is equal to (16). To compute  $Q$  we take a local coordinates  $(x^1, \dots, x^{2m})$  with respect to which  $\omega$  is the standard symplectic form on  $\mathbf{R}^{2m}$ , by using Darboux's theorem. Let  $J_t$  be a family of complex structures with  $J_0 = J$ . Then we have

$$\dot{J}|_{t=0} = 2\sqrt{-1}\mu - 2\sqrt{-1}\bar{\mu}.$$

This follows because, by taking the derivative of

$$J_t(\alpha + \mu(t)\alpha) = \sqrt{-1}(\alpha + \mu(t)\alpha)$$

with  $\dot{\mu}(0) = \mu$ , we have

$$\dot{J}(\alpha) = 2\sqrt{-1}\mu.$$

Let  $g_t = \omega J_t$  be the Riemannian metric induced by  $J_t$ . Then the Christoffel symbols of  $g_t$  are written as

$$\Gamma_{t,jk}^i = \frac{1}{2}g_t^{i\ell} \left( \frac{\partial g_{t,\ell j}}{\partial x^k} + \frac{\partial g_{t,\ell k}}{\partial x^j} - \frac{\partial g_{t,jk}}{\partial x^\ell} \right).$$

At  $p \in M$  we may assume that  $g_{ij}(p) = \delta_{ij}$ ,  $dg_{ij}(p) = 0$ , and

$$J(p) = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$$

where  $g = g_0$ . Then  $\Gamma_{t,jk}^i$  is of order  $t$ , and

$$\begin{aligned} R_{t,ijkl} &= g_t \left( \nabla_{\partial/\partial x^i} \nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^\ell} - \nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right) \\ &= g_{t,sk} \frac{1}{2} \left( \frac{\partial^2 g_{t,pj}}{\partial x^i \partial x^\ell} - \frac{\partial^2 g_{t,j\ell}}{\partial x^i \partial x^p} - \frac{\partial^2 g_{t,pi}}{\partial x^j \partial x^\ell} + \frac{\partial^2 g_{t,i\ell}}{\partial x^j \partial x^p} \right) g_t^{ps} \\ &\quad + \text{quadratic terms in the first derivatives of } g. \end{aligned}$$

Taking the derivative with respect to  $t$  at  $t = 0$ ,

$$\left. \frac{d}{dt} \right|_{t=0} R_{t,ijkl} = \frac{1}{2} (\dot{g}_{kj,i\ell} - \dot{g}_{j\ell,ik} - \dot{g}_{ki,j\ell} + \dot{g}_{i\ell,jk}).$$

Now we compute the right hand side in terms of local holomorphic coordinates  $z^1, \dots, z^m$ . The only terms involved in the integration are  $\dot{g}_{\bar{i},jk}$ 's and their complex conjugates, and we also have

$$\dot{g}_{\bar{i}} = -\sqrt{-1}g_{\bar{i}p}2\sqrt{-1}\mu_1^p = 2\mu_{\bar{i}}.$$

Thus

$$\frac{1}{2}\dot{g}_{\bar{i},jk}\sqrt{-1}dz^k \wedge d\bar{z}^\ell = \mu_{\bar{i},jk}\sqrt{-1}dz^k \wedge d\bar{z}^\ell.$$

Hence we get

$$\begin{aligned} &\langle Q(\mu), u \rangle \\ &= 2\Re \int_M umc_m \left( \frac{\sqrt{-1}}{2\pi} d^{\nabla_j}(\nabla_j \mu), \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta \right). \end{aligned}$$

As in the last part of the previous proof this last term coincides with (16). This completes the alternate proof.

**3. Perturbed extremal Kähler metrics**

For a real or complex valued smooth function  $u$  on a Kähler manifold  $(M, g)$  we put

$$\text{grad}' u = \sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial u}{\partial z^{\bar{j}}} \frac{\partial}{\partial z^i}$$

and call it the gradient vector field of  $u$ . Strictly speaking the real part of  $\text{grad}' u$  is the gradient vector field of  $u$ , but we identify a real vector field with its  $T'M$ -part.

**DEFINITION 3.1.** *A Kähler metric  $g = (g_{i\bar{j}})$  is said to be a perturbed extremal Kähler metric if the gradient vector field*

$$\text{grad}' S(J, t) = \sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial S(J, t)}{\partial z^{\bar{j}}} \frac{\partial}{\partial z^i}$$

*of the perturbed scalar curvature  $S(J, t)$  is a holomorphic vector field.*

**PROPOSITION 3.2.** *Critical points of the functional*

$$J \mapsto \int_M S(J, t)^2 \omega^m$$

*on  $\mathcal{J}$  are perturbed extremal Kähler metrics.*

*Proof.* Let  $J(s)$  be a smooth family of complex structures such that  $J(0) = J$  and  $\dot{J}(0) = \mu$ . By the proof of Theorem 2.2

$$\left. \frac{d}{ds} \right|_{s=0} \int_M u S(J(s), t) \omega^m = 2m\pi \Re(\nabla'' \nabla'' u, \mu)_t$$

for all real smooth function  $u$  with  $\int_M u \omega^m = 0$ . We take  $u$  to be  $v := S(J, t) - \int_M S(J, t) \omega^m / \int_M \omega^m$  and  $\mu$  to be  $(-\sqrt{-1})$ -times the infinitesimal action of the Hamiltonian vector field of  $v$  at  $J$ . Then using the above equality and Lemma 2.3

$$\left. \frac{d}{ds} \right|_{s=0} \int_M v S(J(s), t) = 2m\pi \Re(\nabla'' \nabla'' u, \mu)_t.$$

From this we get

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=0} \int_M S(J(s), t)^2 \omega^m &= 2 \int_M S(J, t) \left. \frac{d}{ds} \right|_{s=0} S(J(s), t) \omega^m \\ &= 2 \int_M v \left. \frac{d}{ds} \right|_{s=0} S(J(s), t) \omega^m \\ &= 4m\pi \Re(\nabla'' \nabla'' u, \mu)_t. \end{aligned}$$

This shows that  $J$  is a critical point if and only if

$$\nabla'' \operatorname{grad}' S(J, t) = 0,$$

i.e. the Kähler metric of  $(M, \omega, J)$  is a perturbed extremal Kähler metric.  $\square$

*Remark 3.3.* In the case of unperturbed extremal Kähler metrics when  $t = 0$ , such Kähler metrics are also the critical points of the functional

$$\omega \mapsto \int_M S(\omega)^2 \omega^m$$

on the space of all Kähler forms  $\omega$  in a fixed Kähler class  $[\omega_0]$  where  $S(\omega)$  denotes the scalar curvature of the Kähler form  $\omega$ , (c.f. [4]). But when  $t \neq 0$  the perturbed extremal Kähler metrics are not the critical points of the functional

$$\omega \mapsto \int_M S(\omega, t)^2 \omega^m$$

on the space of all Kähler forms in a fixed Kähler class where

$$(19) \quad \begin{aligned} \frac{S(\omega, t)}{2m\pi} \omega^m &= c_1(\omega) \wedge \omega^{m-1} + tc_2(\omega) \wedge \omega^{m-2} + \cdots + t^{m-1} c_m(\omega) \\ &= \frac{1}{t} \left( \det \left( \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) - \omega^m \right), \end{aligned}$$

$c_j(\omega)$  being the  $j$ -th Chern form with respect to  $\omega$ :

$$\det \left( 1 + t \frac{\sqrt{-1}}{2\pi} \Theta \right) = 1 + tc_1(\omega) + \cdots + t^{m-1} c_m(\omega).$$

Note that we use the notation  $S(\omega, t)$  instead of  $S(J, t)$  to emphasize that  $\omega$  is varied now.

*Proof of Remark 3.3.* Let  $\omega + \delta\omega$  be a variation of the Kähler form in a fixed Kähler class. Then  $\delta\omega = \sqrt{-1} \partial \bar{\partial} \varphi$  for some real smooth function  $\varphi$ . By (19) the variation  $\delta S(\omega, t)$  of the perturbed scalar curvature is given by

$$\begin{aligned} & \frac{\delta S(\omega, t)}{2m\pi} \omega^m + \frac{S(\omega, t)}{2m\pi} \Delta \varphi \omega^m \\ &= \frac{1}{t} \left( mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I + \frac{\sqrt{-1}}{2\pi} t \delta \Theta, \right. \right. \\ & \quad \left. \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) - \Delta \varphi \omega^m \right) \\ &= mc_m \left( \frac{\sqrt{-1}}{2\pi} \delta \Theta, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \end{aligned}$$

$$\begin{aligned}
& + mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
& + \dots + mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I, \dots, \omega \otimes I \right).
\end{aligned}$$

Thus

$$\begin{aligned}
& \frac{1}{2m\pi} \delta(S(\omega, t)^2 \omega^m) \\
& = 2S(\omega, t) mc_m \left( \frac{\sqrt{-1}}{2\pi} \delta \Theta, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
& \quad + 2S(\omega, t) mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
& \quad + \dots + 2S(\omega, t) mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I, \dots, \omega \otimes I \right) \\
& \quad - \frac{1}{2m\pi} S(\omega, t)^2 \Delta \varphi \omega^m.
\end{aligned}$$

Since  $\delta \Theta = \nabla'' \nabla'(\varphi^i)$  we have

$$\begin{aligned}
(20) \quad & \frac{1}{2m\pi} \delta \int_M S(\omega, t)^2 \omega^m \\
& = 2 \int_M S(\omega, t) mc_m \left( \nabla_{\bar{l}} \nabla_k(\varphi^i) \frac{\sqrt{-1}}{2\pi} dz^{\bar{l}} \wedge dz^k, \right. \\
& \quad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
& \quad + 2 \int_M S(\omega, t) mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \right. \\
& \quad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
& \quad + \dots + 2 \int_M S(\omega, t) mc_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I, \dots, \omega \otimes I \right) \\
& \quad - \frac{1}{2m\pi} \int_M S(\omega, t)^2 \Delta \varphi \omega^m
\end{aligned}$$

But

$$\begin{aligned}
(21) \quad \nabla_{\bar{\ell}} \nabla_k \varphi_j^i &= \nabla_{\bar{\ell}} \nabla_k \nabla^i \varphi_j \\
&= \nabla_{\bar{\ell}} \nabla^i \nabla_k \varphi_j - \nabla_{\bar{\ell}} (R_{jk}^{pi} \varphi_p) \\
&= \nabla_{\bar{\ell}} \nabla^i \nabla_k \varphi_j - (\nabla_{\bar{\ell}} R_{jk}^p i) \varphi_p - R_{jk}^p i \varphi_{p\bar{\ell}} \\
&= \nabla_{\bar{\ell}} \nabla^i \nabla_k \varphi_j - (\nabla^p R_{\bar{\ell}jk}^i) \varphi_p - R_{jk}^p i \varphi_{p\bar{\ell}}
\end{aligned}$$

where we used the second Bianchi identity at the last equality. It follows from (20) and (21) that

$$\begin{aligned}
(22) \quad &\frac{1}{2m\pi} \delta \int_M S(\omega, t)^2 \omega^m \\
&= -2 \int_M S(\omega, t) m c_m \left( (\nabla_{\bar{\ell}} \nabla^i \nabla_k \varphi_j - \varphi_p \nabla^p R_{\bar{\ell}jk}^i - R_{jk}^{pi} \varphi_{p\bar{\ell}}) \frac{\sqrt{-1}}{2\pi} dz^k \wedge dz^{\bar{\ell}}, \right. \\
&\quad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
&+ 2 \int_M S(\omega, t) m c_m \left( \sqrt{-1} \delta \bar{\delta} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \right. \\
&\quad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
&+ \dots + 2 \int_M S(\omega, t) m c_m \left( \sqrt{-1} \delta \bar{\delta} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I, \dots, \omega \otimes I \right) \\
&- \frac{1}{2m\pi} \int_M S(\omega, t)^2 \Delta \varphi \omega^m
\end{aligned}$$

But

$$R_{\bar{\ell}jk}^i \frac{\sqrt{-1}}{2\pi} dz^k \wedge dz^{\bar{\ell}} = R_{k\bar{\ell}j}^i \frac{\sqrt{-1}}{2\pi} dz^k \wedge dz^{\bar{\ell}} = \frac{\sqrt{-1}}{2\pi} \Theta.$$

From this and integration by parts

$$\begin{aligned}
(23) \quad &2 \int_M S(\omega, t) m c_m \left( \varphi_p \nabla^p R_{\bar{\ell}jk}^i \frac{\sqrt{-1}}{2\pi} dz^k \wedge dz^{\bar{\ell}}, \right. \\
&\quad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
&= -\frac{1}{2m\pi} \int_M S(\omega, t)^2 \Delta \varphi \omega^m.
\end{aligned}$$

It follows from (22) and (23) that

$$\begin{aligned}
 & \frac{1}{2m\pi} \delta \int_M S(\omega, t)^2 \omega^m \\
 &= -2 \int_M S(\omega, t) m c_m \left( \left( \nabla_{\bar{z}} \nabla^i \nabla_k \varphi_j \frac{\sqrt{-1}}{2\pi} dz^k \wedge d\bar{z}^{\bar{l}}, \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \right) \\
 (24) \quad & + 2 \int_M S(\omega, t) m c_m \left( R_{jk}^{pi} \varphi_{p\bar{l}} \frac{\sqrt{-1}}{2\pi} dz^k \wedge d\bar{z}^{\bar{l}}, \right. \\
 & \qquad \qquad \qquad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
 (25) \quad & + 2 \int_M S(\omega, t) m c_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \right. \\
 & \qquad \qquad \qquad \left. \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t \Theta \right) \\
 & + \dots + 2 \int_M S(\omega, t) m c_m \left( \sqrt{-1} \partial \bar{\partial} \varphi \otimes I, \frac{\sqrt{-1}}{2\pi} \Theta, \omega \otimes I, \dots, \omega \otimes I \right) \\
 (26) \quad & - \frac{1}{m\pi} \int_m S(\omega, t)^2 \Delta \varphi \omega^m
 \end{aligned}$$

When  $t = 0$  this is equal to

$$\begin{aligned}
 (27) \quad & \frac{1}{2m\pi} \delta \int_M S^2 \omega^m = -2 \int_M S \bar{D} \varphi \omega^m + 2 \int_M S \sum_{i,j=1}^m \frac{1}{2\pi} R_{i\bar{j}} \varphi^{i\bar{j}} \omega^m \\
 & + 2 \int_M S \sum_{i \neq j} \varphi_{i\bar{i}} \frac{1}{2\pi} R_{j\bar{j}} \omega^m - \frac{1}{m\pi} \int_m S^2 \Delta \varphi \omega^m
 \end{aligned}$$

with  $D = \nabla_i \nabla_j \nabla^i \nabla^j$  where  $S = S(\omega, 0)$  is the unperturbed scalar curvature and we used the normal coordinates such that the complex Hessian  $(\varphi_{i\bar{j}})$  is diagonalized. The third term on the right hand side can then be computed using

$$\begin{aligned}
 \sum_{i \neq j} \varphi_{i\bar{i}} \frac{1}{2\pi} R_{j\bar{j}} &= \left( \sum_{i=1}^m \varphi_{i\bar{i}} \right) \left( \sum_{j=1}^m \frac{1}{2\pi} R_{j\bar{j}} \right) - \varphi^{i\bar{j}} \frac{1}{2\pi} R_{i\bar{j}} \\
 &= \Delta \varphi \frac{1}{2m\pi} S - \varphi^{i\bar{j}} \frac{1}{2\pi} R_{i\bar{j}},
 \end{aligned}$$

and we see from this and (27) that

$$\frac{1}{2m\pi} \delta \int_M S^2 \omega^m = -2 \int_M DS\varphi \omega^m.$$

This proves the fact that the critical points in the unperturbed case are the extremal Kähler metrics. We have seen that when  $t = 0$ , (24) + (25) + (26) vanishes. But when  $t \neq 0$ , this is not the case because we have the term with  $t^{m-1}$  only in (24)

$$2 \int_M S(\omega, t) m c_m \left( R_{j\bar{k}}^{p\bar{i}} \varphi_{p\bar{i}} \frac{\sqrt{-1}}{2\pi} dz^k \wedge d\bar{z}^{\bar{l}}, \frac{\sqrt{-1}}{2\pi} t \Theta, \dots, \frac{\sqrt{-1}}{2\pi} t \Theta \right),$$

which does not always vanish. This completes the proof of Remark 3.3.

**4. Kähler metrics of harmonic Chern forms**

Let  $M$  be a compact Kähler manifold with a fixed Kähler class  $[\omega_0]$  and  $\mathfrak{h}(M)$  the complex Lie algebra of all holomorphic vector fields. For any  $\omega \in [\omega_0]$ , let  $c_k(\omega)$  be the  $k$ -th Chern form with respect to  $\omega$  as in Remark 3.3. Let  $Hc_k(\omega)$  be the harmonic part of  $c_k(\omega)$ . Here the harmonic projection  $H$  is taken with respect to the Kähler metric  $\omega$ . Then

$$c_k(\omega) - Hc_k(\omega) = \sqrt{-1} \partial \bar{\partial} F_k$$

for some smooth real  $(k - 1, k - 1)$ -form

$$F_k \in \Omega^{k-1, k-1}(M).$$

For a holomorphic vector field  $X \in \mathfrak{h}(M)$ , define  $f_k : \mathfrak{h}(M) \rightarrow \mathbf{C}$  by

$$f_k(X) = \frac{1}{m - k + 1} \int_M L_X F_k \wedge \omega^{m-k+1}.$$

**THEOREM 4.1** (S. Bando [1]). *The functional  $f_k$  on  $\mathfrak{h}(M)$  is independent of the choice of  $\omega \in [\omega_0]$ , becomes a Lie algebra character and obstructs the existence of Kähler metrics  $\omega$  in  $[\omega_0]$  of harmonic  $k$ -th Chern form.*

In [11] the author gave a larger family of integral invariants including  $f_i$ 's and obstructions to asymptotic Chow semi-stability.

Here again as in Remark 3.3 we are fixing  $J$  and varying  $\omega$ , instead of fixing  $\omega$  and varying  $J$ . So we denote the perturbed scalar curvature by  $S(\omega, t)$  as in (19). If  $X = \text{grad}' u = g^{i\bar{j}} \frac{\partial u}{\partial z^{\bar{j}}} \frac{\partial}{\partial z^i}$  with  $\int_M u \omega^m = 0$  then we see using the integration by parts that

$$(28) \quad \frac{1}{2m\pi} \int_M u S(\omega, t) \omega^m = -f_1(X) - t f_2(X) - \dots - t^{m-1} f_m(X).$$

We put

$$F_t(X) := f_1(X) + tf_2(X) + \dots + t^{m-1}f_m(X).$$

and call it **total Bando character**.

**PROPOSITION 4.2.** *For fixed small  $t \in \mathbf{R}$ ,  $F_t : \mathfrak{h}(M) \rightarrow \mathbf{C}$  is an obstruction to the existence of Kähler metric  $\omega \in [\omega_0]$  of constant perturbed scalar curvature  $S(\omega, t)$ . If there exists a perturbed extremal Kähler metric and the total Bando character vanishes, then the perturbed extremal Kähler metric has constant perturbed scalar curvature.*

*Proof.* If there is a Kähler form  $\omega \in [\omega_0]$  such that  $S(\omega, t)$  is constant. Then the total Bando character has to vanish because of (28) and the normalization  $\int_M u\omega^m = 0$ . If  $\omega$  is a perturbed extremal metric then  $\text{grad}' S(\omega, t)$  is a holomorphic vector field and

$$F_t(\text{grad}' S(\omega, t)) = \frac{1}{2m\pi} \int_M g^{i\bar{j}} \frac{\partial S(\omega, t)}{\partial z^i} \frac{\partial S(\omega, t)}{\partial \bar{z}^j} \omega^m.$$

Thus if  $F_t$  vanishes then  $S(\omega, t)$  is constant. □

Let  $\sigma(t)$  be the topological invariant

$$\sigma(t) = \frac{(c_1(M) \wedge [\omega_0]^{m-1} + tc_2(M)[\omega_0]^{m-2} + \dots + t^{m-1}c_m(M)) [M]}{[\omega_0]^m [M]}.$$

This is obviously the average of the perturbed scalar curvature (with respect to any Kähler form  $\omega \in [\omega_0]$ ). For any two Kähler forms  $\omega'$  and  $\omega''$  we define

$$\mathcal{M}_t(\omega', \omega'') = - \int_0^1 ds \int_M \frac{\partial \varphi_s}{\partial s} (S(\omega_s, t) - \sigma(t)) \omega_s^m$$

where  $\omega_s = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_s$ ,  $0 \leq s \leq 1$ , is a smooth path in  $[\omega_0]$  joining  $\omega'$  and  $\omega''$ . Bando and Mabuchi ([2]) observed that every coefficient of  $t^k$  in  $\mathcal{M}_t(\omega', \omega'')$ , and thus  $\mathcal{M}_t(\omega', \omega'')$ , is independent of the choice of the paths  $\omega_s$  and satisfies the cocycle conditions. Putting  $v_t(\omega) := \mathcal{M}_t(\omega_0, \omega)$ , we get a functional on the space of all Kähler forms in the cohomology class  $[\omega_0]$ . The functional  $v_0$  in the case when  $t = 0$  is the so-called  $K$ -energy or Mabuchi energy. We call  $v_t$  the perturbed Mabuchi energy. It is obvious that the critical points of the perturbed Mabuchi energy are the Kähler metrics of constant perturbed scalar curvature. In the case when  $t = 0$  Chen and Tian [5] proved that the Mabuchi energy is bounded from below if there exists a Kähler metric of constant scalar curvature, and that the infimum of the Mabuchi energy is attained exactly on the space of Kähler metrics of constant scalar curvature, extending earlier result of Bando and Mabuchi [3] for Kähler-Einstein manifolds of positive first Chern class. We hope to discuss for the perturbed case in a later paper.

The proof of the fact that the definition of  $\mathcal{M}_t$  is independent of the paths follows from the fact that  $S(\omega, t)\omega^m$  gives a closed 1-form on the space of Kähler forms. The closedness comes from the symmetry between the endomorphism part and the form part in the definition of  $S(\omega, t)\omega^m$ , as was explained between the equation (9) and Theorem 2.2. The detailed discussion was given in [10] but of course the original idea goes back to Bando [1].

For the identity component  $\text{Aut}^0(M)$  of the group of all holomorphic automorphisms of  $M$ , let  $G$  denote the maximal linear algebraic subgroup. The maximal reductive subgroup  $K^c$  of  $G$  is the complexification of a compact Lie group  $K$ . Taking the average of the Kähler metric by the action of  $K$  we may assume that  $K$  acts as isometries. We denote by  $\omega$  the Kähler form of the averaged Kähler metric. Then the elements of the Lie algebra of  $K$  are Killing vector fields of  $(M, \omega)$  and are thus obtained as the real parts of the gradient vector fields of purely imaginary functions (see e.g. [17]). Therefore as a complex Lie algebra, the Lie algebra  $\mathfrak{k}^c$  is isomorphic to the Lie algebra  $\mathfrak{u}$  spanned over  $\mathbf{C}$  by some real functions  $u_1, \dots, u_d$  with the normalization  $\int_M u_i \omega^m = 0$  where the Lie bracket on  $\mathfrak{u}$  is given by the Poisson bracket

$$\{u, v\} = u^i v_i - v^i u_i = g^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j} \frac{\partial v}{\partial z^i} - g^{i\bar{j}} \frac{\partial v}{\partial \bar{z}^j} \frac{\partial u}{\partial z^i}.$$

**PROPOSITION 4.3.** *Let the situation be as above. If we choose  $\omega_r = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_r$  so that  $\varphi_0 = 0$  and that  $\dot{\varphi}_r|_{r=0} = u$  for some real smooth function  $u$  in  $\mathfrak{u}$ , then*

$$\left. \frac{d}{dr} \right|_{r=0} v_t(\omega_r) = 2m\pi F_t(\text{grad}' u).$$

*Proof.* This is immediate from

$$v_t(\omega_r) = - \int_0^r dq \int_M \frac{\partial \varphi_q}{\partial q} (S(\omega_q, t) - \sigma(t)) \omega_q^m$$

and

$$\begin{aligned} \left. \frac{d}{dr} \right|_{r=0} v_t(\omega_r) &= - \int_M u S(\omega, t) \omega^m \\ &= 2m\pi F_t(\text{grad}' u) \end{aligned}$$

where the last equality follows because  $u$  is a normalized Hamiltonian function for a holomorphic vector field. □

This proposition shows that the perturbed Mabuchi energy is an integral form of the total Bando character. A way of computing the unperturbed Mabuchi energy  $v_0$  without using the path integral was given in [14]. It would be interesting if one can give a formula for  $v_t$  without using path integral. B. Weinkove [23] related the degree 1 and 2 terms in  $t$  of  $\mathcal{M}_t$  to Donaldson's

functional which was used in the proof of the existence of Hermitian-Einstein metrics on stable vector bundles [6].

We also remark that the modified Mabuchi energy to treat the extremal metrics can be also defined in the perturbed case just as defined in [16] and [20]. One can use the proof given in [15].

The results obtained above may be interesting to compare with a results of X. Wang [22] (see also [12]) which we summarize below.

Let  $(Z, \Omega)$  be a Kähler manifold and suppose a compact Lie group  $K$  acts on  $Z$  as holomorphic isometries. Then the complexification  $K^c$  of  $K$  also acts on  $Z$  as biholomorphisms. The actions of  $K$  and  $K^c$  induce homomorphisms of the Lie algebras  $\mathfrak{k}$  and  $\mathfrak{k}^c$  to the real Lie algebra  $\Gamma(TZ)$  of all smooth vector fields on  $Z$ , both of which we denote by  $\rho$ . If  $\xi + i\eta \in \mathfrak{k}^c$  with  $\xi, \eta \in \mathfrak{k}$ , then

$$\rho(\xi + i\eta) = \rho(\xi) + J\rho(\eta),$$

where  $J$  is the complex structure of  $Z$ . Suppose  $[\Omega]$  is an integral class and there is a holomorphic line bundle  $L \rightarrow Z$  with  $c_1(L) = [\Omega]$ . There is an Hermitian metric  $h$  of  $L^{-1}$  such that its Hermitian connection  $\theta$  satisfies

$$-\frac{1}{2\pi} d\theta = \Omega.$$

Suppose we have a lifting of  $K^c$  to  $L^{-1}$ , so that we have a moment map  $\mu : Z \rightarrow \mathfrak{k}^*$  because the lifting of  $K$ -action to  $L$  is equivalent to defining a moment map (see [8], section 6.5). Let  $\pi : L^{-1} \rightarrow Z$  be the projection and  $\pi(p) = x$  with  $p \in L^{-1}$ —zero section,  $x \in Z$ . Denote by  $\Gamma = K^c \cdot x$  the  $K^c$ -orbit of  $x$  in  $Z$ , and  $\tilde{\Gamma} = K^c \cdot p$  be the  $K^c$ -orbit of  $p$  in  $L^{-1}$ . We say that  $x \in Z$  is polystable with respect to the  $K^c$ -action if the orbit  $\tilde{\Gamma}$  is closed in  $L^{-1}$ . Consider the function  $h : \tilde{\Gamma} \rightarrow \mathbf{R}$  defined by

$$h(\gamma) = \log|\gamma|^2.$$

Fundamental facts are

- $h$  has a critical point if and only if the moment map  $\mu : Z \rightarrow \mathfrak{k}^*$  has a zero along  $\Gamma$ ;
- $h$  is a convex function.

For these facts refer again to [8], section 6.5. These imply the following two propositions.

**PROPOSITION 4.4.** *A point  $x \in Z$  is polystable with respect to the action of  $K^c$  if and only if the moment map  $\mu$  has a zero along  $\Gamma$ .*

**PROPOSITION 4.5.** *The set  $\{x \in \Gamma \mid \mu(x) = 0\}$  has only one component, and the orbit  $Stab(x)^c \cdot x$  of the complexification of the stabilizer at  $x$  through  $x$  is connected even if  $Stab(x)^c$  is not connected.*

For a given  $x \in Z$  we extend  $\mu(x) : \mathfrak{k} \rightarrow \mathbf{R}$  complex linearly to  $\mu(x) : \mathfrak{k}^c \rightarrow \mathbf{C}$ . For notational convenience we denote by  $K_x$  (resp.  $(K^c)_x$ ) the stabilizer of  $x$  in

$K$  (resp.  $K^c$ ), and by  $\mathfrak{k}_x$  and  $(\mathfrak{k}^c)_x$  the Lie algebra of  $K_x$  and  $(K^c)_x$ . Define  $f_x : (\mathfrak{k}^c)_x \rightarrow \mathbf{C}$  to be the restriction of  $\mu(x) : \mathfrak{k}^c \rightarrow \mathbf{C}$  to  $(\mathfrak{k}^c)_x$ . Note that  $(K^c)_{gx} = g(K^c)_x g^{-1}$ .

**PROPOSITION 4.6** (Wang [22]). *Fix  $x_0 \in Z$ . Then for  $x \in K^c \cdot x_0$ ,  $f_x$  is  $K^c$ -equivariant in that  $f_{gx}(Y) = f_x(Ad(g^{-1})Y)$ . In particular if  $f_x$  vanishes at some  $x \in K^c \cdot x_0$  it vanishes at all  $x \in K^c \cdot x_0$ . Moreover  $f_x : (\mathfrak{k}^c)_x \rightarrow \mathbf{C}$  is a Lie algebra character.*

For a proof of this proposition, see [22] and also [12]. Suppose now we are given a  $K$ -invariant inner product on  $\mathfrak{k}$ . Then we can identify  $\mathfrak{k} \cong \mathfrak{k}^*$ , and  $\mathfrak{k}^*$  has a  $K$ -invariant inner product. Consider the function  $\phi : K^c \cdot x_0 \rightarrow \mathbf{R}$  defined by  $\phi(x) = |\mu(x)|^2$ . We say that  $x \in K^c \cdot x_0$  is an extremal point if  $x$  is a critical point of  $\phi$ .

**PROPOSITION 4.7** (Wang [22]). *Let  $x \in K^c \cdot x_0$  be an extremal point. Then we have a decomposition*

$$(\mathfrak{k}^c)_x = (\mathfrak{k}_x)^c \oplus \sum_{\lambda > 0} \mathfrak{k}_\lambda^c$$

where  $\mathfrak{k}_\lambda^c$  is  $\lambda$ -eigenspace of  $\text{ad}(i\mu(x))$ , and  $i\mu(x)$  lies in the center of  $(\mathfrak{k}_x)^c$ . In particular  $(\mathfrak{k}_x)^c = (\mathfrak{k}^c)_x$  if and only if  $\mu(x) = 0$ .

For a proof of this proposition, see [22] and also [12]. Let  $(M, \omega_0, J_0)$  be a compact Kähler manifold with a fixed Kähler form  $\omega_0$ . Apply the above results for finite dimensional manifold  $Z$  to the set  $\mathcal{J}$  of all  $\omega$ -compatible integral complex structures  $J$  with respect to which  $(M, \omega_0, J)$  is a Kähler manifold, where the compact Lie group  $K$  is replaced by the group of symplectomorphisms generated by Hamiltonian diffeomorphisms. This explains a relationship between stability and various results about extremal Kähler metrics. For example, Proposition 4.6 explains the total Bando character and Proposition 4.7 of course explains Calabi's decomposition theorem for the Lie algebras of all holomorphic vector fields on compact extremal Kähler manifolds [4] (see the next section).

### 5. Deformations of extremal Kähler metrics

Let  $M$  be a compact complex manifold carrying a Kähler metric. By a ( $t$ -perturbed) extremal Kähler class we mean a de Rham cohomology class which contains the Kähler form of a ( $t$ -perturbed) extremal Kähler metric. In this section we prove the following result which extends the results of LeBrun and Simanca [18], [19].

**THEOREM 5.1.** *For an extremal Kähler class  $[\omega_0]$ , there exists a neighborhood  $U \times (-\varepsilon, \varepsilon)$  of  $([\omega_0], t)$  in  $H_{DR}^{1,1}(M, \mathbf{R}) \times \mathbf{R}$  such that all points of  $U$  are  $t$ -perturbed extremal Kähler classes for all  $t \in (-\varepsilon, \varepsilon)$ .*

The rest of this section is devoted to the proof of this theorem. We first review well known facts on Hamiltonian holomorphic vector fields on compact Kähler manifolds. Let  $(M, g)$  be a compact Kähler manifold. We define a fourth-order elliptic differential operator  $L_g : C_C^\infty(M) \rightarrow C_C^\infty(M)$  by

$$L_g u = \nabla''^* \nabla''^* \nabla'' \nabla'' u,$$

where  $C_C^\infty(M)$  denotes the set of all complex valued smooth functions on  $M$ . More precisely

$$\begin{aligned} L_g u &= \nabla^{\bar{j}} \nabla^{\bar{i}} \nabla_{\bar{j}} \nabla_{\bar{i}} u \\ &= \Delta^2 u + R^{\bar{j}i} \nabla_{\bar{j}} \nabla_i u + \nabla^{\bar{j}} S \nabla_{\bar{j}} u \end{aligned}$$

where  $S$  denotes the unperturbed scalar curvature. Then the kernel of  $L_g$  consists of all smooth functions  $u$  whose gradient vector fields

$$\text{grad}' u := g^{i\bar{j}} \nabla_{\bar{j}} u \frac{\partial}{\partial z^i}$$

are holomorphic vector fields. It is well known that such holomorphic vector fields are exactly those which have zeros (see [18] for a comprehensive proof). Since constant functions correspond to the zero vector field, we only consider the subspace  $(\ker L_g)_0$  consisting of all functions  $u \in \ker L_g$  which are orthogonal to constant functions:

$$\int_M u \omega_g^m = 0.$$

Now we study the behavior of  $u \in (\ker L_g)_0$  when the Kähler metric  $g$  varies in the same Kähler class. The following lemma was used in [13], pp. 208–209, but we will reproduce a proof here for the reader’s convenience.

LEMMA 5.2. *Let  $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} \varphi$  be a Kähler metric in the same Kähler class as  $g_{i\bar{j}}$ . If  $u \in (\ker L_g)_0$ , then  $\tilde{u} := u + \nabla^i u \nabla_i \varphi \in (\ker L_{\tilde{g}})_0$  and  $\text{grad}_{\tilde{g}} \tilde{u} = \text{grad}_g u$ .*

*Proof.* We first show the last equation.

$$\begin{aligned} \text{grad}_{\tilde{g}} \tilde{u} &= \tilde{g}^{i\bar{j}} \frac{\partial \tilde{u}}{\partial \bar{z}^j} \frac{\partial}{\partial z^i} = \tilde{g}^{i\bar{j}} \left( \frac{\partial u}{\partial \bar{z}^j} + \nabla^k u \nabla_k \nabla_{\bar{j}} \varphi \right) \frac{\partial}{\partial z^i} \\ &= \tilde{g}^{i\bar{j}} \nabla^k u (g_{k\bar{j}} + \nabla_k \nabla_{\bar{j}} \varphi) \frac{\partial}{\partial z^i} = \nabla^i u \frac{\partial}{\partial z^i} = \text{grad}_g u. \end{aligned}$$

It remains to see

$$\int_M \tilde{u} \omega_{\tilde{g}}^m = 0.$$

Let  $g_{i\bar{j}} = g_{i\bar{j}} + t \nabla_i \nabla_{\bar{j}} \varphi$  be the line segment of Kähler metrics between  $g$  and  $\tilde{g}$ , and  $u_t = u + t \nabla^i u \nabla_i \varphi$  be the corresponding functions in  $(\ker L_{\tilde{g}})_0$ . It is sufficient to prove

$$\frac{d}{dt} \int_M u_t \omega_{g_t}^m = 0.$$

It is also sufficient to prove this at  $t = 0$ . But

$$\left. \frac{d}{dt} \int_M u_t \omega_{g_t}^m \right|_{t=0} = \int_M (\nabla^i u \nabla_i \varphi + u(\Delta \varphi)) \omega_g^m = 0,$$

where  $\Delta = \nabla^i \nabla_i u$  denotes the complex Laplacian. This completes the proof.  $\square$

Now let  $K$  be the identity component of the isometry group of  $(M, g)$ , and  $\mathfrak{k}$  be its Lie algebra. Hence  $\mathfrak{k}$  consists of all Killing vector fields. On a compact Kähler manifold  $\mathfrak{k}$  can be embedded into the complex Lie algebra  $\mathfrak{h}(M)$  of all holomorphic vector fields on  $M$  by  $X \in \mathfrak{k} \mapsto \frac{1}{2}(X - \sqrt{-1}JX) \in \mathfrak{h}(M)$ . By this  $\mathfrak{k}$  is often identified with the image in  $\mathfrak{h}(M)$  of this embedding. As was explained in the previous section when a holomorphic vector field  $X$  is written as a gradient vector field of a complex valued smooth function,  $X$  is a Killing vector field if and only if the function is a purely imaginary valued function. We choose real valued smooth functions  $u_1, \dots, u_d$  so that the gradient vector fields of  $iu_1, \dots, iu_d$  form a basis of  $\mathfrak{k} \otimes \mathbb{C}$ . We also assume that  $1, u_1, \dots, u_d$  form an  $L^2$ -orthonormal system (under the normalization  $\int_M u_i \omega^m = 0$ ). Let us denote by  $J_g$  the linear span over  $\mathbb{C}$  of  $1, u_1, \dots, u_d$ .

*Remark 5.3.* Since the imaginary part of  $\text{grad}' u_j$  is a Killing vector field,  $(\text{grad}' u_j)\varphi$  is a real function for a  $K$ -invariant real function  $\varphi$ .

*Remark 5.4.* If  $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} \varphi$  is a  $K$ -invariant Kähler metric in the same Kähler class as  $g$ , then the corresponding basis of  $J_{\tilde{g}}$  consisting of real functions are

$$1, \tilde{u}_1 = u_1 + (\text{grad}' u_1)\varphi, \dots, \tilde{u}_d = u_d + (\text{grad}' u_d)\varphi.$$

It is easy to see that they form an  $L^2$ -orthonormal system with respect to  $\tilde{g}$  (see [13], Appendix 2).

Since we assume that there is an extremal Kähler metric, the Lie algebra  $\mathfrak{h}(M)$  has the following structure by a theorem of Calabi [4]. Namely there is a decomposition

$$\mathfrak{h}(M) = \mathfrak{h}_0 + \sum_{\lambda \neq 0} \mathfrak{h}_\lambda,$$

where  $\mathfrak{h}_\lambda$  is a  $\lambda$ -eigenspace of the adjoint action of the extremal vector field

$$\text{ad}(\text{grad}' S) : \mathfrak{h}(M) \rightarrow \mathfrak{h}(M),$$

and further  $\mathfrak{h}_0$  is the complexification of the Lie algebra  $\mathfrak{k}$  consisting of all Killing vector fields on  $(M, g)$ . In particular, it turns out that  $\text{grad}' S$  lies in the center

of  $\mathfrak{h}_0$ . that  $[\mathfrak{h}_\lambda, \mathfrak{h}_\mu] \subset \mathfrak{h}_{\lambda+\mu}$ , and that  $\mathfrak{h}_0$  is a maximal reductive Lie subalgebra of  $\mathfrak{h}(M)$ .

Now we consider the set of all Kähler metrics invariant under the identity component of the isometry group  $K$  of  $(M, g)$  of the form

$$\omega(\alpha, \varphi) = \omega + \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$$

where  $\alpha$  is a  $K$ -invariant real harmonic  $(1,1)$ -form on  $(M, g)$  and  $\varphi$  is a  $K$ -invariant real-valued  $L^2_{k+4}$ -function. Hence the space of such  $K$ -invariant Kähler metrics is identified with an open subset of  $H^{1,1}(M; \mathbf{R}) \times L^2_{k+4, K}$  where  $H^{1,1}(M; \mathbf{R})$  denotes the vector space of all real harmonic  $(1,1)$ -forms on  $M$  and  $L^2_{k+4, K}$  is the vector space of all real valued  $K$ -invariant  $L^2_{k+4}$  functions on  $M$ . Let  $I_{k+4}$  be the orthogonal complement to the subspace spanned by  $1, u_1, \dots, u_d$  in  $L^2_{k+4, K}$ .

Let  $\tilde{g}$  be the Kähler metric corresponding to  $\omega(\alpha, \varphi)$ . Then we obtain, as in Remark 5.4,  $L^2_{k+3}$ -functions  $(1, \tilde{u}_1, \dots, \tilde{u}_d)$  whose gradient vector fields span the Lie algebra  $\mathfrak{k}$ . Let  $\tilde{J}_{k+3}$  be the linear span of  $(1, \tilde{u}_1, \dots, \tilde{u}_d)$ . We put  $\tilde{u}_0 = 1$ . Then for a sufficiently small neighborhood  $U$  of  $g$  in  $H^{1,1}(M; \mathbf{R}) \times L^2_{k+4, K}$ , we have

$$\det(u_i, \tilde{u}_j)_{L^2} \neq 0$$

for all  $\tilde{g} \in U$ . Then it is easy to see

$$\ker(1 - \Pi_g)(1 - \Pi_{\tilde{g}}) = \ker(1 - \Pi_{\tilde{g}})$$

where  $\Pi_g$  and  $\Pi_{\tilde{g}}$  are respectively the  $L^2$  projections of  $L^2_{k, K}$  onto  $J_{k+3} \subset L^2_{k, K}$  and onto  $\tilde{J}_{k+3} \subset L^2_{k, K}$ :

$$\begin{aligned} \Pi_g : L^2_{k, K} &\rightarrow L^2_{k, K}, & \Pi_g(f) &= \sum_{i=0}^d (f, u_i)u_i, \\ \Pi_{\tilde{g}} : L^2_{k, K} &\rightarrow L^2_{k, K}, & \Pi_{\tilde{g}}(f) &= \sum_{i=0}^d (f, \tilde{u}_i)\tilde{u}_i. \end{aligned}$$

Put  $V := U \cap (H^{1,1}(M; \mathbf{R}) \times I_{k+4})$ , and take a neighborhood  $W$  of the origin in  $V \times \mathbf{R}$  such that for every point  $(\tilde{g}, t)$  in  $W$  (identifying  $V$  with the space of Kähler metrics) the inner product (9) makes sense so that one can consider  $t$ -perturbed scalar curvature. Consider the map  $\mathfrak{S} : W \rightarrow I_k$  defined by

$$\mathfrak{S}(\tilde{g}, t) = (1 - \Pi_g)(1 - \Pi_{\tilde{g}})S(\tilde{g}, t).$$

Note that  $\mathfrak{S}(g, 0) = 0$  and that  $\mathfrak{S}^{-1}(0)$  is the set of all perturbed extremal Kähler metrics in  $W$ . To complete the proof of Theorem 5.1, it is sufficient to show, by the implicit function theorem, that the partial derivative

$$D\mathfrak{S}_{(g, 0)} : I_{k+4} \rightarrow I_k$$

at  $(g, 0)$  in the direction of  $I_{k+4}$  is an isomorphism. In the direction of  $\psi \in I_{k+4}$ , the derivative of the scalar curvature is

$$(DS)_g(\psi) = -\Delta^2\psi - R^{\bar{j}i}\nabla_{\bar{j}}\nabla_i\psi,$$

and the derivative of the projection  $\Pi$  is

$$\begin{aligned} (D\Pi)(S(g))_g(\psi) &= \left. \frac{d}{dt} \right|_{t=0} (S + \nabla^i S t \nabla_i \psi) \\ &= \nabla^i S \nabla_i \psi = \nabla^{\bar{j}} S \nabla_{\bar{j}} \psi, \end{aligned}$$

where the last equality follows from Remark 5.3. Combining these two equations, we obtain

$$\begin{aligned} (D\mathfrak{S})_g(\psi) &= (1 - \Pi_g)(-\Delta^2\psi - R^{\bar{j}i}\nabla_{\bar{j}}\nabla_i\psi - \nabla^{\bar{j}}S\nabla_{\bar{j}}\psi) \\ &= (1 - \Pi_g)(-L_g\psi) \end{aligned}$$

If  $(1 - \Pi_g)(L_g\psi) = 0$ , then  $L_g\psi \in J_g$ . But since  $L_g$  is self-adjoint,  $(\text{Image } L_g)^\perp = \ker L_g$  and hence  $L_g\psi = 0$ . Since  $\psi \in I_{k+4}$ , this implies  $\psi = 0$ . Thus  $(D\mathfrak{S})_{(g,0)}$  is injective, which also implies that  $(D\mathfrak{S})_{(g,0)}$  is surjective since  $(D\mathfrak{S})_{(g,0)}$  is self-adjoint. This completes the proof.

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