ON A CERTAIN SEQUENCE OF CHANCE VARIABLES.

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1. Let $\{x_n\}$ be a sequence of chance variables, each of which has an expectation $E(x_n)$, satisfying the following condition;

$$\text{(F)} \qquad E_m(x_n) = x_m \qquad (m \leqq n)$$

with probability 1, where $E_m(x_n)$ denotes the conditional expectation of x_n for given $x_1, x_2, \ldots, x_m$[1]. In the present note, we shall give the sufficient conditions for the strong law of the large number and the central limit theorem in such a sequence of chance variables.

Theorem 1. Let $\{x_n\}$ be a sequence of chance variables satisfying the condition (F). Then the convergence of the following series

$$\sum_{k=0}^{\infty} \frac{1}{2^k} E(|x_{2^{k+1}} - x_{2^k}|)$$

is sufficient for the strong law of the large number, that is,

$$P_r\left\{\lim_{n\to\infty} \frac{x_n}{n} = 0\right\} = 1.$$

Proof. Let $E_m^{(k)}$ denote the set

$$\{x_{2^k+1} - x_{2^k} \leqq \varepsilon 2^k, \ldots, x_{m-1} - x_{2^k} \leqq \varepsilon 2^k, x_m - x_{2^k} > \varepsilon 2^k\}$$

for any $\varepsilon > 0$ and a positive integer m such that $2^k < m \leqq 2^{k+1}$. It is evident that $E_i^{(k)}$ and $E_j^{(k)}$ $(i \neq j)$ are disjoint;

$$(1) \qquad E_i^{(k)} \cdot E_j^{(k)} = 0 \qquad (i \neq j)$$

From the definition of the conditional expectation and the condition (F), we have

$$(2) \int_{E_m^{(k)}} (x_{2^{k+1}} - x_{2^k})dP = \int_{E_m^{(k)}} (x_m - x_{2^k})dP \geqq \varepsilon 2^k P_r\{E_m^{(k)}\}$$

Putting

$$E^{(k)} = \sum_{m=2^k+1}^{2^{k+1}} E_m^{(k)},$$

from (1) and (2), we obtain

$$\int_{E^{(k)}} (x_{2^{k+1}} - x_{2^k})dP \geqq \varepsilon 2^k P_r\{E^{(k)}\}$$

and, a posteriori,

$$E(|x_{2^{k+1}} - x_{2^k}|) \geqq \varepsilon 2^k P_r\{E^{(k)}\}$$

Hence the assumption of the theorem implies the convergence of the series $\sum_{k=0}^{\infty} P_r\{E^{(k)}\}$. It follows that, by the Borel-Canteli's theorem, for sufficiently large k and $2^{k+1} \geqq m > 2^k$, the inequality $x_m - x_{2^k} < \varepsilon 2^k$ holds with the probability 1. Denoting the integral part of $\log n/\log 2$ by p, for an arbitrary $n > 2^k$, we have

$$x_n - x_{2^k} = (x_n - x_{2^p}) + (x_{2^p} - x_{2^{p-1}}) + \cdots + (x_{2^{k+1}} - x_{2^k})$$
$$< \varepsilon(2^p + \cdots + 2^k) < \varepsilon 2^{p+1}$$

For a fixed k, let n tend to ∞, then

$$\overline{\lim} \frac{x_n}{n} \leqq 2\varepsilon.$$

ε being an arbitrary positive number, it follows that

$$\overline{\lim} \frac{x_n}{n} \leqq 0.$$

In the same way, we obtain

$$\underline{\lim} \frac{x_n}{n} \geqq 0$$

and hence $\lim \frac{x_n}{n} = 0$

Thus theorem is proved.

2. Here we shall consider the central limit theorem.

<u>Theorem 2.</u> <u>Let</u> $\{x_n\}$ <u>be a sequence of chance variables satisfying the following conditions;</u>

$$\text{(F)} \qquad E_m(x_n) = x_m \qquad (n \geq m)$$

$$\text{(F}_1\text{)} \qquad E_{m-1}(x_m - x_{m-1})^2 = E(x_m - x_{m-1})^2$$

<u>hold with probability</u> 1, <u>and for any</u> $\varepsilon > 0$,

$$\text{(L)} \qquad \lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{m=1}^{n} E\Big(\int_{|x-x_{m-1}|>\varepsilon\sigma_n} (x-x_{m-1})^2 dF_{m-1}(x)\Big) = 0,$$

<u>where</u> $\sigma_n^2 = E(x_n^2)$, <u>and</u> $F_{m-1}(x)$ <u>being the conditional probability</u> $Pr\{x_1, \cdots, x_{m-1};\ x_m < x\}$ <u>for given</u> $x_1, \cdots, x_{m-1}$. <u>Then, for any</u> x, <u>we have</u>

$$\lim_{n\to\infty} Pr\Big\{\frac{x_n}{\sigma_n} < x\Big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx.$$

It is here observed that the conditional probability $F_{m-1}(x)$ could be considered as a probability measure for $(x_1^0, \cdots, x_{m-1}^0)$ not belonging to the $(x_1, \cdots, x_{m-1})$-set with probability 0. (F), (F$_1$) and (L) correspond to the Lévy's condition (C), (C$_1$) and the Lindeberg's condition respectively.

<u>Proof.</u> In the sequel, let t denote a fixed numerical value and Θ unspecified quantities such that $|\Theta| \leq 1$. Taking into account the identities

$$e^{itx} = 1 + itx + \frac{t^2x^2}{2}\Theta = 1 + itx - \frac{t^2x^2}{2} + \frac{t^3x^3}{6}\Theta$$

we obtain

$$E_{m-1}\Big(e^{it\frac{x_m - x_{m-1}}{\sigma_n}}\Big) = \int_{|x-x_{m-1}|\leq\varepsilon\sigma_n} \Big(1 + \frac{it(x-x_{m-1})}{\sigma_n} - \frac{t^2(x-x_{m-1})^2}{2\sigma_n^2} + \frac{t^3(x-x_{m-1})^3}{6\sigma_n^3}\Theta\Big) \times dF_{m-1}(x)$$

$$+ \int_{|x-x_{m-1}|>\varepsilon\sigma_n} \Big(1 + \frac{it(x-x_{m-1})}{\sigma_n} - \frac{t^2(x-x_{m-1})^2}{2\sigma_n^2}\Theta\Big) dF_{m-1}(x),$$

$$(m = 1, 2, \cdots, n)$$

From this expression, the conditions (F) and (F$_1$) imply that

$$\text{(3)} \quad E_{m-1}\Big(e^{it\frac{x_m - x_{m-1}}{\sigma_n}}\Big) = 1 - \frac{t^2}{2\sigma_n^2} E(x_m - x_{m-1})^2 + \frac{t^2\Theta}{\sigma_n^2} \int_{|x-x_{m-1}|>\varepsilon\sigma_n} (x-x_{m-1})^2 dF_{m-1}(x) + \frac{|t^3|\varepsilon\Theta}{6\sigma_n^2} E(x_m - x_{m-1})^2.$$

putting $g_m(t) = E\big(e^{it(x_m - x_{m-1})}\big)$, $(m = 1, 2, \cdots, n)$

then $g_m\big(\frac{t}{\sigma_n}\big) = E\Big(E_m\Big(e^{it\frac{x_m - x_{m-1}}{\sigma_n}}\Big)\Big)$; from (3) we find

$$\text{(4)} \quad g_m\Big(\frac{t}{\sigma_n}\Big) = 1 - \frac{t^2}{2\sigma_n^2} E(x_m - x_{m-1})^2 + \frac{t^2\Theta}{\sigma_n^2} E\int_{|x-x_{m-1}|>\varepsilon\sigma_n} (x-x_{m-1})^2 dF_{m-1}(x) + \frac{|t^3|\varepsilon\Theta}{6\sigma_n^2} E(x_m - x_{m-1})^2$$

By virtue of the condition (L), it is easy to see that, for $m = 1, 2, \cdots, n$, $E(x_m^2 - x_{m-1}^2)/\sigma_n^2$ tends to zero uniformly as $n \to \infty$. Hence, for sufficiently large n, we have

$$\log g_m\Big(\frac{t}{\sigma_n}\Big) = g_m\Big(\frac{t}{\sigma_n}\Big) - 1 + o\Big(\Big|g_m\Big(\frac{t}{\sigma_n}\Big) - 1\Big|\Big).$$

(4) and the equality $E(x_m - x_{m-1})^2 = E(x_m^2) - E(x_{m-1}^2)$, which is easily verified by the condition (F), imply that

$$\sum_{m=1}^{n} \Big|g_m\Big(\frac{t}{\sigma_n}\Big) - 1\Big| \leq \frac{3}{2}t^2 + \frac{|t^3|}{6}\varepsilon$$

and

$$\sum_{m=1}^{n} \log g_m\Big(\frac{t}{\sigma_n}\Big) = -\frac{t^2}{2} + \frac{\Theta t^2}{2\sigma_n^2} \sum_{m=1}^{n} E\Big\{\int_{|x-x_{m-1}|>\varepsilon\sigma_n} (x-x_{m-1})^2 dF_{m-1}(x)\Big\} + \frac{|t^3|}{6}\varepsilon + o\Big(\frac{3}{2}t^2 + \frac{|t^3|}{6}\varepsilon\Big).$$

Therefore, from (L), we have

$$\overline{\lim_{n\to\infty}} \Big|\sum_{m=1}^{n} \log g_m\Big(\frac{t}{\sigma_n}\Big) + \frac{t^2}{2}\Big| \leq \frac{|t^3|}{6}\varepsilon,$$

ε being arbitrary,

$$\overline{\lim_{n\to\infty}} \Big|\sum_{m=1}^{n} \log g_m\Big(\frac{t}{\sigma_n}\Big) + \frac{t^2}{2}\Big| \leq \frac{|t^3|}{6}\varepsilon,$$

that is,

$$\lim_{n\to\infty} \prod_{m=1}^{n} g_m\left(\frac{t}{\sigma_n}\right) = e^{-\frac{t^2}{2}} \tag{5}$$

Putting

$$f_m(t) = e^{itX_m} \prod_{\nu=m+1}^{n} g_\nu(t) \qquad (m=0, 1, \cdots, n),$$

(5) shows that $\lim_{n\to\infty} f_0(t/\sigma_n) = \exp(-t^2/2)$.
Next we shall estimate the absolute value of the expectation of $f_m(t/\sigma_n) - f_{m-1}(t/\sigma_n)$.

$$\left| E\left(f_m\left(\frac{t}{\sigma_n}\right) - f'_{m-1}\left(\frac{t}{\sigma_n}\right)\right)\right|$$

$$= \left|\prod_{\nu=m+1}^{n} g_\nu\left(\frac{t}{\sigma_n}\right)\right| \left| E\left\{ e^{it\frac{X_{m-1}}{\sigma_n}} \left(e^{it\frac{X_m - X_{m-1}}{\sigma_n}} - g_m\left(\frac{t}{\sigma_n}\right)\right)\right\}\right|$$

$$\leqq \left| E\left\{ e^{it\frac{X_{m-1}}{\sigma_n}} E_{m-1}\left(e^{it\frac{X_m - X_{m-1}}{\sigma_n}} - g_m\left(\frac{t}{\sigma_n}\right)\right)\right\}\right|$$

$$\leqq E\left\{\left| E_{m-1}\left(e^{it\frac{X_m - X_{m-1}}{\sigma_n}} - g_m\left(\frac{t}{\sigma_n}\right)\right)\right|\right\}$$

From (3) and (4), we find that

$$\left| E\left(f_m\left(\frac{t}{\sigma_n}\right) - f_{m-1}\left(\frac{t}{\sigma_n}\right)\right)\right|$$

$$\leqq \frac{2t^2}{\sigma_n^2} E \int_{|x-X_{m-1}|>\varepsilon\sigma_n} (x - X_{m-1})^2 dF_{m-1}(x) + \frac{|t^3|\varepsilon}{3\sigma_n^2} E(X_m - X_{m-1})^2,$$

hence,

$$\left| E\left(f_n\left(\frac{t}{\sigma_n}\right)\right) - f_0\left(\frac{t}{\sigma_n}\right)\right|$$

$$\leqq \sum_{m=1}^{n} \left| E\left\{ f_m\left(\frac{t}{\sigma_n}\right) - f_{m-1}\left(\frac{t}{\sigma_n}\right)\right\}\right|$$

$$\leqq 2t^2 \frac{1}{\sigma_n^2} \sum_{m=1}^{n} E \int_{|x-X_{m-1}|>\varepsilon\sigma_n} (x - X_{m-1})^2 dF_{m-1}(x) + \frac{|t|^3}{3}\varepsilon,$$

and

$$\overline{\lim_{n\to\infty}} \left| E\left(f_n\left(\frac{t}{\sigma_n}\right)\right) - f_0\left(\frac{t}{\sigma_n}\right)\right| \leqq \frac{|t|^3}{3}\varepsilon,$$

ε being arbitrary,

$$\lim_{n\to\infty} \left\{ E\left(f_n\left(\frac{t}{\sigma_n}\right)\right) - f_0\left(\frac{t}{\sigma_n}\right)\right\} = 0 \tag{6}$$

(5) and (6) imply that

$$E\left(e^{it\frac{X_n}{\sigma_n}}\right) = E\left(f_n\left(\frac{t}{\sigma_n}\right)\right) \to e^{-\frac{t^2}{2}}, \quad (n\to\infty). \tag{7}$$

We have assumed t to b fixed, but since $\exp(-t^2/2)$ is a characteristic function, by (7) we have the theorem.

(*) Received July 28, 1949.

1) J.L.Doob, Regularity properties of certain families of chance variables. Trans. of Amer. Math. Soc. Vol.47, (1940).

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