ON AN INVARIANT TENSOR UNDER A *CL*-TRANSFORMATION

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Tashiro and Tachibana showed some characteristic properties of Fubinian and C-Fubinian manifolds in their paper [6], where the notion of C-loxodromes was introduced in an almost contact manifold with affine connection.

The purpose of the present paper is to obtain an invariant tensor, that is, a tensor which is left invariant under a CL-transformation between two almost contact manifolds with symmetric affine connections. And Takamatsu and Mizusawa have performed the similar consideration about infinitesimal CL-transformations. [2].

§1. **Preliminaries.** [4, 5, 7, 8].

Let there be given, in an N-dimensional differentiable manifold M of class C^∞, a non-null tensor field f of type (1, 1) and of class C^∞ satisfying $f^3+f=0$. When the rank of f is constant everywhere and is equal to r, such a structure is called an f-structure of rank r. r is necessarily even.

Now, let M be a $(2n+1)$-dimensional differentiable manifold of class C^∞ for which the second axiom of countability holds true. If there exist a mixed tensor $f_j{}^i$, a contravariant vector field f^i and a covariant vector field f_j, all of which are of class C^∞, satisfying the conditions:

$$f^i f_i=1, \qquad f_j{}^i f_k{}^j=-\delta_k^i+f^i f_k,$$

then such a manifold M is said to have an almost contact structure $(f_j{}^i, f^i, f_j)$ of class C^∞ and we call the manifold an almost contact manifold of class C^∞.

It is well-known that in a manifold with an almost contact structure $(f_j{}^i, f^i, f_j)$ of class C^∞, there exists a positive definite Riemannian metric g_{ji}, which is called a Riemannian metric associated with the almost contact structure, such that

$$f_i=g_{ij}f^j, \qquad g_{ji}f_h{}^j f_k{}^i=g_{hk}-f_h f_k.$$

We call the set $(f_j{}^i, f^i, f_j, g_{ji})$ an almost contact metric structure and a manifold with an almost contact metric structure $(f_j{}^i, f^i, f_j, g_{ji})$ of class C^∞ is called an almost contact metric (or Riemannian) manifold of class C^∞.

In a $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(f_j{}^i, f^i, f_j)$, the following properties are satisfied:

$$f^i f_i=1, \tag{1.1}$$

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(1. 2) $$f_k{}^j f_j{}^i = -\delta_k^i + f^i f_k,$$

(1. 3) $$f_j{}^i f^j = 0,$$

(1. 4) $$f_j{}^i f_i = 0,$$

(1. 5) $$\text{rank } (f_j{}^i) = 2n.$$

Therefore, the almost contact structure is an f-structure of rank $2n$, where $f_j{}^i$ are components of f.

Furthermore, if this manifold has an associated metric and f_{ji} is defined as $f_j{}^h g_{hi}$, then in addition to (1. 1)∼(1. 5) the following relations hold true:

(1. 6) $$f_{ji} = -f_{ij},$$

(1. 7) $$\text{rank } (f_{ji}) = 2n,$$

(1. 8) $$f_i = g_{ij} f^j,$$

(1. 9) $$g_{ji} f_h{}^j f_k{}^i = g_{hk} - f_h f_k.$$

If, in a $(2n+1)$-dimensional differentiable manifold M, there exists a differentiable 1-form f such that $f \wedge (df)^n \neq 0$ everywhere, then such a manifold is called to have a contact structure f and we call the manifold a contact manifold.

It is well-known that in any contact manifold with a contact structure f there exists always an almost contact metric structure $(f_j{}^i, f^i, f_j, g_{ji})$ such that

$$f_j{}^h g_{hi} = f_{ji} \equiv \frac{1}{2}(\partial_j f_i - \partial_i f_j),$$

where, in terms of a local coordinate system x^i, f is expressed as $f = f_i dx^i$ and ∂_i denotes $\partial/\partial x^i$. Such an almost contact (metric) structure is simply called a contact (metric) structure. If a $(2n+1)$-dimensional differentiable manifold has a contact metric structure $(f_j{}^i, f^i, f_j, g_{ji})$ in the above sense, then the following relations hold true,

(1. 10) $$f_j{}^h g_{hi} = f_{ji} \equiv \frac{1}{2}(\partial_j f_i - \partial_i f_j),$$

(1. 11) $$\nabla_k f_{ji} + \nabla_j f_{ik} + \nabla_i f_{kj} = 0,$$

(1. 12) $$f^j \nabla_j f_i = 0,$$

(1. 13) $$f^{ji} \nabla_j f_{ih} = 0,$$

(1. 14) $$\nabla_j f^j = 0,$$

(1. 15) $$\nabla_i f_j{}^i = 2n f_j,$$

where ∇_j denotes the covariant differentiation with respect to the Riemannian connection.

Next, an almost contact or a contact manifold are called to be normal if the tensor

$$S_{jk}{}^{i}=N_{jk}{}^{i}-f^{i}\,(\partial_j f_k-\partial_k f_j)$$

vanishes, where $N_{jk}{}^{i}$ is the Nijenhuis tensor defined by $f_j{}^{i}$. If a contact metric manifold is normal, then the following equations are satisfied:

$$\nabla_j f_i=f_{ji}, \tag{1. 16}$$

$$\nabla_k f_{ji}=f_j g_{ki}-f_i g_{kj}, \tag{1. 17}$$

$$K_{kji}{}^{h} f_h=f_k g_{ji}-f_j g_{ki} \tag{1. 18}$$

$$K_{ji} f^{i}=2n f_j, \tag{1. 19}$$

where $K_{kji}{}^{h}$ and K_{ji} denote the Riemannian curvature tensor and the Ricci tensor respectively.

Now if we put

$$H_{ji}=\frac{1}{2}K_{jiab}f^{ab}=-\frac{1}{2}K_{jia}{}^{b}f_b{}^{a}, \tag{1. 20}$$

then H_{ji} is skew symmetric and we have

$$H_{ji}=K_{ajib}f^{ba}=K_{aji}{}^{b}f_b{}^{a}. \tag{1. 21}$$

Moreover operating ∇_l to (1. 17) and taking use of the Ricci's formula we get

$$f_j{}^{a}K_{ai}=(2n-1)f_{ji}-H_{ji}, \tag{1. 22}$$

and hence

$$f_j{}^{a}K_{ai}+f_i{}^{a}K_{ja}=0. \tag{1. 23}$$

Multiplying (1. 22) by $f_l{}^{j}$ and summing for j we have

$$f_j{}^{a}H_{ai}=K_{ji}-(2n-1)g_{ji}-f_j f_i, \tag{1. 24}$$

$$f_j{}^{a}H_{ai}+f_i{}^{a}H_{ja}=0. \tag{1. 25}$$

§ 2. Manifolds with corresponding *C*-loxodromes.

Let M be a $(2n+1)$-dimensional differentiable normal contact manifold with an associated almost contact metric structure $(f_j{}^{i}, f^{i}, f_j, g_{ji})$ and with the Riemannian connection $\Gamma_{ji}{}^{h}$.

The equation of a C-loxodrome in the manifold M in terms of any parameter t is

$$\frac{\delta^2 x^h}{dt^2}=\alpha\frac{dx^h}{dt}+af_j f_i{}^{h}\frac{dx^j}{dt}\frac{dx^i}{dt}, \tag{2. 1}$$

where δ indicates the covariant differentiation along the curve, α is a function of t and a is a constant. [6].

If $'M$ is a second $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(f_j{}^i, f^i, f_j)$ and $'\Gamma_{ji}{}^h$ is its symmetric affine connection, then the equation of its C-loxodrome is analogous to (2. 1) and is obtained by replacing $\Gamma_{ji}{}^h$, α and a in (2. 1) by $'\Gamma_{ji}{}^h$, $'\alpha$ and $'a$ respectively.

Suppose that there exists a CL-transformation (correspondence), that is, to C-loxodromes in M there correspond C-loxodromes in $'M$. Then the equation

$$\{\delta_l^k('\Gamma_{ji}{}^h-\Gamma_{ji}{}^h)-\delta_l^h('\Gamma_{ji}{}^k-\Gamma_{ji}{}^k)\}\frac{dx^j}{dt}\frac{dx^i}{dt}\frac{dx^l}{dt}=0$$

must be satisfied identically. By the usual process it follows that their connections are in the relation

$$'\Gamma_{ji}{}^h=\Gamma_{ji}{}^h+\delta_j^h p_i+\delta_i^h p_j+c(f_jf_i{}^h+f_if_j{}^h), \tag{2. 2}$$

where the vector field p_i is equal to $('\Gamma_{ai}{}^a-\Gamma_{ai}{}^a)/2(n+1)$ and a constant c is equal to $('a-a)/2$.

Let $K_{kji}{}^h$ and $'K_{kji}{}^h$ be the curvature tensors for the connections $\Gamma_{ji}{}^h$ and $'\Gamma_{ji}{}^h$ respectively. Then the respective curvature tensors are related to each other by the relation

$$\begin{aligned}'K_{kji}{}^h=K_{kji}{}^h-\delta_k^h P_{ji}+\delta_j^h P_{ki}+\delta_i^h(P_{kj}-P_{jk})\\ +c\nabla_k(f_jf_i{}^h+f_if_j{}^h)-c\nabla_j(f_kf_i{}^h+f_if_k{}^h),\end{aligned} \tag{2. 3}$$

where we put

$$P_{ji}=\nabla_j p_i-p_jp_i-c(f_jf_i{}^l+f_if_j{}^l)p_l-c^2f_jf_i \tag{2. 4}$$

and ∇_j denotes the covariant differentiation with respect to the connection $\Gamma_{ji}{}^h$. [6].

Contracting h and k in (2. 3), we have

$$'K_{ji}=K_{ji}-2nP_{ji}+(P_{ij}-P_{ji})+c\nabla_l(f_jf_i{}^l+f_if_j{}^l). \tag{2. 5}$$

Contracting h and i in (2. 3), we have

$$'K_{kja}{}^a=2(n+1)(P_{kj}-P_{jk}).$$

Since $'K_{ji}+'K_{jia}{}^a$, K_{ji}, $f_jf_i{}^l+f_if_j{}^l$ are symmetric in j and i, it follows easily that P_{ji} is symmetric in j and i. [1]. Accordingly it follows that the tensor $'K_{ji}='K_{aji}{}^a$ formed by the connection $'\Gamma_{ji}{}^h$ must be symmetric in this case. Consequently we have instead of (2. 3) and (2. 5)

$$'K_{kji}{}^h=K_{kji}{}^h-\delta_k^hP_{ji}+\delta_j^hP_{ki}+c\nabla_k(f_jf_i{}^h+f_if_j{}^h)-c\nabla_j(f_kf_i{}^h+f_if_k{}^h), \tag{2. 6}$$

$$'K_{ji}=K_{ji}-2nP_{ji}+c\nabla_l(f_jf_i{}^l+f_if_j{}^l). \tag{2. 7}$$

Substituting (2. 7) into (2. 6) to eliminate P_{ji}, we get

$$\begin{aligned}2n'W_{kji}{}^h=&2nW_{kji}{}^h+2nc\{\nabla_k(f_jf_i{}^h+f_if_j{}^h)\\&-\nabla_j(f_kf_i{}^h+f_if_k{}^h)\}-c\{\delta_k^h\nabla_l(f_jf_i{}^l+f_if_j{}^l)-\delta_j^h\nabla_l(f_kf_i{}^l+f_if_k{}^l)\},\end{aligned} \tag{2. 8}$$

where $W_{kji}{}^h$ is the so-called Weyl's projective curvature tensor, i.e. [1]

$$W_{kji}{}^h=K_{kji}{}^h-\frac{1}{2n}(\delta_k^hK_{ji}-\delta_j^hK_{ki}). \tag{2. 9}$$

In the following, if X is a quantity in M, then we denote the corresponding quantity in $'M$ as $'X$. Since the manifold M is normal contact, we see that the equation (2. 8) is rewritten as follows:

$$\begin{aligned}n'W_{kji}{}^h=&nW_{kji}{}^h+c[f_i(\delta_j^hf_k-\delta_k^hf_j)\\&+n(2f_i{}^hf_{kj}+f_j{}^hf_{ki}-f_k{}^hf_{ji})+(\delta_k^h+nf^hf_k)g_{ji}-(\delta_j^h+nf^hf_j)g_{ki}].\end{aligned} \tag{2. 10}$$

Transvecting on both sides of this equation with f^kf_h, we have

$$n'W_{kji}{}^hf^kf_h=nW_{kji}{}^hf^kf_h+c(n+1)(g_{ji}-f_jf_i). \tag{2. 11}$$

Substituting (2. 11) into (2. 10) to eliminate g_{ji}, we have

$$'W_{kji}{}^h+'X_{kji}{}^h=W_{kji}{}^h+X_{kji}{}^h+c(2f_i{}^hf_{kj}+f_j{}^hf_{ki}-f_k{}^hf_{ji}), \tag{2. 12}$$

where we put for simplicity

$$X_{kji}{}^h=\frac{1}{n+1}[(\delta_j^h+nf^hf_j)W_{aki}{}^b-(\delta_k^h+nf^hf_k)W_{aji}{}^b]f^af_b. \tag{2. 13}$$

Further, transvecting on both sides of (2. 12) with $f_h{}^k$, we get

$$('W_{kji}{}^h+'X_{kji}{}^h)f_h{}^k=(W_{kji}{}^h+X_{kji}{}^h)f_h{}^k+c(2n+1)f_{ji}. \tag{2. 14}$$

Lastly, substituting (2. 14) into (2. 12) to eliminate f_{ji}, we obtain

$$'L_{kji}{}^h=L_{kji}{}^h, \tag{2. 15}$$

where we put

$$\begin{aligned}L_{kji}{}^h=&W_{kji}{}^h+X_{kji}{}^h-\frac{1}{2n+1}[2f_i{}^h(W_{akj}{}^b+X_{akj}{}^b)\\&+f_j{}^h(W_{aki}{}^b+X_{aki}{}^b)-f_k{}^h(W_{aji}{}^b+X_{aji}{}^b)]f_b{}^a.\end{aligned}$$

Substituting (2. 9) and (2. 13) into this equation, we obtain

$$
\begin{aligned}
L_{kji}{}^{h} = K_{kji}{}^{h} &- \frac{1}{2n}(\delta_k^h K_{ji} - \delta_j^h K_{ki}) \\
&- \frac{1}{n+1}\Big[(\delta_k^h + n f^h f_k)\Big(K_{aji}{}^{b} f^a f_b + \frac{1}{2n} f_j{}^b f_b{}^a K_{ai}\Big) \\
&- (\delta_j^h + n f^h f_j)\Big(K_{aki}{}^{b} f^a f_b + \frac{1}{2n} f_k{}^b f_b{}^a K_{ai}\Big)\Big] \\
&- \frac{1}{2n+1}\Big[2 f_i{}^h\Big(H_{kj} + \frac{1}{n+1} f_k{}^c K_{acj}{}^{b} f^a f_b + \frac{1}{2(n+1)} f_k{}^a K_{aj}\Big) \\
&+ f_j{}^h\Big(H_{ki} + \frac{1}{n+1} f_k{}^c K_{aci}{}^{b} f^a f_b + \frac{1}{2(n+1)} f_k{}^a K_{ai}\Big) \\
&- f_k{}^h\Big(H_{ji} + \frac{1}{n+1} f_j{}^c K_{aci}{}^{b} f^a f_b + \frac{1}{2(n+1)} f_j{}^a K_{ai}\Big)\Big].
\end{aligned} \tag{2. 16}
$$

Thus if there exists a CL-correspondence between two manifolds M and $'M$, then the tensor $L_{kji}{}^{h}$ has the same components for them. In this sense we shall call the tensor $L_{kji}{}^{h}$ defined by (2. 16) the CL-curvature tensor. Consequently, we obtain the following

THEOREM 1. *Let M be a $(2n+1)$-dimensional differentiable normal contact manifold with an associated almost contact metric structure $(f_j{}^i, f^i, f_j, g_{ji})$ and with the Riemanian connection Γ_{ji}^h. And let $'M$ be a $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(f_j{}^i, f^i, f_j)$ and with a symmetric affine connection $'\Gamma_{ji}^h$. If the two manifolds M and $'M$ are related to each other under a CL-transformation, then their CL-curvature tensors have the same components.*

§ 3. *CL*-flat manifolds.

The CL-curvature tensor $L_{kji}{}^{h}$, which was obtained in the preceding section, is able to be defined in an almost contact manifold with a symmetric affine connection. Now, if the tensor $L_{kji}{}^{h}$ vanishes identically, then we shall call such a manifold to be CL-flat.

Let M be a normal contact manifold with an associated almost contact metric structure $(f_j{}^i, f^i, f_j, g_{ji})$ and with the Riemannian connection $\{{}^{h}_{ji}\}$. In the manifold M, on account of (1. 18), (1. 19) and (1. 22) we have

$$K_{aji}{}^{b} f^a f_b = g_{ji} - f_j f_i,$$

$$f_j{}^b f_b{}^a K_{ai} = -K_{ji} + 2n f_j f_i,$$

$$f_j{}^c K_{aci}{}^b f^a f_b = f_{ji},$$

$$f_j{}^a K_{ai} = (2n-1) f_{ji} - H_{ji}.$$

Therefore the CL-curvature tensor of the manifold M is expressible in the form

$$\begin{aligned} L_{kji}{}^h = K_{kji}{}^h &- \frac{1}{2(n+1)}[(\delta_k^h - f^h f_k)K_{ji} - (\delta_j^h - f^h f_j)K_{ki}] \\ &- \frac{1}{n+1}[(\delta_k^h + n f^h f_k)g_{ji} - (\delta_j^h + n f^h f_j)g_{ki}] \\ &- \frac{1}{2(n+1)}[2f_i{}^h(H_{kj}+f_{kj}) + f_j{}^h(H_{ki}+f_{ki}) - f_k{}^h(H_{ji}+f_{ji})]. \end{aligned} \tag{3. 1}$$

And if the manifold M is CL-flat, then we have

$$\begin{aligned} K_{kji}{}^h = &\frac{1}{2(n+1)}[(\delta_k^h - f^h f_k)K_{ji} - (\delta_j^h - f^h f_j)K_{ki}] \\ &+ \frac{1}{n+1}[(\delta_k^h + n f^h f_k)g_{ji} - (\delta_j^h + n f^h f_j)g_{ki}] \\ &+ \frac{1}{2(n+1)}[2f_i{}^h(H_{kj}+f_{kj}) + f_j{}^h(H_{ki}+f_{ki}) - f_k{}^h(H_{ji}+f_{ji})]. \end{aligned}$$

Lowering the index h, we have

$$\begin{aligned} K_{kjih} = \frac{1}{2(n+1)} &[g_{kh}(2g_{ji}+K_{ji}) - g_{jh}(2g_{ki}+K_{ki}) \\ &+ f_k f_h(2n g_{ji} - K_{ji}) - f_j f_h(2n g_{ki} - K_{ki}) \\ &+ 2f_{ih}(H_{kj}+f_{kj}) + f_{jh}(H_{ki}+f_{ki}) - f_{kh}(H_{ji}+f_{ji})]. \end{aligned} \tag{3. 2}$$

Since K_{kjih} is skew symmetric in i and h, we have the identity

$$(K_{kjih} + K_{kjhi})g^{kh} = 0.$$

Substituting (3. 2) into this identity and making use of (1. 19) and (1. 25), we have

$$K_{ji} = \left(\frac{K}{2n} - 1\right)g_{ji} + \left(2n+1-\frac{K}{2n}\right)f_j f_i, \tag{3. 3}$$

where K is the scalar curvature of the manifold M. Therefore it follows that the manifold M is η-Einstein and hence K=const. [3].

Substituting (3. 3) into (1. 22), we have

$$H_{ji}=\left(2n-\frac{K}{2n}\right)f_{ji}. \tag{3.4}$$

If the expressions (3. 3) and (3. 4) are substituted in (3. 2), the resulting equation is reducible to

$$\begin{aligned} K_{kjih}=&(k+1)(g_{kh}g_{ji}-g_{jh}g_{ki}) \\ &+k(f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{kj}f_{ih}) \\ &-k(g_{kh}f_jf_i+g_{ji}f_kf_h-g_{jh}f_kf_i-g_{ki}f_jf_h), \end{aligned} \tag{3.5}$$

where

$$k=\frac{K-2n(2n+1)}{4n(n+1)}.$$

Therefore it follows that the manifold M is locally C-Fubinian. Thus we have the

THEOREM 2. *If a normal contact metric manifold is CL-flat, then the manifold is locally C-Fubinian.*

When an almost contact metric manifold is of constant curvature, then from (2. 16) it is easily seen that the CL-curvature tensor vanishes identically, that is, the manifold is CL-flat. Therefore we have the

THEOREM 3. *An almost contact metric manifold of constant curvature is CL-flat.*

THEOREM 4. *A normal contact metric manifold related to an almost contact metric manifold of constant curvature under a CL-transformation is locally C-Fubinian.*

In particular, we have the

THEOREM 5. *A normal contact metric manifold related to a locally Euclidean almost contact metric manifold under a CL-transformation is locally C-Fubinian.*

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