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ON DETERMINING THE POINTS OF THE SECOND COEFFICIENT BODY (a_4, a_3, a_2) FOR BOUNDED REAL UNIVALENT FUNCTIONS

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1. Introduction

In studying classes of univalent functions attempts to determine the complete range of the coefficients have been of primary importance. This, again, has been connected with efforts to extremalize singular coefficients. As is well known, in the general class S of normalized univalent functions f,

$$f(z)=z+a_2z^2+\cdots, \qquad |z|<1,$$

this trend is not relevant any more. In the class $S(b)$ of bounded univalent functions f,

$$f(z)=b(z+a_2z^2+\cdots),\ |z|<1,\ |f(z)|<1;\ 0<b<1,$$

maximizing the coefficients $a_n=b_n/b$ is essentially open from $n=4$ upwards. Moreover, the complicated structure of the coefficient bodies (a_3, a_2) and (a_4, a_3, a_2) suggests the evident fact that the coefficient body problem itself is a very difficult one and can not give much help in maximizing singular a_n-coefficients in general. Thus, the problem of determining the first coefficient bodies must be interpreted as a central problem itself without any special trends for further applications.

The first characterizations for coefficient bodies in $S(b)$ are obtained by aid of the variational method. For (a_3, a_2) this is done by Charzyński and Janowski [1]. The results, however, remain in implicit form yielding information mainly of the general type of the extremal functions. The next body (a_4, a_3, a_2) is closely related with studies concerning a_4 in the class of real bounded univalent functions $S_R(b)\subset S(b)$, $a_n\in \boldsymbol{R}$ [6]. In [6] the types of the extremal functions are classified by aid of the symbol $\alpha:\beta$. Here α denotes the amount of starting points and β that of the endpoints of the slit system for the extremal image $f(U)$; U: $|z|<1$. However, also in this case the results only classify the extremal cases.

The turning point in obtaining explicit results for bodies is the work done

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together with Kortram by using Grunsky type inequalities. The idea of this was first applied for (a_3, a_2) in $S(b)$ [4] and was tested successfully for the second coefficient body in $S_R(b)$ yielding first algebraic parts of it [5]. However, in both cases the results remained partial and their completion needed much additional work and above all improving the inequalities available. This was achieved by using so called Löwner-identities invented by Haario and Jokinen. The results of their works are described in the compendiums [7] and [8]. Especially the missing algebraic part of (a_4, a_3, a_2) in $S_R(b)$ was found by Jokinen by deriving a sharp inequality by aid of a proper Löwner expression [3].

Until now a great difficulty has been encountered when trying to process elliptic parts of the body (a_4, a_3, a_2). This was first discussed in [6] but, as mentioned above, without final success. In the present paper this part will finally be completed by combining the result of Schiffer's differential equation to the working idea of G. Goodman [2]. According to it extremal Löwner-expressions remain constant in the Löwner parameter $u=e^{-t}(t\in]0, \infty[)$. Apply this to the left side of Schiffer's equation for obtaining, in the extremal case, the elliptic specifications $T(u)$ and $R(u)$, to be defined later on. Thus, the finding of the elliptic cases will be based on the fact that they are solutions of Schiffer's equation and best presented in the characterizing function $\kappa(u)$ of Löwner. Solving this κ appears to be possible by aid of the constancy idea mentioned.

Observe that in this final stage we have to get along without the convenience of sharp inequality which in all the former cases was at our disposal. This implies that we loose the direction of estimation and do not know if the extremal points belong to the upper or lower part of the surface of the body. As will be observed in Section 5 this, however, has no essential meaning because these parts are easily converted with each other. The main problem will be finding the proper technique of handling the elliptic integrals involved. It appears that for developing this technique the method left, characterizing extremal function by aid of differential equation, offers all the information to be needed.

2. The algebraic parts

For the sake of completeness, let us repeat here the results concerning the algebraic parts of the coefficient body (a_4, a_3, a_2). The results are those collected in [7] and [8].

In [8], p. 149, there is a map in the a_2a_3-plane where different boundary function types are listed. On pp. 150–154 the curves on which these types are interchanged are given in explicit form.

The first algebraic part with the extremal domains of the types 3: 3 and 1: 3 was first determined by using the Grunsky-type condition. The result is in [7], p. 285 and the curve separating these two parts is given in equation (9), p. 276 of [7]. This result was later refined and extended by estimating a properly chosen Löwner expression, a Löwner-identity, discussed in [8] pp. 122-136. The main advancement was reached by finding the exact form for extremal $\kappa=\kappa(u)$. The result is in Theorem 1, p. 133 of [8] where the points of the upper surface of the coefficient body, connected with 3: 3- and 1: 3-functions are determined by aid of two parameters σ and λ:

$$(1)\qquad \begin{cases} a_2=2(\sigma-b)-\dfrac{2}{3}(1-\sigma)+\dfrac{2}{3}(1-3\lambda)(1-1\sqrt{\sigma}), \\ a_3=a_2^2+1-b^2-2(\sigma^2-b^2)-\dfrac{2}{9}(1-\sigma^2)-\dfrac{8}{9}(1-3\lambda)(1-\sqrt{\sigma}) \\ \qquad +\dfrac{1}{9}(1-3\lambda)^2(1-1/\sigma), \\ a_4=2a_2a_3-\dfrac{13}{12}a_2^3+\dfrac{2}{3}(1-b^3)-\dfrac{1}{2}ba_2^2-\lambda\left(a_3-\dfrac{3}{4}a_2^2+ba_2\right); \\ b\leqq\sigma\leqq 1,\ \dfrac{1}{3}-\dfrac{4}{3}\sigma^{3/2}\leqq\lambda\leqq\dfrac{1}{3}+\dfrac{8}{3}\sigma^{3/2}. \end{cases}$$

The range of σ and λ is given in Figure 32, p. 130 of [8]. The dividing parabola between the types 3: 3 and 1: 3, mentioned above, has the equation

$$(2)\qquad a_3=\frac{5}{4}a_2^2-(1+3b)a_2+6b(1-b).$$

This together with the above expressions of a_2 and a_3 determine the connection of σ and λ on this dividing line. That connection, however, is quite complicated and will not be discussed here.

The idea of determining the extremal $\kappa(u)$ was successfully used by Jokinen in finding the second algebraic part of the body, where the extremal functions are of the type 2: 3 [3]. The results are described in [8], pp. 144-155. The points of the surface of the coefficient body with boundary functions 2: 3 are again expressable in two parameters σ_1 and σ_2:

$$(3)\qquad \begin{cases} a_2=-\dfrac{2}{3}-2b+4\sigma_1-4\sigma_2+\dfrac{8}{3}\sigma_2^{3/2}, \\ a_3=a_2^2+\dfrac{7}{9}+b^2-\dfrac{32}{9}\sigma_2^{3/2}+\dfrac{16}{9}\sigma_2^3, \\ a_4=2a_2a_3-a_2^3+b^2a_2+2(a_3-a_2^2+1-b^2)-\dfrac{4}{3}(a_3-a_2^2+1-b^2)^{3/2}; \\ b\leqq\sigma_1\leqq\sigma_2\leqq 1. \end{cases}$$

In [8], pp. 149-154, the boundary arcs of the algebraic parts of the body are given in explicit form. This is due to the fact that the concrete expressions of the exremal $\kappa(u)$ are available. Next, when turning to the remaining elliptic cases we will keep this in mind and will try, again, to find information about $\kappa(u)$ yielding the elliptic boundary functions. Clearly, $\kappa(u)$ can not be expressed in explicit form, and we have to confine ourselves to the determining of certain elliptic differential equations.

3. The elliptic boundary functions 2: 2

As mentioned in Section 1, the necessary condition for the boundary function is derived in [1] and is of the type of Schiffer's differential equation, an example of which is e.g. that in [7], p. 142, belonging to maximal $a_n>0$ in $S(b)$. Because we are dealing with the type 2: 2 we can factorize the equation according to the structure of the slit-system [6]:

$$\left(z\frac{f'}{f}\right)^2\frac{b^4}{f^3}(f^2-Tf+1)^2(f^2-Rf+1)=\frac{b}{z^3}(z^2-\tau z+1)^2(z^2-Pz+1). \tag{4}$$

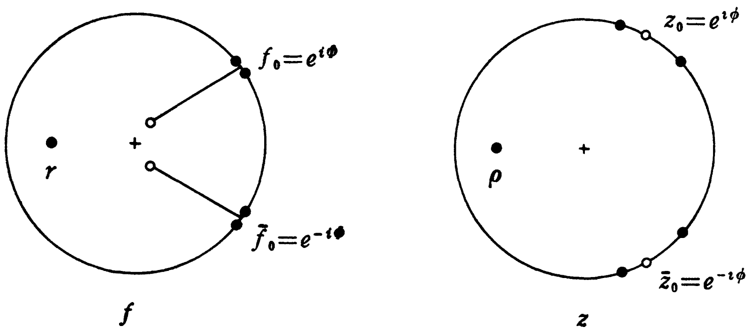

Figure 1.

Here is

$$R=r+1/r,\quad T=2\cos\Phi\,;\ P=\rho+1/\rho,\quad \tau=2\cos\phi,$$

with the four parameters r, $\rho\in]0, 1]$ and Φ, $\phi\in[0, 2\pi[$ (Figure 1).

Suppose now that $f\in S_R(b)$ is a solution of Löwner's differential equation [7], p. 27 and 68. Denote $u=e^{-t}\in[b, 1]$, $\kappa=\kappa(u)$ and $f=f(u)$. The solution sought will be $f=f(b)$. Write (4) for $b=u$ in the integrated form; the upper limit $f=f(u)$:

$$\int_1^f u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}df=\int_1^z\frac{(z^2-\tau z+1)(z^2-Pz+1)^{1/2}}{z^{5/2}}dz. \tag{5}$$

According to Goodman's principle the left side is constant for $u\in[b, 1]$ and hence the extremal $\kappa(u)$ satisfies the necessary condition

$$\frac{d}{du}\int_1^f u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}df=0.$$

$\Rightarrow$

$$A(u)=u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}\cdot\frac{\partial f}{\partial u}$$

$$+\int_1^f\frac{\partial}{\partial u}u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)}{f^{5/2}}df=0.$$

According to (11), p. 68 of [7]

$$\frac{\partial f}{\partial u}=u^{-1}\frac{f-f^3}{1-Tf+f^2}$$

and hence

$$A(u)=u^{3/2}(f-f^3)\frac{(f^2-Rf+1)^{1/2}}{f^{5/2}}$$

$$+\int_1^f u\frac{\partial}{\partial u}u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}df$$

$$=\int_1^f\Big[u^{3/2}\frac{\partial}{\partial f}\frac{(f^2-Rf+1)^{1/2}(f-f^3)}{f^{5/2}}$$

$$+u\frac{\partial}{\partial u}u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}\Big]df=0.$$

This condition is now processed as follows:

$$u^{3/2}\frac{\partial}{\partial f}\frac{(f^2-1)(f^2-Rf+1)^{1/2}}{f^{3/2}}=u\frac{\partial}{\partial u}u^{3/2}\frac{(f^2-Tf+1)(f^2-Rf+1)^{1/2}}{f^{5/2}}$$

$\Rightarrow$

$$u^{3/2}\frac{\partial}{\partial f}(f-f^{-1})(f+f^{-1}-R)^{1/2}=u\frac{\partial}{\partial u}u^{3/2}(f+f^{-1}-T)(f+f^{-1}-R)^{1/2}f^{-1}$$

Performing the differentiations yields:

$$u^{3/2}[(1+f^{-2})(f+f^{-1}-R)^{1/2}+1/2\cdot(f-f^{-1})(f+f^{-1}-R)^{-1/2}(1-f^{-2})]$$

$$=u^{5/2}f^{-1}[-T'(f+f^{-1}-R)^{1/2}+1/2\cdot(f+f^{-1}-T)(f+f^{-1}-R)^{-1/2}(-R')]$$

$$+3/2\cdot u^{3/2}(f+f^{-1}-T)(f+f^{-1}-R)^{1/2}f^{-1}$$

$\Rightarrow$

$$(f+f^{-1})(f+f^{-1}-R)^{1/2}+1/2\cdot(f-f^{-1})^2(f+f^{-1}-R)^{-1/2}$$

$$=-uT'(f+f^{-1}-R)^{1/2}-1/2\cdot uR'(f+f^{-1}-T)(f+f^{-1}-R)^{-1/2}$$

$$+3/2\cdot(f+f^{-1}-T)(f+f^{-1}-R)^{1/2}.$$

Rewrite this by using the abbreviation

$$x=f+f^{-1};$$

$$x(x-R)^{1/2}+1/2\cdot(x^2-4)(x-R)^{-1/2}$$

$$=-uT'(x-R)^{1/2}-1/2\cdot uR'(x-T)(x-R)^{-1/2}+3/2\cdot(x-T)(x-R)^{1/2}$$

$\Rightarrow$

$$(-1/2\cdot x+3/2\cdot T+uT')(x-R)=2+1/2\cdot uR'T-1/2\cdot uR'x-1/2\cdot x^2$$

$\Rightarrow$

$$-3/2\cdot RT-uT'R+(1/2\cdot R+3/2\cdot T+uT')x-1/2\cdot x^2$$

$$=2+1/2\cdot uR'T-1/2\cdot uR'x-1/2\cdot x^2$$

$\Rightarrow$

$$\begin{cases} -3/2\cdot RT-uT'R=2+1/2\cdot uR'T, \\ 1/2\cdot R+3/2\cdot T+uT'=-1/2\cdot uR', \end{cases}$$

i.e.

$$\begin{cases} 4+uR'T+2uT'R+3RT=0, \\ R+uR'+3T+2uT'=0. \end{cases} \tag{6}$$

This is a system of differential equations for R and T which is simplified to the form

$$\begin{cases} 2uT'=\dfrac{3T^2-2RT-4}{R-T}, \\ uR'=\dfrac{4-R^2}{R-T}. \end{cases} \tag{7}$$

This system must be integrated by starting from the initial values $T=T(b)$, $R=R(b)$ and yielding the solution $\tau=T(1)$, $P=R(1)$. Clearly, in this way the original condition (5) is satisfied by the $\kappa(u)$ found. The existence of the extremal function 2: 2 as a solution of (5) thus guarantes that the $\kappa(u)$ found determines the elliptic solution sought.

The coefficients needed to fix the boundary points of the body (a_4, a_3, a_2) are obtained from the formulae (15), p. 70, of [7]. If $\kappa(u)=e^{-i\vartheta}$, we denote $c=\cos\vartheta(u)$. Hence $T=2c$. For brevity, shorten the integral-notation: $\int_b^1 du=\int$. Thus

$$\begin{cases} -a_2=2\int c=\int T, \\ a_3=4\left(\int c\right)^2-2\int u(2c^2-1)=\left(\int T\right)^2-\int u(T^2-2), \\ -a_4=8\left(\int c\right)^3-8\int c\cdot\int u(2c^2-1)-4\int u(2c^2-1)\int_u^1 c\,du_1+2\int u^2(4c^3-3c) \\ \quad =\left(\int T\right)^3-2\int T\cdot\int u(T^2-2)-\int u(T^2-2)\int_u^1 T\,du_1+\int u^2(T^3-3T). \end{cases}$$

From the latter equation (6) we obtain

$$(uR+2uT)'=-T\,;\ \int_u^1 T=\Big/_u^1(2uT+uR).$$

Hence

$$a_2=\Big/_b^1(2T+R)u\,.$$

For a_3 we had above the formula

$$a_3-a_2^2=-\int u(T^2-2).$$

The right side can be integrated by using the following identity:

$$\begin{aligned}[u^2(T^2+2TR+3)]'&=2u^2(T+R)T'+2u^2TR'+2u(T^2+2TR+3)\\ &=u(T+R)\frac{3T^2-2RT-4}{R-T}+2uT\frac{4-R^2}{R-T}+2u(T^2+2TR+3)\\ &=u\frac{T^3-RT^2-2T+2R}{R-T}=-u(T^2-2)\end{aligned}$$

$\Longrightarrow$

$$a_3-a_2^2=\Big/_b^1 u^2(T^2+2TR+3)$$

Finally, for a_4 there follows

$$\begin{aligned}-a_4&=-a_2^3+2a_2(a_2^2-a_3)+\int u(T^2-2)\Big/_u^1(2uT+uR)+\int u^2(T^3-3T)\\ &=a_2^3-2a_2a_3-(2\tau+P)(a_3-a_2^2)+\int[u^2(2-T^2)(2T+R)+u^2(T^3-3T)]\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}a_4&=-a_2^3+2a_2a_3+(2\tau+P)(a_3-a_2^2)+\int u^2(T^3+T^2R-2R-T)\\ &=-a_2^3+2a_2a_3+(2\tau+P)(a_3-a_2^2)-\int u^2(-T^3-T^2R)-1/2\cdot\int 2u^2(2R+T).\end{aligned}$$

In order to perform the integrations left two identities are to be considered.

$$[u^3(T^2R+4T+2R)]'=3u^2(T^2R+4T+2R)+u^3(2TR+4)T'+u^3(T^2+2)R'$$

$$=u^2\left[3T^2R+12T+6R+\frac{(TR+2)(3T^2-2RT-4)}{R-T}+\frac{(T^2+2)(4-R^2)}{R-T}\right]$$

$$=u^2\frac{4R^2-2RT-2T^2}{R-T}=2u^2(2R+T);$$

$$[u^3(T^3+R^2T)]'=3u^2(T^3+R^2T)+u^3(3T^2+R^2)T'+u^3 2RTR'$$

$$=\left[3T^3+3R^2T+1/2\cdot(3T^2+R^2)\frac{3T^2-2RT-4}{R-T}+2RT\frac{4-R^2}{R-T}\right]$$

$$=u^2(-3/2\cdot T^3-3/2\cdot RT^2+6T-2R).$$

For brevity, write temporarily $\Big/_b^1=\Big/$;

$$3/2\int u^2(-T^3-RT^2)+\int u^2(6T-2R)=\Big/ u^3(T^3+R^2T)$$

$\Longrightarrow$

$$\int u^2(-T^3-RT^2)=2/3\cdot\Big/ u^3(T^3+R^2T)+\int[2/3\cdot u^2(2R+T)+7/6\cdot(-4u^2T)]$$

$$=2/3\cdot\Big/ u^3(T^3+R^2T)+1/3\cdot\Big/ u^3(T^2R+4T+2R)+7/6\cdot\int -4u^2T.$$

The last integral is obtained from

$$(u^3T^2R)'=3u^2T^2R+u^3 2T'TR+u^3T^2R'$$

$$=3u^2T^2R+u^2T(2uT'R+uTR')$$

$$=3u^2T^2R-4u^2T-3u^2RT^2=-4u^2T$$

$\Longrightarrow$

$$\int u^2(-T^3-RT^2)=1/3\cdot\Big/ u^3[2T^3+2R^2T+T^2R+4T+2R]+7/6\cdot\Big/ u^3T^2R$$

$$=1/6\cdot\Big/ u^3[4T^3+9T^2R+4R^2T+8T+4R].$$

Thus we have the expression for $\int u^2(-T^3-T^2R)+1/2\cdot\int 2u^2(2R+T)$ needed for determining a_4. Let us collect the integrated expressions of the coefficients. Remember that in the substitutions involved we denote

$$T=T(b),\quad R=R(b) \quad \text{and} \quad T(1)=\tau,\quad R(1)=P;$$

$$
(8) \quad \begin{cases} a_2 = \int_b^1 u(2T+R), \\ a_3 = a_2^2 + \int_b^1 u^2(T^2+2TR+3), \\ a_4 = 2a_2a_3 - a_2^3 + (2\tau+P)(a_3-a_2^2) - \dfrac{1}{3}\int_b^1 u^3(2T^3+6T^2R+2R^2T+10T+5R). \end{cases}
$$

The two parameters available are $T \in [-2, 2]$ and $R \in [-2, -\infty[$. By integrating the system (7) (numerically) with given initial values T and R the corresponding values τ and P are determined and hence the point (8) of the elliptic part $2:2$ of the cofficient body.

When taking $R \equiv -2$ and letting T cover the interval $[-2, 2]$ we are dealing with the limit case called the curve $2'$ in Figure 38, p. 149 of [8]. This case can be used as a test of the sharpness of the numerical integration oi (7). Observe, that in this special case (7) can also be integrated in exact form because on $2'$, obviously, $R(u) \equiv -2$. Thus

$$2u\frac{dT}{du} = \frac{3T^2+4T-4}{-2-T}$$

$\Longrightarrow$

$$\frac{du}{u} = -2\frac{T+2}{3(T+2)(T-2/3)}dT = -2/3 \cdot \frac{dT}{T-2/3}$$

$\Longrightarrow$

$$T = 2/3 + 2lu^{-3/2}.$$

Hence $\tau = 2/3+2k$ and $P=-2$ and we obtain from (8)

$$
\begin{cases} a_2 = -2/3 \cdot (1-b) + 4l(1-b^{-1/2}), \\ a_3 = a_2^2 + 7/9 \cdot (1-b^2) - 16/3 \cdot l(1-b^{1/2}) + 4l^2(1-b^{-1}). \end{cases}
$$

The notation $l = (1-3\lambda)/6$ yields the formulae of $2'$ on p. 152 of [8].

When fixing $-R$ sufficiently large and letting T cover the interval $[-2, 2]$ the upper boundary arc of the body emerges. In Figure 38, p. 149 of [8] the projection is visible without any name. According to pp. 226–227 of [7] we can find out the exact coordinates of this curve on which the boundary functions are of the algebraic type $2:2$;

$$
(9) \quad \begin{cases} |a_2| \leqq 2b|\ln b|, \\ a_3 = 1-b^2 + (1+1/\ln b)a_2^2, \\ a_4 = 2a_2a_3 - a_2^3 - b^2a_2 + \dfrac{a_2}{\ln b}(a_3 - a_2^2/2) \\ \quad = \left(2-3b^2+\dfrac{1-b^2}{\ln b}\right)a_2 + (1+1/\ln b)(1+2/\ln b)a_2^3. \end{cases}
$$

Because a_2 and a_3 in the present case belong to the uniquely determined extremal function of the first body (a_3, a_2) the same uniqueness remains to hold for the boundary function of the body (a_4, a_3, a_2). Hence, the part of the body shrinks to the curve (9) in the present case.

Table 1.

T	R	a_2	a_3	a_4
2	−2	−0.782	1.289	−1.082
2	−4	−0.746	1.265	−1.034
2	−8	−0.693	1.232	−0.967
2	−16	−0.628	1.195	−0.892
2	−32	−0.565	1.163	−0.825
2	−64	−0.519	1.140	−0.778
2	−128	−0.490	1.126	−0.755
2	−256	−0.474	1.119	−0.757
2	−512	−0.464	1.114	−0.824

$b=0.1, \quad T\equiv 2$

There are two more boundary curves worth considering; those on which the elliptic types 2:2 and 1:2 unite. The left border line is obtained for $T=2$, $R\in]-\infty, -2]$ and the right one for $T=-2$, $R\in]-\infty, -2]$. Unfortunately, the functions $R(u)$ and $T(u)$ are more complicated on these border lines as was the case on $2'$. Therefore we must confine ourselves on examples given in the above Table 1.

Table 2.

T	R	a_2	a_3	a_4
2	−2	−0.509	1.063	−0.576
2	−4	−0.441	1.033	−0.480
2	−8	−0.345	1.006	−0.357
2	−16	−0.233	0.992	−0.231
2	−32	−0.136	0.989	−0.129
2	−64	−0.071	0.990	−0.066
2	−128	−0.035	0.990	−0.032
2	−256	−0.016	0.990	−0.012
2	−512	−0.004	0.990	0.002

$b=0.1, \quad T\equiv 0$

The boundary arcs mentioned above define a quadrilateral inside of which one can determine elliptic points of proper 2:2-types. For example, by fixing $T \in [-2, 2]$ and letting R cover the interval $]-\infty, -2]$ one obtains a curve located in the quadrilateral, e. g. that of Table 2.

In order to understand the above choices of T and R observe that all the elliptic cases 2:2 emerge when $T \in [-2, 2]$ and $R \in]-\infty, -2]$ or $\in [2, +\infty[$. The cases $R>0$ and $R<0$ are obtained from each other by rotation (cf. Section 5). Comparison of the corresponding a_4-values shows that the former yields points on the lower part and the latter those on the upper part of the surface of the coefficient body.

4. The elliptic boundary functions 1:2

Above we have discussed the algebraic parts of the coefficient body and considered also the transition curves on which the extremal type degenerates and is then transformed to another type. Thus, e. g. on the curve 2, Figure 38, p. 149 of [8], there holds the limit case of 2:2 with the degenerated fork on the right side of the image $f(U)$ and a radial slit on the left side of it. It is not difficult to check that the proper extremal type 2:3 holding in II (cf. Figure 38) is obtained by superimposing the above-mentioned mapping with the right radial-slit mapping. This very idea can be utilized when constructing the remaining elliptic mappings of the type 1:2.

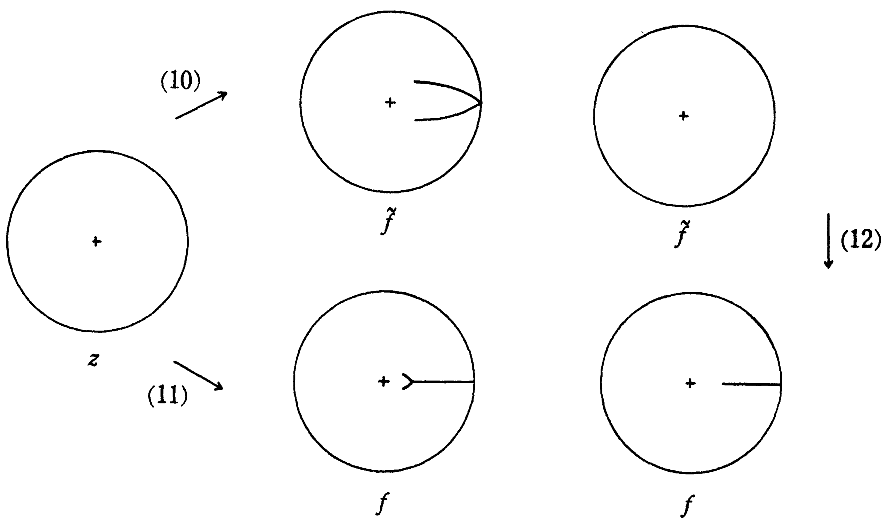

Figure 2.

Consider the schematic presentation of the mappings to be applied (Figure 2). The connection $\tilde{f}=\tilde{f}(z)$ defines a degenerated right(-hand) elliptic mapping

$1:2$, $f=f(z)$ is the proper elliptic mapping $1:2$ sought and $\tilde{f}\to f$ is a right (-hand) radial-slit mapping. By using a parameter $a\in[b, 1]$ we write the following connections (cf. (4) for $T\equiv 2$ and (6)):

$$a^3\left(z\frac{\tilde{f}'}{\tilde{f}}\right)^2(\tilde{f}+\tilde{f}^{-1}-2)^2(\tilde{f}+\tilde{f}^{-1}-\tilde{R})=(z+z^{-1}-\tau)^2(z+z^{-1}-P), \tag{10}$$

$$b^3\left(z\frac{f'}{f}\right)^2(f+f^{-1}-2)(f+f^{-1}-R)(f+f^{-1}-S)=(z+z^{-1}-\tau)^2(z+z^{-1}-\Sigma), \tag{11}$$

$$\frac{b}{a}(f+f^{-1}+2)=\tilde{f}+\tilde{f}^{-1}+2;\ f=K_r(\tilde{f}). \tag{12}$$

In (11) $S=s+s^{-1}$ and $\Sigma=\sigma+\sigma^{-1}$. These numbers characterize the singularities of the mapping $f=f(z)$ in the same sence as R and P before (Figure 3).

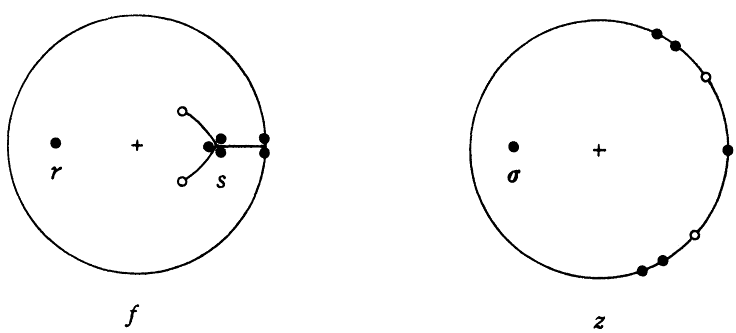

Figure 3.

By aid of (12) compare the left sides of (10) and (11). Introduce the abbreviations

$$x=f+f^{-1}, \qquad \tilde{x}=\tilde{f}+\tilde{f}^{-1}.$$

Thus

$$b^3\left(z\frac{f'}{f}\right)^2(f+f^{-1}-2)(f+f^{-1}-R)(f+f^{-1}-S)=b^3\left(z\frac{f'}{f}\right)^2(x-2)(x-R)(x-S).$$

From (12) there follows

$$\frac{b}{a}(x+2)=\tilde{x}+2;\ x=\frac{a}{b}(\tilde{x}+2)-2. \tag{13}$$

Differentiate (12) and utilize (13):

$$\frac{b}{a}(1-f^{-2})f'=(1-\tilde{f}^{-2})\tilde{f}'$$

$\Longleftrightarrow$

$$\frac{b}{a}(f-f^{-1})\frac{f'}{f}=(\tilde{f}-\tilde{f}^{-1})\frac{\tilde{f}'}{\tilde{f}}$$

$\Longrightarrow$

$$\frac{b^2}{a^2}(x^2-4)\left(\frac{f'}{f}\right)^2=(\tilde{x}^2-4)\left(\frac{\tilde{f}'}{\tilde{f}}\right)^2$$

$\Longrightarrow$

$$\left(\frac{f'}{f}\right)^2=\frac{a^2}{b^2}\frac{(\tilde{x}+2)(\tilde{x}-2)}{(x+2)(x-2)}\left(\frac{\tilde{f}'}{\tilde{f}}\right)^2=\frac{a}{b}\cdot\frac{\tilde{x}-2}{x-2}\left(\frac{\tilde{f}'}{\tilde{f}}\right)^2.$$

Hence

$$\begin{aligned}
b^3\left(z\frac{f'}{f}\right)^2(x-2)(x-R)(x-S)&=b^3\left(z\frac{\tilde{f}'}{\tilde{f}}\right)^2\frac{a}{b}(\tilde{x}-2)(x-R)(x-S)\\
&=b^3\left(z\frac{\tilde{f}'}{\tilde{f}}\right)^2\frac{a}{b}(\tilde{x}-2)\left[\frac{a}{b}(\tilde{x}+2)-2-R\right]\left[\frac{a}{b}(\tilde{x}+2)-2-S\right]\\
&=a^3\left(z\frac{\tilde{f}'}{\tilde{f}}\right)^2(\tilde{x}-2)\left[\tilde{x}+2-(2+R)\frac{b}{a}\right]\left[\tilde{x}+2-(2+S)\frac{b}{a}\right].
\end{aligned}$$

Require here that

$$\begin{cases}\tilde{R}=-2+(2+R)\cdot b/a\,,\\ 2=-2+(2+S)\cdot b/a\,;\end{cases}$$

$$(14)\qquad\begin{cases}a=\dfrac{2+S}{4}b\,,\\ \tilde{R}=4\dfrac{2+R}{2+S}-2\,.\end{cases}$$

The limits for a are $b\leqq a\leqq 1$ which implies for S:

$$2\leqq S\leqq\frac{4}{b}-2\,.$$

Because $\tilde{R}\leqq-2$ we have for R similarly

$$R\leqq-2\,.$$

The connection (14) guarantees the transformation sought:

$$b^3\left(z\frac{f'}{f}\right)^2(f+f^{-1}-2)(f+f^{-1}-R)(f+f^{-1}-S)=a^3\left(z\frac{\tilde{f}'}{\tilde{f}}\right)^2(\tilde{f}+\tilde{f}^{-1}-2)^2(\tilde{f}+\tilde{f}^{-1}-\tilde{R}).$$

Thus by starting from the two characterizing parameters R and S we determine a and $\tilde{R}$ from (14). The solution of (10) yields $\tau=T(1)$ and $P=R(1)=\Sigma$ and thus $\tilde{f}(z)$. The solution f of (11), which was to be determined, is finally (cf. (12))

$$f(z)=K_\tau(\tilde{f}(z))\,.$$

In order to find the coefficients a_ν of $f(z)=b(z+a_2z^2+\cdots)$ consider the right radial-slit mapping $f(z)=K_r(z)=k(z+a_2(k)z^2+\cdots)$ defined by

$$k(f+f^{-1}+2)=z+z^{-1}+2\,.$$

For the coefficients of $K_r(z)$ we have

$$\left\{\begin{aligned} -a_2(k)&=2-2k\,,\\ a_3(k)&=3-8k+5k^2\,,\\ -a_4(k)&=4-20k+30k^2-14k^3\,.\end{aligned}\right. \tag{15}$$

The connection (12) implies

$$f=K_r(\tilde f)=k(\tilde f+a_2(k)\tilde f^2+\cdots)\,;\ \ k=b/a\,.$$

In this substitute, according to (10),

$$\tilde f(z)=a(z+\tilde a_2z^2+\cdots)\,.$$

Comparing the coefficients we obtain finally:

$$\left\{\begin{aligned} &a_2=\tilde a_2+a\cdot a_2(k)\,,\\ &a_3=\tilde a_3+a\cdot a_2(k)\cdot 2\tilde a_2+a^2\cdot a_3(k)\,,\\ &a_4=\tilde a_4+a\cdot a_2(k)(2\tilde a_3+\tilde a_2^2)+a^2\cdot a_3(k)\cdot 3\tilde a_2+a^3\cdot a_4(k)\,;\\ &a=\frac{2+S}{4}b,\ \ k=\frac{b}{a}=\frac{4}{2+S}\,;\\ &a_\nu(k)\ \text{from (15).}\end{aligned}\right. \tag{16}$$

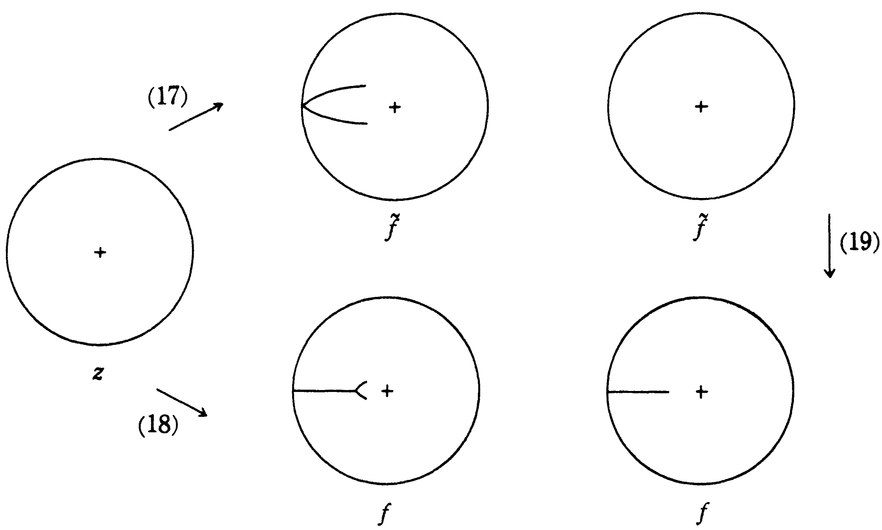

Figure 4.

There remains the left elliptic mapping 1 : 2 for which the previous procedure is to be repeated according to Figure 4 and the formulae (17)-(19):

$$a^3\Big(z\frac{\tilde{f}'}{\tilde{f}}\Big)^2(\tilde{f}+\tilde{f}^{-1}+2)^2(\tilde{f}+\tilde{f}^{-1}-\tilde{R})=(z+z^{-1}-\tau)(z+z^{-1}-P), \tag{17}$$

$$b^3\Big(z\frac{f'}{f}\Big)^2(f+f^{-1}+2)(f+f^{-1}-R)(f+f^{-1}-S)=(z+z^{-1}-\tau)^2(z+z^{-1}-\Sigma), \tag{18}$$

$$\frac{b}{a}(f+f^{-1}-2)=\tilde{f}+\tilde{f}^{-1}-2\,;\ f=K_l(\tilde{f}). \tag{19}$$

Again, $S=s+s^{-1}$ and $R=\sigma+\sigma^{-1}$ characterize the mapping sought, according to the notations in Figure 5.

Repeat the above transformation;

$$b^3\Big(z\frac{f'}{f}\Big)^2(f+f^{-1}+2)(f+f^{-1}-R)(f+f^{-1}-S)=b^3\Big(z\frac{f'}{f}\Big)^2(x+2)(x-R)(x-S).$$

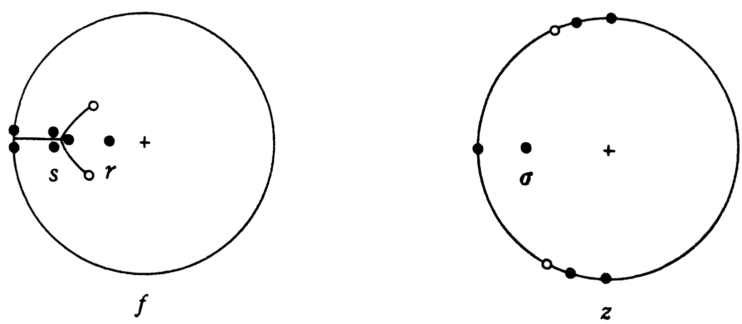

Figure 5.

The formula (19) yields

$$\frac{b}{a}(x-2)=\tilde{x}-2\,;\ x=\frac{a}{b}(\tilde{x}-2)+2 \tag{20}$$

$\Longrightarrow$

$$\frac{b^2}{a^2}(x^2-4)\Big(\frac{f'}{f}\Big)^2=(\tilde{x}^2-4)\Big(\frac{\tilde{f}'}{\tilde{f}}\Big)^2$$

$\Longrightarrow$

$$\Big(\frac{f'}{f}\Big)^2=\frac{a^2}{b^2}\frac{(\tilde{x}+2)(\tilde{x}-2)}{(x+2)(x-2)}\Big(\frac{\tilde{f}'}{\tilde{f}}\Big)^2=\frac{a}{b}\cdot\frac{\tilde{x}+2}{x+2}\Big(\frac{\tilde{f}'}{\tilde{f}}\Big)^2$$

$\Longrightarrow$

$$b^3\Big(z\frac{f'}{f}\Big)^2(x+2)(x-R)(x-S)=b^3\Big(z\frac{\tilde{f}'}{\tilde{f}}\Big)^2\frac{a}{b}(\tilde{x}+2)(x-R)(x-S)$$

$$=b^3\Big(z\frac{\tilde{f}'}{\tilde{f}}\Big)^2\frac{a}{b}(\tilde{x}+2)\Big[\frac{a}{b}(\tilde{x}-2)+2-R\Big]\Big[\frac{a}{b}(\tilde{x}-2)+2-S\Big]$$

$$=a^3\Big(z\frac{\tilde{f}'}{\tilde{f}}\Big)^2(\tilde{x}+2)\Big[\tilde{x}-2+(2-R)\frac{b}{a}\Big]\Big[\tilde{x}-2+(2-S)\frac{b}{a}\Big].$$

For the parameters put

$$\begin{cases} -\tilde{R}=-2+(2-R)\cdot b/a\,, \\ 2=-2+(2-S)\cdot b/a\,; \end{cases}$$

$$\begin{cases} a=\dfrac{2-S}{4}b\,, \\ \tilde{R}=-\Big(4\dfrac{2-R}{2-S}-2\Big). \end{cases} \tag{21}$$

The limits of a and $\tilde{R}$ are $b\leqq a\leqq 1$, $\tilde{R}\leqq -2$ which imply for S and R:

$$\begin{cases} 2-\dfrac{4}{b}\leqq S\leqq -2\,, \\ R\leqq S\leqq -2\,. \end{cases}$$

Thus, according to (21),

$$b^3\Big(z\frac{f'}{f}\Big)^2(f+f^{-1}+2)(f+f^{-1}-R)(f+f^{-1}-S)=a^3\Big(z\frac{\tilde{f}'}{\tilde{f}}\Big)^2(\tilde{f}+\tilde{f}^{-1}+2)^2(\tilde{f}+\tilde{f}^{-1}-\tilde{R}).$$

As before, given R and S determine the numbers a and $\tilde{R}$ and further $\tau=T(1)$, $P=R(1)=\Sigma$ yield $\tilde{f}(z)$ from (16). The solution f of (18) is

$$f(z)=K_l(\tilde{f}(z)).$$

This gives the coefficients a_ν of $f(z)=b(z+a_2z^2+\cdots)$. For the left radial-slit mapping $f(z)=K_l(z)$ we have

$$k(f+f^{-1}-2)=z+z^{-1}-2$$

$$\Longrightarrow$$

$$\begin{cases} a_2(k)=2-2k\,, \\ a_3(k)=3-8k+5k^2\,, \\ a_4(k)=4-20k+30k^2-14k^3\,. \end{cases} \tag{22}$$

From (19) there follows now

$$f=K_l(\tilde{f})=k(\tilde{f}+a_2(k)\tilde{f}^2+\cdots)\,;\ k=b/a\,.$$

The previous calculations can be repeated and they give the formulae (16) with $a_\nu(k)$ from (22):

$$(23)\quad \begin{cases} a_2=\tilde{a}_2+a\cdot a_2(k), \\ a_3=\tilde{a}_3+a\cdot a_2(k)\cdot 2\tilde{a}_2+a^2\cdot a_3(k), \\ a_4=\tilde{a}_4+a\cdot a_2(k)(2\tilde{a}_3+\tilde{a}_2^2)+a^2\cdot a_3(k)\cdot 3\tilde{a}_2+a^3\cdot a_4(k); \\ a=\dfrac{2-S}{4}b,\ k=\dfrac{b}{a}=\dfrac{4}{2-S}; \\ a_\nu(k) \text{ from (22).} \end{cases}$$

On pp. 9 and 10 the limit cases of 2:2 were discussed. Consider the projection of the body schematically given in Figure 38, p. 149 of [8]. The curves 1′ and 3′ in it yield the lower boundary arcs for 1:2-mappings. The latter is obtained when putting $R\equiv -2$. The former requires $S=R$ and thus does not emerge without preliminary limit processes in our formulae.

The upper boundary arcs of 1:2-regions are continuations of the curve defined in (9). These are related with the formulae (20) and (21), p. 234 of [7] and with those on p. 10 of [4]. The coordinates involved appear to be the following:

$$(24)\quad \begin{cases} \pm a_2=2\sigma\ln\sigma-2(\sigma-b);\ b\leqq\sigma\leqq 1;\ 2b|\ln b|\leqq|a_2|\leqq 2(1-b), \\ a_3=1-2b\sigma+(\sigma-b)^2+(\sigma-|a_2|)^2, \\ \pm a_4=2a_2a_3-a_2^3+b^2a_2+2(\sigma-\sigma^3)+2\sigma^2a_2(\sigma)+\sigma a_2(\sigma)^2;\ a_2(\sigma)=2\sigma\ln\sigma. \end{cases}$$

The signs + yield the left boundary arc and the signs — the right one.

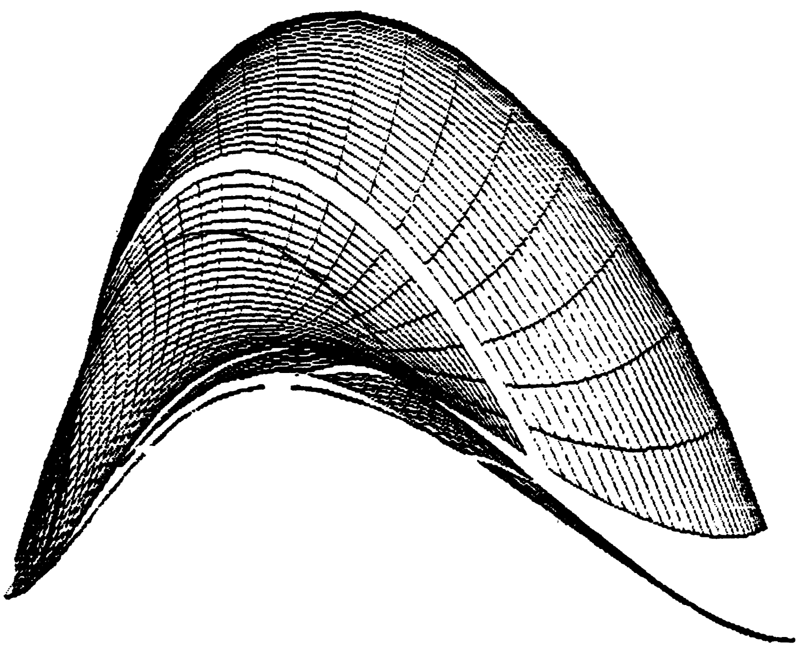

Figure 6.

5. The image of the coefficient body

The collection of the formulae, (1), (3), (8), (9), (16), (23) and (24) obtained, determines the boundary points of the coefficient body (a_4, a_3, a_2). Those found by using sharp inequalities can be connected, at (a_2, a_3), with the highest or lowest point in the direction of the a_4-axis. Those found by using differential equation may belong to any of these categories. The upper and lower surfaces, however, are closely connected with each other. This is seen by aid of the rotation

$$\tau^{-1}f(\tau z)=b(z+\tau a_2z^2+\tau^2a_3z^3+\tau^3a_4z^4+\cdots),$$

with $\tau=-1$, which changes the signs of a_4 and a_2 simultanously, preserving that of a_3 (cf. [7], pp. 285-286).

In Figure 6 there is a central projection of the upper surface of the coefficient body (a_4, a_3, a_2) for $b=0.1$ looked from a point $a_2=2$, $a_3=4$, $a_4=6$. The regions connected with points for which the coefficient expressions have denominators very close to zero are characterized by caps in the picture.

In the lower edge of the surface in Figure 6 there lies also the corresponding elliptic strip. The net on the compartement $2:2$ is obtained by aid of lines on which $T=$constant $\in]-2, 2[$. On the left $1:2$-triangle the net is obtained by using the lines $S=$constant $\in]2, +\infty[$ and on the right $1:2$-triangle, simimilarly, $S=$constant $\in]-\infty, -2[$.

In Figure 7 there is finally a stereoscopic pair of the previous surface, looked from different direction. The projection centers are $(0, -4, 1)$ for the right eye picture and $(-0.1, -4, 1)$ for the left one. The 3-dimensional impression is achieved either by readjusting the looking directions of eyes to unite the black spots in Figure 7 or more easyly, by superimposing the two components correspondingly by aid of some proper picture shifting devise.

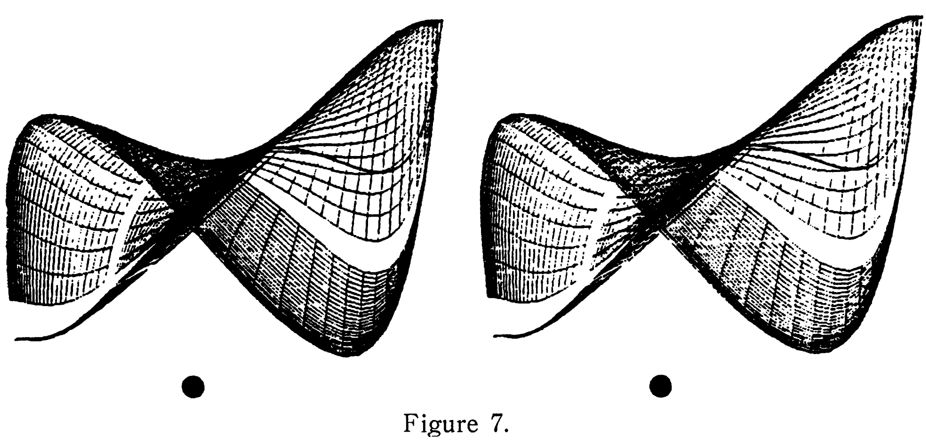

Figure 7.

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