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COMPLETE SPACE-LIKE HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A LORENTZ SPACE FORM OF DIMENSION 4

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Abstract

On complete space-like hypersurfaces with constant mean curvature in a Lorentz space form of dimension 4, we study the case that the scalar curvature is constant and that the Ricci curvature is bounded from above.

1. Introduction.

Let $\boldsymbol{R}_1^{n+1}$ be an $(n+1)$-dimensional Minkowski space and $\boldsymbol{S}_1^{n+1}(c)$ (resp. $\boldsymbol{H}_1^{n+1}(c)$) be an $(n+1)$-dimensional de Sitter space (resp. anti-de Sitter space) of constant curvature c. Considered collectively, a Lorentz manifold of these kinds is called a *Lorentz space form* of constant curvature c, which is denoted by $M_1^{n+1}(c)$.

Since Calabi [4] and S. Y. Cheng and Yau [7] proved the Bernstein type theorem in $\boldsymbol{R}_1^{n+1}$, complete space-like hypersurfaces with constant mean curvature in a Lorentz space form $M_1^{n+1}(c)$ have been studying by many mathematicians. On the other hand, space-like hypersurfaces with constant mean curvature in spacetimes get interested in relativity theory.

It is well known that totally umbilical hypersurfaces $M^n(c')$ $(c'<c)$ and hypersurfaces in the form of $\boldsymbol{H}^k(c_1)\times M^{n-k}(c_2)$ $[k=1, \cdots, n-1,\ c_1<0,\ c(c_1+c_2)=c_1c_2]$ are standard models of complete space-like hypersurfaces with constant mean curvature in $M_1^{n+1}(c)$. Here $M^n(c)$ means an n-dimensional space form with constant curvature c, that is, a Riemannian sphere $\boldsymbol{S}^n(c)$, a hyperbolic space $\boldsymbol{H}^n(c)$ or a Euclidean space $\boldsymbol{R}^n$.

Let M be a complete space-like hypersurface with constant mean curvature h/n in $M_1^{n+1}(c)$. In a de Sitter space $\boldsymbol{S}_1^{n+1}(c)$, M is nothing but totally umbilical if $n=2$ and $h^2\leqq 4c$ or if $n>2$ and $h^2<4(n-1)c$ (cf. Akutagawa [3], Ramanathan [12] or Cheng [5]).

In the other case, there are many examples in $M_1^{n+1}(c)$ which are not standard models (cf. Treibergs [13], Ishihara and Hara [8], Akutagawa [3] and others). But we have known some characterizations of standard models with respect to

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the squared norm S of the second fundamental form α on M. We define numbers S_0, S_- and S_+ by $S_0=h^2/n$ and $S_\pm=-nc+\{nh^2\pm(n-2)\sqrt{h^4-4(n-1)ch^2}\}/2(n-1)$, respectively. Then $S_0\leqq S_-\leqq S_+$. In [9], Ki, Kim and Nakagawa proved that

$$S_0\leqq S\leqq S_+,$$

where $S\equiv S_+\neq S_0$ only when M is a hyperbolic cylinder $\boldsymbol{H}^1(c_1)\times M^{n-1}(c_2)$. Also we remark that $S\equiv S_0$ only when M is a totally umbilical hypersurface $M^n(c')$ in $M_1^{n+1}(c)$. Furthermore, in a de Sitter space $\boldsymbol{S}_1^{n+1}(c)$, the second author and Nakagawa [6] proved that if $h^2\leqq n^2c$ and $\sup S<S_-$ then M is nothing but a totally umbilical hypersurface.

In the case of $n=2$, the hyperbolic cylinder $\boldsymbol{H}^1(c_1)\times M^1(c_2)$ is the only complete space-like surface in $M_1^3(c)$ with constant mean curvature $h/2$ on which S satisfies $\inf S>S_0$ (cf. Aiyama [2]).

However, in the case of $n=3$, we have an example on which S is constant and satisfies $S_0<S<S_+$, that is, $\boldsymbol{H}^2(c_1)\times M^1(c_2)$ in $M_1^4(c)$ $(c\geqq 0)$ satisfies $S\equiv S_-$. So the first purpose of this paper is to study the 3-dimensional complete space-like hypersurfaces with constant mean curvature and constant S in Lorentz space forms.

THEOREM 3.1. *Let M be a complete space-like hypersurface with non-zero constant mean curvature and constant scalar curvature in $M_1^4(c)$. If $S>S_-$ then M is nothing but a hyperbolic cylinder.*

In particular, we can completely classify complete space-like hypersurfaces with constant mean curvature and constant scalar curvature in $\boldsymbol{S}_1^4(c)$ if $h^2\leqq 9c$ (Theorem 3.2).

The paper is organized as follows. In Section 2 we give the basic concepts and prove some local formulae. In Section 3 we study the 3-dimensional complete space-like hypersurfaces with constant mean curvature and constant scalar curvature, and prove Theorem 3.1 and Theorem 3.2. At last, in Section 4 we consider the case that the Ricci curvature is bounded from above by $3(c-h^2/n^2)$.

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2. Local formulae.

Throughout this paper, we assume manifolds to be connected and geometric objects to be smooth.

Let (M, g) be a space-like hypersurface in an $(n+1)$-dimensional Lorentz space form $M_1^{n+1}(c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n$ on M adapted to the Riemannian metric g induced from the indefinite Riemannian metric on the ambient space, and $\omega_1, \cdots, \omega_n$ denote the dual coframes on M. The connection forms ω_{ij} are characterized by the structure equations

$$(2.1)\qquad \begin{cases} d\omega_i+\sum\omega_{ij}\wedge\omega_j=0\,, \quad \omega_{ij}+\omega_{ji}=0\,, \\ d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}\,, \\ \Omega_{ij}=-1/2\sum R_{ijkl}\omega_k\wedge\omega_l\,, \end{cases}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. The second fundamental form α and the mean curvature H of M are denoted by

$$\alpha=-\sum h_{ij}\omega_i\omega_j e_0\,, \quad \text{and} \quad nH=\sum h_{ii}=h\,,$$

respectively. Since α is symmetric tensor,

$$h_{ij}=h_{ji}\,.$$

If we think about hypersurfaces with constant mean curvature H, we may assume that H is non-negative.

The Gauss equation, the Codazzi equation and the Ricci formulae for the second fundamental form and its covariant derivatives are given by

$$(2.2)\qquad R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})-(h_{il}h_{jk}-h_{ik}h_{jl})\,,$$

$$(2.3)\qquad h_{ijk}-h_{ikj}=0\,,$$

$$(2.4)\qquad h_{ijkl}-h_{ijlk}=-\sum h_{mj}R_{mikl}-\sum h_{im}R_{mjkl}\,,$$

$$(2.5)\qquad h_{ijklm}-h_{ijkml}=-\sum h_{tjk}R_{tilm}-\sum h_{itk}R_{tjlm}-\sum h_{ijt}R_{tklm}\,,$$

where h_{ijk}, h_{ijkl} and h_{ijklm} denote the components of the covariant derivatives $\nabla\alpha$, $\nabla\nabla\alpha$ and $\nabla^3\alpha$ of α, respectively.

The components of the Ricci curvature *Ric* and the scalar curvature r are given by

$$(2.6)\qquad R_{ij}=(n-1)c\delta_{ij}-hh_{ij}+\sum_k h_{ik}h_{kj}\,,$$

$$(2.7)\qquad r=n(n-1)c-h^2+\sum_{i,j} h_{ij}{}^2\,.$$

Now we compute some local formulae under the assumption that the mean curvature of M is constant. For arbitrary fixed point p in M we choose a local frame field $e_1, \cdots, e_n$ such that

$$h_{ij}=\lambda_i\delta_{ij}\,.$$

We define functions S and f_k as follows:

$$S=|\alpha|^2=\sum\lambda_i^2\,, \qquad f_k=\sum\lambda_j^k\,.$$

The Laplacians of these functions and $|\nabla\alpha|^2$ are calculated by using suitably the equations (2.1)-(2.5).

Then we have the following equations:

(2.8)
$$\frac{1}{2}\Delta S=|\nabla\alpha|^2+S(S+nc)-hf_3-ch^2,$$

(2.9)
$$\frac{1}{2}\Delta|\nabla\alpha|^2=|\nabla\nabla\alpha|^2+\{S+(2n+3)c\}|\nabla\alpha|^2+\frac{3}{2}|\nabla S|^2$$
$$-3h\sum\lambda_i h_{ijk}{}^2+3[\sum\lambda_i{}^2 h_{ijk}{}^2-2\sum\lambda_i\lambda_j h_{ijk}{}^2],$$

(2.10)
$$\frac{1}{3}\Delta f_3=-hf_4+(S+nc)f_3-chS+2\sum\lambda_i h_{ijk}{}^2,$$

(2.11)
$$\frac{1}{4}\Delta f_4=-hf_5+(S+nc)f_4-chf_3+2\sum\lambda_i{}^2 h_{ijk}{}^2+\sum\lambda_i\lambda_j h_{ijk}{}^2.$$

Next we only consider the case that $n=3$. In this case, the functions f_4 and f_5 are described by f_3 as follows:

(2.12)
$$f_4=\frac{1}{6}h^4+\frac{4}{3}hf_3+\frac{1}{2}S^2-h^2S,$$

(2.13)
$$f_5=\frac{5}{6}(S+h^2)f_3+\frac{1}{6}h^5-\frac{5}{6}h^3S.$$

Now we define functions μ_i $(i=1, 2, 3)$ as $\mu_i=\lambda_i-H$. So we have

(2.14)
$$\mu_1+\mu_2+\mu_3=0,$$

(2.15)
$$(\mu_1)^2+(\mu_2)^2+(\mu_3)^2=S-\frac{h^2}{3},$$

(2.16)
$$B_3\equiv(\mu_1)^3+(\mu_2)^3+(\mu_3)^3=f_3-hS+\frac{2}{9}h^3.$$

Next, assuming that S is constant, we get the following useful equations.

PROPOSITION 2.1. *Let M be a 3-dimensional space-like hypersurface in a Lorentz space form $M_1^4(c)$ with constant mean curvature $H=h/3$. If S is constant, then we have*

(2.17)
$$|\nabla\alpha|^2=hB_3-S^2+(h^2-3c)S+ch^2-\frac{2}{9}h^4,$$

(2.18)
$$|\nabla\nabla\alpha|^2=-\frac{h}{2}\Delta B_3+\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2 h_{ijk}{}^2$$
$$+11h\sum\mu_i h_{ijk}{}^2+3(S-S_0)(S-S_-)(S-S_+).$$

Here,

$$S_0=\frac{h^2}{3} \quad \textit{and}$$

$$S_{\pm}=-3c+\frac{3}{4}h^2\pm\frac{1}{4}\sqrt{h^4-8ch^2}=\frac{h^2}{3}+\frac{3}{8}\left(\frac{h}{3}\pm\sqrt{h^2-8c}\right)^2.$$

Proof. It follows from $\Delta S=0$ that the equation (2.8) implies

$$|\nabla\alpha|^2=hf_3-S(S+3c)+ch^2. \tag{2.19}$$

From this equation (2.19) combined with (2.16), we get the equation (2.17).

Also it follows from $\nabla S=0$ that the equation (2.9) implies

$$|\nabla\nabla\alpha|^2=\frac{1}{2}\Delta|\nabla\alpha|^2-(S+9c)|\nabla\alpha|^2+3hA-3(B-2C), \tag{2.20}$$

where $A=\sum\lambda_i h_{ijk}{}^2$, $B=\sum\lambda_i{}^2 h_{ijk}{}^2$ and $C=\sum\lambda_i\lambda_j h_{ijk}{}^2$.

First we remark that replacing λ_i in the functions A, B and C with μ_i implies

$$A=\sum\mu_i h_{ijk}{}^2+\frac{h}{3}|\nabla\alpha|^2, \tag{2.21}$$

$$B+2C=\frac{1}{3}\sum(\mu_i+\mu_j+\mu_k)^2 h_{ijk}{}^2+2h\sum\mu_i h_{ijk}{}^2+\frac{h^2}{3}|\nabla\alpha|^2. \tag{2.22}$$

On the other hand, from the relations (2.10) and (2.12), we can describe the function A with f_3:

$$6A=\Delta f_3-3\left(S-\frac{4}{3}h^2+3c\right)f_3+\frac{3}{2}hS^2+3h(c-h^2)S+\frac{1}{2}h^5.$$

From this equation combined with (2.19), we get

$$2hA=\frac{1}{3}\Delta|\nabla\alpha|^2-\frac{1}{3}(3S-4h^2+9c)|\nabla\alpha|^2-(S-S_0)(S-S_-)(S-S_+). \tag{2.23}$$

Also it follows from the equation (2.11) combined with (2.12), (2.13) and (2.19) that we have

$$\begin{aligned}2B+C&=\frac{1}{4}\Delta f_4+hf_5-(S+3c)f_4+chf_3\\&=\frac{1}{3}h\Delta f_3-\frac{1}{6}(3S-5h^2+18c)hf_3\\&\quad-\frac{1}{6}[3(S+3c)(S^2-2h^2S+h^4/3)+h^4(5S-h^2)]\\&=\frac{1}{3}\Delta|\nabla\alpha|^2-\frac{1}{6}(3S-5h^2+18c)|\nabla\alpha|^2-(S-S_0)(S-S_-)(S-S_+).\end{aligned} \tag{2.24}$$

Then, from the equations (2.22) and (2.24), we get

(2.25) $$3(B-2C)=\frac{4}{3}\Delta|\nabla\alpha|^2-\Big(2S-\frac{5}{3}h^2+12c\Big)|\nabla\alpha|^2$$
$$-10h\sum\mu_i h_{ijk}{}^2-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2 h_{ijk}{}^2-4(S-S_0)(S-S_-)(S-S_+).$$

At last, computing $|\nabla\nabla\alpha|^2$ from (2.20) combined with (2.21), (2.23) and (2.25), we have proved the equation (2.18). ■

The following generalized maximum principle due to Omori [11] and Yau [14] will play a major part in this paper.

THEOREM 2.1 (cf. Omori [11] and Yau [14]). *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from below on M, then for any $\varepsilon>0$ there exists a point p in M such that*

$$F(p)<\inf F+\varepsilon,\quad |\nabla F|(p)<\varepsilon,\quad \Delta F(p)>-\varepsilon.$$

3. Proof of Theorems.

In this section, we consider that M is a 3-dimensional complete space-like hypersurface with constant mean curvature and constant scalar curvature in a Lorentz space form $M_1^4(c)$ and we prove the theorems stated in the introduction.

For that purpose, we need the proposition below.

PROPOSITION 3.1. *Let M be a complete space-like hypersurface with constant mean curvature $H=h/3$ and constant scalar curvature. Also we define S and B_3 as in Section 2. Then S is constant, and the function B_3 satisfies*

(3.1) $$|B_3|\leqq\frac{1}{\sqrt{6}}\Big(S-\frac{h^2}{3}\Big)^{3/2}.$$

When M is not totally umbilical, $B_3\equiv\frac{1}{\sqrt{6}}\Big(S-\frac{h^2}{3}\Big)^{3/2}$ if and only if M is congruent to a hyperbolic cylinder $\boldsymbol{H}^1(c_1)\times M^2(c_2)$, and also, $B_3\equiv-\frac{1}{\sqrt{6}}\Big(S-\frac{h^2}{3}\Big)^{3/2}$ if and only if M is congruent to $\boldsymbol{H}^2(c_1)\times M^1(c_2)$.

Here we remark that if M has an umbilical point then M is totally umbilicalin in this case.

Proof. According to (2.7), we know that the scalar curvature r is constant if and only if S is constant.

The inequality (3.1) follows from (2.14) and (2.15) by solving the problem for the conditional extremum (cf. Okumura [10]), and the equality holds if and only if $(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_2-\mu_3)=0$. Then the equality $|B_3|=\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$ means that

$$\mu_1=\mu_2=\pm\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{1/2} \quad \text{and} \quad \mu_3=\mp\frac{2}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{1/2}$$

except the order. So $|B_3|\equiv\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}\neq 0$ if and only if M has two distinct constant principal curvatures. Therefore this proposition proved by the use of a theorem due to Abe, Koike and Yamaguchi [1]. ∎

Here we describe our main theorem and its proof. This theorem characterizes the hyperbolic cylinder $\boldsymbol{H}^1(c_1)\times M^2(c_2)$ in $M_1^4(c)$ when the constant mean curvature $h/3$ of complete space-like hypersurfaces in $M_1^4(c)$ satisfies $h^2\geqq 8c$. As explained in the introduction, it is known that complete space-like hypersurfaces with constant mean curvature $h/3$ in $M_1^4(c)$ are totally umbilical if $h^2<8c$ (cf. Akutagawa [3]). Throughout this section, we assume that $h^2\geqq 8c$. Then we can define real numbers S_- and S_+ by

$$S_\pm=\frac{h^2}{3}+\frac{3}{8}\left(\frac{h}{3}\pm\sqrt{h^2-8c}\right)^2.$$

THEOREM 3.1. *The hyperbolic cylinder is the only complete space-like hypersurface with constant mean curvature $h/3$ and constant scalar curvature in $M_1^4(c)$, whose squared norm S of the second fundamental form is greater than* $S_-=\frac{h^2}{3}+\frac{3}{h}\left(\frac{h}{3}-\sqrt{h^2-8c}\right)^2$.

Proof. Since M is not totally umbilical under the assumption $S>S_-(\geqq S_0)$, by virtue of Proposition 3.1, it is sufficient to show that $B_3\equiv\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$. So we shall prove some contradictions when we assume that $\inf B_3<\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$.

The function B_3 on M is smooth and bounded. Also the formula (2.6) implies that the Ricci curvature of M is bounded from below by $2c-h^2/4$. These means that Theorem 2.1 can be applied to the function B_3. Let ε be any positive number that is small enough to be less than $\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}-\inf B_3(>0)$. There exists a point p in M, at which B_3 satisfies the following:

$$B_3(p)<\inf B_3+\varepsilon<\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}, \tag{3.2}$$

$$|\nabla B_3|(p)<\varepsilon, \qquad \Delta B_3(p)>-\varepsilon. \tag{3.3}$$

Our proof is divided into the following two cases:

(I) The case of that $\inf B_3=-\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$,

(II) The case of that $\inf B_3>-\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$.

In the case of (I), it follows from (2.17) and (3.2) that

$$|\nabla\alpha|^2(p)<K+h\varepsilon,$$

where

$$K=-\left(S-\frac{h^2}{3}\right)\left[\sqrt{S-\frac{h^2}{3}}+\frac{3}{2\sqrt{6}}\left(\frac{h}{3}-\sqrt{h^2-8c}\right)\right]$$
$$\times\left[\sqrt{S-\frac{h^2}{3}}+\frac{3}{2\sqrt{6}}\left(\frac{h}{3}+\sqrt{h^2-8c}\right)\right].$$

Since $S>S_-$, we have $K<0$. Accordingly, for an enough small positive number ε, there exists a point p in M such that $|\nabla\alpha|^2(p)<0$. This is a contradiction.

Next, we consider the case of (II). In this case, we make use of the equation (2.18) in Proposition 2.1.

Since h and S are constant, at any point q in M, we have

$$h_{11k}+h_{22k}+h_{33k}=0, \tag{3.4}$$

$$\mu_1 h_{11k}+\mu_2 h_{22k}+\mu_3 h_{33k}=0, \tag{3.5}$$

where $k=1, 2, 3$. Also we define the numbers $\delta_k(q)$ $(k=1, 2, 3)$ by

$$[(\mu_1)^2 h_{11k}+(\mu_2)^2 h_{22k}+(\mu_3)^2 h_{33k}](q)=\delta_k(q). \tag{3.6}$$

From (3.2) and the assumption of (II), we have $|B_3|(p)\lneqq\frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}$. Then the proof of Proposition 3.1 asserts that $\mu_1(p)$, $\mu_2(p)$ and $\mu_3(p)$ are distinct number with each other. So, when the equations (3.4), (3.5) and (3.6) at p are considered as a system of equations with 3 unknowns $h_{11k}(p)$, $h_{22k}(p)$ and $h_{33k}(p)$, they can be solved uniquely:

$$h_{iik}(p)=a^i(p)\delta_k(p) \qquad (i, k=1, 2, 3).$$

Since (2.15) means that the coefficients of the system of equations are bounded, there is a positive number a such that $|a^i(p)|<a (i=1, 2, 3)$ for any point p in M satisfying (3.2) and (3.3). Furthermore, $|\nabla B_3|(p)<\varepsilon$ in (3.3) implies that $|\delta_k(p)|<\varepsilon$, and also,

$$h_{iik}(p)<a\varepsilon \qquad (i, k=1, 2, 3). \tag{3.7}$$

Accordingly, from this (3.7) and (2.14), we have positive constant numbers K_1

and K_2 such that

$$[\sum \mu_i h_{ijk}^2](p) < K_1 \varepsilon^2, \tag{3.8}$$

$$[\sum (\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2](p) < K_2 \varepsilon^2.$$

Then it follows from (2.18) combined with (3.3) and (3.8) that

$$|\nabla\nabla\alpha|^2(p) < 3(S-S_0)(S-S_-)(S-S_+) + \frac{h}{2}\varepsilon + \left(11hK_1 + \frac{5}{3}K_2\right)\varepsilon^2.$$

Since it is known that $S_0 \leqq S \leqq S_+$ by Ki, Kim and Nakagawa [9] and $S > S_-$ from the assumption, $(S-S_0)(S-S_-)(S-S_+)$ is a negative constant number. Accordingly, for an enough small positive number ε, there exists a point p in M such that $|\nabla\nabla\alpha|^2(p) < 0$. This is a contradiction, too.

Hence, we get that $B_3 \equiv \frac{1}{\sqrt{6}}\left(S - \frac{h^2}{3}\right)^{3/2}$, and complete the proof of Theorem 3.1. ■

In a general case, we do not know whether or not there are examples which the squared norm S of the second fundamental form is in the region $S_0 < S \leqq S_-$. However, when $c > 0$ and $h^2 \leqq 9c$, we have a nonexistence theorem due to the second author and Nakagawa [6]: Let M be a complete space-like hypersurface with constant mean curvature $H = h/3$ in $\boldsymbol{S}_1^4(c)$. If $h^2 \leqq 9c$ and $\sup S < S_-$, then M is totally umbilical. Furthermore, when S is constant, we can prove nonexistence in the case of which $S \leqq S_-$.

PROPOSITION 3.2. *There are no complete space-like hypersurfaces with constant mean curvature $H = h/3$ and constant scalar curvature in $\boldsymbol{S}_1^4(c)$, on which the squared norm S of second fundamental form satisfies that*

(1) $S_0 < S \leqq S_-$ *if* $8c < h^2 \leqq 9c$,

(2) $S_0 < S < S_-(=S_+)$ *if* $h^2 = 8c$.

Proof. Let M be a complete space-like hypersurface in $\boldsymbol{S}_1^4(c)$ satisfying the assumption of Proposition 3.2.

It follows from (2.17) combined with (3.1) that we get

$$\begin{aligned} |\nabla\alpha|^2 &\leqq -S^2 + (h^2 - 3c)S + ch^2 - \frac{2}{9}h^4 + \frac{h}{\sqrt{6}}\left(S - \frac{h^2}{3}\right)^{3/2} \\ &= -\left(S - \frac{h^2}{3}\right)\left[\sqrt{S - \frac{h^2}{3}} - \frac{3}{2\sqrt{6}}\left(\frac{h}{3} - \sqrt{h^2 - 8c}\right)\right] \\ &\quad \times \left[\sqrt{S - \frac{h^2}{3}} - \frac{3}{2\sqrt{6}}\left(\frac{h}{3} + \sqrt{h^2 - 8c}\right)\right]. \end{aligned} \tag{3.9}$$

Hence, we remark that if $h^2 \leqq 9c$ then the condition $S \leqq S_-$ implies

$$\sqrt{S-\frac{h^2}{3}} \leqq \frac{3}{2\sqrt{6}}\left(\frac{h}{3}-\sqrt{h^2-8c}\right).$$

Then the fact $S_0 \leqq S \leqq S_+$ means that the right side of the above inequality (3.9) is non-positive. Accordingly, in the inequality (3.9), the equality has to hold. So we get

$$B_3 \equiv \frac{1}{\sqrt{6}}\left(S-\frac{h^2}{3}\right)^{3/2}.$$

By virtue of Proposition 3.1, this means that M is either a totally umbilical hypersurface or a hyperbolic cylinder. However, the assumption $S_0 < S < S_+$ implies that M is not these standard models by the theorem due to Ki, Kim and Nakagawa [9]. ∎

Combined with Theorem 3.1 and Proposition 3.2, we can completely classify complete space-like hypersurfaces with constant mean curvature $H=h/3$ and constant scalar curvature in $S_1^4(c)$ if $h^2 \leqq 9c$.

THEOREM 3.2. *Let M be a complete space-like hypersurface with constant mean curvature $H=h/3$ and constant scalar curvature in $S_1^4(c)$. If $h^2 \leqq 9c$, then M is congruent to R^3, $S^3(c_1)$ or a hyperbolic cylinder $H^1(c_1) \times S^2(c_2)$.*

Proof. It follows from Theorem 3.1 and Proposition 3.2 that M must be a totally umbilical hypersurface or a hyperbolic cylinder. In $S_1^4(c)$, a complete totally umbilical space-like hypersurface is congruent to $H^3(c_2)$, R^3 or $S^3(c_1)$. However we can easily check that the hypersurfaces in the form of $H^3(c_2)$ do not satisfy the assumption $h^2 \leqq 9c$. ∎

4. Hypersurfaces with Ricci curvature bounded from above.

In this section we study that M is a 3-dimensional complete space-like hypersurface in a Lorentz space form $M_1^4(c)$ with constant mean curvature and with Ricci curvature bounded from above.

THEOREM 4.1. *The totally umbilical hypersurface $S^3(c_1)$ in a de-Sitter space $S_1^4(c_2)$ $(c_2 > c_1 > 0)$ is the only complete space-like hypersurface in a Lorentz space form $M_1^4(c)$ with constant mean curvature H whose Ricci curvature is bounded from above by some number δ less than $3(c-H^2)$.*

Proof. Let M be a 3-dimensional complete space-like hypersurface with constant mean curvature H in a Lorentz space form $M_1^4(c)$. Assume that the Ricci curvature is bounded from above by some number δ less than $3(c-H^2)$.

From (2.8) we have

$$\frac{1}{2}\Delta S=|\nabla\alpha|^2+S(S+3c)-hf_3-ch^2$$

$$=|\nabla\alpha|^2+\frac{1}{2}\sum(c-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2$$

$$=|\nabla\alpha|^2+\frac{1}{2}\sum R_{ijji}(\lambda_i-\lambda_j)^2.$$

On the other hand, since M is a 3-dimensional submanifold, its Weyl conformal curvature tensor vanishes, i.e.,

$$R_{ijkl}=R_{il}\delta_{jk}-R_{ik}\delta_{jl}+R_{jk}\delta_{il}-R_{jl}\delta_{ik}-\frac{r}{2}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}).$$

Hence we get

$$R_{ijji}=R_{ii}+R_{jj}-\frac{r}{2}$$

for any distinct indices i and j. By $R_{11}+R_{22}+R_{33}=r$, we have

$$R_{ijji}=-R_{kk}+\frac{r}{2}$$

for any distinct indices. Hence we get

$$\frac{1}{2}\Delta S=|\nabla\alpha|^2+\frac{1}{2}\sum\left(\frac{r}{2}-R_{kk}\right)(\lambda_i-\lambda_j)^2$$

$$\geqq\frac{1}{4}(r-2\delta)\sum(\lambda_i-\lambda_j)^2$$

$$=\frac{1}{2}(3S-h^2)(6c-h^2+S-2\delta)$$

$$\geqq\frac{1}{2}(3S-h^2)\left(6c-\frac{2}{3}h^2-2\delta\right).$$

Applying Theorem 2.1 to the function $F=-S$, we have

$$0\geqq\frac{1}{2}(3\sup S-h^2)\left(6c-\frac{2}{3}h^2-2\delta\right).$$

Hence $\sup S\leqq(1/3)h^2$. Thus M is totally umbilical.

On the other hand, the Ricci curvature tensor of a totally umbilical hypersurface $M^3(c')$ in $M_1^4(c)$ is given by $R_{ij}=2c'\delta_{ij}=2(c-H^2)\delta_{ij}$. In order for the totally umbilical hypersurface to satisfy the assumption, $c'=c-H^2$ must be positive.

We have completed the proof of Theorem 4.1.

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